Probing Isospin Symmetry Breaking from Lattice QCD+QED

Xin-Yu Tuo

BNL HET lunch time talk

Nov 13th , 2025

Outline

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Isospin Breaking: A Tiny Difference That Shapes the Universe

- Proton and neutron mass difference: isospin breaking effects

$$\boldsymbol{m_p = 938.2720882 \pm 0.0000003\ \mathrm{MeV/c^2}}$$
$$\boldsymbol{m_n = 939.5654205 \pm 0.0000005\ \mathrm{MeV/c^2}}$$

$$\boldsymbol{m_n - m_p \approx 1.293\mathrm{MeV}, \qquad \frac{m_n - m_p}{m_p} \approx 0.14\%}$$

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Strong Isospin Breaking

$q_u \neq q_d$
QED interaction

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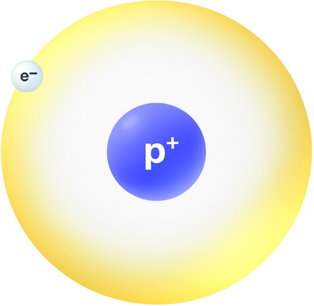
$m_u \neq m_d$ Strong Isospin Breaking	$q_u \neq q_d$ QED interaction

- Why $m_n > m_p$ matters:

$$n \to p + e^- + \bar{\nu}_e \qquad \text{~~}p \to n + e^+ + \nu_e\text{~~}$$

Hydrogen stability → star burning → elements, chemistry, life

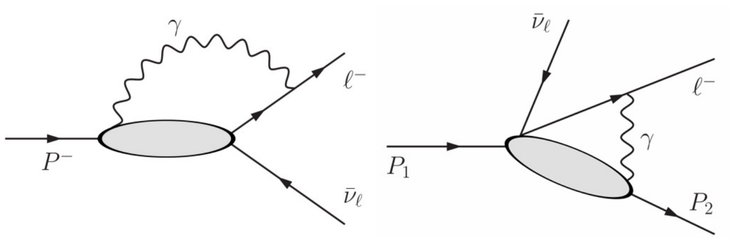


Hydrogen-1
mass number: 1

A 0.14% mass difference creates the visible Universe

Isospin Breaking Across the Standard Model

Leptonic and semi-leptonic decays



CKM unitarity: 2.4σ tension

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9983(6)_{V_{ud}}(4)_{V_{us}}$$

Isospin breaking: dominant source of hadronic uncertainty

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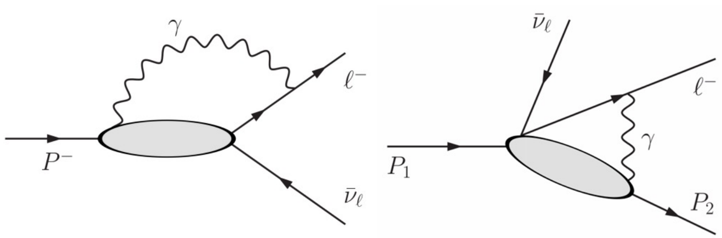

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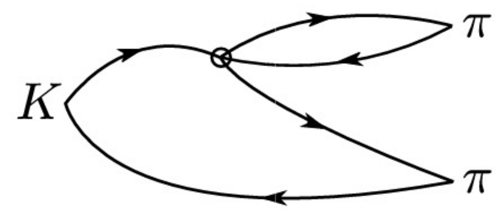
$K \to \pi\pi$ and direct CPV ε'/ε



$\Delta I = \frac{1}{2}$ rule: $\frac{A_0}{A_2} \approx 22.5$

Isospin Breaking could be amplified

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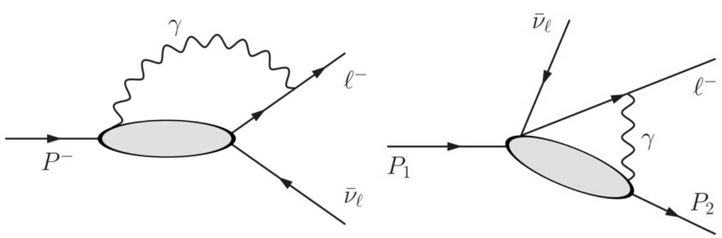

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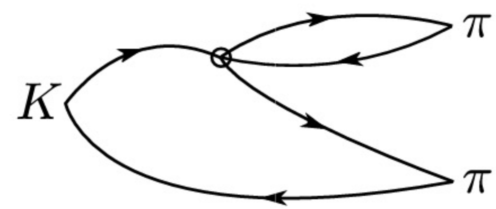

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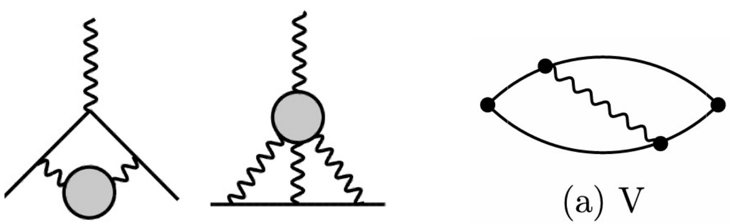
HVP and HLbL function in muon g-2



Isospin breaking $\delta a_\mu^{\mathrm{HVP,LO}} \sim 1\%$

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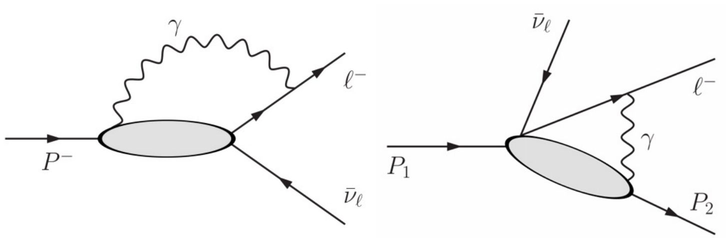

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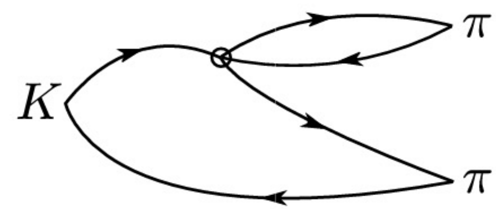

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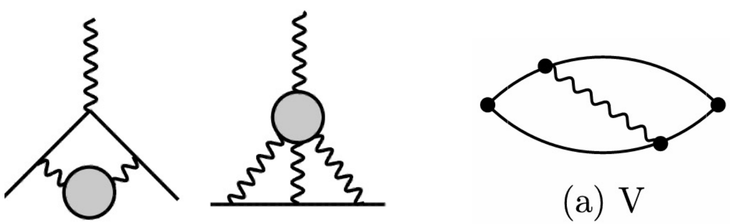

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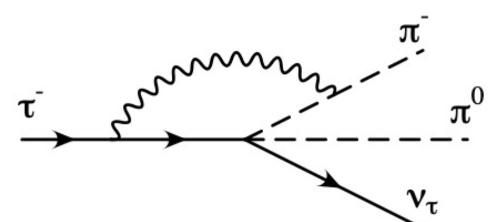
Hadronic τ decay and $e^+e^- \to \pi\pi$



Related to $e^+e^- \to \pi\pi$ by isospin breaking

Important input to HVP function

Isospin breaking: dominant source of hadronic uncertainty

Challenges on lattice

- $m_u \neq m_d$, strong isospin breaking effects: scheme dependence

  different lattice groups define 'isosymmetric world' in different ways,

  direct comparison of IB results across collaborations is not straightforward.

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- QED effects: Long-distance photon propagator
  - Non-local hadronic matrix elements, four-point function
  - Large finite volume effects

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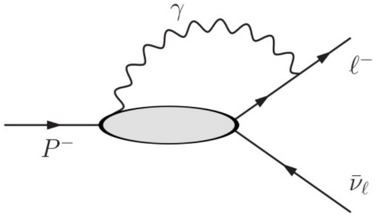
- QED effects: Long-distance photon propagator
  - Non-local hadronic matrix elements, four-point function
  - Large finite volume effects

- This talk: introducing techniques to overcome these challenges, and then apply them to two examples:
  - Baryon mass splitting
  - Light-meson leptonic decays



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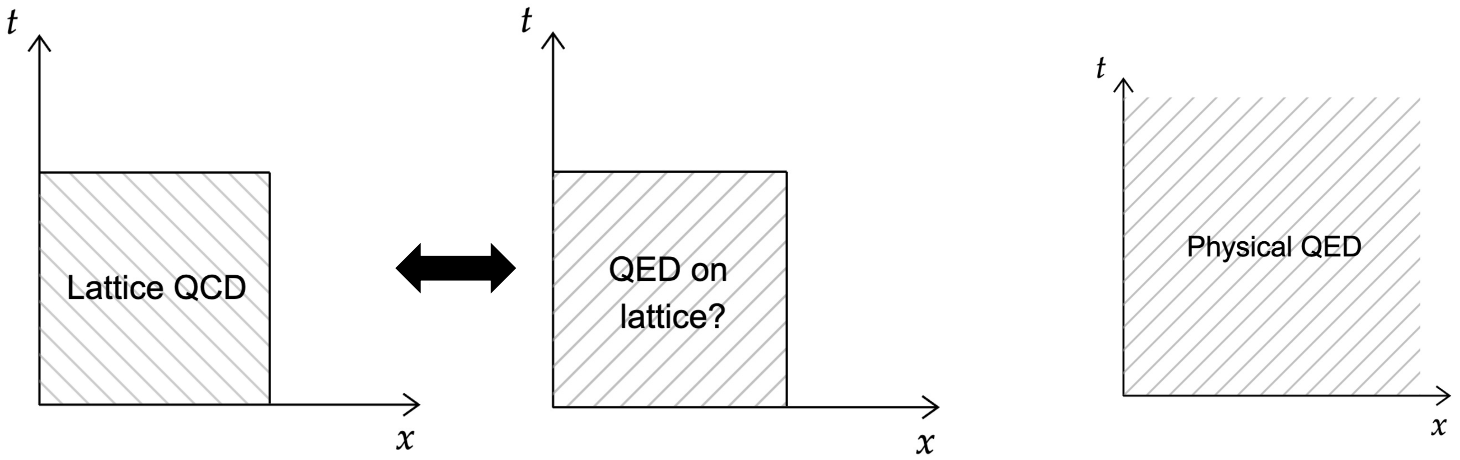
Idea of QED_∞

- Traditional QED_L method:



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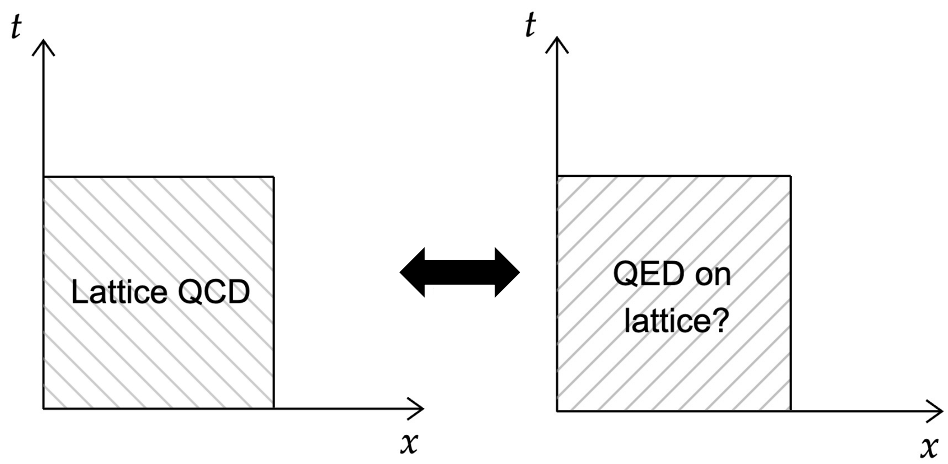

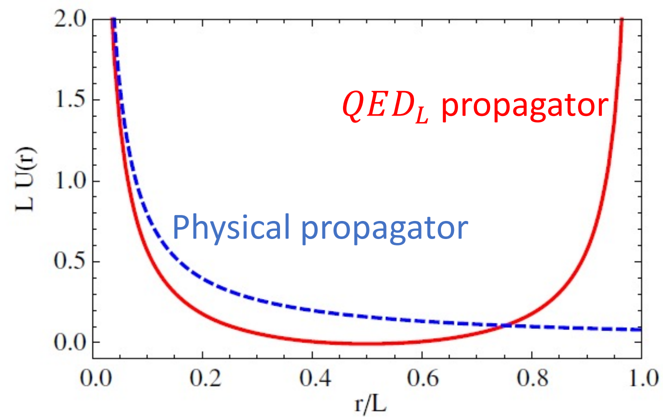

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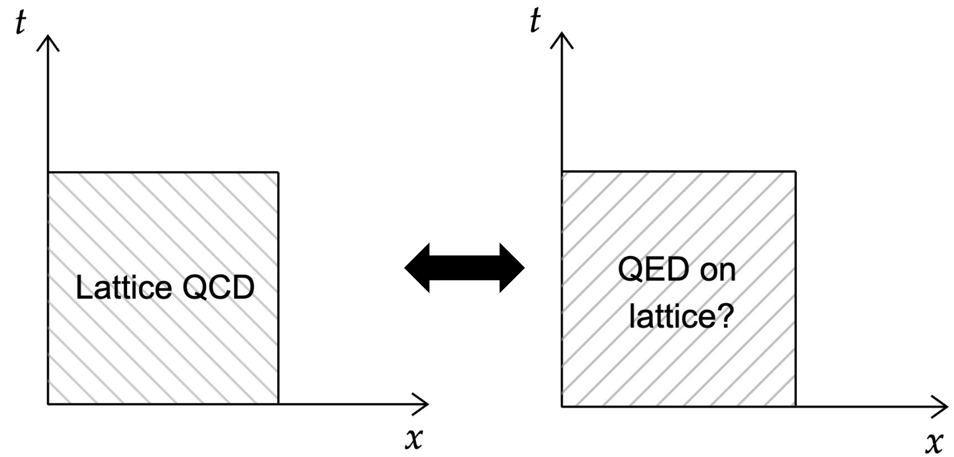
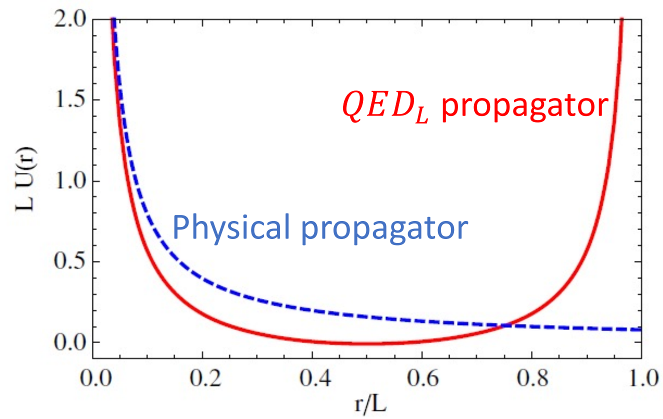
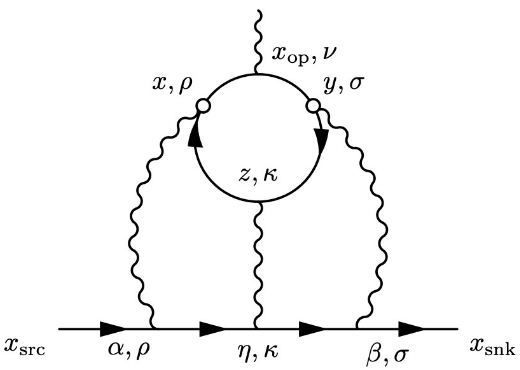
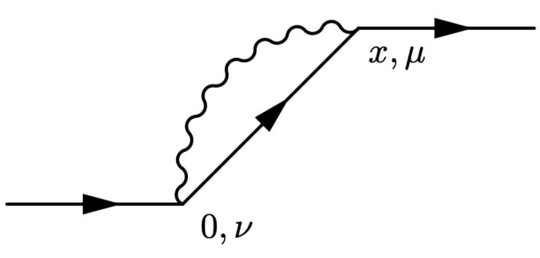




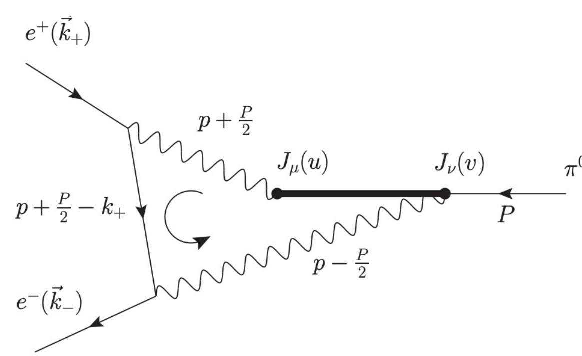
- Idea of QED_∞: photon (or lepton) propagators in infinite volume



HLbL



QED self energy



$\pi^0 \to e^+e^-$

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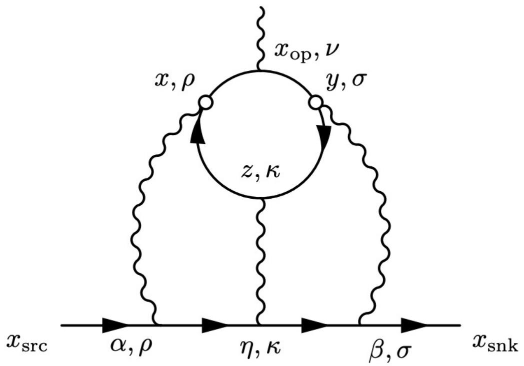

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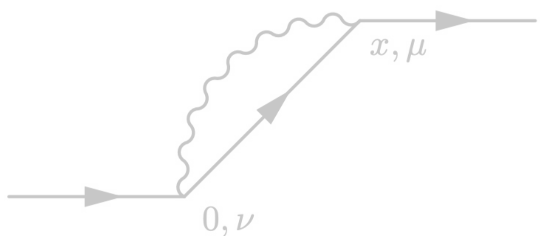

QED self energy: IVR
without power-law FV errors

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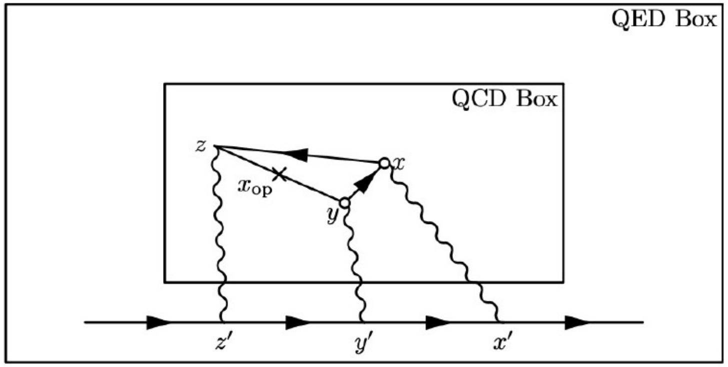

Thomas Blum, Norman Christ, Masashi Hayakawa, Taku Izubuchi, Luchang Jin, Chulwoo Jung, Christoph Lehner, PRD 96 (2017) 3, 034515, arxiv:1705.01067

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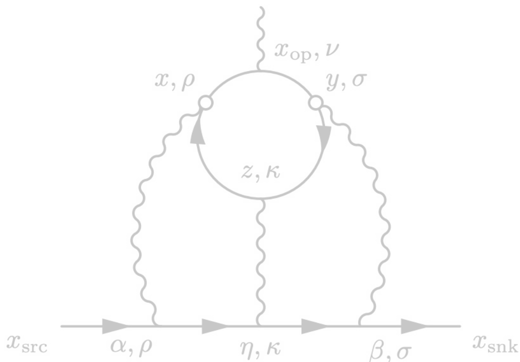

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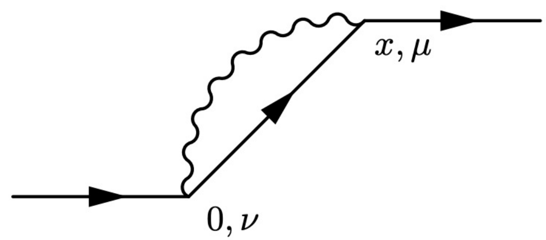

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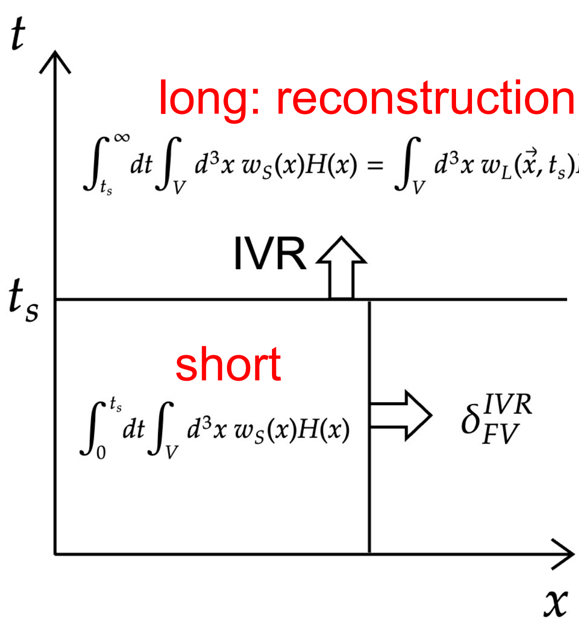
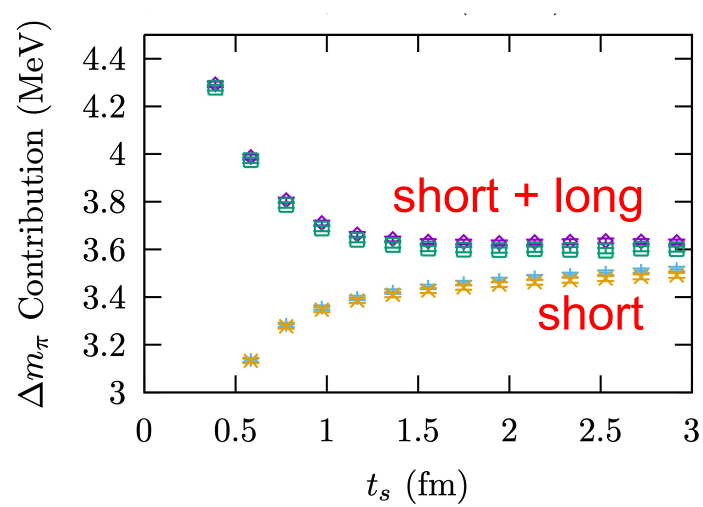
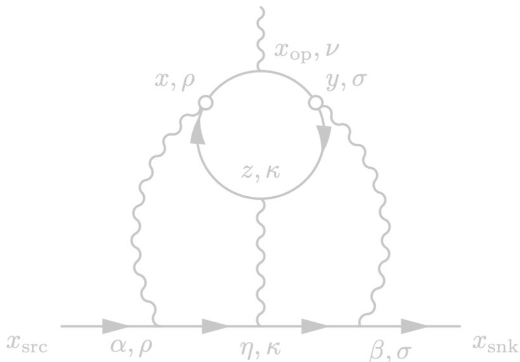




X. Feng, L. Jin., M. Roberdy, PRL 2022 [arXiv:2108.05311]

➢ Idea of QED_∞: photon (or lepton) propagators in infinite volume



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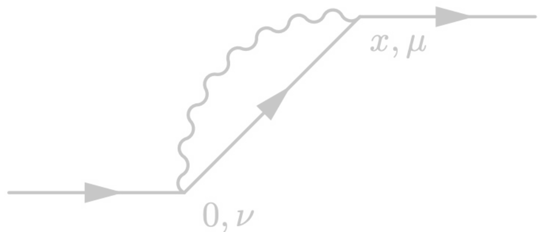

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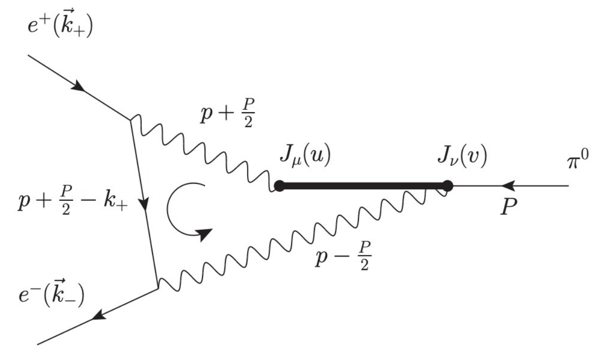

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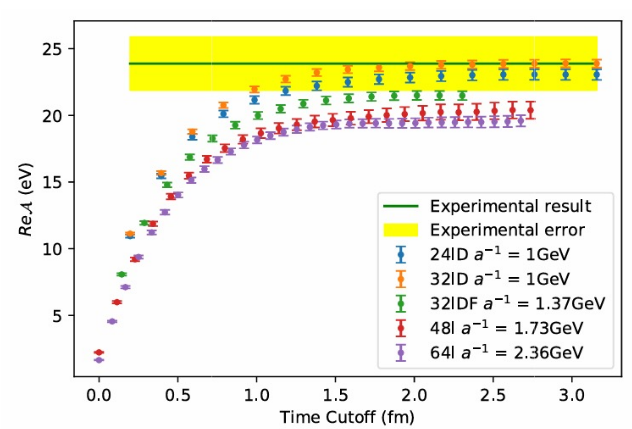

(b) Real part

Generalization of QED_∞ idea:
Not only the photon propagators, but also the lepton propagators are defined in infinite volume

$$\mathcal{A} = \int \mathrm{d}^4 w \; L_{\mu\nu}(w) H_{\mu\nu}(w)$$

Norman Christ, Xu Feng, Luchang Jin, Cheng Tu, Yidi Zhao,
PRL 130 (2023) 19, 191901, arxiv:2208.03834

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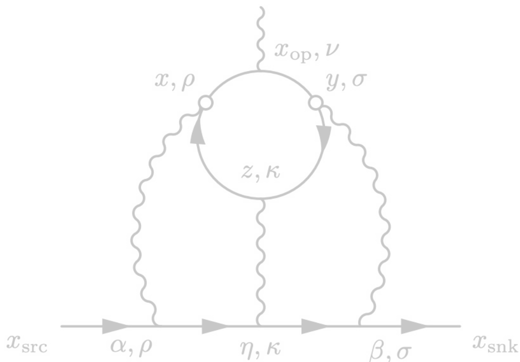

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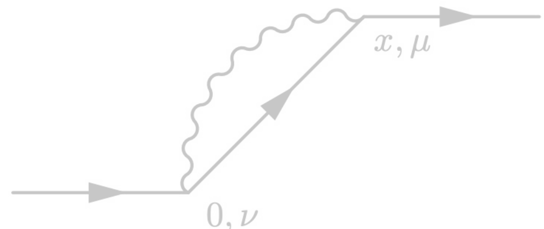

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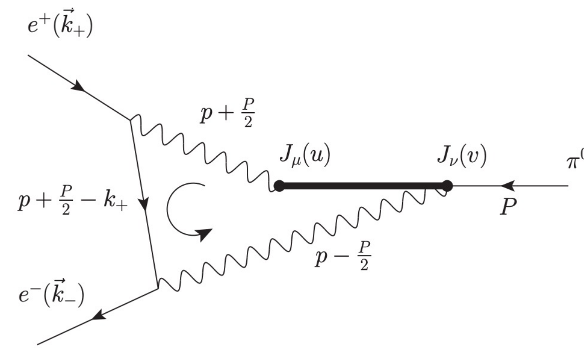
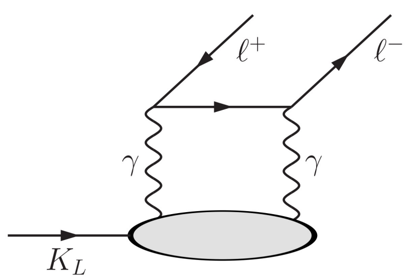


$\pi^0 \to e^+e^-$

Can we use the same method in $K_L \to \mu^+\mu^-$?

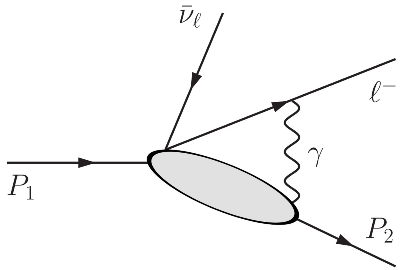


$K_L \to \mu^+\mu^-$

Generalization: 2π intermediate states

- Large finite volume effects can arise from both the hadronic part (2π states), and photon/lepton propagators

RC in semileptonic decays



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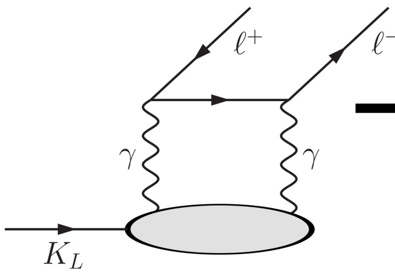

massless photons, IR singularities

hadronic part:
$\pi\pi$ intermediate states

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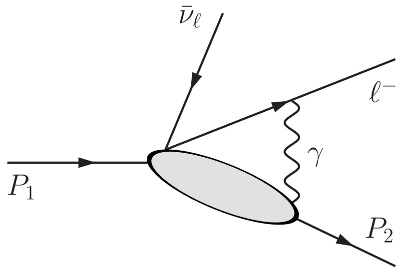

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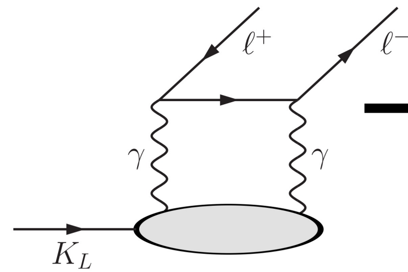

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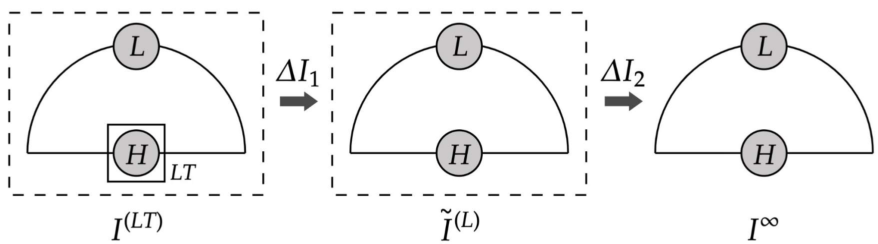
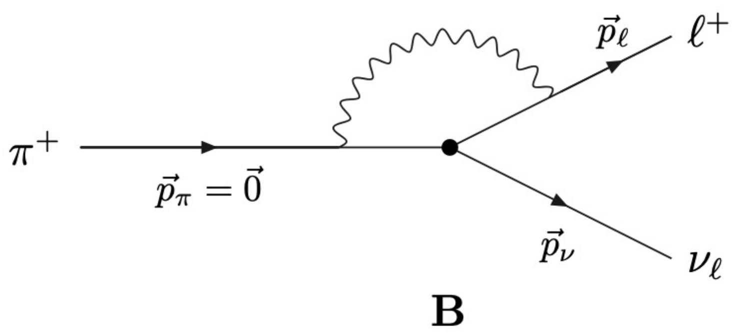
- Generalization to 2π intermediate states: new finite-volume formalism

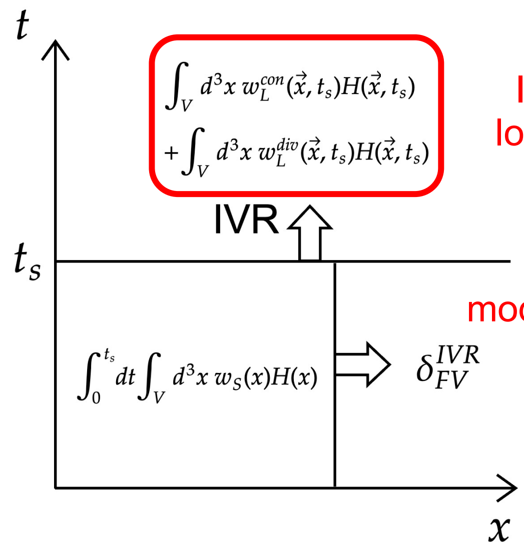


Xin-Yu Tuo, Xu Feng, PRD 112 (2025) 034512

Generalization: decays with IR divergence

- Example: RC in leptonic decays





IR div. is related to long-distance physics

modify the long-distance weight function

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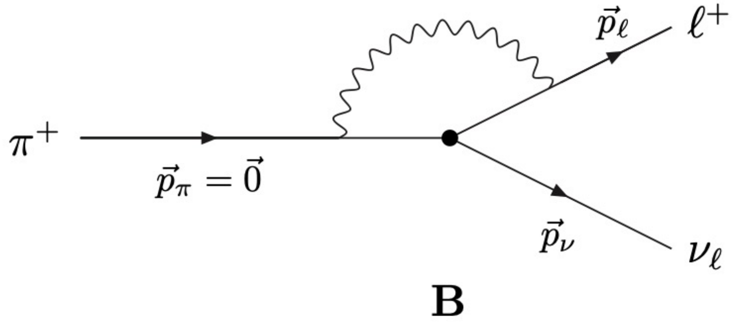
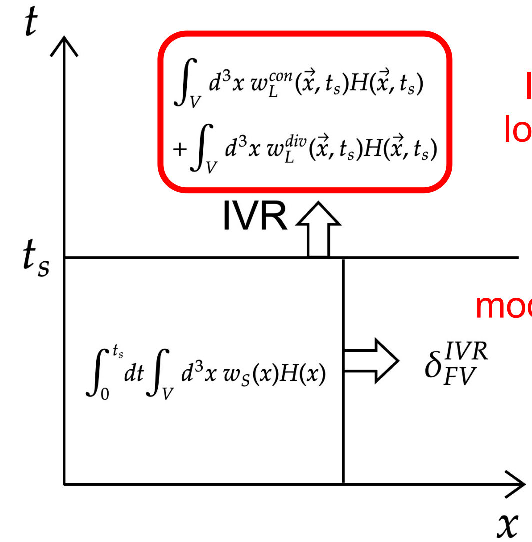




IR div. is related to long-distance physics

modify the long-distance weight function

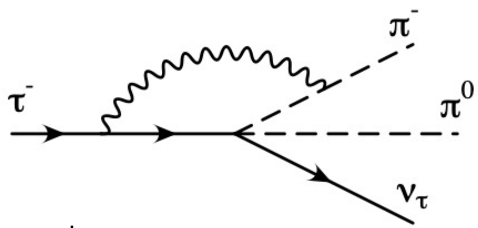
- Subtraction of IR div.: $e^{i\vec{k}\cdot\vec{x}} = (e^{i\vec{k}\cdot\vec{x}} - 1) + 1$ trick

$$w_L^{\mu\rho}(t, \vec{x}) = i\frac{G_F V_{ud}^* e^2}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_E^0}{2\pi} \frac{\bar{L}^{\mu\rho}(k, p)}{((k_E^0)^2 + E_\gamma^2 - i\epsilon)((p_{\ell,E}^0 - k_E^0)^2 + E_\ell^2 - i\epsilon)} \times \left[\left(e^{-i\vec{k}\cdot\vec{x}} - 1\right) + 1\right] \frac{e^{ik_E^0 t_s}}{-ik_E^0 + E_\pi(\vec{k}) - m_\pi}$$

w_L^{con} $\quad$ w_L^{div}

New frontier: 2π final states

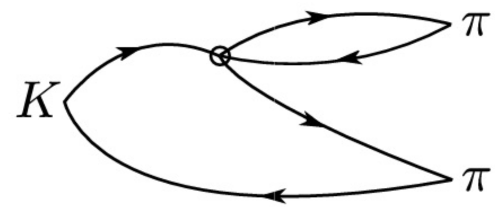
Hadronic τ decay and $e^+e^- \to \pi\pi$



Related to $e^+e^- \to \pi\pi$ by isospin breaking

Important input to HVP function

$K \to \pi\pi$ and direct CPV ε'/ε



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Isospin Breaking could be amplified

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- The finite-volume formalism is much more challenging.

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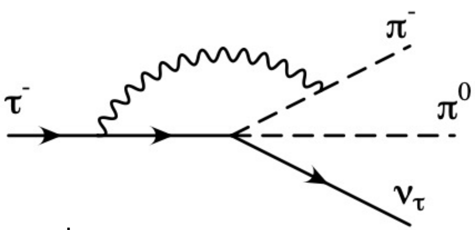

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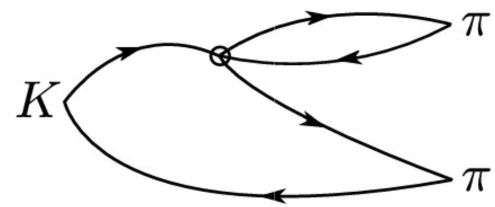

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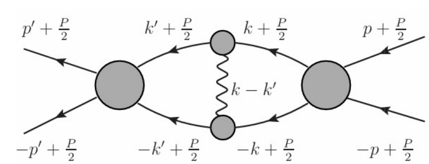
- The finite-volume formalism is much more challenging.
- Several attempts for ε'/ε:
  - Simplified QED: coulomb potential effects to $\pi\pi$ scattering

    Norman Christ, Xu Feng, Joseph Karpie, Tuan Nguyen, PRD 106 (2022) 014508



  - Using ChPT to estimate the size of QED effects beyond the coulomb potential (from transverse photons)

    Erik Lundstrum and Norman Christ

Outline

Strong IB corrections: scheme dependence

- For any quantity $X(\alpha_{QED};\{m_q\})$ as a function of α_{QED} and quark masses $\{m_q\} = \{m_u, m_d, m_s\}$ with mass dimension n,

$$X(0;\{m_q^{\text{iso}}\})\Big|_{m_u^{\text{iso}}=m_d^{\text{iso}}}$$

isosymmetric QCD world

$$X(\alpha_{QED};\{m_q^{\text{phy}}\})$$

physical QCD+QED world

difference →

$$\delta X_{IB} = \frac{X(\alpha_{QED};\{m_q^{\text{phy}}\})}{X(0;\{m_q^{\text{iso}}\})} - 1$$

IB corrections

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IB corrections

- However, the quark masses on lattice $\{m_q^{\text{lat}}\}$ are not same as $\{m_q^{\text{iso}}\}$, $\{m_q^{\text{phy}}\}$
- Traditional method: determine the quark masses and their differences in each "world", which depend on the specific isosymmetric QCD theory.
- Different lattice group might use different isosymmetric world, and thus different quark masses.

Strong IB corrections: new strategy

$$X\left(0;\{m_q^{\rm iso}\}\right)\Big|_{m_u^{\rm iso}=m_d^{\rm iso}}$$

isosymmetric QCD world

$$X(\alpha_{QED};\{m_q^{\rm phy}\})$$

physical QCD+QED world

goal

Strong IB corrections: new strategy

➢ Define a new quark-mass-insensitive quantity

$$X^{\mathrm{sub}} = X - m_\Omega^n \left(c_1 \frac{m_{\pi^\pm}^2}{m_\Omega^2} + c_2 \frac{m_{K^\pm}^2}{m_\Omega^2} + c_1 \frac{m_{K^0}^2}{m_\Omega^2} \right)$$

with constraints $\frac{\partial X^{\mathrm{sub}}}{\partial m_{q_i}} \Big|_{m_{q_i}=m_{q_i}^{\mathrm{lat}}} = 0$

independent of quark masses
direct calculation on lattice

reconstruct →

$X(0;\{m_q^{\mathrm{iso}}\})\Big|_{m_u^{\mathrm{iso}}=m_d^{\mathrm{iso}}}$
isosymmetric QCD world

$X(\alpha_{QED};\{m_q^{\mathrm{phy}}\})$
physical QCD+QED world

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$X(0; \{m_q^{\text{iso}}\}) \Big|_{m_u^{\text{iso}} = m_d^{\text{iso}}}$
isosymmetric QCD world

$X(\alpha_{QED}; \{m_q^{\text{phy}}\})$
physical QCD+QED world

goal

- Strong IB correction as a function of meson masses

$$X\left(\alpha_{QED}; \left\{m_q^{\text{phy}}\right\}\right) - X(0; \{m_q^{\text{iso}}\}) = \delta X_{\text{sub}}^{QED} - \sum_{i=1}^{3} c_i \left[\frac{\left(m_{P_i}^{\text{phy}}\right)^2}{\left(m_\Omega^{\text{phy}}\right)^{2-n}} - \frac{\left(m_{P_i}^{\text{iso}}\right)^2}{\left(m_\Omega^{\text{iso}}\right)^{2-n}} \right]$$

QED   strong IB

- In order to get strong IB, we only need to know the scheme-independent coefficient c_i. Physical meaning: the dependence on meson masses.

Lattice scale setting with the new quantity

- The first application of this new quantity, is to set the lattice spacing in isosymmetric QCD world, or QCD+QED world.

- Choose X to be omega mass m_Ω. In pure QCD theory:

$$am_\Omega^{\mathrm{sub}} = am_\Omega^{\mathrm{lat}} - am_\Omega^{\mathrm{lat}} \sum_{i=1}^{3} c_i \left(\frac{m_{P_i}^{\mathrm{lat}}}{m_\Omega^{\mathrm{lat}}}\right)^2$$

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- For any given isosymmetric QCD world

$$a^{\mathrm{QCD}} = \frac{am_\Omega^{\mathrm{sub}}}{m_\Omega^{\mathrm{sub,iso}}}$$

- For QCD+QED world

$$a^{\mathrm{QCD+QED}} = \frac{a(m_\Omega^{\mathrm{sub}} + \delta_{\mathrm{QED}} m_\Omega^{\mathrm{sub}})}{m_\Omega^{\mathrm{sub,phy}}}$$

New quantity is independent of quark masses

↓

the scale setting is not sensitive to quark mass mistuning

Outline

Application I: Baryon mass splitting

$$\frac{m_p - m_n}{m_p} \approx 0.14\%$$

- Based on our unpublished work, in collaboration with Xu Feng and Chenfei Lu. Choose X to be $m_n, m_p, m_\Sigma, m_\Xi, m_\Lambda$. Lattice spacing is set by m_Ω.

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$$m_B^{\text{sub}} = m_B - m_\Omega\left(c_1\frac{m_{\pi^\pm}^2}{m_\Omega^2} + c_2\frac{m_{K^\pm}^2}{m_\Omega^2} + c_1\frac{m_{K^0}^2}{m_\Omega^2}\right)$$

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reconstruct →

Baryon masses in QCD+QED world

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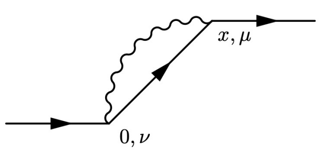
with constraints $\frac{\partial m_B^{\text{sub}}}{\partial m_{q_i}}\Big|_{m_{q_i}=m_{q_i}^{\text{lat}}} = 0$

reconstruct → Baryon masses in QCD+QED world

- Isospin breaking correction to baryon masses:

$$m_B\left(\alpha_{QED};\{m_q^{\text{phy}}\}\right) - m_B\left(0;\{m_q^{\text{iso}}\}\right) = \delta m_{B,\text{sub}}^{QED} - \sum_{i=1}^{3} c_i\left[\frac{\left(m_{P_i}^{\text{phy}}\right)^2}{\left(m_\Omega^{\text{phy}}\right)^{2-n}} - \frac{\left(m_{P_i}^{\text{iso}}\right)^2}{\left(m_\Omega^{\text{iso}}\right)^{2-n}}\right]$$

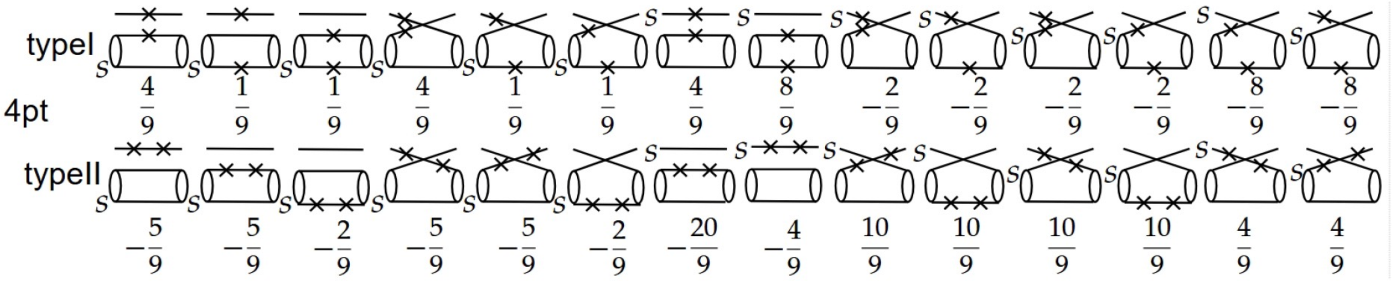


IVR method
lattice 4-pt function

fixed by constraint
lattice 3-pt function

Challenge: complicated contractions

- Example: Λ baryon 4-pt functions, 28 diagrams after contraction



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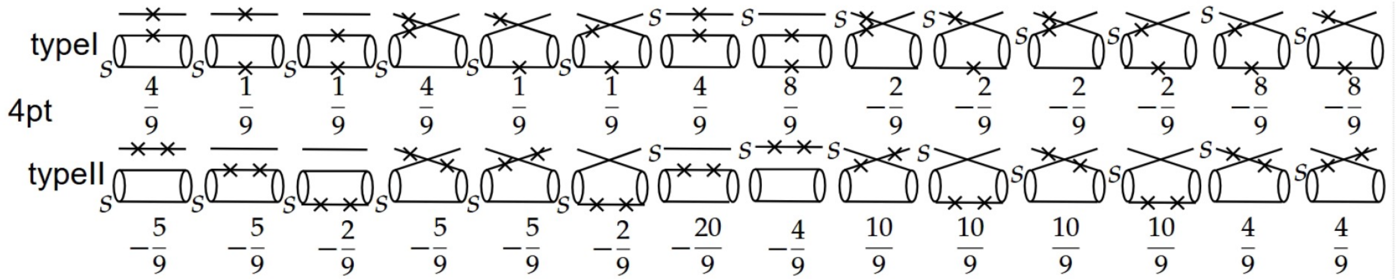
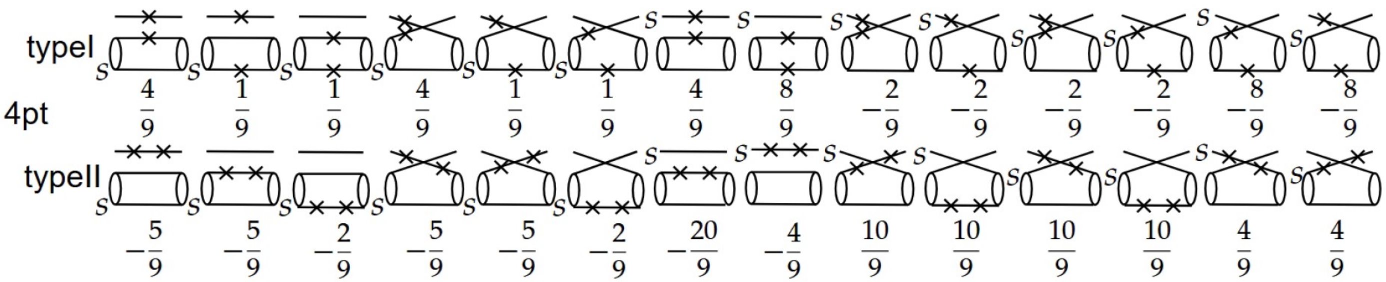


- To obtain reliable coefficients for each diagram, I developed an automatic diagrammatic contraction tool, consisting of a backend contraction module and a diagrammatic frontend.

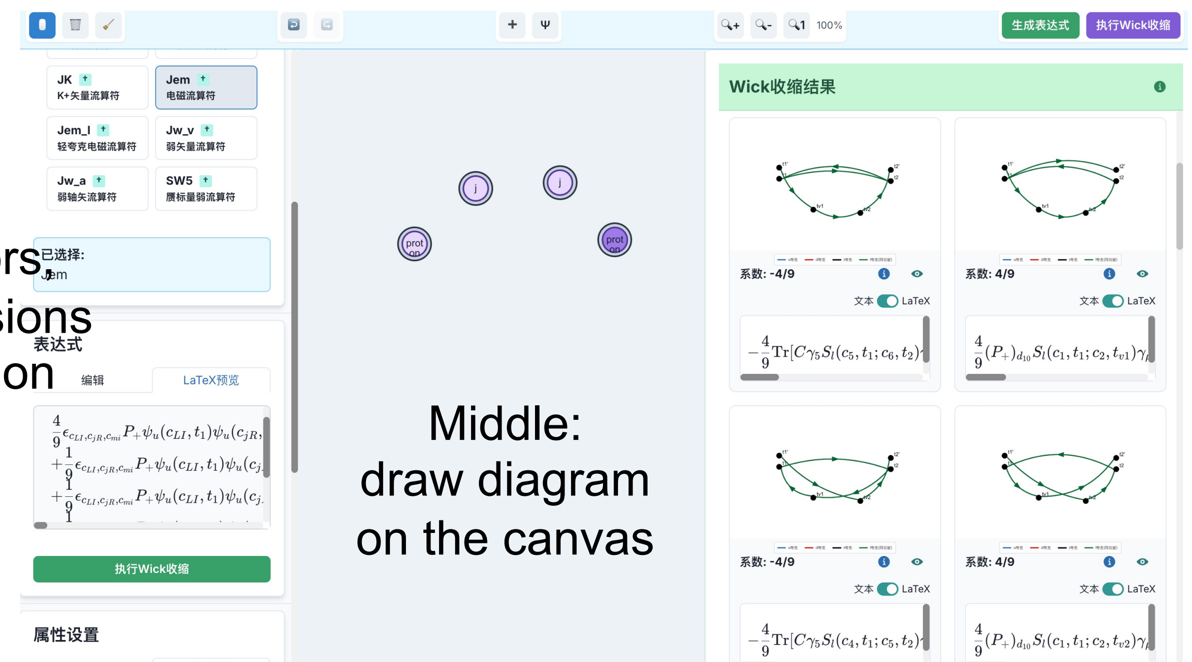
Challenge: complicated contractions

- Example: Λ baryon 4-pt functions, 28 diagrams after contraction



- To obtain reliable coefficients for each diagram, I developed an automatic diagrammatic contraction tool, consisting of a backend contraction module and a diagrammatic frontend.

Left: choose operators generate expressions before contraction

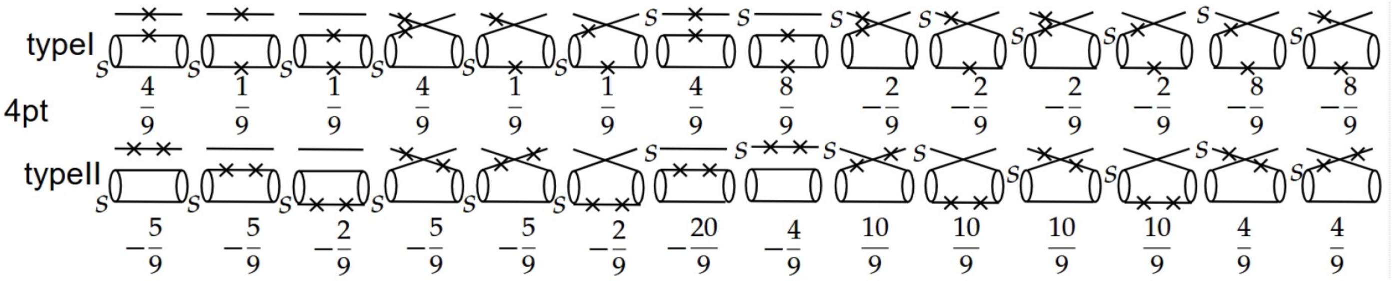


Middle: draw diagram on the canvas

Right: automatic contractions (diagrams and expressions)

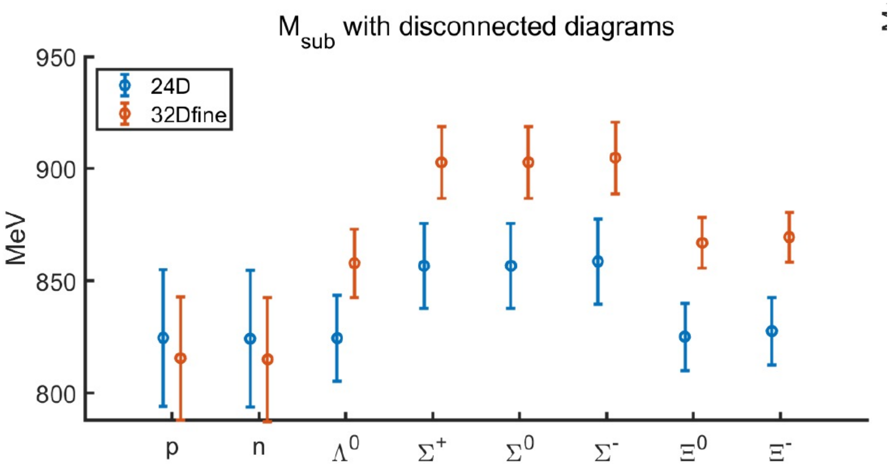
Challenge: complicated contractions

- Example: Λ baryon 4-pt functions, 28 diagrams after contraction

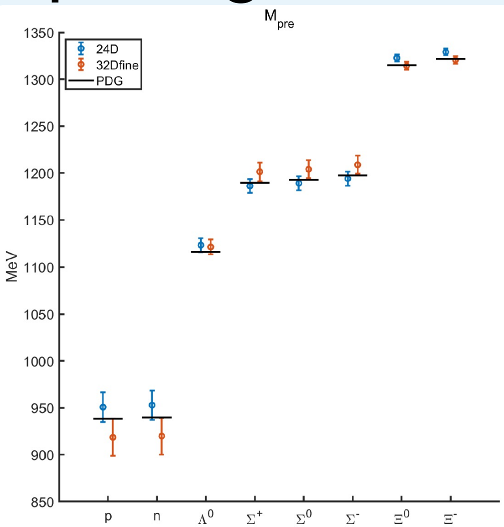


- Calculation of contraction diagrams:
  - Sparsed, smeared propagators in 24D and 32Dfine
  - Calculated by *qlattice* package developed by Luchang Jin.

Results: baryon mass splitting

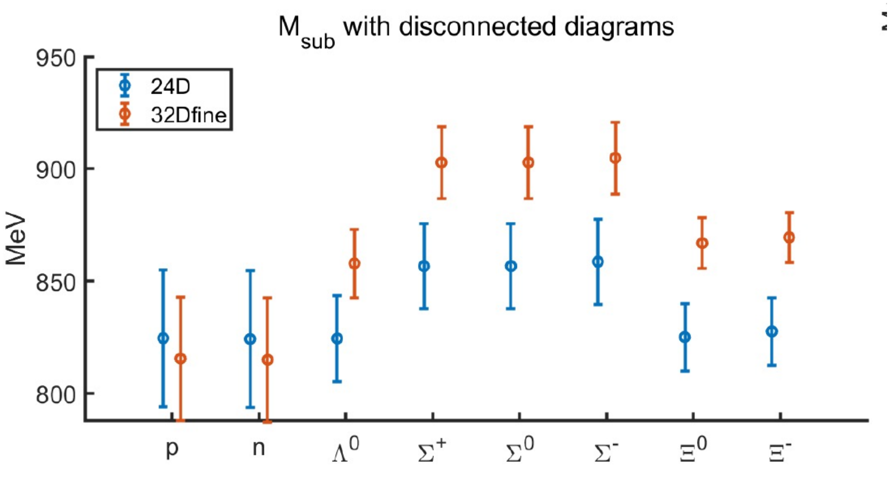


New quantity $m_{B,\mathrm{sub}}$ for baryon masses



Prediction of physical baryon masses

Results: baryon mass splitting



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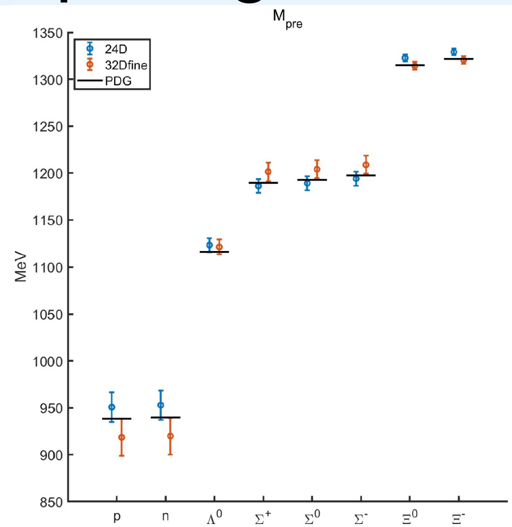

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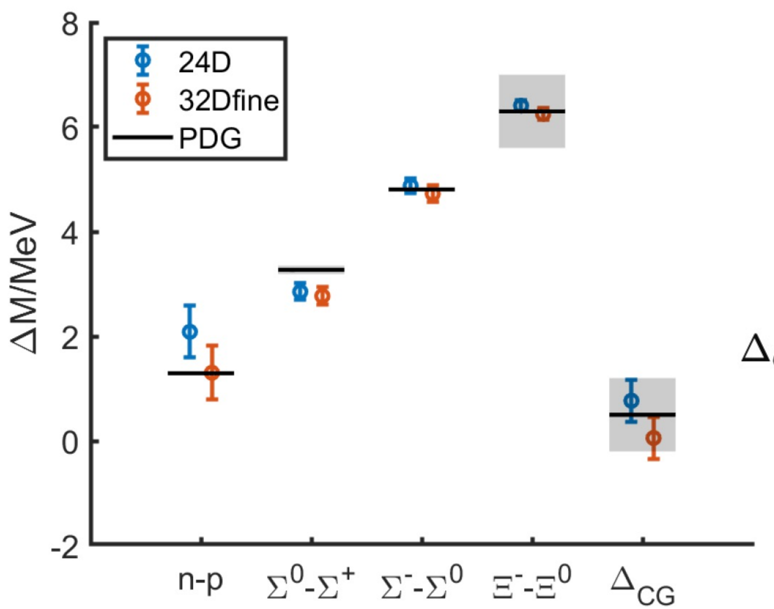

Baryon mass differences due to isospin breaking corrections

Coleman–Glashow mass relation

$$\Delta_{CG} \equiv (M_n - M_p) - (M_{\Sigma^-} - M_{\Sigma^+}) + (M_{\Xi^-} - M_{\Xi^0}) = 0$$

Outline

Isospin-breaking corrections in $K_{\ell 2}/\pi_{\ell 2}$

- Define X as quantity $R_P(\alpha_{QED}; \{m_q\})$

$$R_P(\alpha_{\rm QED}; \{m_{q_i}\}) = \frac{1}{C} \frac{\Gamma_{P_{\ell 2(\gamma)}}(\alpha_{\rm QED}; \{m_{q_i}\})}{m_P(\alpha_{\rm QED}; \{m_{q_i}\})}$$

$$C = G_F^2 |V_P|^2 m_\ell^2 (1 - m_\ell^2/m_P^2)/(4\pi)$$

$$R_P(0; \{m_{q_i}^{\rm iso}\}) = (F_P^{\rm iso})^2$$

isosymmetric QCD

$$R_P(\alpha_{QED}; \{m_{q_i}^{\rm phy}\})$$

physical QCD+QED

$$R_P(\alpha_{\rm QED}; \{m_{q_i}^{\rm phy}\}) - R_P(0; \{m_{q_i}^{\rm iso}\}) = \delta R_{P,\rm sub}^{\rm QED} + \sum_{i=1}^{3} c_i \left((m_{P_i}^{\rm phy})^2 - (m_{P_i}^{\rm iso})^2 \right)$$

QED strong IB

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$R_P(0;\{m_{q_i}^{\rm iso}\}) = (F_P^{\rm iso})^2$	$R_P(\alpha_{QED};\{m_{q_i}^{\rm phy}\})$
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QED　　strong IB

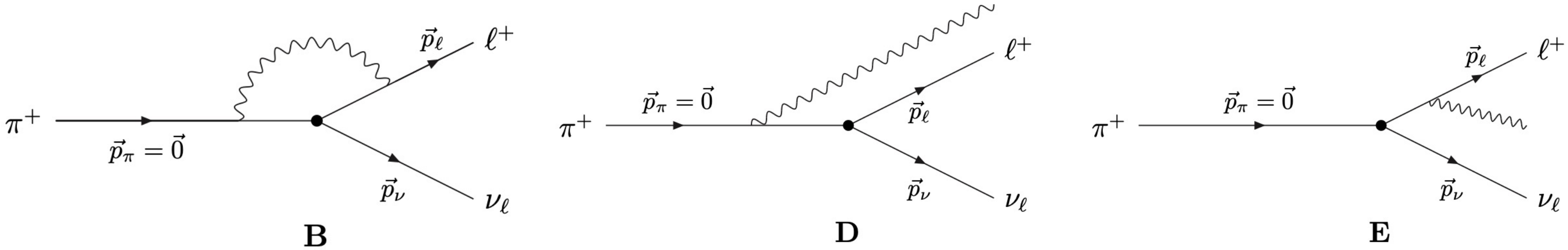
- c_i are determined on lattice. (meaning: dependence of F_P^2 on meson masses)

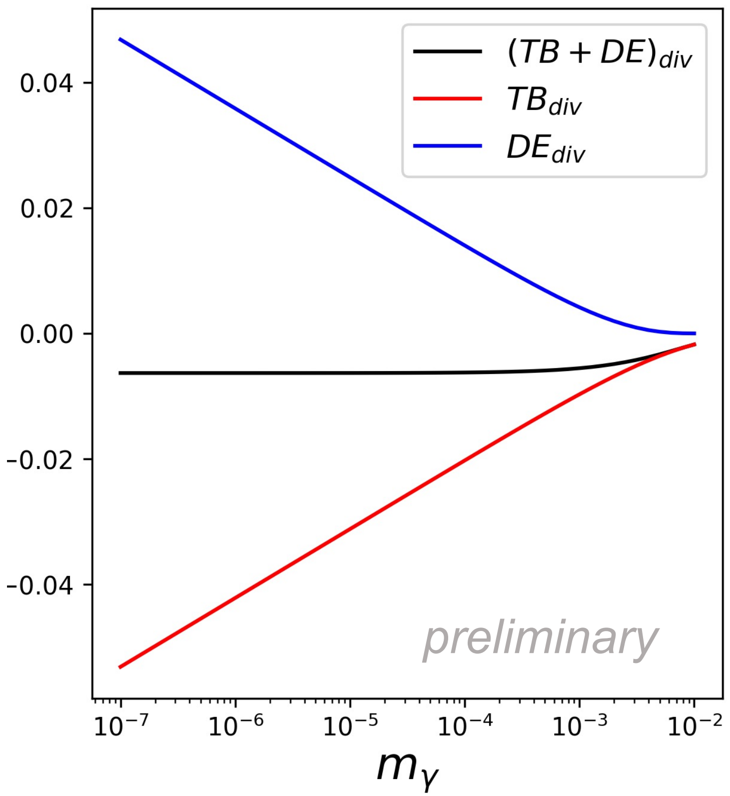
meson	c_1 (for $m_{\pi^\pm}$)	c_2 (for $m_{K^\pm}$)	c_3 (for m_{K^0})
$P=\pi^\pm$	0.0149(15)	0.0011(9)	0.0011(9)
$P=K^\pm$	0.0014(10)	0.0147(9)	-0.0041(9)

48I results

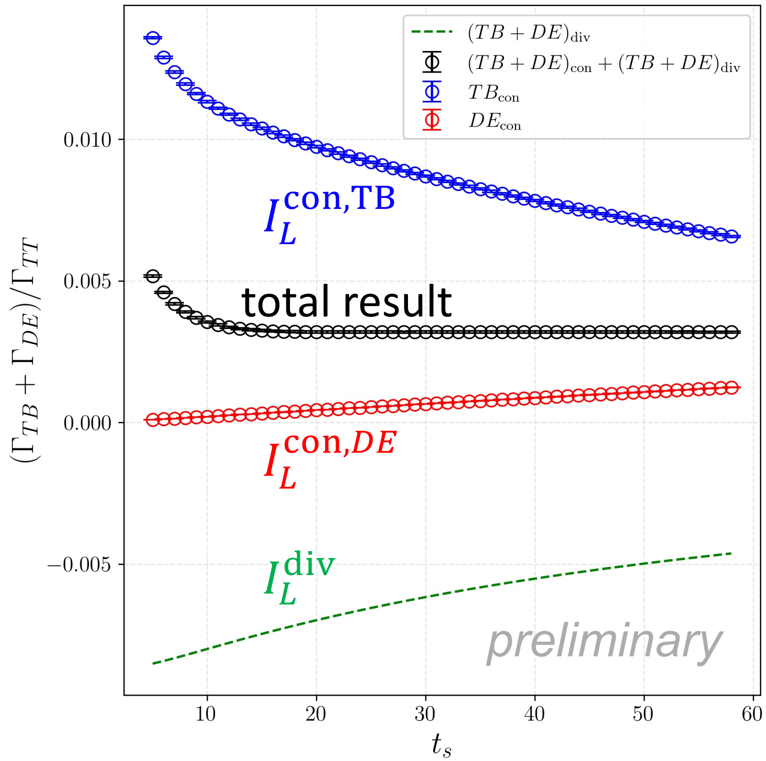
In ChPT, c_1 for pion and c_2 for kaon are given by the same LEC $L_5^r(\mu)$.

QED calculation example: TB+DE





I_L^{div}: analytical calculation



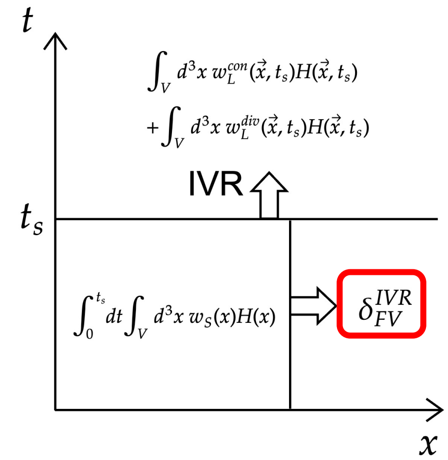
total result for TB+DE independent of t_s

Finite-volume effects

- FV effects in IVR method (48I, $m_\pi L \sim 3.8$):

$\delta_{\mathrm{FV}}^{\mathrm{IVR,pt}}$: point-particle approximation, $F^{(\pi)}(q^2) = 1$

$\delta_{\mathrm{FV}}^{\mathrm{IVR,SD}}$: structure-dependent, $F^{(\pi)}(q^2) = 1 + \frac{\langle r_\pi^2 \rangle}{6} \mathrm{q}^2$



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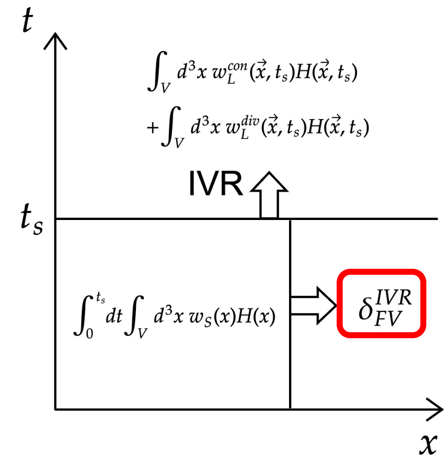
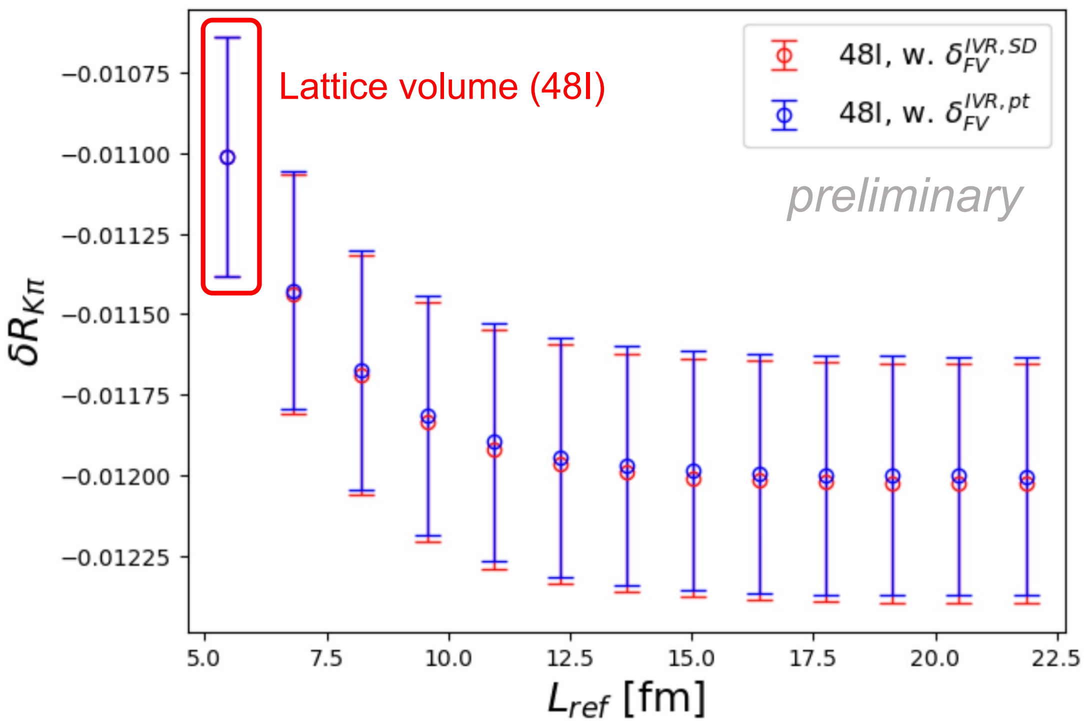
- $O\left(e^{-mL/2}\right)$ convergence with volume $V = L_{\mathrm{ref}}^3$:



	π	K
$\delta_{\mathrm{FV}}^{\mathrm{IVR,\,pt}}/0.01$	$+7.90\%$	$+0.004\%$
$\delta_{\mathrm{FV}}^{\mathrm{IVR,\,SD}}/0.01$	$+7.82\%$	$+0.008\%$



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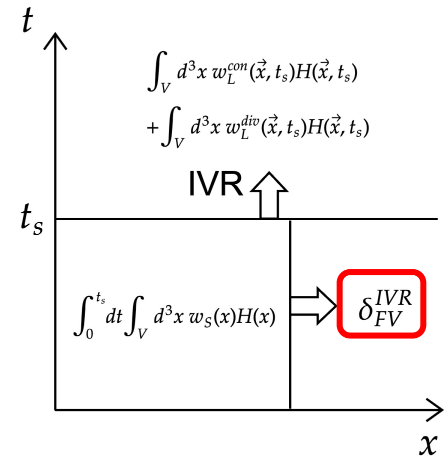
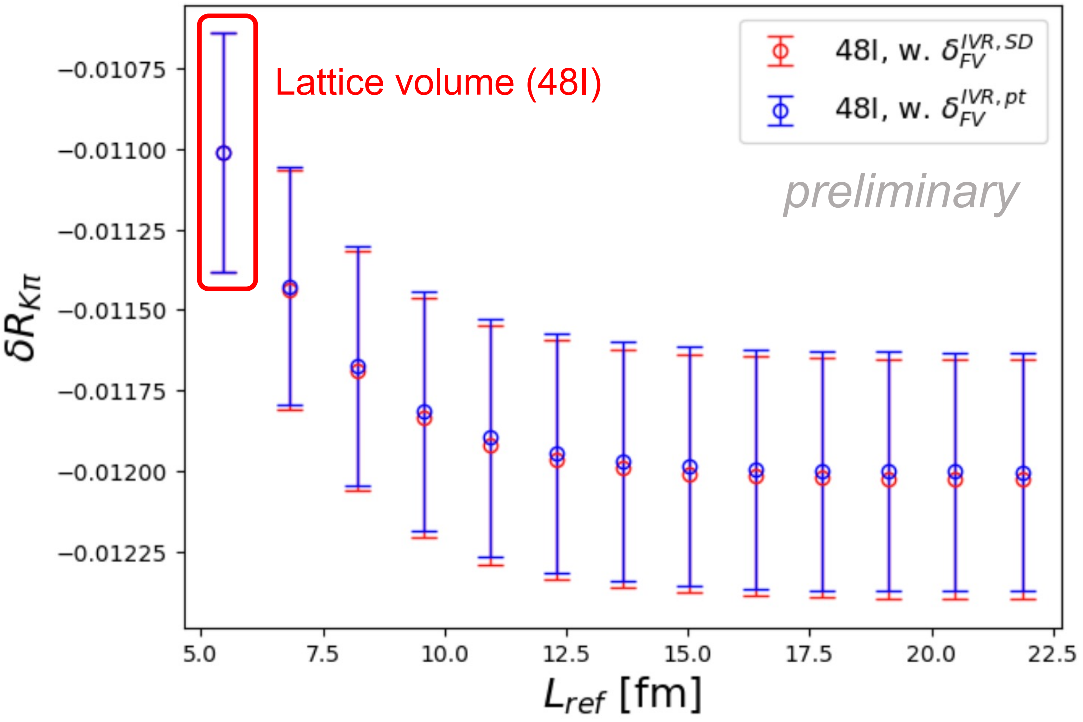




	π	K
$\delta_{\mathrm{FV}}^{\mathrm{IVR,\,pt}}/0.01$	$+7.90\%$	$+0.004\%$
$\delta_{\mathrm{FV}}^{\mathrm{IVR,\,SD}}/0.01$	$+7.82\%$	$+0.008\%$

Comparison:

$\delta_{\mathrm{FV}}^{\mathrm{QED_L,pt}}$ at $O\left(\frac{1}{L^3}\right)$ ~45% error

⬇

$\delta_{\mathrm{FV}}^{\mathrm{IVR,pt}}$~8% error

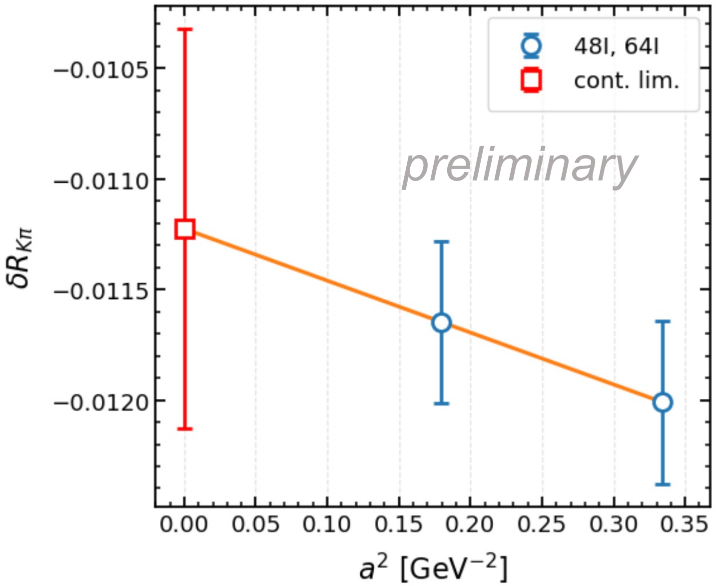
$\delta_{\mathrm{FV}}^{\mathrm{IVR,SD}} - \delta_{\mathrm{FV}}^{\mathrm{IVR,pt}}$~0.08% error

Result for $\delta R_{K\pi}$

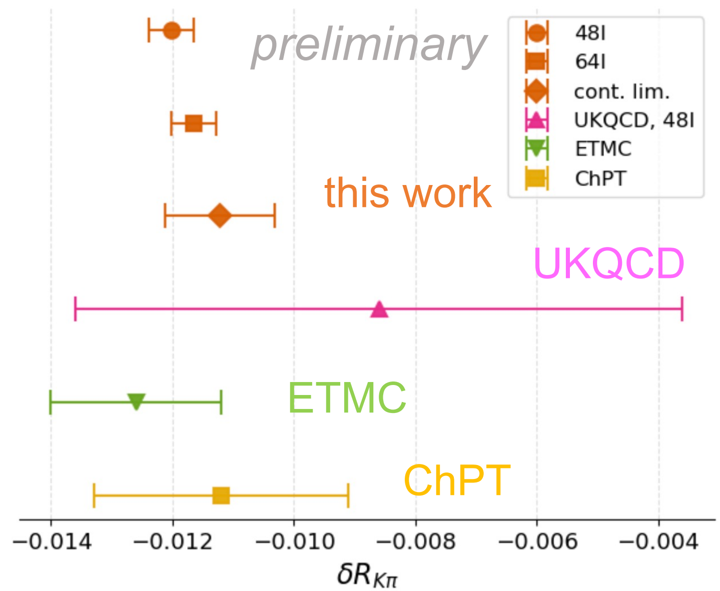
$$R_P(\alpha_{\rm QED};\{m_{q_i}^{\rm phy}\}) - R_P(0;\{m_{q_i}^{\rm iso}\}) = \delta R_{P,\rm sub}^{\rm QED} + \sum_{i=1}^{3} c_i \left((m_{P_i}^{\rm phy})^2 - (m_{P_i}^{\rm iso})^2\right)$$

- First lattice result with errors below $O(10\%)$

Continuum extrapolation



Comparison with literature



$$\delta R_{K\pi} = -0.01123(91)$$

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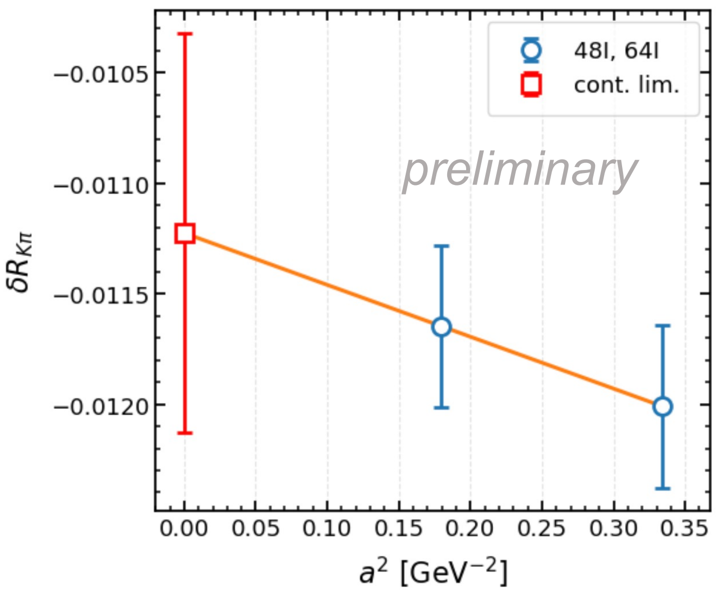

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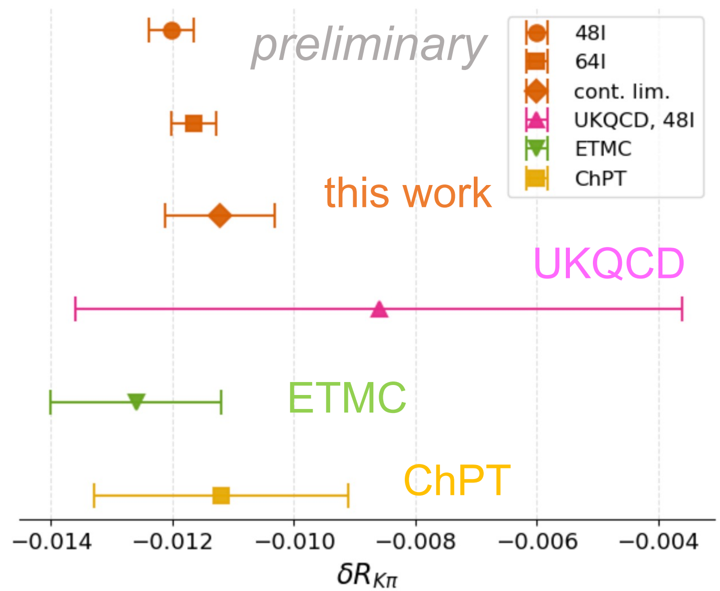

$$\delta R_{K\pi} = -0.01123(91)$$

$$|V_{us}|/|V_{ud}| = 0.23184(28)_{\rm exp}\,(10)_{\delta R_{K\pi}}(65)_{f_P^{\rm iso}}$$

- Compare: previous lattice result

$$|V_{us}|/|V_{ud}| = 0.23154(28)_{\rm exp}\,(15)_{\delta R_{K\pi}}(45)_{\delta R_{K\pi},\rm vol}(65)_{f_P^{\rm iso}}$$

- From $0^+ \to 0^+$ β decay: $|V_{us}|/|V_{ud}| = 0.2341(15)$

Outline

Conclusions

- Isospin breaking, though small, plays a crucial role in precision frontier — from baryon mass splitting to weak decays.
- The IVR method enables QED corrections without large finite-volume effects. The similar idea can be extended to general physical processes with photon and lepton propagators.
- A new scheme-independent approach for strong IB allows consistent comparison across lattice calculations.
- Applied to baryon mass differences, our method successfully predict the mass differences for n, p, Σ, Ξ, and check the Coleman–Glashow mass relation.
- Applied to leptonic decays, we obtain the first lattice result with sub-10% accuracy.