Matching calculations for semi-leptonic decays and ε_K

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Standard Model (SM) & New Physics (NP)

- Shortcomings of the SM: Origin of Flavour, Origin & stability of electroweak scale; Strong CP problem; Neutrino masses; Dark matter; Origin of the matter-antimatter asymmetry; Dark energy; Quantum gravity
- SM: Yukawa only source of flavour & CP violation
  - Predictive but not revealing
- NP competes with SM $\frac{X_{SM}}{v_{ew}^2} \leftrightarrow \frac{X_{NP}}{\Lambda_{NP}^2}$
- Search for NP, where SM is suppressed ($X_{SM} \ll 1$)
  - Precision in X_{SM} increases reach in Λ_{NP}

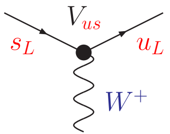
SM: Higgs only source of Flavour Violation

- $\mathcal{L}_{SM,Yukawa} \supset -\bar{d} Y_d Q_L H^\dagger - \bar{u} Y_u Q_L \cdot H$
- $Y_u \propto \mathrm{diag}(m_u, m_c, m_t)$
- $Y_d \propto \mathrm{diag}(m_d, m_s, m_b) V_{CKM}^\dagger$
- $V_{\mathrm{CKM}} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho+i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$
- $\lambda \sim 0.2$, A ~ 0.8, $\rho \sim 0.15$, $\eta \sim 0.35$
- CKM: misalignment of $u_L \leftrightarrow d_L$ mass and gauge states

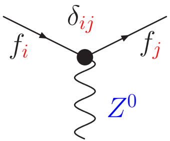
Precision Tests: Flavour Example

- Tree level: only $W^{\pm}$ test misalignment

Mass $\neq$ flavour eigenstates



SM: Only charged currents change the flavour ($\propto V_{us}$)



SM: Neutral currents do not change the flavour (i=j) at tree-level

- Tree: $\Delta_{CKM} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1$

- Loop: FCNCs & CP violation in mixing and decays

Matrix elements & EFT

Integrating out W: Effective Field Theory

- $\mathcal{L}_{\mathrm{Fermi}} \propto G_\mu \left(\bar{\mu}\gamma^\mu P_L e\right)\left(\bar{\nu}\gamma_\mu P_L \nu\right) + h.c.$

- $\mathcal{L}_{\mathrm{Fermi}} \propto G_\mu C V_{ud_i} \left(\bar{d}_i\gamma^\mu P_L u\right)\left(\bar{\nu}\gamma_\mu P_L \ell\right) + h.c.$

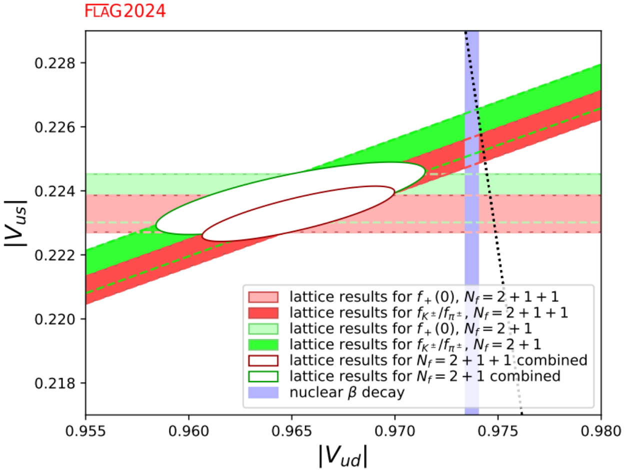
- We need to evaluate $\langle \ell\nu m| \left(\bar{d}_i\gamma^\mu P_L u\right)\left(\bar{\nu}\gamma_\mu P_L \ell\right)|M\rangle$

    - SM: $\Delta_{CKM} = 0$ we can test C, i.e. semi-leptonic/leptonic decays

    - Requires precise calculations of Matrix elements

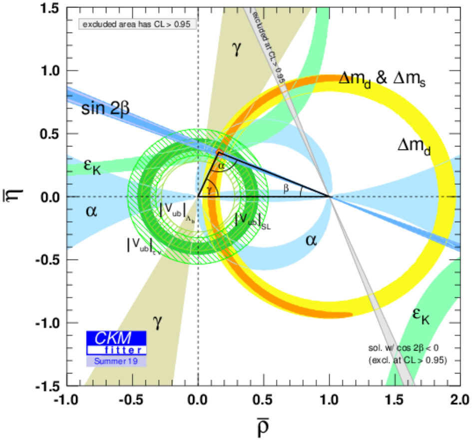
    - QED x QCD: Semi-leptonic operator renormalises

Lattice input FLAG 2024



- SMEFT constraints on $(\bar{L}\gamma_\mu L)(\bar{L}\gamma^\mu L)$, $(\bar{Q}\gamma_\mu \tau^I Q)(\bar{L}\gamma^\mu \tau^I L)$, ... at the multi TeV range

ε_K: CP violation in Kaons



- ε_K is highly suppressed in SM: SMEFT at 10^5 TeV
- Requires $\langle \overline{K_0} | (\overline{s}_L \gamma_\mu d_L) \otimes (\overline{s}_L \gamma^\mu d_L) | K_0 \rangle$

Content

- $\overline{MS}$ and Lattice schemes
- mass normalization
- V_{us} & V_{ud}
- ε_K and B_K

Non-perturbative & Continuum Input

- QCD has asymptotic freedom:
- Factorise short distance from long distance physics
- presence of UV divergences: Factorisation is scheme dependent
- a and $\epsilon = (4 - d)/2$ as Lattice / Continuum UV regulator
- Flavour physics: $\overline{\mathrm{MS}}$ scheme
- Need schemes, that can be implemented with both methods

Renorm non-singlet $O_\Gamma(x) = \bar{\psi}(x)\Gamma\psi(x)$

x-space

$$\lim_{a\to 0}\langle O^X_\Gamma(x) O^X_\Gamma(0)\rangle|_{x^2=x_0^2} = \langle O_\Gamma(x_0) O_\Gamma(0)\rangle^{\text{free}}_{\text{cont}}$$

- $O^X_\Gamma(x, x_0) = Z^X_\Gamma(x_0) O_\Gamma(x)$ and $a \ll x_0 \ll \Lambda^{-1}_{\text{QCD}}$

g. flow

$$Z_\chi \bar{\chi}(t,x)\Gamma\chi(t,x) \overset{t\to 0}{\sim} \zeta_1(t) O_\Gamma(x) + \mathcal{O}(t)$$

- Lattice: a $\to$ 0 then t $\to$ 0
- Continuum: modified feynman rules (exponential)

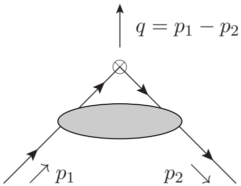
iMOM

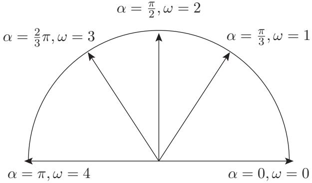
$$\Lambda_\Gamma = S^{-2}\int d^4x_1 d^4x_2 e^{i(p_1x_1 - p_2x_2)}\langle T\psi(x_1) O_\Gamma(0)\psi(x_2)\rangle$$

- Project Λ: $\lambda_{\Gamma,B}(p_1^2, p_2^2, (p_1 - p_2)^2) Z_q^{-1} Z_\Gamma \to \text{tree}$

Light Quark Masses: iMOM $\leftrightarrow \overline{\mathrm{MS}}$

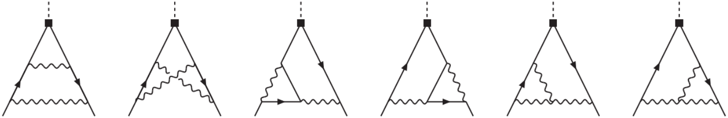
- $\Lambda_\Gamma = S^{-2} \int d^4x_1 d^4x_2 e^{i(p_1x_1 - p_2x_2)} \langle T\psi(x_1) O_\Gamma(0) \psi(x_2) \rangle$
- Ward Identity for non-singlet bilinear: $Z_P = Z_m^{-1}$
- fixed by: $\lambda_R(p_1^2, p_2^2, (p_1 - p_2)^2) = Z_q^{-1} Z_P \mathrm{tr}\left[\Lambda_P \gamma_5\right] = 12$
- For $p_i^2 = -\mu^2$ & $(p_1 - p_2)^2 = -\omega\mu^2$





Perturbation Theory

$$C_m^{(\gamma)} = Z_m^{\overline{\mathrm{MS}}}/Z_m^{(\gamma)}(\omega) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_m^{(\gamma,1)} + \frac{\alpha_s^2(\mu)}{(4\pi)^2} C_m^{(\gamma,2)} + \ldots$$



ω	$C_m^{(\gamma,1)}$	$C_m^{(\gamma,2)}$	$C_m^{(\gamma,3)}$
.5	-3.248	-89.07 + 7.571 N_f	
1	-1.979	-55.032 + 6.162 N_f	-2086 -362.6 N_f + 6.7220 N_f^2
2	-0.098	-6.829 + 4.072 N_f	
4	2.575	62.576 + 1.102 N_f	

NNLO result for $\omega = 0 \ldots 4$ 1004.3997
N^{3LO} result for $\omega = 1$ 2002.12758

Momentum Source propagator

- $G_x(p) = \sum_y D^{-1}(x,y)e^{ip\cdot(y-x)}$ Momentum source propagator
  (From solving $\sum_x D(y,x)\tilde{G}_x(p) = e^{ip\cdot y}$)

- Amputed Green's function for $O_\Gamma = \bar{\psi}\Gamma\psi$
  $\Pi_\Gamma = \langle G^{-1}(-p_2)\rangle \sum_x \langle G_x(-p_2)\Gamma G_x(p_1)\rangle\langle G^{-1}(p_1)\rangle$

- Renormalisation conditions on projected result

Lattice Interpolating Momentum calc Garron, MG, et.al. PRD23

Landau gauge fixed 2+1 domain-wall configuration

- 24^3: a^{-1} = 1.785(5)GeV and $Z_V = Z_A = 0.71651(46)$
- 32^3: a^{-1} = 2.383(9)GeV and $Z_V = Z_A = 0.74475(12)$
- $p_1 = (\mu{=}2\pi\, l/L,0,0,0)$ & $p_2 = \mu(\cos\alpha, \sin\alpha,0,0)$

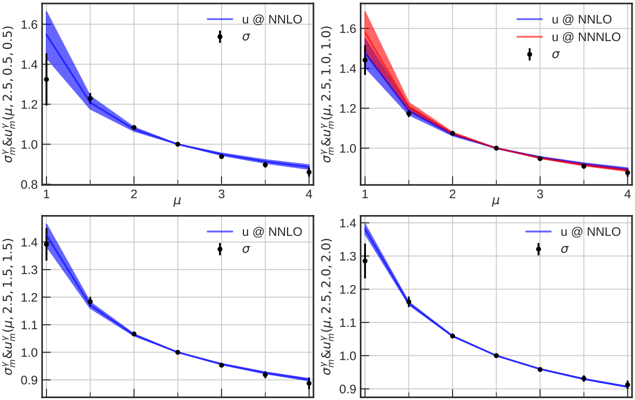
$$Z_q(\mu,\omega) = Z_V \lim_{m\to 0} [\lambda_V]_{\mathrm{IMOM}} \quad Z_m(\mu,\omega) = \frac{1}{Z_V} \lim_{m\to 0} \left[\frac{\lambda_S}{\lambda_V}\right]_{\mathrm{IMOM}}$$

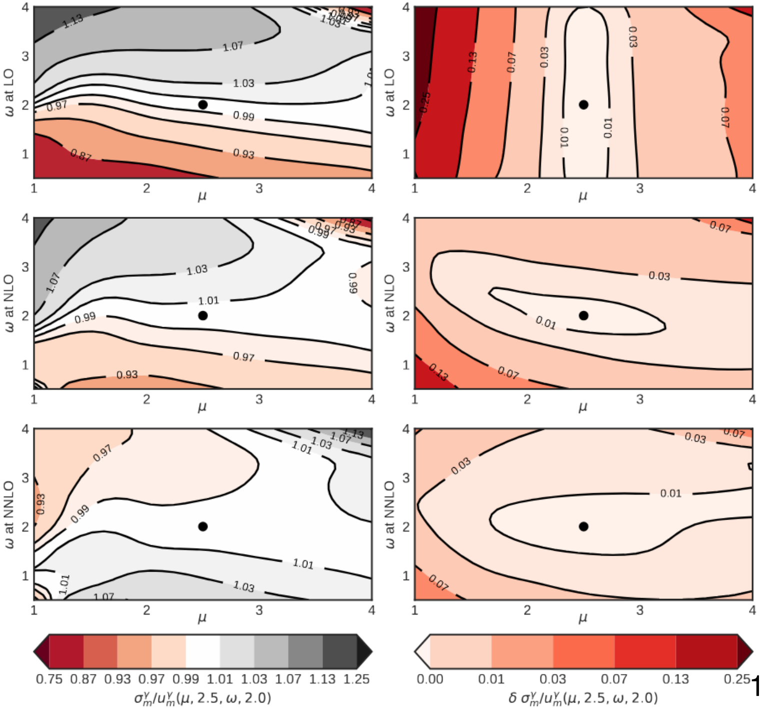
with projectors:

$$\lambda_V^{(\gamma_\mu)} = \frac{1}{48}\mathrm{Tr}[\gamma_\mu \Lambda_{V^\mu}] \qquad \lambda_V^{(\not{q})} = \frac{q^\mu}{12q^2}\mathrm{Tr}[\Lambda_{V^\mu}]$$

Lattice & Continuum

- Lattice: $\sigma_i^\gamma(\mu,\mu_0,\omega,\omega_0) = \lim_{a^2\to 0}\lim_{m\to 0}\dfrac{Z_i^\gamma(a,\mu,\omega)}{Z_i^\gamma(a,\mu_0,\omega_0)}$
- Continuum: $u_i^\gamma(\mu,\mu_0,\omega,\omega_0) = \lim_{\epsilon\to 0}\dfrac{Z_i^\gamma(\epsilon,\mu,\omega)}{Z_i^\gamma(\epsilon,\mu_0,\omega_0)}$ + RGE





ω at LO
ω at NLO
ω at NNLO
μ
0.75 0.87 0.93 0.97 0.99 1.01 1.03 1.07 1.13 1.25
$\sigma_m^\gamma/u_m^\gamma(\mu, 2.5, \omega, 2.0)$
0.00 0.01 0.03 0.07 0.13 0.25
$\delta\ \sigma_m^\gamma/u_m^\gamma(\mu, 2.5, \omega, 2.0)$

Charged Current decays

- $\mathrm{K}_{\ell 2}$ and $\mathrm{K}_{\ell 3}$ extraction of $\lambda = |V_{us}|$:

$$\Gamma(K^0 \to \pi^- \ell^+ \nu_\ell(\gamma)) = \frac{G_F^2 m_K^5}{128\pi^3} |V_{us}|^2 S_{EW} |f_+^{K^0\pi^-}(0)|^2 I_{K^0\ell}^{(0)} (1 + \delta_{EM}^{K^0\ell} + \delta_{SU(2)}^{K^0\pi^-})$$

- QED: χPT Seng et.al.'2019,Cirigliano et.al.'23
  and Lattice Carrasco et.al.'15,DiCarlo et.al.'19
- $|V_{ud}|$, extracted from nuclear β decays Hardy,Towner'20,
  - $|V_{ud}|\tau_n(1 + 3\lambda^2)(1 + \Delta_f)(1 + \Delta_R) = 5283.321(5)s$
  - λ, Δ_f & Δ_R ratio of (axial-)vector coupling, phase space & radiative corrections
- EW corrections in W-Mass scheme [Marciano, Sirlin]
  - No scale separation and α^2 log and $1/s_w{}^2$ effects

(Effective) Interaction

- $\mathcal{H}(x) = 4\,\frac{G_F}{\sqrt{2}} C_O\; V^*_{us} \left(\bar{d}(x)\gamma^\mu P_L u(x)\right)\left(\bar{\nu}_l(x)\gamma_\mu P_L l(x)\right)$
- $O(x) = \left(\bar{d}(x)\gamma^\mu P_L u(x)\right)\left(\bar{\nu}_l(x)\gamma_\mu P_L l(x)\right)$
- SD in W-Mass scheme: $\frac{1}{k^2} \to \frac{1}{k^2 - M_W^2} - \frac{M_W^2}{k^2 - M_W^2}\frac{1}{k^2} = \gamma_> - \gamma_<$
  - UV poles $\to$ Absorbed into G_F from muon decay
  - Combining with SU(3) current algebra $\to$ QCD corrections to S_{EW}
  - No scale separation and α^2 log and $1/s_w{}^2$ effects
- EFT: scale separation, $\mathcal{O}(\alpha^2)$, match to Lattice or χ PT

EFT Approach 2209.05289,EPS-HEP2023,2510.27648

- Sum higher α_s logs: $\mu \frac{d}{d\mu} C(\mu) = \gamma C(\mu)$
- $\gamma = \frac{\alpha(\mu)}{4\pi}\gamma_e + \frac{\alpha^2(\mu)}{(4\pi)^2}\gamma_{ee} + \frac{\alpha}{4\pi}\frac{\alpha_s(\mu)}{4\pi}\left(\gamma_{es} + \frac{\alpha_s(\mu)}{4\pi}\gamma_{ess}\right)$
- Solution: $C(\mu) = J(\mu)u(\mu)u^{-1}(\mu_0)J^{-1}(\mu_0)C(\mu_0)$
  - $u(\mu) = \alpha(\mu)^{\frac{\gamma_e}{2\beta_0}}\alpha_s(\mu)^{-\frac{\alpha}{4\pi}\frac{\gamma_{es}}{2\beta_{0,s}}} \rightarrow (\alpha \ln)^n$ and $\alpha\,(\alpha_s \ln)^n$
  - $J(\mu) = 1 + \frac{\alpha(\mu)}{4\pi}\delta J_e + \frac{\alpha}{4\pi}\frac{\alpha_s(\mu)}{4\pi}\delta J_s$
- Match to non-perturbative scheme
  - Check scale and scheme cancellation

Scheme independence

- Scheme dependence through use of naive dimensional regularisation and choice of
  - $E = (\bar{d}\gamma^\mu\gamma^\nu\gamma^\lambda P_L u)(\bar{\nu}_l\gamma_\mu\gamma_\nu\gamma_\lambda P_L l) - 4(4 - a\epsilon - b\epsilon^2)O$
  - $O = (\bar{d}\gamma^\mu P_L u)(\bar{\nu}_l\gamma_\mu P_L l)$
- Scheme independent quantities
  - Wilson coefficient: $\hat{C} = u^{-1}(\mu)J^{-1}(\mu)C(\mu)$
  - Flavour decoupling: $\hat{M}_{f\downarrow} = u_{f-1}^{-1}(\mu)J_{f-1}^{-1}(\mu)J_f(\mu)u_f(\mu)$
  - Matrix element: $\hat{m} = m(\mu)J(\mu)u(\mu)$

Anomalous dimensions & SM matching

- 3 loop anomalous dimensions:

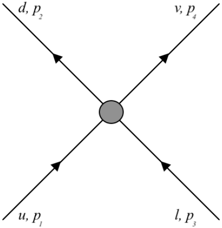
$$\gamma_e = -4, \quad \gamma_{es} = 4, \quad \gamma_{ess} = a\left(\frac{308}{9} - \frac{56f}{27}\right) + \frac{10f}{9} - 3$$

- scheme (a) dependence cancels in

$$\hat{C}_{es} = \frac{\alpha\alpha_s^{(5)}(\mu)}{(4\pi)^2 u_{(5)}(\mu)}\left[\frac{1972}{529} + 2\log\left(\frac{\mu^2}{M_Z^2}\right) - \frac{4}{s_W^2} - \frac{2c_W^2}{s_W^4}\log\left(\frac{M_W^2}{M_Z^2}\right)\right]$$

- $\hat{C} = u_{(5)}^{-1}(\mu) + \hat{C}_e(\mu) + \hat{C}_{es}(\mu)$ is μ independent
- Perform matching to Lattice or χPT
  - Has QED running and m_γ logs cancel
  - Has no QCD running
  - For V_{ud}, long distance absorbed into $\overline{\Box}_{\gamma W}^{V}$

Lattice Renormalisation



- RI-MOM and -SMOM schemes
- MOM: only 1 momentum p
- Pure QCD only couple to quarks

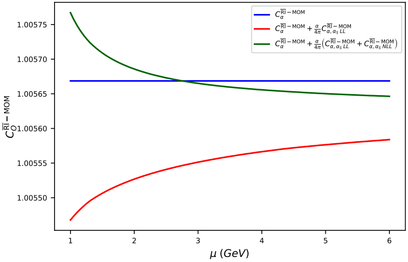
- $\Lambda^b = (F_1(p)\gamma^\mu P_L + F_2(p)p^\mu \not{p}/p^2 P_L) \otimes \gamma_\mu P_L$
- Ward identity $F_1 \leftrightarrow$ field renormalisation

Lattice Renormalisation

- $\sigma^A \equiv \frac{1}{4\,p^2}\mathrm{Tr}(S_A^{-1}(p)\not{p}) \overset{\text{A=RI}}{=} 1, \quad \lambda^A \equiv \Lambda^A_{\alpha\beta\gamma\delta}\mathcal{P}^{\alpha\beta\gamma\delta}$
- Choose $\mathcal{P} = -\frac{1}{12\,p^2}(\not{p}P_R \otimes \not{p}P_R + p^2/2\gamma^\nu P_R \otimes \gamma_\nu P_R)$
  - Projects out F_1(p) → no pure QCD corrections (F_1 canceled by Ward Identity) [MG, Moretti et.al. JHEP23]
- Lattice used $\gamma^\nu P_R \otimes \gamma_\nu P_R$
  - might be OK, where QED was treated perturbatively
  - problematic for full QED × QCD

$\overline{RI}$ and $\overline{MS}$ Wilson coefficients

- 2-loop scheme conversion factor [MG, Moretti et.al. JHEP23] plus 2-loop EW × QCD matching and 3-loop RGE
  $C^{RI-MOM} = \hat{m}_{\overline{\mathrm{MS}}\to RI-MOM}\hat{m}_{4\to 3}\hat{m}_{5\to 4}\hat{C}$
- In combination scale and scheme independent
- Small impact for V_{us}, but larger for V_{ud} for $\Box_{\gamma W}$ + NLO



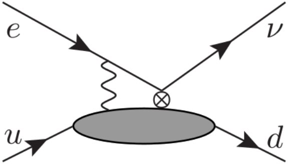
Neutron Decay in HBχPT

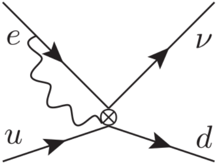
$$\mathcal{H}_{\not\pi} = \sqrt{2} G_F V_{ud} \bar{e} \gamma_\mu P_L \nu_e \bar{N}_v \left(g_V v^\mu - 2 g_A S^\mu \right) \tau^+ N_v$$

- HBChPT: proton neutron vector coupling g_V is product of Low Energy Constants and high-scale Wilson coefficient $g_V = \hat{m}_{\mu_0}(\mu_\chi) \hat{M}_{4\downarrow} \hat{M}_{5\downarrow} \hat{C}^{(5)}$
- Introduce scheme that matches to $\overline{\Box}(\mu_0) = \Box_{\gamma W}^{V<}(Q_0^2) + \frac{\alpha}{8\pi} \left[1 - \frac{\alpha_s(\mu_0)}{\pi} \right] \log \frac{\mu_0^2}{Q_0^2}$
- $\hat{m}_{\mu_0}(\mu_\chi) = u_3(\mu) \left(1 + \frac{\alpha^{(3)}(\mu)}{4\pi} f_e(\mu, \mu_\chi, \mu_0) + \frac{\alpha \alpha_s^{(3)}(\mu)}{(4\pi)^2} f_{es}(\mu, \mu_0) + \overline{\Box}(\mu_0) \right)$
  α terms in agreement with 2306.03138

Beyond LL α_s MG, Moretti, Jager 2510.27648

- Isoscalar OP: $T_0^{\mu\nu}(q^2) = \int \mathrm{d}^d x\, e^{iq\cdot x} \mathrm{T}\left[J_W^\mu(x) J_{V,0}^\nu(0)\right]$
- μ_0^2 scheme: Subtract OPE with $-q^2 \to -q^2 + \mu_0^2$
  - Proper dim-reg recovers 4-D OPE and Nachtmann moment
- $\alpha\, \alpha_s$ scheme change $\leftrightarrow$ 3-loop ADM





- Matrix element gives $\overline{\Box}(\mu_0) \leftarrow$ Lattice

OPE subtraction

- OPE subtraction in d-dimensions

$$T^{\mu\nu}_{S,0}(q^2) = \frac{Q_+ q_\rho}{2(q^2-\mu^2)} \left\{ \left[1 - \frac{\alpha_s}{\pi} \left(\frac{\mu^2}{\mu_0^2 - q^2} \right)^{\epsilon} \right] o^{\mu\rho\nu} + \left[1 - \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{\mu_0^2 - q^2} \right)^{\epsilon} \left(\frac{28}{3} + 12\epsilon \right) \right] e^{\mu\rho\nu} \right\}$$

- Recovers 4-d OPE for $o^{\mu\rho\nu} = 2i\varepsilon^{\mu\nu\rho\sigma}\bar{d}\gamma^\sigma P_L u$

- with evanescent

$$e^{\mu\rho\nu} = \bar{d}(\gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\rho\gamma^\mu)P_L u - (1 + \frac{3\alpha}{3\pi})o^{\mu\rho\nu}$$

- Scheme change $m_{\mu_0}(\mu) = (Z^{s.c.}_{O,i})^{-1}$ from $Z^{s.c.}_{O,i} Z_{i,j}\langle Q_j\rangle = \langle O\rangle_{\mu_0}$

Matrix element in subtracted scheme

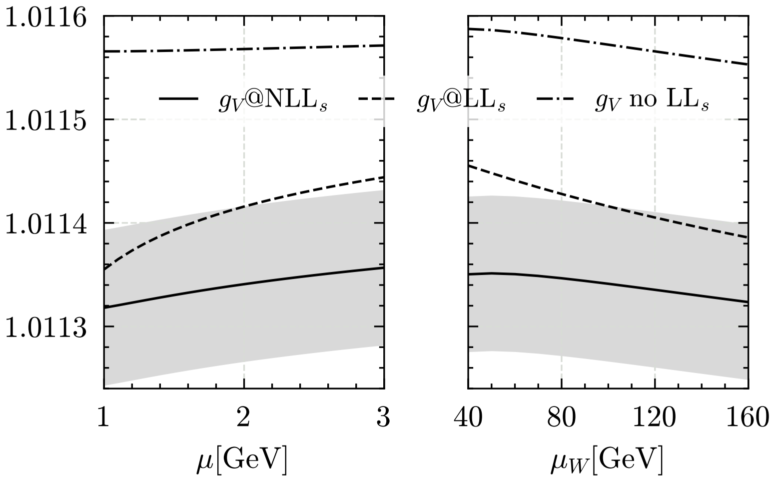
$$\overline{\Box}^{V}_{\mathrm{Had}}(\mu_0) = -e^2 \int \frac{i d^4 q}{(2\pi)^4} \frac{\nu^2 + Q^2}{Q^4} \frac{T_3(\nu, Q^2)}{2 m_N \nu}$$
$$+ e^2 \int \frac{i d^4 q}{(2\pi)^4} \frac{2}{3} \frac{\nu^2 + Q^2}{Q^6 + \mu_0^2 Q^4} \left(1 - \frac{\alpha_s(\mu_0^2)}{\pi}\right)$$

- 1. Integral: 1-d integral over Nachtmann moment 1812.03352 split into high and low $Q^2 = -q^2$ (ν = v · q)
- High Q^2 → DIS – combine with 2. Integral

$$\overline{\Box}^{V}_{\mathrm{Had}}(\mu_0) = \Box^{V<}_{\gamma W}(Q_0^2) + \frac{\alpha}{8\pi}\left[1 - \frac{\alpha_s(\mu_0)}{\pi}\right] \log \frac{\mu_0^2}{Q_0^2},$$

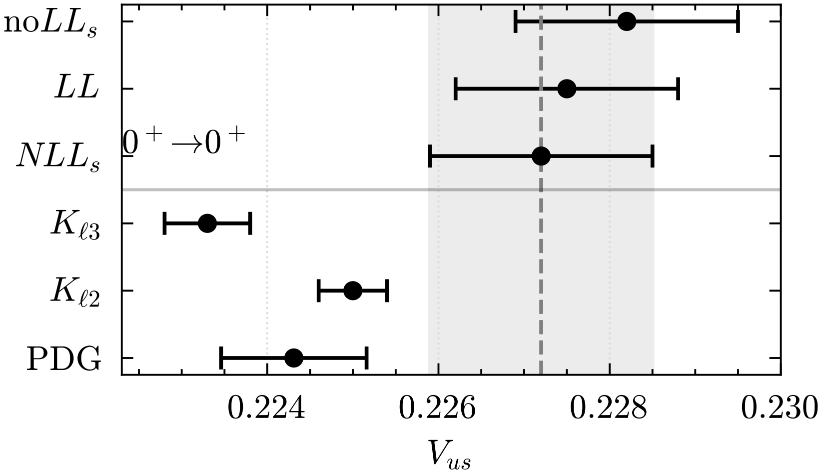
- Use lattice calc 2308.16755 of $\Box^{V<}_{\gamma W}(Q_0^2) = 1.490(73) \cdot 10^{-3}$

Residual scale dependence of $g_V(m_n)$



V_{us} from V_{ud} + unitarity

- Use nuclear input from Hardy,Towner'20 (2308.16755)



CP violation in $K \to \pi\pi$

- Experimental definition using $\eta_{ij} = \dfrac{\langle \pi^i \pi^j | K_L \rangle}{\langle \pi^i \pi^j | K_S \rangle}$

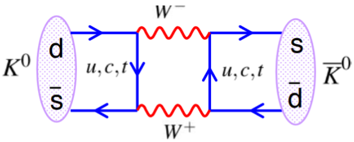
$$\epsilon_K = (2\eta_{+-} + \eta_{00})/3\,, \qquad \epsilon' = (\eta_{+-} - \eta_{00})/3$$

- ϵ_K theory expression $\epsilon_K \simeq \dfrac{\langle (\pi\pi)_{I=0} | K_L \rangle}{\langle (\pi\pi)_{I=0} | K_S \rangle} =$

$$e^{i\phi_\epsilon} \sin\phi_\epsilon \frac{1}{2} \arg\left(\frac{-M_{12}}{\Gamma_{12}}\right) = e^{i\phi_\epsilon} \sin\phi_\epsilon \left(\frac{\mathrm{Im}(M_{12})^{Dis}}{\Delta M_K} + \xi\right)$$

$$\langle K^0 | H^{|\Delta S|=2} | \bar{K}^0 \rangle \to \mathrm{Im}(M_{12})^{Dis}\,,\ \frac{\mathrm{Im}\langle (\pi\pi)_{I=0} | K^0 \rangle}{\mathrm{Re}\langle (\pi\pi)_{I=0} | K^0 \rangle} \to \xi\,,\ \phi_\epsilon \equiv \arctan \frac{\Delta M_K}{\Delta\Gamma_K/2}$$

Kaon Mixing: CKM Structure



matches onto $Q_{S2} = (\bar{s}_L \gamma_\mu d_L) \otimes (\bar{s}_L \gamma^\mu d_L)$

	Im	Re	$\mathcal{O}$
λ_t^2	$\sim \lambda^{10}$	$\sim \lambda^{10}$	m_t^2/M_W^2
$\lambda_c \lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$	$m_c^2/M_W^2 \ln(m_t/m_c)$
λ_c^2	$\sim \lambda^6$	$\sim \lambda^2$	m_c^2/M_W^2
$\lambda_u \lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$	$m_c^2/M_W^2 \ln(m_t/m_c)$
λ_u^2	0	$\sim \lambda^2$	m_c^2/M_W^2

Where $\lambda_i = V_{id} V_{is}^*$, $\lambda \equiv |V_{us}| \sim 0.2$ and we eliminated either:
$\lambda_u = -\lambda_c - \lambda_t$ or $\lambda_c = -\lambda_u - \lambda_t$.

$\Delta S = 2$ Hamiltonian - Phase (In)Dependence

- Recall $\epsilon_K \propto \arg(-M_{12}/\Gamma_{12})$
- Trick: pull out λ_u^* and $(\lambda_u^*)^2$ from $H^{\Delta S=1}$ and $H^{\Delta S=2}$:
- Rephaseing invariant: $\lambda_i \lambda_j^* = V_{id} V_{is}^* V_{jd}^* V_{js}$
- One Operator: $Q_{S2} = (\bar{s}_L \gamma_\mu d_L) \otimes (\bar{s}_L \gamma^\mu d_L)$

$$\mathcal{H}_{f=3}^{\Delta S=2} = \frac{G_F^2 M_W^2}{4\pi^2 (\lambda_u^*)^2} Q_{S2} \Big\{ f_1 C_1(\mu) + iJ \left[f_2 C_2(\mu) + f_3 C_3(\mu) \right] \Big\} + \text{h.c.}$$

- $f_1 = |\lambda_u|^4$, $f_2 = 2\mathrm{Re}(\lambda_t \lambda_u^*)$ and $f_3 = |\lambda_u|^2$

Kaon mixing and ε_K

$$\mathcal{H}^{\Delta=2}_{f=3} = \frac{G_F^2 M_W^2}{4\pi^2}\Big[\lambda_u^2 C_{S2}^{uu}(\mu) + \lambda_t^2 C_{S2}^{tt}(\mu) + \lambda_u\lambda_t C_{S2}^{ut}(\mu)\Big] Q_{S2} + \text{h.c.}$$

- Phase convention independent form Brod, MG, Stamou PRL'20:

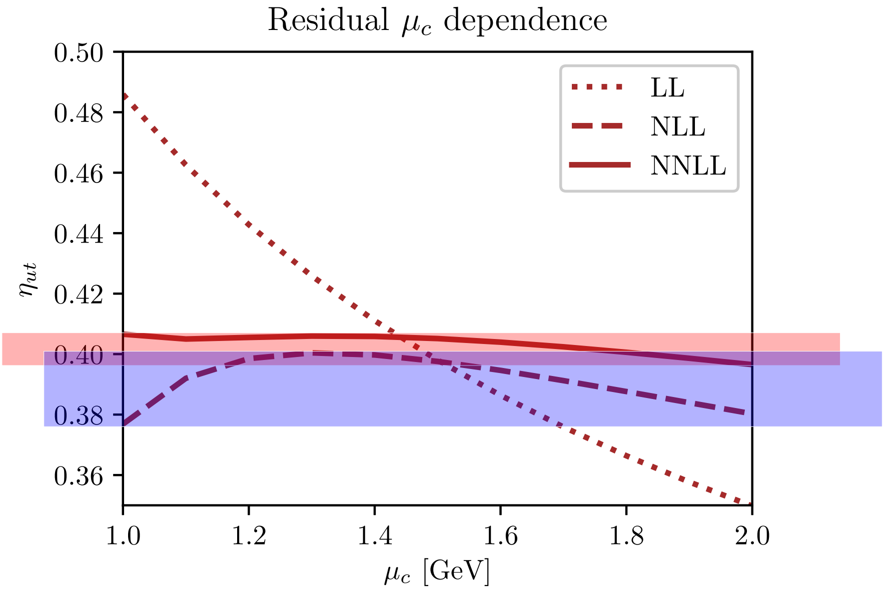
  $$|\varepsilon_K| = \kappa_\epsilon C_\epsilon \hat{B}_K |V_{cb}|^2 \lambda^2 \bar{\eta} \times \Big(|V_{cb}|^2(1-\bar{\rho})\eta_{tt} S_{tt}(x_t) - \eta_{ut} S_{ut}(x_c, x_t)\Big)$$

  apart from $\eta_{\rm tt} \leftrightarrow \eta_{\rm ut}$ rotation

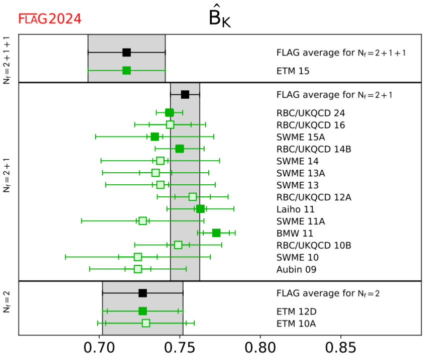
- $\hat{B}_K^f \propto \langle \bar{K}^0 | Q^{|\Delta S=2|} | K^0 \rangle_f(\nu) J_f(\nu) u_f(\nu)$ is fn of # of active f

- Perturbative matching between f=5 $\leftrightarrow$ f=4 $\leftrightarrow$ f=3

Residual scale dependence Brod, MG, Stamou PRL'20



$\hat{B}_K$, FLAG24



FLAG24 only has NLO corrections

- NNLO should increase consistency
- Allows for comparison of f=3 ↔ f=4 results

Combine with Lattice values

- $\hat{B}_K = \frac{3}{2f_K^2 M_K^2}\langle \bar{K}^0 | Q^{|\Delta S=2|} | K^0 \rangle u^{-1}(\mu_{\rm had})$ from Latticen

- The result should be independent of the matching scale

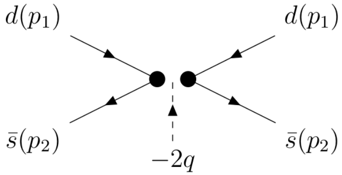
$$B_K^{(X,Y)}(|p|) C_{B_K}^{(X,Y)}(|p|,\mu) u^{-1}(\mu) u(\mu_0)$$

- study scale variation setting $\mu_0 = |p|$

- Compute $\hat{B}_K$ for f=3,4 for SMOM, RIMOM, RI'MOM

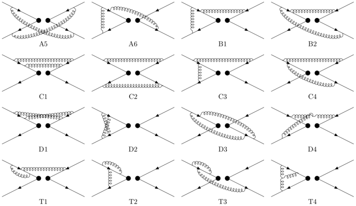
MOM scheme

SMOM kinematics used for the most recent results



- Projected renormalised Green's function $P_{(\gamma_\mu)}(\Lambda_R) \to$
- $Z_{Q_{S2}}^{(\gamma_\mu,\gamma_\mu)} = \left(Z_q^{(\gamma_\mu)}\right)^2 \dfrac{1}{P_{(\gamma_\mu)}(\Lambda_B)}$
- $Z_{Q_{S2}}^{(\gamma_\mu,\gamma_\mu)} / Z_{Q_{S2}}^{\overline{\mathrm{MS}}}$ converts between Lattice and continuum

SMOM $\hat{B}_K$ @ NNLO 2411.19861

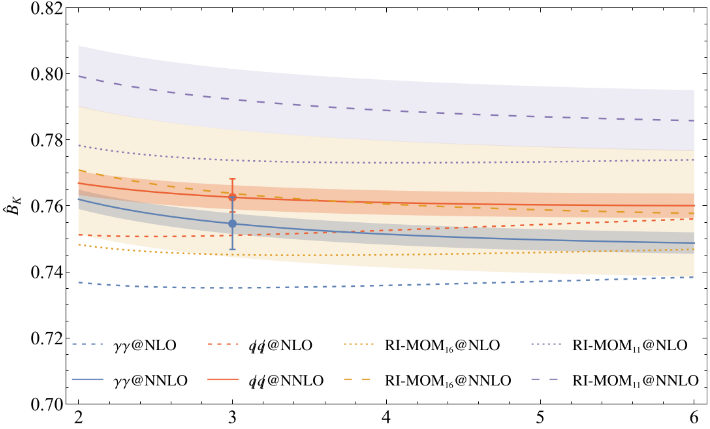


- Use projectors to find $\Lambda^{ijkl}_{\alpha\beta\gamma\delta}$ at 2-loops
- Integrals reduce to scalar off-shell 4-point functions

$$C^{(X,Y)}_{B_K}(|p|, \mu = |p|) = \sum_i \frac{\alpha_s^i}{(4\pi)^i} C_i$$

scheme	C_0	C_1	C_2
$(\gamma, \not{q})$	1	-2.45482	-36.2(1)
(γ, γ)	1	0.211844	16.2(1)
$(\not{q}, \not{q})$	1	-0.454823	-13.4(1)
$(\not{q}, \gamma)$	1	2.21184	44.4(1)

Residual Scale Dependence at NNLO



Combine $\hat{B}_K$ at NNLO MG, Jager, Kvedaraite 2411.19861

- Compute $\hat{B}_K$ for f=3,4 for SMOM, RIMOM, RI'MOM

f	Ref.	Scheme	$\hat{B}_K^{(f=3)}$	$\hat{B}_K^{(f=4)}$
4	ETM 15	RI′-MOM	0.733(26)	0.745(25)
3	RBC/ UKQCD 24	$(\not{q}, \not{q})$	0.7626(56)	0.7749(93)
		(γ_μ, γ_μ)	0.7546(80)	0.767(11)
3	BMW 11	RI-MOM	0.790(13)	0.804(15)

- New world average $\hat{B}_K^{(f=3)} = 0.7627(60)$
  - Compare with FLAG'24: $\hat{B}_K = 0.7533(91)$
  - and DQCD 1401.1385 $\hat{B}_K^{(f=3)} = 0.7300(200)$
- $|\epsilon_K| = 2.171(65)_{\text{pert.}}(71)_{\text{non-pert.}}(153)_{\text{param.}} \times 10^{-3}$

Conclusion

- Precision to Constrain New Physics
- Combination of Perturbative and Non-Perturbative Methods
- Interface Scheme
  - accessible on Lattice and Continuum
  - MOM scheme $\langle Q \rangle_a \; c_a(p^2) \; c_\epsilon(p^2) \; C(\epsilon)$
  - scheme change absorbs regulator dependence
  - Requires consistent expansion
- Good convergence
- Higher order QED corrections important for V_{ud}
- Essential for Precision Flavour Physics