Gluon saturation and exclusive vector meson production at small-x

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[Boussarie, M.F., Szymanowski, Wallon. Phys. Rev. Lett. 134 (2025) 4]

[Boussarie, M.F., Szymanowski, Wallon. Phys. Rev. D 111 (2025) 1]

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

ePIC and EIC Physics Readiness Workshop,
UNICAL, 17 March 2026

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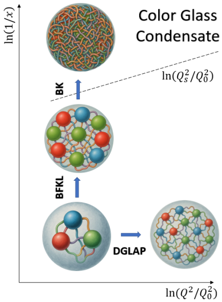
PROGRAMME

Potential of the Electron-Ion Collider (EIC)

- **Dense gluonic matter**



Small-x → High-density regime

Saturation ↓

Collective & non-linear dynamics

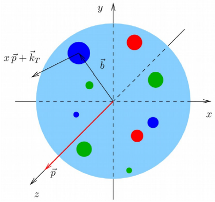
↓

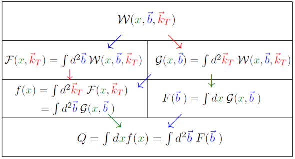
Color Glass Condensate

- **Multi-dimensional structure**

(PDF) [Maria Pia's talk]

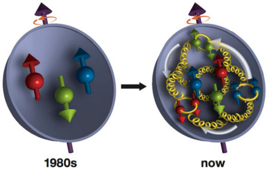
(TMD) [Marco's talk]





$\mathcal{W}(x,\vec{b},\vec{k}_T)$	
$\mathcal{F}(x,\vec{k}_T)=\int d^2\vec{b}\,\mathcal{W}(x,\vec{b},\vec{k}_T)$	$\mathcal{G}(x,\vec{b})=\int d^2\vec{k}_T\,\mathcal{W}(x,\vec{b},\vec{k}_T)$
$f(x)=\int d^2\vec{k}_T\,\mathcal{F}(x,\vec{k}_T)$ $=\int d^2\vec{b}\,\mathcal{G}(x,\vec{b})$	$F(\vec{b})=\int dx\,\mathcal{G}(x,\vec{b})$
$Q=\int dx\,f(x)=\int d^2\vec{b}\,F(\vec{b})$	

- **Proton spin**



Ji's decomposition

$$\frac{1}{2}=\frac{1}{2}\Sigma+L_q+J_g$$

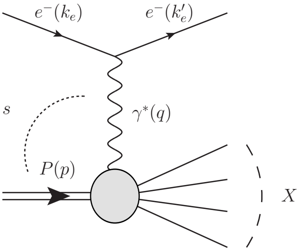
$\Sigma \rightarrow$ Quark spin

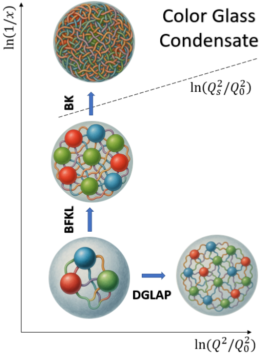
$L_q \rightarrow$ Orbit. angular momenta

$J_g \rightarrow$ Tot. angular momenta

Deep Inelastic scattering

- **Deep Inelastic Scattering (DIS)**

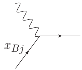




- DIS variables

$$Q = \sqrt{-q^2} \longrightarrow \text{related to the transverse resolution } \Delta x_\perp = \frac{1}{Q}$$

$$x_{Bj} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{Q^2 + s}$$



(naive parton model)

Saturation physics

- **DIS in k_T-factorization**

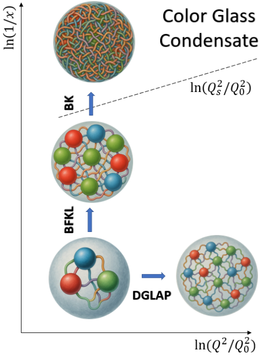
$$\sigma_{\gamma^* P}(x_{Bj}) = \Phi_{\gamma^*\gamma^*}(\vec{k}\,) \otimes_{\vec{k}} \mathcal{F}(x_{Bj}, \vec{k})$$

$$\downarrow$$

$$\sigma_{\gamma^* P}(x_{Bj}) \sim \left(\frac{s}{Q^2}\right)^{\omega_0} = \left(\frac{1}{x_{Bj}}\right)^{\omega_0}$$

- Martin-Froissart bound

$$\sigma_{tot} \lesssim c \ln^2 s$$

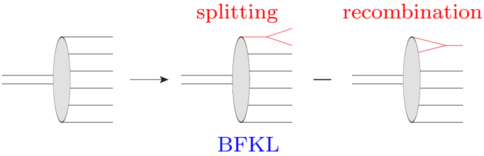


- Violation of Martin-Froissart bound → breakdown of the **unitarity**
- Physically interpretable as a infinite growth of the unintegrated gluon density at small value of the Bjorken-x

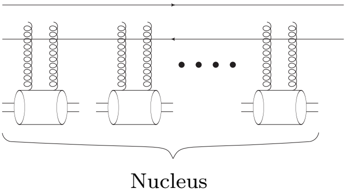
$$\Delta x_\perp = \frac{1}{Q}$$

Saturation physics

- **Saturation effects**
  - *i.* Very dense system $\Longrightarrow$ *Recombination effects*
    [Gribov, Levin, Ryskin (1981-1983)] [Mueller and Qiu (1985)]



  - *ii.* In large nuclei $\Longrightarrow$ *Multiple re-scattering* ($\alpha_s^2 A^{1/3}$ resummation)
    [Glauber (1959)—Gribov (1969)] [Kovchegov (1999)]



- **Dense QCD** system → *collective* and *non-linear* dynamics

Color glass condensate

- Characteristic **saturation scale**

  $$Q_s^2 \sim \left(\frac{A}{x}\right)^{1/3} \Lambda^2_{\text{QCD}} \qquad \alpha_s(Q_s^2) \ll 1 \implies \textit{Weakly coupled QCD}$$

  Saturation window: $Q^2 \lesssim Q_s^2$

- The small-x gluon field can be obtained by solving the classical Yang-Mills equation with a model for the color charge density

  [McLerran, Venugopalan (1994)]

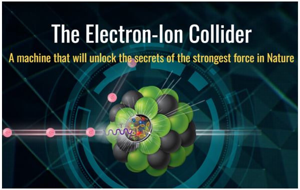
- *Quantum corrections* to MV model → non-linear small-x evolution
- The solution that is obtained has three main properties:

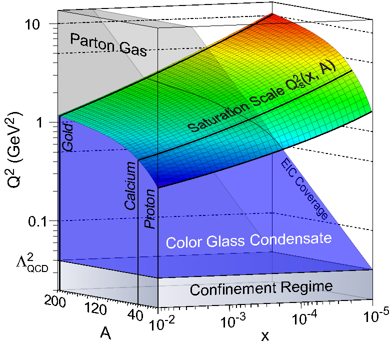
  i. **Color** → dominated by colored particle (gluons)

  ii. **Glass** → well-separated time scales between small-x and large-x, with this latter appearing as "frozen"

  iii. **Condensate** → very high-density of gluons

Saturation at the EIC





- Saturation scale

$$Q_s^2(A,x) \sim \left(\frac{A}{x}\right)^{1/3} \Lambda^2_{\rm QCD}$$

- **Perturbative control on gluonic saturation**

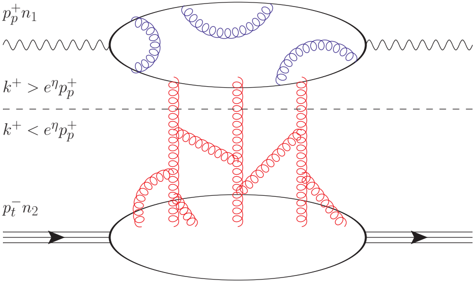
$$\Lambda^2_{\rm QCD} \ll Q^2 \lesssim Q_s^2$$

- At the **EIC** the saturation scale Q_s will be in the perturbative range

- At the **LHC** the saturation can be tested in Ultra Peripheral Collision

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$



$$p_p = p_p^+ n_1 - \frac{Q^2}{2p_p^+} n_2$$

$$p_t = \frac{m_t^2}{2p_t^-} n_1 + p_t^- n_2$$

$$p_p^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$$

$$n_1^2 = n_2^2 = 0 \quad n_1 \cdot n_2 = 1$$

- Separation of the gluonic field into "fast" (quantum) part and "slow" (classical) part through a rapidity parameter $\eta < 0$

[McLerran, Venugopalan (1994)] [Balitsky (1996-2001)]

$$\mathcal{A}^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k}) \qquad e^\eta \ll 1$$

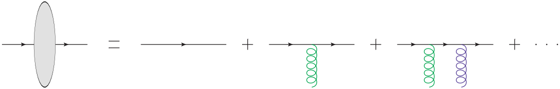
- Large longitudinal Boost: $\Lambda = \sqrt{(1+\beta)/(1-\beta)} \sim \sqrt{s}$

$$b^\mu(x^+, x^-, \vec{x}) = \delta(x^+)\mathbf{B}(\vec{x}) n_2^\mu + \mathcal{O}\left(\Lambda^{-1}\right) \quad \leftarrow \textbf{Shockwave} \text{ approximation}$$

Shockwave approach

- Multiple interactions with the target → **path-ordered Wilson lines**

$$V^{\eta}_{\vec{z}_i} = \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dz_i^{+} b_{\eta}^{-} \left(z_i^{+}, \vec{z}_i \right) \right]$$



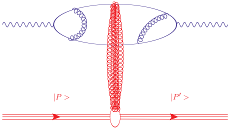
$$V_{\vec{z}_i} = 1 + ig \int_{-\infty}^{+\infty} dz_i^{+} {\color{green} b_{\eta}^{-} \left(z_i^{+}, \vec{z}_i \right)} + (ig)^2 \int_{-\infty}^{+\infty} dz_i^{+} dz_j^{+} {\color{green} b_{\eta}^{-} \left(z_i^{+}, \vec{z}_i \right)} {\color{blue} b_{\eta}^{-} \left(z_j^{+}, \vec{z}_i \right)} \theta \left(z_{ij}^{+} \right) + \cdots$$

- Factorization in the Shockwave approximation

$$\mathcal{M}^{\eta} = N_c \int d^d z_{1\perp} d^d z_{2\perp} {\color{blue} \Phi^{\eta}(z_{1\perp}, z_{2\perp})} {\color{red} \left\langle P' \left| \left[\frac{1}{N_c} \mathrm{Tr} \left(V^{\eta}_{\vec{z}_1} V^{\eta\dagger}_{\vec{z}_2} \right) - 1 \right] (\vec{z}_1, \vec{z}_2) \right| P \right\rangle}$$

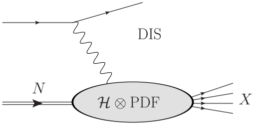
- **Dipole operator**

$$\mathcal{U}^{\eta}_{ij} = \frac{1}{N_c} \mathrm{Tr} \left(V^{\eta}_{\vec{z}_i} V^{\eta\dagger}_{\vec{z}_j} \right) - 1$$

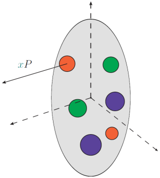


- Renormalization group equation of $\mathcal{U}^{\eta}_{ij}$ (B-JIMWLK, BK, BFKL equations)

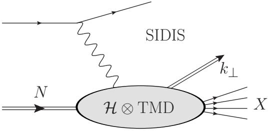
Inclusive and exclusive processes



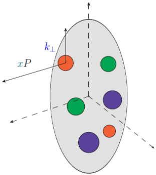
Parton Distribution Function



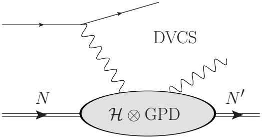
- Moderate x, $Q^2 \to \infty$ (Bjorken)
  $\Longrightarrow$ Collinear Factorization
  [Maria Pia's talk]
- Q^2-fixed, $x \to 0$ (Regge)
  $\Longrightarrow$ small-x improved PDF
  [Bonvini (2018)]



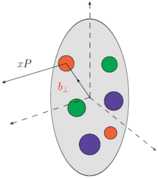
Transverse Momentum Distribution



- Moderate x, small $k_\perp$
  $\Longrightarrow$ TMD Factorization
  [Marco's talk]
- Small x
  $\Longrightarrow$ k_T-factorization



Generalized Parton Distribution



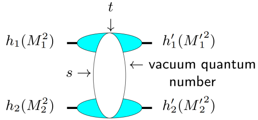
- Exclusive Collinear Factorization
  [Ji (1997)] [Radyushkin (1997)]
- Small-x description of exclusive processes $\Longrightarrow$ This talk

Diffractive processes

- Exclusive processes at small-x

$\downarrow$

Rapidity gap



- **Hard Pomeron** exchange at the **amplitudes** level

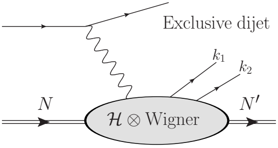
$\Longrightarrow$ **Enhanced sensitivity** to small-x effects

- Solid framework to investigate at high-energy Collinear-factorization-breaking exclusive processes (i.e. with GPD)

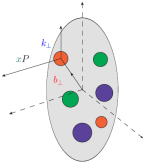
e.g. $\gamma N \rightarrow \pi^0 \gamma N'$ [Nabeebaccus, Schoenleber, Szymanowski, Wallon (2025)]

- Exclusive processes at small-x are sensitive to the Wigner distribution

Diffractive dijet & diffractive SIDIS [Hatta, Xiao, Yuan (2016, 2020)]
(NLO) [Boussarie, Fucilla, Grabovsky, Li, Szymanowski, Wallon (2016, 2023)]



Wigner Distribution



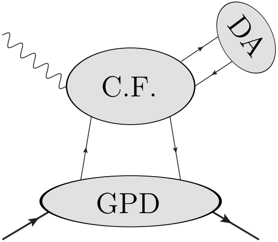
Deeply virtual meson production (DVMP)

- Exclusive vector meson leptoproduction

  $\gamma^{(*)}(p_\gamma) + P(p_0) \to \rho(p_\rho) + P(p_0')$

- Extensively studied at HERA

  [Kowalski, Motyka, Watt (2006)]



- Collinear factorization proven at all order for

$$\gamma^{(*)}(p_\gamma) + P(p_0) \to \rho_L(p_\rho) + P(p_0')$$

[Collins, Frankfurt, Strikman (1997)] [Radyushkin (1997)]

- NLO corrections in Collinear Factorization

  [D.Yu. Ivanov, L. Szymanowski, G. Krasnikov (2004)]

- NLO corrections to the production of a longitudinally polarized ρ-meson at small-x

  [Ivanov, Kotsky, Papa (2004)]

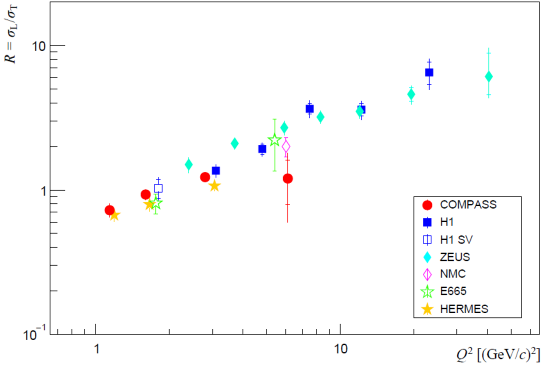
  [Boussarie, Grabovsky, Ivanov, Szymanowski, Wallon (2017)]

  [Mäntysaari, Pentalla (2022)]

Transversely vs longitudinally polarized exclusive ρ-meson leptoproduction

- Exclusive ρ-meson leptoproduction at different experiments

[G. D. Alexeev et al. (2023)]



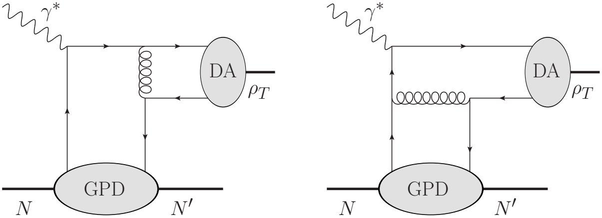
- σ_T starts at the **twist-3**
- Size of σ_T **far larger than expected** (by naive twist-counting)

Transversely polarized ρ-meson production

- The leading DA (twist 2) of ρ_T is **chiral odd** ($\sigma^{\mu\nu}$ coupling)
- The amplitude for $\gamma^* N \to \rho_T N'$ is zero to all order in perturbation theory at the leading twist

[Diehl, Gousset, Pire (1999)] [Collins, Diehl (2000)]

- Lowest order diagrammatic argument:



$$\gamma^\alpha[\gamma^\mu, \gamma^\nu]\gamma_\alpha = 0$$

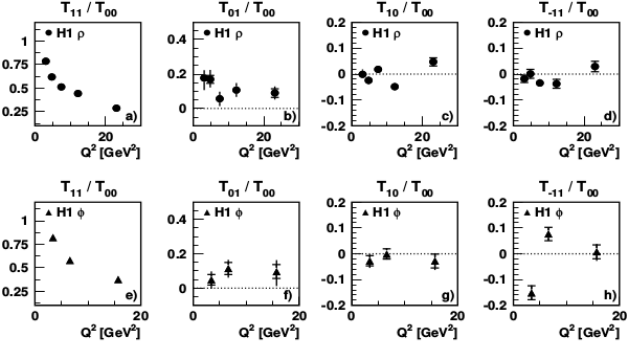
- Transversally polarized vector meson production start at the **twist-3**
- Collinear treatment at the twist-3 leads to **end point singularities**

[Mankiewicz, Piller (2000)] [Anikin, Teryaev (2002)]

Transversely polarized vector meson production

- HERA data for the ρ and ϕ meson [F.D. Aaron et al. (2010)]

$$\gamma^*(\lambda_\gamma)p \to V(\lambda_V)p \qquad \lambda_\gamma = 0, 1, -1 \quad \text{and} \quad \lambda_V = 0, 1, -1$$



"Leading twist description brought the knowledge that hadrons are made of quarks and gluons; if we want to learn *how are they made*, we have to understand higher-twist effects" [David Politzer]

Possible solutions

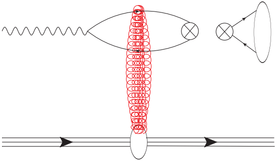
- To solve this issues one should go **beyond collinear factorization**
- One possibility is allowing for a relative transverse momentum between the outgoing quark and anti-quark [Goloskokov, Kroll (2007)]

  Distribution amplitude $\phi(x) \longrightarrow$ Light-cone wavefunction $\phi(x, \vec{k}_T)$
- Alternative: moving to (t-channel) k_T-factorization

  [Anikin, Ivanov, B. Pire, L. Szymanowski, S. Wallon (2010)]

- Small-x EFT

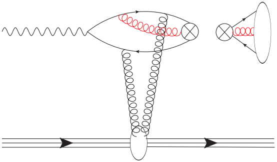
  [Balitsky (1996-2001)]



- It is an all-twist factorization in the t-channel

- Higher-twist collinear factorization

  [Ball, Braun, Koike, Tanaka (1998)]



- Systematic inclusion of higher-twist effects in the s-channel

↘ ↙

- Most general description of the DVMP at high-energy

  [Boussarie, M.F., Szymanowski, Wallon. Phys. Rev. Lett. 134 (2025) 4]

Spin-density matrix in the CGC approach

- Expansion around the forward direction ($\boldsymbol{\Delta} = 0$)

$$A^{L,L} \sim \frac{1}{Q} \qquad A^{T,T}|_{\text{no flip}} \sim \frac{1}{Q}\frac{m_M}{Q} \qquad A^{T,L} \sim \frac{1}{Q}\frac{|\boldsymbol{\Delta}|}{Q}$$

$$A^{L,T} \sim \frac{1}{Q}\frac{m_M}{Q}\frac{|\boldsymbol{\Delta}|}{Q} \qquad A^{T,T}|_{\text{flip}} \sim \frac{1}{Q}\frac{m_M}{Q}\frac{|\boldsymbol{\Delta}|^2}{Q^2}$$

- Hierarchy observed at HERA [F. Aaron et al., H1 coll. (2009)]

$$A^{L,L} > A^{T,T}|_{\text{no flip}} > A^{T,L} > A^{L,T}, A^{T,T}|_{\text{flip}}$$

- Initial condition → **MV model** [McLerran, Venugopalan (1994)]

$$\mathcal{U}^{\eta=0}(\boldsymbol{r}) = 1 - \exp^{-\frac{\boldsymbol{r}^2 Q_0^2}{4}\ln\left(\frac{1}{|\boldsymbol{r}|\Lambda}+e\right)}$$

- Numerical LO BK evolution

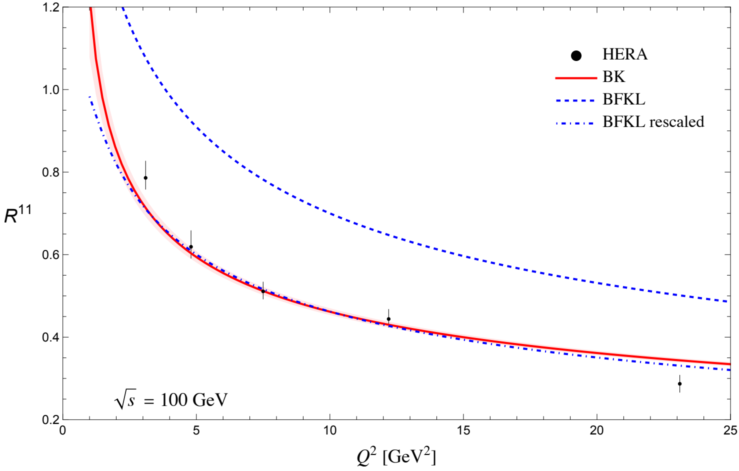
$$\frac{\partial \mathcal{U}^\eta_{12}}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_3 \left(\frac{\vec{z}_{12}^{\,2}}{\vec{z}_{23}^{\,2}\vec{z}_{31}^{\,2}}\right)\left[\underbrace{\mathcal{U}^\eta_{13} + \mathcal{U}^\eta_{32} - \mathcal{U}^\eta_{12}}_{\text{BFKL}} - \delta\, \mathcal{U}^\eta_{13}\mathcal{U}^\eta_{32}\right]$$

- $\delta = 1$ for BK and $\delta = 0$ to investigate the dilute (BFKL) regime
- We also implement running coupling effects and collinear improvement [Ducloue, Iancu, Mueller, Soyez, Triantafyllopoulos (2019)] (comparing [Cougoulic, Korcyl, Stebel (2024)])

Preliminary results: Predictions vs HERA data

- Ratio of helicity amplitudes $R^{11} = \mathcal{A}^{11}/\mathcal{A}^{00}$

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

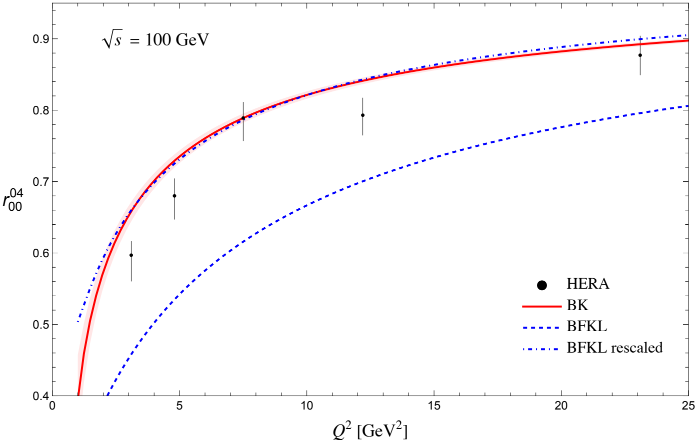


Preliminary results: Predictions vs HERA data

- Spin-density matrix element r_{00}^{04} (ε = photon polarization parameter)

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

$$r_{00}^{04} = \frac{\varepsilon + R_{10}^2}{R_{11}^2 + \varepsilon + R_{10}^2 + R_{-11}^2 + 2\varepsilon R_{01}^2} \simeq \frac{\varepsilon}{R_{11}^2 + \varepsilon}$$



Summary and prospects

Summary

- The EIC will provide a unique opportunity to explore the small-x regime of QCD and the gluon structure of hadrons and nuclei
- Diffractive processes offer **several opportunities** (5-D hadron/ nuclear structure, gluon saturation, higher-twist effects)
- All helicity amplitudes of **exclusive vector meson production**, at small-x, in arbitrary kinematics (including saturation)

  [Boussarie, M.F., Szymanowski, Wallon. Phys. Rev. Lett. 134 (2025) 4]
  [Boussarie, M.F., Szymanowski, Wallon. Phys. Rev. D 111 (2025) 1]
- Partially NLO predictions in the CGC show qualitative agreement with HERA measurements of R^{11} and r_{00}^{04}
- Coming soon: predictions for electron-lead collisions

  [Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

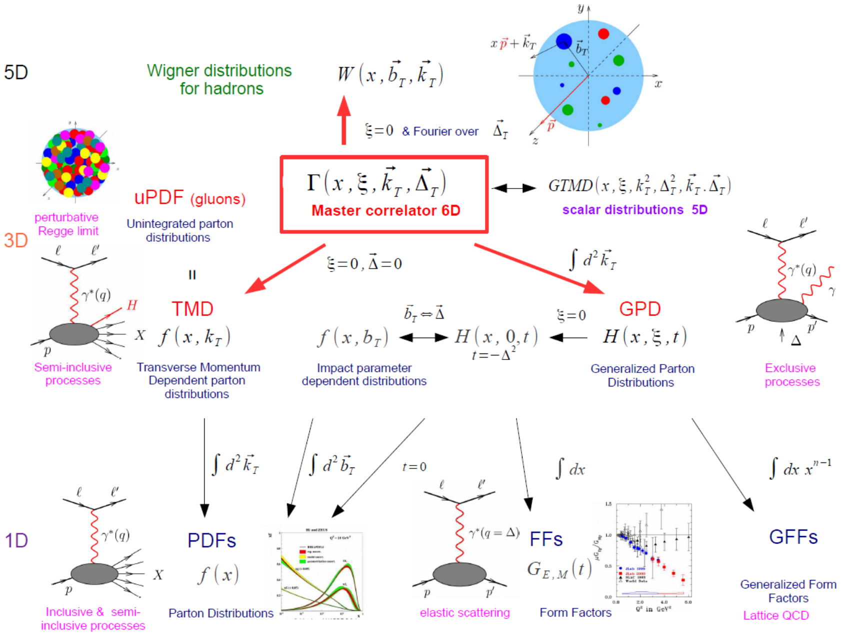
Future prospects

- Study of the total cross-section and full **spin density matrix** of ρ-meson leptoproduction
- Full **NLO corrections** (adding NLO from the impact factors)

Thank you for the attention!

Backup

Multi-dimensional structure of the proton



5D

Wigner distributions for hadrons

$W(x,\vec{b}_T,\vec{k}_T)$

$\xi=0$ & Fourier over $\vec{\Delta}_T$

$\Gamma(x,\xi,\vec{k}_T,\vec{\Delta}_T)$

Master correlator 6D

$\longleftrightarrow$ $GTMD(x,\xi,k_T^2,\Delta_T^2,\vec{k}_T.\vec{\Delta}_T)$

scalar distributions 5D

perturbative Regge limit

uPDF (gluons)

Unintegrated parton distributions

3D

=

$\xi=0,\vec{\Delta}=0$

$\int d^2\vec{k}_T$

TMD

$f(x,k_T)$

Transverse Momentum Dependent parton distributions

$\vec{b}_T \Leftrightarrow \vec{\Delta}$

$f(x,b_T) \longleftrightarrow H(x,0,t)$

$t=-\Delta^2$

Impact parameter dependent distributions

$\xi=0$

GPD

$H(x,\xi,t)$

Generalized Parton Distributions

Semi-inclusive processes

Exclusive processes

$\int d^2\vec{k}_T$

$\int d^2\vec{b}_T$

$t=0$

$\int dx$

$\int dx\, x^{n-1}$

1D

PDFs

$f(x)$

Parton Distributions

Inclusive & semi-inclusive processes

elastic scattering

FFs

$G_{E,M}(t)$

Form Factors

GFFs

Generalized Form Factors

Lattice QCD

The dipole gluon Wigner function

- **Diffractive processes** → golden channels to probe the GTMD gluon distribution $xW\left(x,\vec{q}_\perp,\vec{\Delta}_\perp\right)$
- Dipole gluon Wigner function

$$xW\left(x,\vec{q}_\perp,\vec{b}_\perp\right)=\frac{2}{P^+(2\pi)^3}\int dz^+d^2\vec{z}_\perp\int\frac{d^2\vec{\Delta}_\perp}{(2\pi)^2}e^{i\vec{q}_\perp\cdot\vec{z}_\perp-ixP^-z^+}$$
$$\times\left\langle P+\frac{\vec{\Delta}_\perp}{2}\middle|\mathrm{Tr}\left[U_+F_a^{+i}\left(\vec{b}_\perp+\frac{z}{2}\right)U_-F_a^{+i}\left(\vec{b}_\perp-\frac{z}{2}\right)\right]\middle|P-\frac{\vec{\Delta}_\perp}{2}\right\rangle$$

- Gluon **GTMD distribution** at small-x

[Dominguez, Marquet, Xiao Yuan (2011)]
[Hatta, Xiao, Yuan (2016)]

$$xW\left(x,\vec{q}_\perp,\vec{\Delta}_\perp\right)\approx\frac{2N_c}{\alpha_s}\left(q_\perp^2-\frac{\Delta_\perp^2}{4}\right)S_Y\left(\vec{q}_\perp,\vec{\Delta}_\perp\right)$$

- Fourier transform of the dipole S-matrix

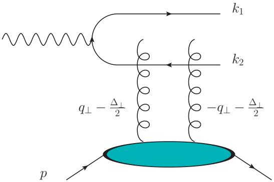
$$S_Y\left(\vec{q}_\perp,\vec{\Delta}_\perp\right)=\int\frac{d^2\vec{r}_\perp d^2\vec{b}_\perp}{(2\pi)^4}e^{i\vec{\Delta}_\perp\cdot\vec{b}_\perp+i\vec{q}_\perp\cdot\vec{r}_\perp}\left\langle\frac{1}{N_c}\mathrm{Tr}\,V\left(\vec{b}_\perp+\frac{\vec{r}_\perp}{2}\right)V^\dagger\left(\vec{b}_\perp-\frac{\vec{r}_\perp}{2}\right)\right\rangle_Y$$

$Y=\ln(1/x)$

Diffractive dijet production

- **Diffractive dijet electroproduction** can be sensitive to both $q_\perp$ and $\Delta_\perp$ dependence of $W\left(x, \vec{q}_\perp, \vec{\Delta}_\perp\right)$

[Hatta, Xiao and Yuan (2016)]



- Possibility of investigating angular correlations between impact parameter and dipole size predicted by small-x evolutions
- Enhanced sensitivity to **gluonic saturation** from contributions beyond the leading order (e.g. two hard quark anti-quark jets accompanied by a softer gluon jet, in the *correlation limit*)

[Iancu, Mueller, Triantafyllopoulos (2022)]

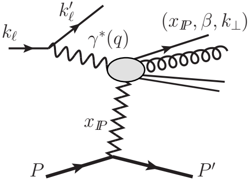
- Even better access to the Wigner distribution is possible via **diffractive dijet photoproduction** in ultraperipheral pA collision

[Hagiwara, Hatta, Pasechnik, Tasevsky, Teryaev (2017)]

Semi-inclusive diffractive DIS (SIDDIS)

- At small-x, the quark and gluon *TMD distribution functions* are directly related to the *color dipole* S-matrix in the CGC formalism
- A similarly connection between the **diffractive parton distribution functions (DPDFs)** and the color dipole exists
- One of the best processes to investigate this connection is the **semi-inclusive diffractive DIS (SIDDIS)**

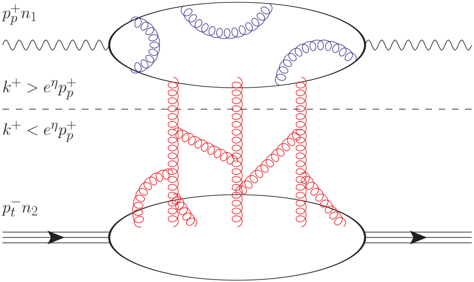
[Hatta, Xiao and Yuan (2022)]



- pQCD motivated initial inputs for collinear QCD evolution of DPDFs
- Study the matching between small-x and moderate-x regime
- **Goal**: extend all these studies at the full NLO level

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$
- n_1^μ, n_2^μ are light-cone vectors (+/- directions)



$$p_p = p_p^+ n_1 - \frac{Q^2}{2p_p^+} n_2$$

$$p_t = \frac{m_t^2}{2p_t^-} n_1 + p_t^- n_2$$

$$p_p^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$$

- Separation of the gluonic field into “fast" (quantum) part and “slow" (classical) part through a rapidity parameter $\eta < 0$

[McLerran, Venugopalan (1994)] [Balitsky (1996-2001)]

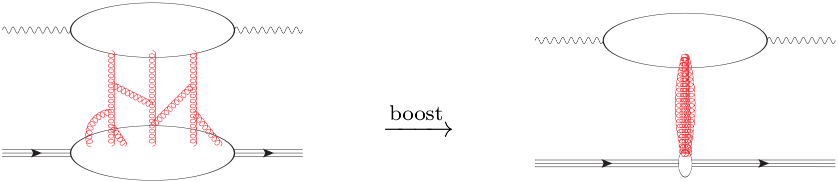
$$\mathcal{A}^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k})$$

$e^\eta \ll 1$

Shockwave approach

- Large longitudinal Boost: $\Lambda = \sqrt{\frac{1+\beta}{1-\beta}} \sim \frac{\sqrt{s}}{m_t}$

$$\begin{cases} b^+(x^+,x^-,\vec{x}) & = \Lambda^{-1} b_0^+(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \\ b^-(x^+,x^-,\vec{x}) & = \Lambda b_0^-(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \\ b^i(x^+,x^-,\vec{x}) & = b_0^i(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \end{cases}$$



$b_0^\mu(x)$

$b^\mu(x^+,x^-,\vec{x}) = \delta(x^+)\mathbf{B}(\vec{x})n_2^\mu + \mathcal{O}\left(\Lambda^{-1}\right)$

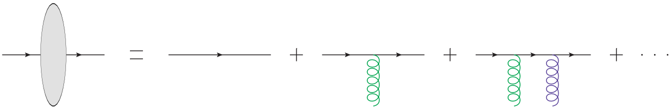
Shockwave approximation

- Independence from x^- $\implies$ conservation of p^+ (**eikonal approximation**)
- Light-cone gauge $A \cdot n_2 = 0 \implies A \cdot b = 0 \implies$ *Simple effective Lagrangian*

Shockwave approach

- Multiple interactions with the target → ***path-ordered Wilson lines***

$$V^{\eta}_{\vec{z}_i} = \mathcal{P}\exp\left[ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+,\vec{z}_i\right)\right]$$



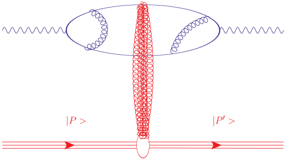
$$V_{\vec{z}_i} = 1 + ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+,\vec{z}_i\right) + (ig)^2\int_{-\infty}^{+\infty} dz_i^+ dz_j^+ b_\eta^-\left(z_i^+,\vec{z}_i\right) b_\eta^-\left(z_j^+,\vec{z}_i\right)\theta\left(z_{ij}^+\right) + \cdots$$

- Factorization in the Shockwave approximation

$$\mathcal{M}^\eta = N_c\int d^d z_{1\perp} d^d z_{2\perp}\,\Phi^\eta(z_{1\perp},z_{2\perp})\left\langle P'\left|\left[\frac{1}{N_c}\mathrm{Tr}\left(V^\eta_{\vec{z}_1}V^{\eta\dagger}_{\vec{z}_2}\right)-1\right](\vec{z}_1,\vec{z}_2)\right|P\right\rangle$$

- ***Dipole operator***

$$\mathcal{U}^\eta_{ij} = \frac{1}{N_c}\mathrm{Tr}\left(V^\eta_{\vec{z}_i}V^{\eta\dagger}_{\vec{z}_j}\right) - 1$$



Balitsky-JIMWLK evolution equations

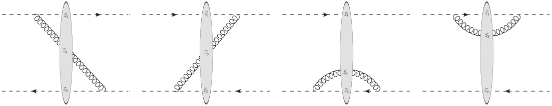
- ***Balitsky-JIMWLK evolution equations*** for the dipole

  [Balitsky — Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

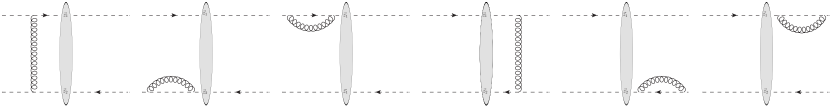
$$\frac{\partial \mathcal{U}^{\eta}_{12}}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \vec{z}_3 \left(\frac{\vec{z}_{12}^{\,2}}{\vec{z}_{23}^{\,2}\, \vec{z}_{31}^{\,2}} \right) \left[\underbrace{\mathcal{U}^{\eta}_{13} + \mathcal{U}^{\eta}_{32} - \mathcal{U}^{\eta}_{12}}_{\text{BFKL}} - \mathcal{U}^{\eta}_{13} \mathcal{U}^{\eta}_{32} \right]$$

$$\frac{\partial \mathcal{U}^{\eta}_{13} \mathcal{U}^{\eta}_{32}}{\partial \eta} = \cdots \qquad \longleftarrow \quad \textit{Balitsky hierarchy}$$

- Double dipole contribution and Dipole contribution



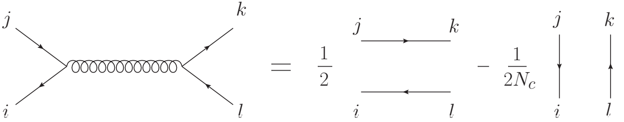
- Dipole contribution



- Gluon with rapidity slightly above the cut-off: $\eta + \Delta\eta$

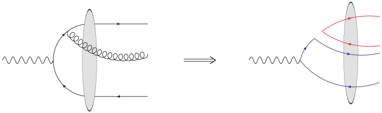
Balitsky-Kovchegov evolution equation

- Large-N_c limit ['t Hooft (1974)]



$$\boxed{t^a_{ij}t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)}$$

- Double dipole → Dipole × dipole



$$\langle \mathcal{U}^\eta_{13}\mathcal{U}^\eta_{32}\rangle \to \langle \mathcal{U}^\eta_{13}\rangle\,\langle \mathcal{U}^\eta_{32}\rangle$$

- Hierarchy of equations broken → closed non-linear BK equation

[Balitsky (1995)] [Mueller (1994-1995)] [Kovchegov (1999)]

$$\frac{\partial\langle \mathcal{U}^\eta_{12}\rangle}{\partial\eta} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_3\left(\frac{\vec{z}^{\,2}_{12}}{\vec{z}^{\,2}_{23}\vec{z}^{\,2}_{31}}\right)\left[\langle \mathcal{U}^\eta_{13}\rangle + \langle \mathcal{U}^\eta_{32}\rangle - \langle \mathcal{U}^\eta_{12}\rangle - \langle \mathcal{U}^\eta_{13}\rangle\langle \mathcal{U}^\eta_{32}\rangle\right]$$

with $\langle \mathcal{U}^\eta_{12}\rangle \equiv \langle P'|\mathcal{U}^\eta_{12}|P\rangle$

Saturation scale

- Cross-section for $gg \to g$

$$\sigma_{gg\to g}\left(x,Q^2\right) \sim \alpha_s \frac{xg\left(x,Q^2\right)}{Q^2}$$

- **Recombination probability**

$$\Gamma(x,Q^2) \sim \alpha_s \frac{xg\left(x,Q^2\right)}{Q^2}\frac{1}{\pi R_p^2}$$

- **Saturation scale** → Recombination probability of order one

$$\Gamma(x,Q_s^2) \sim 1 \longrightarrow Q_s^2(x) \sim \alpha_s \frac{xg\left(x,Q^2\right)}{\pi R_p^2}$$

- $1/x$-dependence of the gluon distribution coming from LLA BFKL

$$Q_s^2(x) \propto \left(\frac{1}{x}\right)^{\omega_0} \implies \ln Q_s^2\left(Y\right) = \omega_0 Y + K$$

- Nucleus

$$xg\left(x,Q_s^2\right) \longrightarrow xg_A\left(x,Q_s^2\right) \sim Axg\left(x,Q_s^2\right) \sim A(1/x)^{\omega_0}$$

- **Nucleus saturation scale**

$$Q_s^2(x) \sim A^{1/3}x^{-\omega_0}\Lambda_{QCD}^2 \sim \left(\frac{A}{x}\right)^{1/3}\Lambda_{QCD}^2$$

Higher-twist ρ-meson DAs

- Twist and chiral classification of the ρ-meson distribution amplitudes

[Ball, Braun, Koike, Tanaka (1998)]

- Two body distribution amplitudes

Twist	2 $O(1)$	3 $O(1/Q)$	4 $O(1/Q^2)$
$e_\parallel$	$\phi_\parallel$	$\underline{h_\parallel^{(t)}}, \underline{h_\parallel^{(s)}}$	g_3
$e_\perp$	$\underline{\phi_\perp}$	$g_\perp^{(v)}, g_\perp^{(a)}$	$\underline{h_3}$

- $g_\perp^{(v)}$ and $g_\perp^{(a)}$ are the vector and axial two-body twist-3 DA

$$\langle M(p_M)|\overline{\psi}(z)\Gamma_\lambda\,[z,0]\,\psi(0)|0\rangle$$

- Vector and axial three-body twist-3 DA

$$\langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\,[z,tz]\,gF^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle$$

$$\langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\,[z,tz]\,g\tilde{F}^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle$$

Theoretical framework

- **Effective shockwave background field operators**

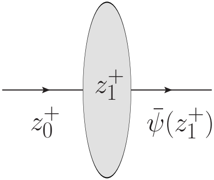
$$\left[\psi_{\mathrm{eff}}\ (z_0)\right]_{z_0^+<0} = \psi\left(z_0\right) - \int \mathrm{d}^D z_2 G_0\left(z_{02}\right)\left(V^{\dagger}_{\boldsymbol{z}_2} - 1\right)\gamma^+\psi\left(z_2\right)\delta(z_2^+)$$

$$\left[\bar{\psi}_{\mathrm{eff}}\ (z_0)\right]_{z_0^+<0} = \bar{\psi}\left(z_0\right) + \int \mathrm{d}^D z_1 \bar{\psi}\left(z_1\right)\gamma^+\left(V_{\boldsymbol{z}_1} - 1\right)G_0\left(z_{10}\right)\delta(z_1^+)$$

$$\left[A^{\mu a}_{\mathrm{eff}}\ (z_0)\right]_{z_0^+<0} = A^{\mu a}\left(z_0\right) + 2i\int \mathrm{d}^D z_3 \delta(z_3^+) F^b_{-\sigma}\left(z_3\right) G^{\mu\sigma_\perp}\left(z_{30}\right)\left(U^{ab}_{\boldsymbol{z}_3} - \delta^{ab}\right)$$

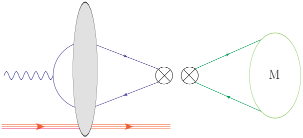
Such operators serve to construct amplitudes involving non-perturbative matrix elements of general off light-cone correlators, i.e. **without any reference to the twist-expansion**

- Example: Antiquark effective operator
  - A fermionic line starts at the light-cone time $z_0^+ < 0$
  - Freely propagates to $z_1^+ = 0$
  - It interacts eikonally at z_1^+ with the background field



ρ_T-meson production: factorization

- **Two-body contribution**

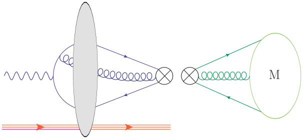


$$\mathcal{A}_2 = ie_f \int \mathrm{d}^D z_0 \int \mathrm{d}^D z_1 \int \mathrm{d}^D z_2 \theta(-z_0^+)\delta(z_1^+)\delta(z_2^+)\left\langle M\left(p_M\right)\left|\overline{\psi}\left(z_1\right)\Gamma^\lambda \psi\left(z_2\right)\right|0\right\rangle$$

$$\times\left\langle P\left(p'\right)\left|1-\frac{1}{N_c}\mathrm{tr}\left(V_{\boldsymbol{z}_1}V^\dagger_{\boldsymbol{z}_2}\right)\right|P\left(p\right)\right\rangle \frac{1}{4}\mathrm{tr}_D\left[\gamma^+ G_0\left(z_{10}\right)\hat{\varepsilon}_q \mathrm{e}^{-i(q\cdot z_0)} G_0\left(z_{02}\right)\gamma^+\Gamma_\lambda\right]$$

Hard part

- **Three-body contribution**



$$\mathcal{A}_{q3} = -ie_q \int \mathrm{d}^D z_4 \mathrm{d}^D z_3 \mathrm{d}^D z_2 \mathrm{d}^D z_1 \mathrm{d}^D z_0 \theta(-z_4^+)\delta(z_3^+)\delta(z_2^+)\delta(z_1^+)\theta\left(-z_0^+\right)\mathrm{e}^{-i(q\cdot z_0)}$$

$$\times\left\langle M\left(p_M\right)\left|\overline{\psi}\left(z_1\right)\Gamma^\lambda g F_{-\sigma}\left(z_3\right)\psi\left(z_2\right)\right|0\right\rangle\left\langle P\left(p'\right)\left|\mathrm{tr}\left(V_{\boldsymbol{z}_1}t^a V^\dagger_{\boldsymbol{z}_2}t^b U^{ab}_{\boldsymbol{z}_3}\right)\right|P\left(p\right)\right\rangle$$

$$\times\frac{1}{N_c^2-1}\mathrm{tr}_D\left[\gamma^+ G_0\left(z_{14}\right)\gamma_\mu G^{\mu\sigma_\perp}\left(z_{34}\right)G_0\left(z_{40}\right)\hat{\varepsilon}_q G_0\left(z_{02}\right)\gamma^+\Gamma_\lambda\right] - \text{n.i.}$$

Results: two-body contribution

- **Dipole amplitude**

$$\mathcal{A}_2 = \int_0^1 \mathrm{d}x \int \mathrm{d}^2\boldsymbol{r}\Psi\left(x,\boldsymbol{r}\right) \int \mathrm{d}^d\boldsymbol{b}\, \mathrm{e}^{i(\boldsymbol{q}-\boldsymbol{p}_M)\cdot\boldsymbol{b}} \left\langle P\left(p'\right) \middle| 1 - \frac{1}{N_c}\mathrm{tr}\left(V_{\boldsymbol{b}+\bar{x}\boldsymbol{r}}V^{\dagger}_{\boldsymbol{b}-x\boldsymbol{r}}\right) \middle| P\left(p\right)\right\rangle$$

- **Coordinate-space impact factor**

$$\Psi_2\left(x,\boldsymbol{r}\right) = e_q\delta\left(1-\frac{p_M^+}{q^+}\right)\left(\varepsilon_{q\mu}-\frac{\varepsilon_q^+}{q^+}q_\mu\right)$$

$$\times\left[\phi_{\gamma^+}(x,\boldsymbol{r})\left(2x\bar{x}q^\mu - i(x-\bar{x})\frac{\partial}{\partial r_{\perp\mu}}\right) + \epsilon^{\mu\nu+-}\phi_{\gamma^+\gamma^5}(x,\boldsymbol{r})\frac{\partial}{\partial r_\perp^\nu}\right]K_0\left(\sqrt{x\bar{x}Q^2\boldsymbol{r}^2}\right)$$

- Two-body vacuum to meson matrix elements

$$\phi_{\gamma^+}(x,\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr^- e^{ixp_M^+r^-}\left\langle M\left(p_M\right)\middle|\overline{\psi}\left(r\right)\gamma^+\psi\left(0\right)\middle|0\right\rangle_{r^+=0}$$

$$\phi_{\gamma^+\gamma^5}(x,\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr^- e^{ixp_M^+r^-}\left\langle M\left(p_M\right)\middle|\overline{\psi}\left(r\right)\gamma^+\gamma^5\psi\left(0\right)\middle|0\right\rangle_{r^+=0}$$

At this stage, r^2 is arbitrary, in principle off the light-cone.

Results: three-body contribution

- Three-body amplitude: involves **dipole** and **double dipole** contributions

$$\mathcal{A}_3 = \left(\prod_{i=1}^{3} \int dx_i \theta(x_i)\right) \delta(1 - x_1 - x_2 - x_3) \int d^2\boldsymbol{z}_1 d^2\boldsymbol{z}_2 d^2\boldsymbol{z}_3 e^{i\boldsymbol{q}(x_1\boldsymbol{z}_1 + x_2\boldsymbol{z}_2 + x_3\boldsymbol{z}_3)}$$

$$\times \Psi_3\left(x_1, x_2, x_3, \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3\right) \left\langle P\left(p'\right) \middle| \mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3}\mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2} - \mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3} - \mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2} + \frac{1}{N_c^2}\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_2} \middle| P\left(p\right) \right\rangle$$

- **Coordinate-space impact factor** (with $Z = \sqrt{x_1 x_2 \boldsymbol{z}_{12}^2 + x_2 x_3 \boldsymbol{z}_{23}^2 + x_1 x_3 \boldsymbol{z}_{31}^2}$)

$$\Psi_3\left(x_1, x_2, x_3, \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3\right) = \frac{e_q q^+}{2(4\pi)} \frac{N_c^2}{N_c^2 - 1} \left(\varepsilon_{q\rho} - \frac{\varepsilon_q^+}{q^+} q_\rho\right)$$

$$\times \left\{ \chi_{\gamma^+\sigma} \left[\left(4 i g_{\perp\perp}^{\rho\sigma} \frac{x_1 x_2}{1 - x_2} \frac{Q}{Z} K_1(QZ) + T_1^{\sigma\rho\nu}(x_1, x_2, x_3) \frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2} K_0(QZ) \right) - (1 \leftrightarrow 2) \right] \right.$$

$$\left. - \chi_{\gamma^+\gamma^5\sigma} \left[\left(4 \epsilon^{\sigma\rho+-} \frac{x_1 x_2}{1 - x_2} \frac{Q}{Z} K_1(QZ) + T_2^{\sigma\rho\nu}(x_1, x_2, x_3) \frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2} K_0(QZ) \right) + (1 \leftrightarrow 2) \right] \right\}$$

- Three-body vacuum to meson matrix elements

$$\chi_{\Gamma^\lambda,\sigma} \equiv \chi_{\Gamma^\lambda,\sigma}(x_1, x_2, x_3, \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3) =$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}z_1^-}{2\pi} \frac{\mathrm{d}z_2^-}{2\pi} \frac{\mathrm{d}z_3^-}{2\pi} \mathrm{e}^{-ix_1 q^+ z_1^- - ix_2 q^+ z_2^- - ix_3 q^+ z_3^-} \left\langle M\left(p_M\right) \middle| \overline{\psi}\left(z_1\right) \Gamma^\lambda g F_{-\sigma}\left(z_3\right) \psi\left(z_2\right) \middle| 0 \right\rangle_{z_{1,2,3}^+ = 0}$$

At this stage, $z_{13}^2, z_{32}^2, z_{21}^2$ are arbitrary, in principle off the light-cone.

Covariant collinear factorization

- Covariant collinear factorization is based on the non-local operator product expansion (OPE)

  [Balitsky, Braun (1989)]

  - Expansion in powers of the hard scale = expansion of M.E. of **string operators** in powers of the deviation from the light-cone
  - OPE expansion → finite sum of M.E. of on-light-cone **non-local operator**
- For each term in this Taylor expansion: vacuum-to-meson matrix elements contribute to different kinematic twist
- Up to twist 3: only the first term in the Taylor expansion of the off-light-cone matrix elements survives (next one is r^2 suppressed, i.e. twist 4)
- Example: parametrization of the 2-body vector matrix element

$$\left\langle M(p_M) \left| \overline{\psi}(r)\, \gamma^\mu [r,0] \psi(0) \right| 0 \right\rangle \Big|_{r^2=0}$$

$$\sim f_M m_M \int_0^1 \mathrm{d}x e^{ix(p_M \cdot r)} \Bigg[\underbrace{p_M^\mu \frac{(\varepsilon_M^* \cdot r)}{(p_M \cdot r)} \phi_\parallel(x)}_{\substack{\sim Q \\ \text{twist 2}}} + \underbrace{\varepsilon_{M,T}^{*\mu}\, g_\perp^{(v)}(x)}_{\substack{\sim 1 \\ \text{twist 3}}} - \underbrace{\frac{1}{2} r^\mu \frac{(\varepsilon_M^* \cdot r)}{(p_M \cdot r)^2} m_\rho^2 g_3(x)}_{\substack{\sim 1/Q \\ \text{twist 4}}} \Bigg]$$

Covariant collinear factorization

- **Covariant collinear factorization**

[Braun, Filyanov (1990)]

[Ball, Braun, Koike, Tanaka (1998)]

  i. Minimal basis of *independent distributions* (twist-3 collinear DAs)
  ii. *Minimal numbers of non-perturbative parameters*
  iii. Easy to perform the calculation directly in coordinate space

- 2- and 3-body operators in gauge invariant form, on the light-cone $z^2 = 0$

$$\boxed{\begin{array}{c} \langle M(p_M)|\overline{\psi}(z)\Gamma_\lambda\,[z,0]\,\psi(0)|0\rangle \\ \langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\,[z,tz]\,gF^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle \\ \langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\gamma^5\,[z,tz]\,g\tilde{F}^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle \end{array}}$$

where

$$[z,0] = \mathcal{P}_{\exp}\left[ig\int_0^1 dt A^\mu\left(tz\right) z_\mu\right]$$

- Gauge invariant matrix elements can be related to the non-gauge invariant one within a fixed twist

$$\left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)[r,0]\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0} \longleftrightarrow \left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0}$$

Covariant collinear factorization

- Before twist expansion, our result does not contain gauge links between fields
- Three-body matrix element

$$\left\langle M(p)\left|\overline{\psi}\left(z_q\right)\gamma^\lambda gF^{\mu\nu}\left(z_g\right)\psi\left(z_{\overline{q}}\right)\right|0\right\rangle$$
$$\simeq\left\langle M(p)\left|\overline{\psi}(z_q)[z_q,z_g]\gamma^\lambda gF^{\mu\nu}(z_g)[z_g,z_{\overline{q}}]\psi(z_{\overline{q}})\right|0\right\rangle_{z_i^+=0}+\text{twist four}$$

- Gauge link effects do not contribute to the 3-body result within twist 3
- Two-body matrix element

$$\left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)[r,0]\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0}\longleftrightarrow\left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0}$$

- Expansion of the gauge link

$$[z,0]=\mathcal{P}\exp\left[ig\int_0^1 dtA^\mu(tz)z_\mu\right]\simeq 1+ig\int_0^1 dtA^\mu\left(tz\right)z_\mu+\text{higher twist}$$

- In a given n light-cone gauge

$$A^\mu(z)=\int_0^\infty \mathrm{d}\sigma\,\mathrm{e}^{-\epsilon\sigma}n_\nu F^{\mu\nu}(z+\sigma n)$$

- Gauge link effect does contribute to the 2-body twist-3 result

Twist-expanded results: three-body contribution

- Twist-expanded results: three-body contribution

$$\begin{aligned}
&\Psi_3\left(x_1, x_2, x_3, \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3\right) \\
&= \frac{e_q m_M c_f}{8\pi}\delta\left(1-\frac{p_M^+}{q^+}\right)\left(\varepsilon_{q\rho}-\frac{\varepsilon_q^+}{q^+}q_\rho\right)\left(\varepsilon_M^{*\mu}-\frac{p_M^\mu}{p_M^+}\varepsilon_M^{*+}\right)\left(\prod_{j=1}^{3}\theta(x_j)\theta(1-x_j)e^{-ix_j\boldsymbol{p}_M\boldsymbol{z}_j}\right) \\
&\times\Bigg\{-if_{3M}^V g_{\sigma\mu}V(x_1,x_2)\left[\left(4ig_\perp^{\rho\sigma}\frac{x_1x_2}{1-x_2}\frac{Q}{Z}K_1(QZ)+T_1^{\sigma\rho\nu}(\{x_i\})\frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2}K_0(QZ)\right)-(1\leftrightarrow 2)\right] \\
&-\epsilon_{-+\sigma\beta}f_{3M}^A g_{\perp\mu}^\beta A(x_1,x_2)\left[\left(4\epsilon^{\sigma\rho+-}\frac{x_1x_2}{1-x_2}\frac{Q}{Z}K_1(QZ)+T_2^{\sigma\rho\nu}(\{x_i\})\frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2}K_0(QZ)\right)+(1\leftrightarrow 2)\right]\Bigg\}
\end{aligned}$$

- Collinear ingredients:

[Ball, Braun, Koike, Tanaka (1998)]

$V(x_1, x_2)$ = genuine twist-3 vector DA

$A(x_1, x_2)$ = genuine twist-3 axial DA

f_{3M}^V and f_{3M}^A = normalization constant

Twist-expanded results: two-body contribution

- Twist-expanded results: two-body contribution

$$\Psi_2\left(x,\boldsymbol{r}\right)=e_q m_M f_M \delta\left(1-\frac{p_M^+}{q^+}\right)\left(\varepsilon_{q\mu}-\frac{\varepsilon_q^+}{q^+}q_\mu\right)\left(\varepsilon^*_{M\alpha}-\frac{\varepsilon_M^{*+}}{p_M^+}p_{M\alpha}\right)$$
$$\times\left[-ir_\perp^\alpha(h(x)-\tilde{h}(x))\left(2x\bar{x}q^\mu+(x-\bar{x})\frac{-i\partial}{\partial r_{\perp\mu}}\right)+\right.$$
$$\left.\epsilon^{\mu\nu+-}\epsilon^{+\alpha-\delta}r_{\perp\delta}\left(\frac{g_\perp^{(a)}(x)-\tilde{g}_\perp^{(a)}(x)}{4}\right)\frac{\partial}{\partial r_\perp^\nu}\right]K_0\left(\sqrt{x\bar{x}Q^2\boldsymbol{r}^2}\right)$$

- Collinear ingredients: [Ball, Braun, Koike, Tanaka (1998)]

$h(x)=$ kinematic twist-3 vector DA

$g_\perp^{(a)}(x)=$ kinematic twist-3 axial DA

- Collinear ingredients: gauge link related terms

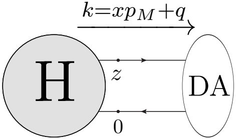
$$\widetilde{h}\left(x\right)=\frac{f_{3M}^V}{f_M}\int_0^x \mathrm{d}x_q\int_0^{1-x}\mathrm{d}x_{\overline{q}}\,\frac{V\left(x_q,x_{\overline{q}}\right)}{\left(1-x_q-x_{\overline{q}}\right)^2}$$
$$\widetilde{g}_\perp^{(a)}\left(x\right)=4\frac{f_{3M}^A}{f_M}\int_0^x \mathrm{d}x_q\int_0^{1-x}\mathrm{d}x_{\overline{q}}\,\frac{A\left(x_q,x_{\overline{q}}\right)}{(1-x_q-x_{\overline{q}})^2}$$

$f_M=$ vector decay constant

Light-cone collinear factorization

- Previous work was based on light-cone collinear factorization (LCCF)
  [Ellis, Furmanski, Petronzio (1982)]
- Most general two-body amplitude

$$\mathcal{A}_2 = \int \frac{d^4k}{(2\pi)^4} \int d^4z e^{-ik\cdot z} \langle M(p_M)|\overline{\psi}^{\,i}_{\alpha}(z)\,\psi^{\,j}_{\beta}(0)\,|0\rangle H^{ij}_{2,\alpha\beta}$$



- The separation is choosen as $z^\mu = \lambda n_2^\mu$ $\Longrightarrow$ no Wilson line in the n_2 **light-cone gauge**
- Sudakov decomposition: $k = k^+ n_1 + q = xp_M + q$ and small q expansion
- Two-body amplitude after Fierz decomposition

$$\mathcal{A}_2 = \frac{1}{4N_c} p_M^+ \int \frac{\mathrm{d}x}{2\pi} \int \frac{\mathrm{d}q^-}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d} \int \mathrm{d}^D z\; \mathrm{e}^{-ixp_M^+ z^- - iq^- z^+ + i(\boldsymbol{q}\cdot\boldsymbol{z})}$$
$$\times \left\langle M(p_M) \left| \overline{\psi}(z)\, \Gamma_\lambda \psi(0) \right| 0 \right\rangle \mathrm{tr}\left[H_2(xp_M+q)\, \Gamma^\lambda \right]$$

Light-cone collinear factorization

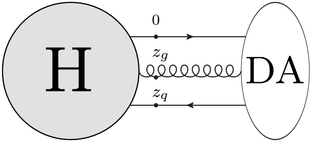
- **Taylor expansion** of the hard part

$$H_2\left(xp_M+q\right)=H_2\left(xp_M\right)+q_{\perp\mu}\left[\frac{\partial}{\partial q_{\perp\mu}}H_2\left(xp_M+q\right)\right]_{k=xp_M}+\text{h.t.}$$

- Two-body amplitude factorized form up to the **twist-3**

$$\mathcal{A}_2=\frac{1}{4N_c}\int \mathrm{d}x\, p_M^+\int\frac{\mathrm{d}z^-}{2\pi}\mathrm{e}^{-ixp_M^+z^-}\left\{\left\langle M\left(p\right)\middle|\overline{\psi}\left(z^-\right)\Gamma_\lambda\psi\left(0\right)\middle|0\right\rangle \mathrm{tr}\left[H_2\left(xp_M\right)\Gamma^\lambda\right]\right.$$
$$\left.+i\left\langle M\left(p_M\right)\middle|\overline{\psi}\left(z^-\right)\overleftrightarrow{\partial}_{\perp\mu}\Gamma_\lambda\psi\left(0\right)\middle|0\right\rangle \mathrm{tr}\left[\partial_\perp^\mu H_2\left(xp_M\right)\Gamma^\lambda\right]\right\}$$

- *Gauge invariance* is broken $\implies$ need to include a **3-body contribution**



- Three-body contribution factorized

$$\mathcal{A}_3=\frac{1}{2(N_c^2-1)}\int \mathrm{d}x_q\mathrm{d}x_g\left(p_M^+\right)^2\int\frac{\mathrm{d}z_q^-}{2\pi}\frac{\mathrm{d}z_g^-}{2\pi}\mathrm{e}^{-ix_qp_M^+z_q^--ix_gp_M^+z_g^-}$$
$$\times\left\langle M\left(p\right)\middle|\overline{\psi}\left(z_q^-\right)\Gamma_\lambda gA_\mu\left(z_g^-\right)\psi\left(0\right)\middle|0\right\rangle \mathrm{tr}\left[t^bH_3^{\mu,b}\left(x_qp_M,x_gp_M\right)\Gamma^\lambda\right]$$

Light-cone collinear factorization

- **Overcomplete set of distributions** must be reduced exploiting QCD equations of motion

$$\langle i(\hat{D}(0)\psi(0))_\alpha \bar{\psi}_\beta(z)\rangle = 0 \qquad \langle i\psi_\alpha(0)(\bar{\psi}(z)\overleftarrow{\hat{D}}(z))_\beta\rangle = 0$$

- Invariance of the amplitude under **rotation on the light-cone**

  [Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

  i. Independence of the amplitude from the choice of n

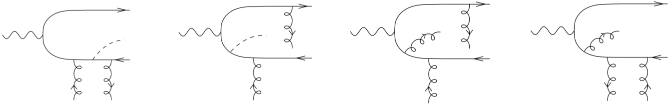
  ii. Given a "natural" choice n_0, we can define

$$n^\mu = \alpha p^\mu + \beta n_0^\mu + n_\perp^\mu$$

  iii. Imposing $p \cdot n = 1$ and $n^2 = 0 \longrightarrow \beta = 1\,, \alpha = -n_\perp^2/2$

  iiii. Freedom parametrized in terms of the transverse component $\frac{\partial \mathcal{A}}{\partial n_\perp^\mu} = 0$

- **Feynman diagrams**

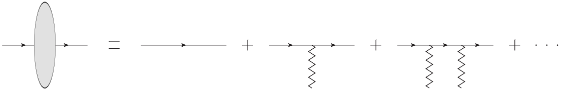


Total: 12 (2-body) + 28 (3-body)

Dilute regime: two-body contribution

- **Reggeon** definition [Caron-Huot (2013)] $R^a(\boldsymbol{z}) \equiv \frac{f^{abc}}{gC_A} \ln\left(U^{bc}_{\boldsymbol{z}}\right)$
- Expansion of the *Wilson line* in Reggeized gluons

$$V_{\boldsymbol{z}_1} = 1 + ig\boldsymbol{t}^a R^a(\boldsymbol{z}_1) - \frac{1}{2} g^2 \boldsymbol{t}^a \boldsymbol{t}^b R^a(\boldsymbol{z}_1) R^b(\boldsymbol{z}_1) + O\left(g^3\right)$$



- **BFKL $\boldsymbol{k}_{\mathrm{T}}$-factorization**

$$\mathcal{A}_2^{\text{dilute}} = \frac{g^2}{4N_c} (2\pi)^d \delta^d(\boldsymbol{q} - \boldsymbol{p}_M - \boldsymbol{\Delta}) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \mathcal{U}(\ell) \int_0^1 \mathrm{d}x$$

$$\times \underbrace{\left[\Phi_2\left(x, \boldsymbol{\ell} - \frac{x-\bar{x}}{2}\boldsymbol{\Delta}\right) + \Phi_2\left(x, -\boldsymbol{\ell} - \frac{x-\bar{x}}{2}\boldsymbol{\Delta}\right) - \Phi_2(x, \bar{x}\boldsymbol{\Delta}) - \Phi_2(x, -x\boldsymbol{\Delta})\right]}_{\Phi_{2,\mathrm{BFKL}}(x, \boldsymbol{l}, \boldsymbol{\Delta})}$$

- $\mathcal{U}(\boldsymbol{l}) \to k_T$-unintegrated gluon density (UGD) in the BFKL sense

$$\mathcal{U}(\boldsymbol{\ell}) \equiv \int \mathrm{d}^d \boldsymbol{v} \mathrm{e}^{-i(\boldsymbol{\ell}\cdot\boldsymbol{v})} \left\langle P\left(p'\right) \middle| R^a\left(\frac{\boldsymbol{v}}{2}\right) R^a\left(-\frac{\boldsymbol{v}}{2}\right) \middle| P(p) \right\rangle \ ,$$

- The 3-body BFKL impact factor is a combination of 12 BK impact factors (dilute, $\boldsymbol{\Delta} = 0$, $T \to T$) [Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

Explicit two-body term in the dilute and $\Delta = 0$ limit

- **BK impact factor**

$$\Phi_{2,\boldsymbol{\Delta}=0}\left(x,\boldsymbol{l}\right) = 2\pi m_M f_M e_q \delta(1-p_M^+/q^+)$$
$$\times \left[\frac{2\boldsymbol{l}^2}{\left[\boldsymbol{l}^2+x\bar{x}Q^2\right]^2} T_{\text{f.}}\,\phi_{2,\text{f.}}(x) - \frac{x\bar{x}Q^2}{\left[\boldsymbol{l}^2+x\bar{x}Q^2\right]^2} T_{\text{n.f.}}\,\phi_{2,\text{n.f.}}(x)\right]$$

- Helicity structures and DAs combinations

$$T_{\text{n.f.}} = \boldsymbol{\varepsilon}_q \cdot \boldsymbol{\varepsilon}_M^* \qquad \phi_{2,\text{n.f.}}(x) = (2x-1)(h(x)-\tilde{h}(x)) + \frac{g_\perp^{(a)}(x)-\tilde{g}_\perp^{(a)}(x)}{4}$$

$$T_{\text{f.}} = \frac{(\boldsymbol{\varepsilon}_q\cdot\boldsymbol{l})(\boldsymbol{\varepsilon}_M^*\cdot\boldsymbol{l})}{\boldsymbol{l}^2} - \frac{\boldsymbol{\varepsilon}_q\cdot\boldsymbol{\varepsilon}_M^*}{2} \qquad \phi_{2,\text{f.}}(x) = (2x-1)(h(x)-\tilde{h}(x)) - \frac{g_\perp^{(a)}(x)-\tilde{g}_\perp^{(a)}(x)}{4}$$

- **Forward limit matching**

$$\Phi^{\text{BFKL}}_{2,\boldsymbol{\Delta}=0}\left(x,\boldsymbol{l}\right) = 2\left(\Phi_{2,\boldsymbol{\Delta}=0}\left(x,\boldsymbol{l}\right) - \Phi_{2,\boldsymbol{\Delta}=0}\left(x,\boldsymbol{0}\right)\right)$$

- **BFKL impact factor**

$$\Phi^{\text{BFKL}}_{2,\boldsymbol{\Delta}=0}\left(x,\boldsymbol{l}\right) = 4\pi m_M f_M e_q \delta(1-p_M^+/q^+)$$
$$\times \left[\frac{2\boldsymbol{l}^2}{\left[\boldsymbol{l}^2+x\bar{x}Q^2\right]^2} T_{\text{f.}}\,\phi_{\text{f.}}(x) + \frac{\boldsymbol{l}^2(\boldsymbol{l}^2+2x\bar{x}Q^2)}{x\bar{x}Q^2\left[\boldsymbol{l}^2+x\bar{x}Q^2\right]^2} T_{\text{n.f.}}\,\phi_{\text{n.f.}}(x)\right]$$

- From BFKL to BK $\longrightarrow$ subtraction of the $|\boldsymbol{l}| \to \infty$ limit

Explicit three-body term in the dilute and $\Delta = 0$ limit

- The 3-body BFKL impact factor is a combination of 12 BK impact factors

$$\Phi_3\left(\{x\},\{\boldsymbol{p}\}\right) = \left(\prod_{j=1}^{3}\int d^2\boldsymbol{z}_j e^{-i\boldsymbol{z}_j\boldsymbol{p}_j}\right)\Psi_3\left(\{x\},\{\boldsymbol{z}\}\right)$$

- Transverse to transverse transition in the **forward** and **dilute** limit

$$\begin{aligned}
\mathcal{A}^{\text{dilute}}_{3T,\boldsymbol{\Delta}=\boldsymbol{0}} &= e_q m_M \frac{g^2}{N_c}(2\pi)\delta\left(1-\frac{p_M^+}{q^+}\right)(2\pi)^2\delta^2\left(\boldsymbol{q}-\boldsymbol{p}_M\right)\int\frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d}\mathcal{U}(\boldsymbol{\ell})\\
&\times\left(\prod_{i=1}^{3}\int_0^1\frac{dx_i}{x_i}\right)\frac{\delta(1-x_1-x_2-x_3)}{x_3}\frac{\boldsymbol{\ell}^2}{Q^2}\Big\{T_{\text{f.}}\left[f^V_{3M}V\left(x_1,x_2\right)-f^A_{3M}A\left(x_1,x_2\right)\right]\\
\times 2x_1&\left(\frac{x_3c_f}{\boldsymbol{\ell}^2+\frac{x_2x_3}{x_2+x_3}Q^2}+\frac{x_3c_f}{\boldsymbol{\ell}^2+\frac{x_1x_3}{x_1+x_3}Q^2}-\frac{\bar{x}_3\left(1-c_f\right)}{\boldsymbol{\ell}^2+\frac{x_1x_2}{x_1+x_2}Q^2}+\frac{x_2-\bar{x}_1c_f}{\boldsymbol{\ell}^2+x_1\bar{x}_1Q^2}+\frac{x_1-\bar{x}_2c_f}{\boldsymbol{\ell}^2+x_2\bar{x}_2Q^2}\right)\\
&-T_{\text{n.f.}}\left[f^V_{3M}V\left(x_1,x_2\right)+f^A_{3M}A\left(x_1,x_2\right)\right]\\
&\times\left(\frac{\left(1-c_f\right)x_1\bar{x}_3}{\bar{x}_3\boldsymbol{\ell}^2+x_1x_2Q^2}-\frac{c_fx_3^2}{\bar{x}_1\boldsymbol{\ell}^2+x_2x_3Q^2}-\frac{\left(x_2-\bar{x}_1c_f\right)x_1x_2}{\bar{x}_1\left(\boldsymbol{\ell}^2+x_1\bar{x}_1Q^2\right)}-\frac{\left(x_1-\bar{x}_2c_f\right)\bar{x}_2}{\left(\boldsymbol{\ell}^2+x_2\bar{x}_2Q^2\right)}\right)\Big\}
\end{aligned}$$

- **The forward and dilute limit matches the previous result**

[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

BFKL approach + twist-expansion via **light-cone collinear factorization**

Result for the Wandzura-Wilczek part

- Two-body amplitude

$$\mathcal{A}_2^{\lambda_\gamma,\lambda_V} = (2\pi)^4\delta^{(4)}(q+p_t-p_M-p_{t'})\; s\int d^2\boldsymbol{r}\,\mathcal{U}\left(\boldsymbol{r},\boldsymbol{\Delta}\right)\int_0^1 dz\; e^{iz\boldsymbol{\Delta}\cdot\boldsymbol{r}}\;\psi_2^{\lambda_\gamma,\lambda_V}\left(z,\boldsymbol{r}\right)$$

- Different transition amplitudes

$$\psi_2^{L,L}\left(z,\boldsymbol{r}\right) = \frac{e_q m_M f_M}{2\pi}2z\bar{z}Q^2K_0\left(\sqrt{z\bar{z}Q^2\boldsymbol{r}^2}\right)\left[\frac{\varepsilon_\gamma^+}{q^+}\frac{\varepsilon_M^{+*}}{p_M^+}\right]\varphi_\parallel(z)$$

$$\psi_2^{T,L}\left(z,\boldsymbol{r}\right) = \frac{e_q m_M f_M}{2\pi}(2z-1)\sqrt{\frac{z\bar{z}Q^2}{\boldsymbol{r}^2}}K_1\left(\sqrt{z\bar{z}Q^2\boldsymbol{r}^2}\right)\left[i\frac{\varepsilon_M^{+*}}{p_M^+}(\boldsymbol{r}\cdot\boldsymbol{\varepsilon}_\gamma)\right]\varphi_\parallel(z)$$

$$\psi_2^{L,T}\left(z,\boldsymbol{r}\right) = \frac{e_q m_M f_M}{2\pi}2z\bar{z}Q^2K_0\left(\sqrt{z\bar{z}Q^2\boldsymbol{r}^2}\right)\left[i\frac{\varepsilon_\gamma^+}{q^+}(\boldsymbol{r}\cdot\boldsymbol{\varepsilon}_M^*)\right]\varphi_1^T(z)$$

$$\psi_2^{T,T}\left(z,\boldsymbol{r}\right) = -\frac{e_q m_M f_M}{2\pi}\sqrt{z\bar{z}Q^2\boldsymbol{r}^2}K_1\left(\sqrt{z\bar{z}Q^2\boldsymbol{r}^2}\right)\left[T_{\text{f.}}(\boldsymbol{r})\;\phi_{2,\text{f.}}(z)+\frac{T_{\text{n.f.}}}{2}\;\phi_{2,\text{n.f.}}(z)\right]$$

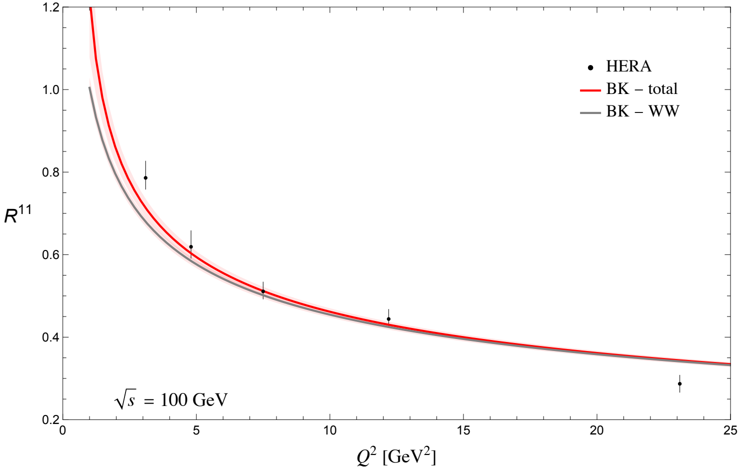
- Helicity structures

$$T_{\text{f.}}(\boldsymbol{r}) = \frac{(\boldsymbol{\varepsilon}_\gamma\cdot\boldsymbol{r})\left(\boldsymbol{\varepsilon}_M^*\cdot\boldsymbol{r}\right)}{\boldsymbol{r}^2}-\frac{\boldsymbol{\varepsilon}_\gamma\cdot\boldsymbol{\varepsilon}_M^*}{2} \qquad T_{\text{n.f.}} = \boldsymbol{\varepsilon}_\gamma\cdot\boldsymbol{\varepsilon}_M^*$$

Preliminary results: Predictions vs HERA data

- Ratio of helicity amplitudes $R^{11} = \mathcal{A}^{11}/\mathcal{A}^{00}$

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

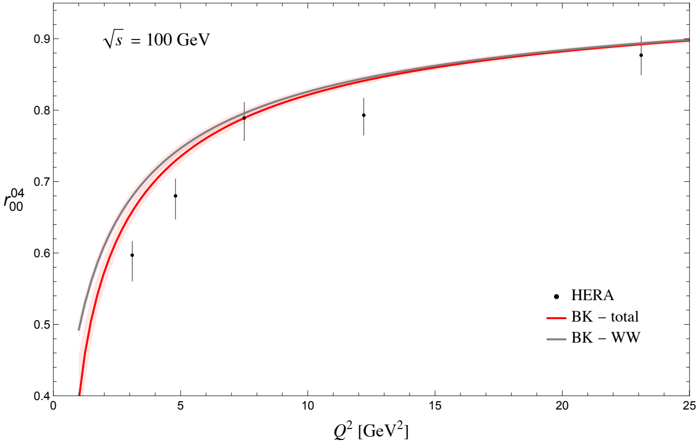


Preliminary results: Predictions vs HERA data

- Spin-density matrix element r_{00}^{04}

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]

$$r_{00}^{04} = \frac{\varepsilon + R_{10}^2}{R_{11}^2 + \varepsilon + R_{10}^2 + R_{-11}^2 + 2\varepsilon R_{01}^2} \simeq \frac{\varepsilon}{R_{11}^2 + \varepsilon}$$



Behavior in energy

- Ratio of helicity amplitudes $R^{11} = \mathcal{A}^{11}/\mathcal{A}^{00}$

[Boussarie, Delle Rose, M.F., Papa, Szymanowski, Wallon (to appear)]