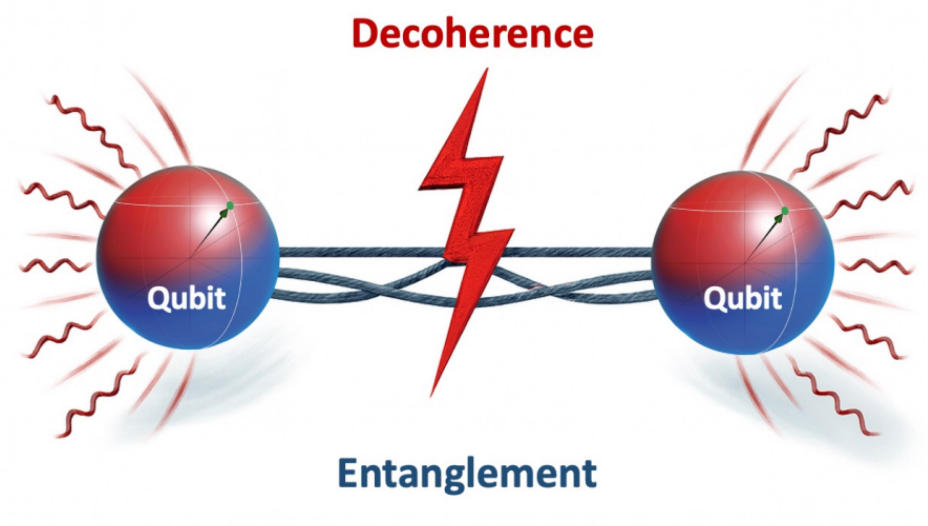
Decoherence and entanglement in high energy collisions



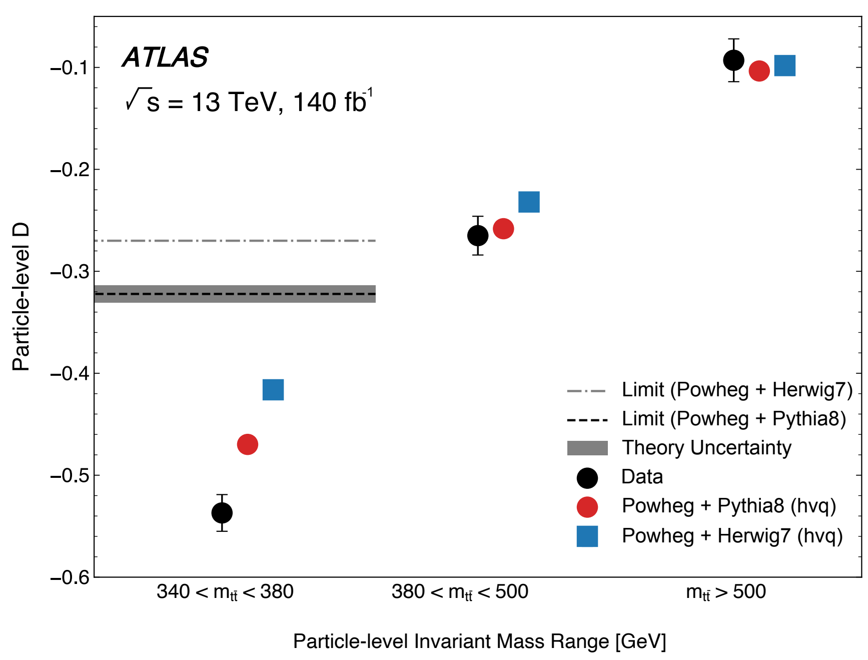
LianTao Wang
University of Chicago

Based on work in collaboration with
Jiayin Gu, Shijia Lin, Dingyu Shao, Sixiang Yang, 2510.13951

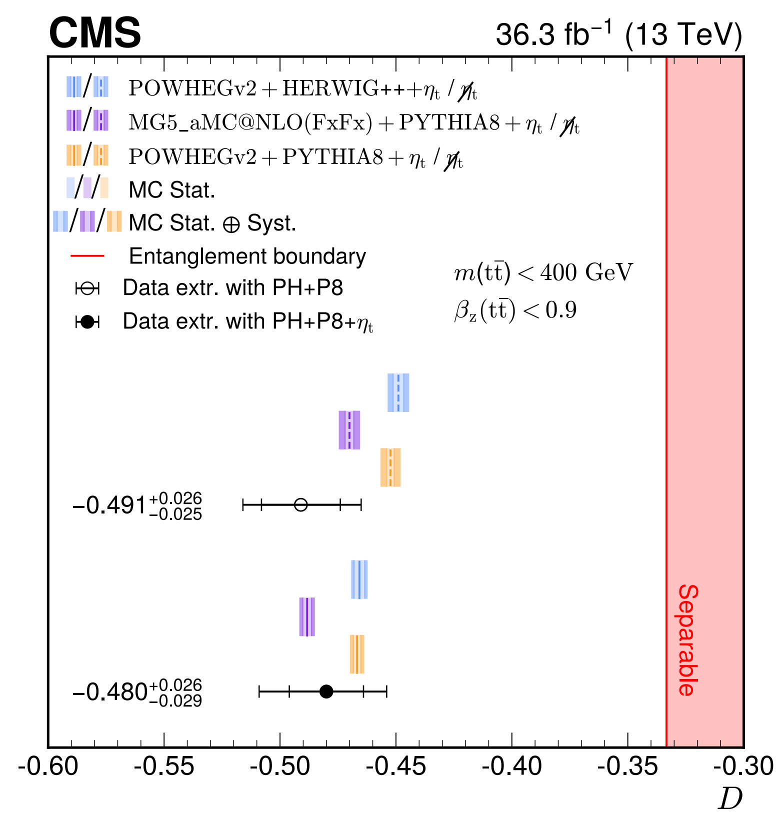
Seminar, Feb 17, BNL

Entanglement measurement

- Measurement of entanglement in high energy collisions has become a new frontier.



Atlas, Nature 2024



CMS, Rept. Prog. Phys. 2024

A lot of recent work

- Prospect of measuring a broader range of quantum entanglement and quantum information observables.
- Applied to different processes, and colliders.
- In this talk, I will focus on a different question.

Decoherence

- High energy collisions are complicated. Measurement often can’t capture everything.
  - For example, additional radiation can be missed due to acceptance and thresholds.
- Incomplete information decoherence, which needs to be taken into account for an accurate interpretation of the results.

Double slit

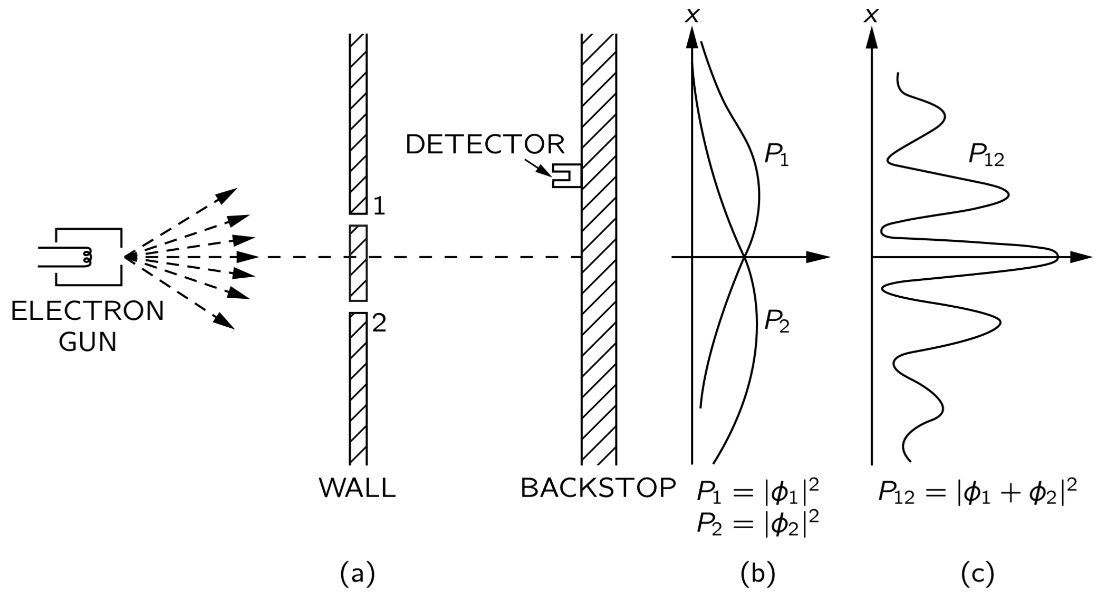


Fig. 1-3. Interference experiment with electrons.

Pure state

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

$$|\psi_{\text{pure}}\rangle = \frac{1}{\sqrt{2}}(|\text{Slit }1\rangle + |\text{Slit }2\rangle)$$

$$\rho_{\text{pure}} = |\psi_{\text{pure}}\rangle\langle\psi_{\text{pure}}| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$$

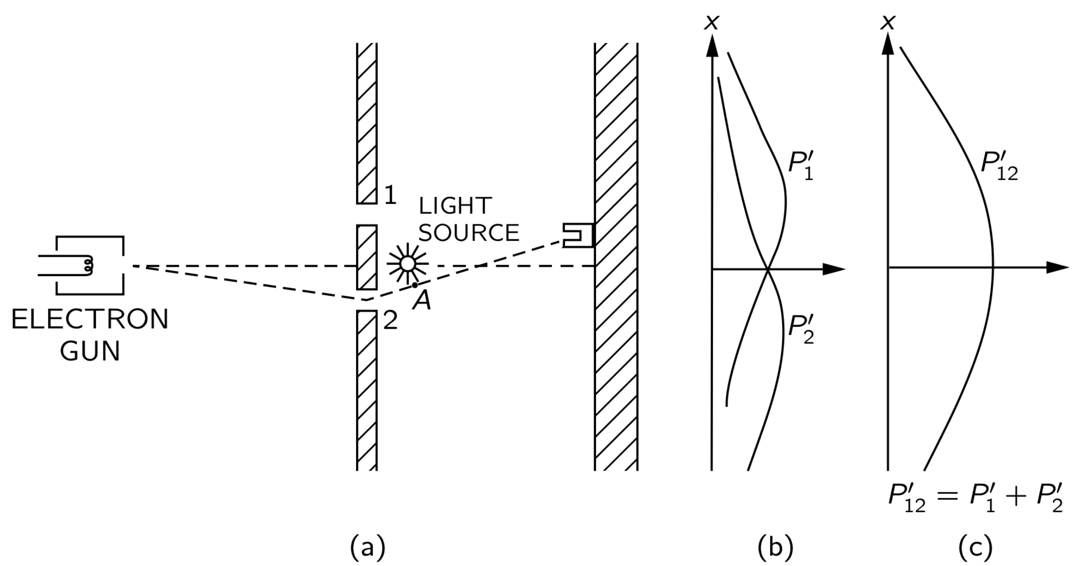


Fig. 1-4. A different electron experiment.

《The Feynman Lectures on Physics》

Mixed state

$$\rho_{\text{mixed}} = \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}$$

Pure dephasing quantum channel

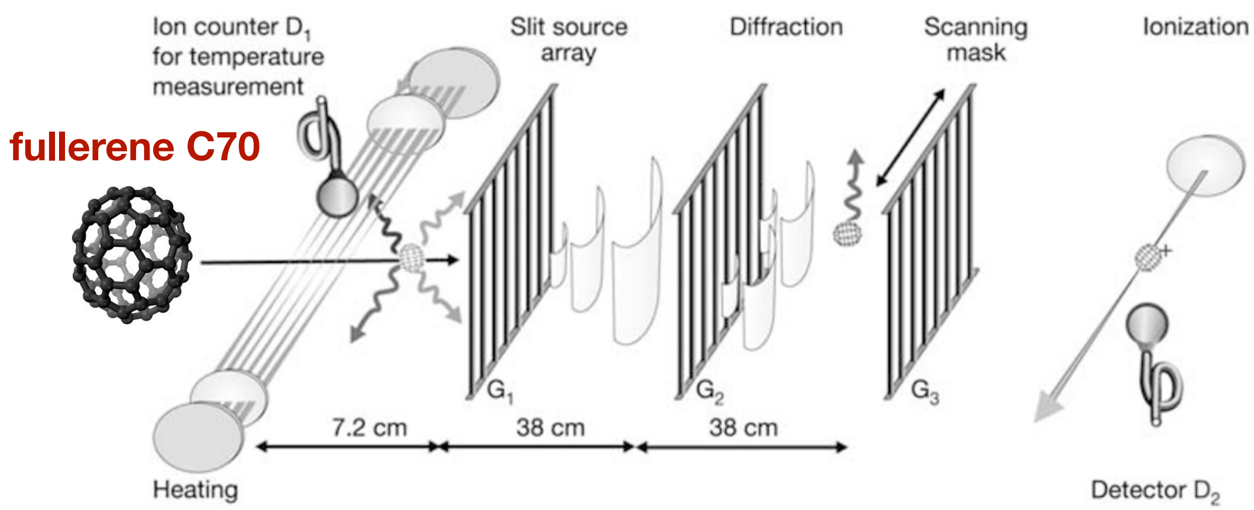
$$\rho(t) = \begin{pmatrix}1/2 & 1/2\cdot e^{-\Lambda t}\\1/2\cdot e^{-\Lambda t} & 1/2\end{pmatrix}$$

Λ is the decoherence rate. It describes how fast the quantum coherence (the off-diagonal terms) disappears; decoherence time *t* ≈ 1/Λ

Slide of D. Y. Shao

Decoherence by thermal emission of photons

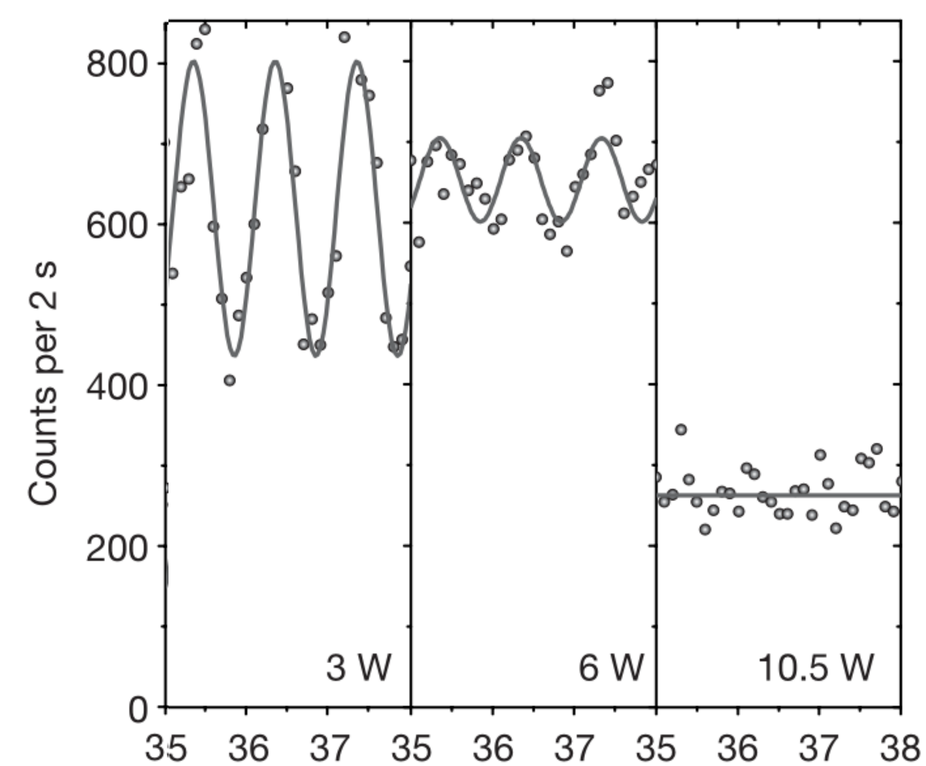
Hackermuller, Hornberger, Brezger, Zeilinger, Arndt Nature ‘04



Setup: Heating & Interferometry

- **A Talbot-Lau interferometer**
- **Before entering the interferometer, C70 molecules pass through a laser beam to increase their internal temperature**
- **The internal temperature is varied from below 1,000 K (cold) up to 3,000 K (hot)**

- **Cold Molecules:**
  - **Molecules emit negligible radiation.**
  - **Result: Strong quantum interference is observed with a fringe visibility.**
- **Hot Molecules:**
  - **Molecules emit thermal photons that carry path information**
  - **Result: The interference fringe visibility drops to 0%.**
- **The decoherence mechanism is the thermal emission of photons by the molecule itself**



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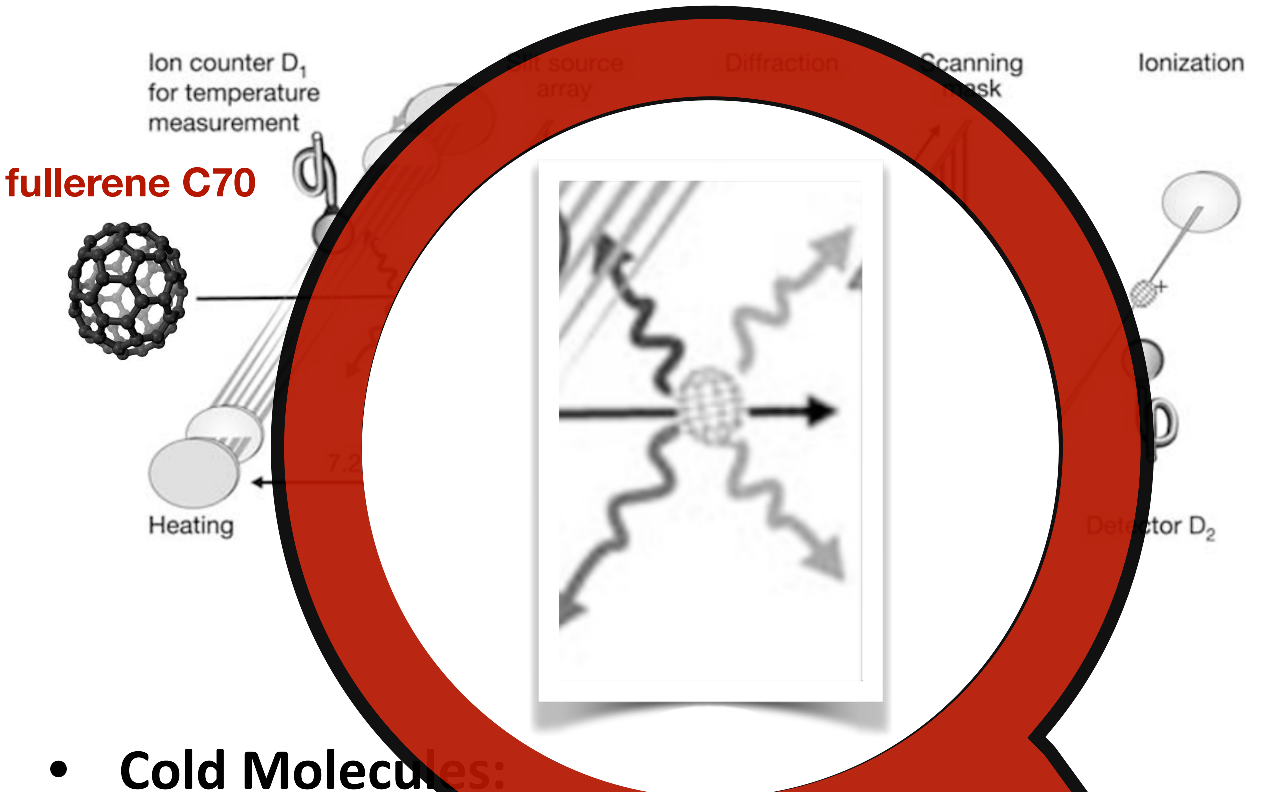

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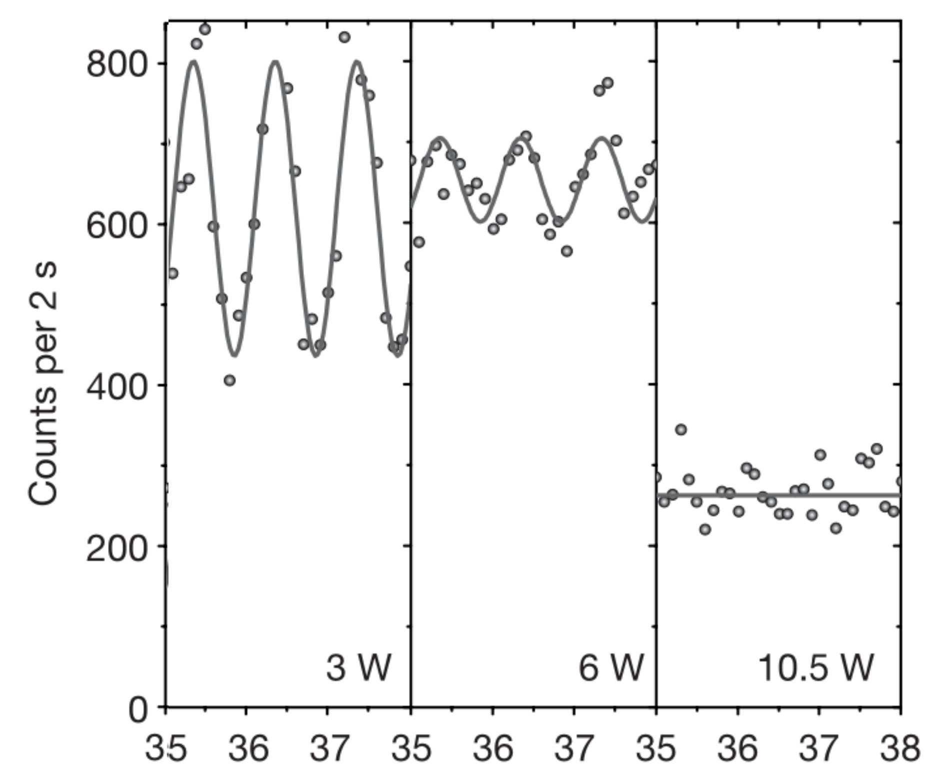

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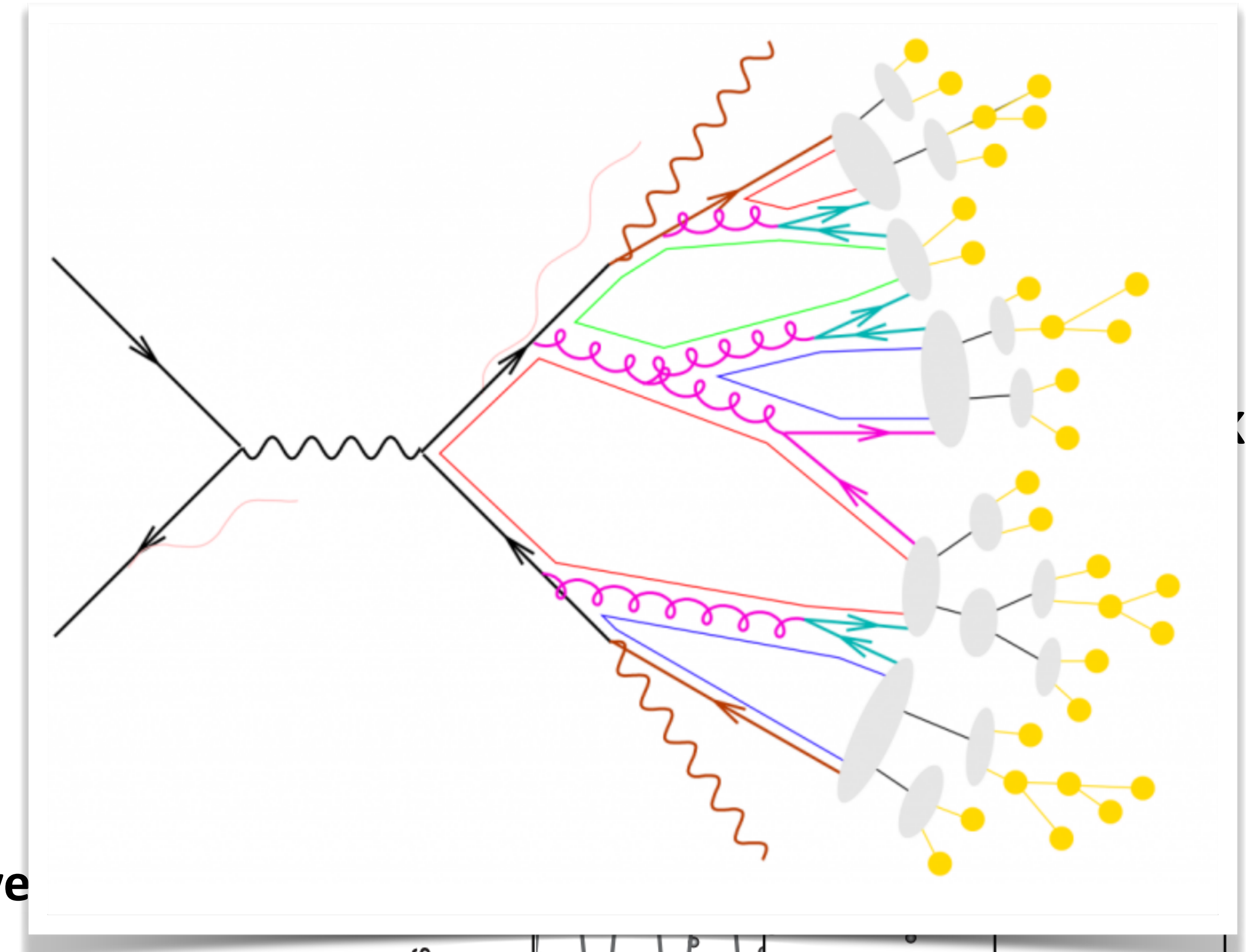

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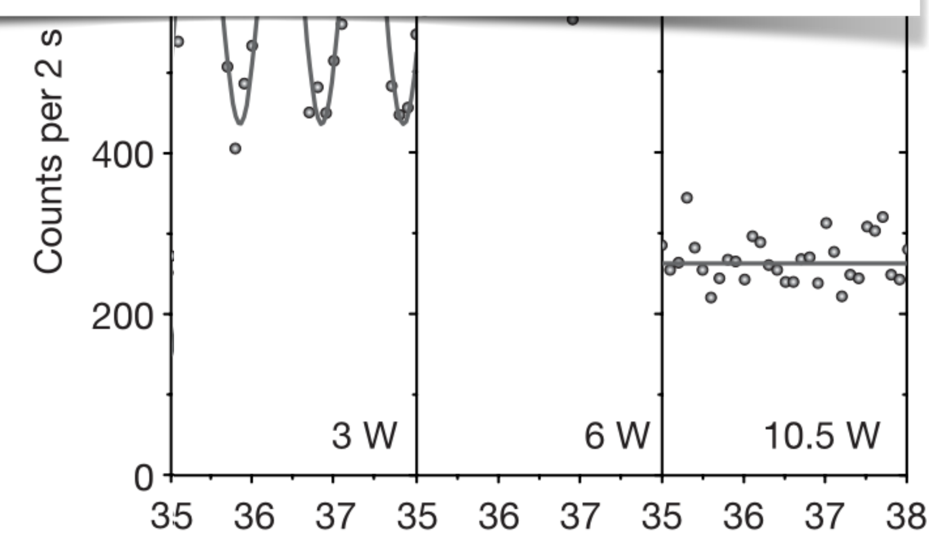
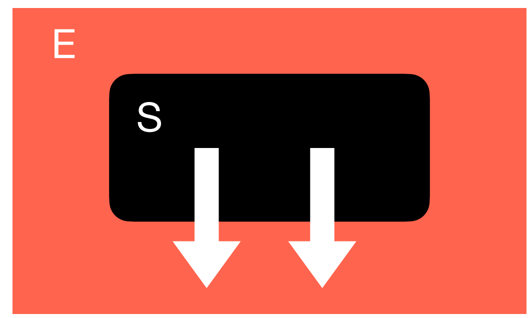


Slide of D. Y. Shao

Decoherence and quantum channel

Quantum channel



$$\begin{tikzcd} \rho_A(t_0) \otimes \rho_B(t_0) \arrow[r, "U(t_1,t_0)"] \arrow[d, "\mathrm{Tr}_B"'] & \rho_{AB}(t_1) \arrow[d, "\mathrm{Tr}_B"'] \\ \rho_A(t_0) \arrow[r, "\mathcal{E}_{(t_1,t_0)}"] & \rho_A(t_1) \end{tikzcd}$$

Representation by Kraus operator:

$$\mathscr{E}[\rho] = \sum_k E_k^\dagger \rho E_k, \qquad \sum_k E_k^\dagger E_k = 1$$

Decoherence Models for a Single Qubit

- Bit-flip Channel

$$|0\rangle \leftrightarrow |1\rangle$$

$$\mathcal{E}(\rho) = |\alpha|^2 \sigma_x \rho \sigma_x^\dagger + (1 - |\alpha|^2)\rho$$

- Phase-flip Channel

$$\mu|0\rangle + \nu|1\rangle \leftrightarrow \mu|0\rangle - \nu|1\rangle$$

$$\mathcal{E}(\rho) = |\gamma|^2 \sigma_z \rho \sigma_z^\dagger + (1 - |\gamma|^2)\rho$$

Master equation

Consider the time evolution of density matrix

$$\rho(t+\delta t) = \mathcal{S}(t+\delta t, t)\rho(t) = \sum_{k=0}^{N^2-1} E_k \rho(t) E_k^\dagger$$

Write infinitesimal Kraus operator:

$$E_0 = I + (-iH + K)\delta t \qquad E_k = L_k\sqrt{\delta t} \quad (\text{for } k \geq 1)$$

L_k are the **Lindblad operators**

Assuming Markovian evolution:

$$\tau_B \ll \delta t \ll \tau_R$$

- τ_B (Bath Correlation Time): The environment "forgets" information instantly.
- δt (Coarse-Graining): Large enough that the bath relaxes, but small enough to define a derivative.
- τ_R (Relaxation Time): The system evolves slowly.

Master equation

Consider the time evolution of density matrix

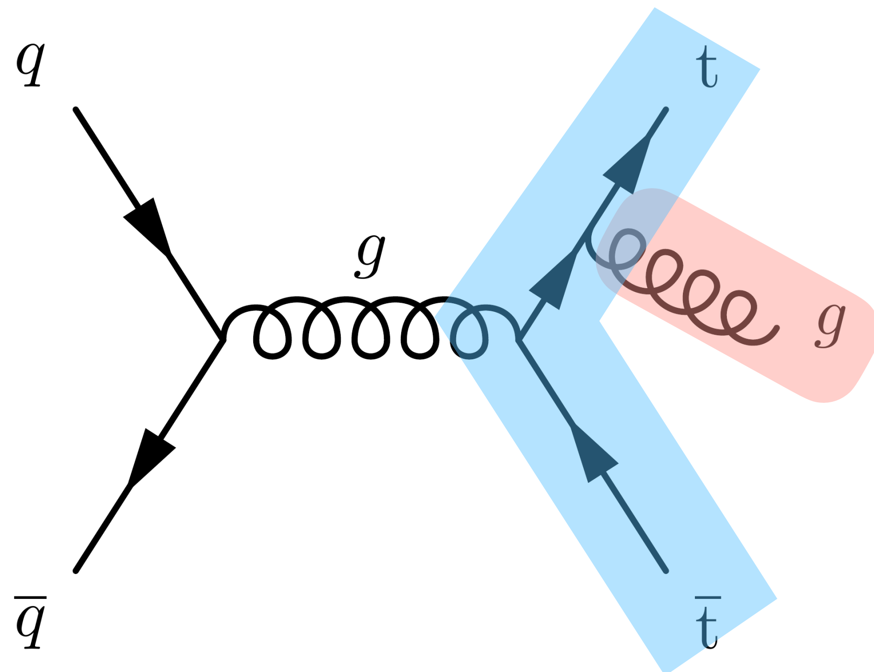
$$\rho(t+\delta t) = \mathcal{S}(t+\delta t, t)\rho(t) = \sum_{k=0}^{N^2-1} E_k \rho(t) E_k^\dagger$$

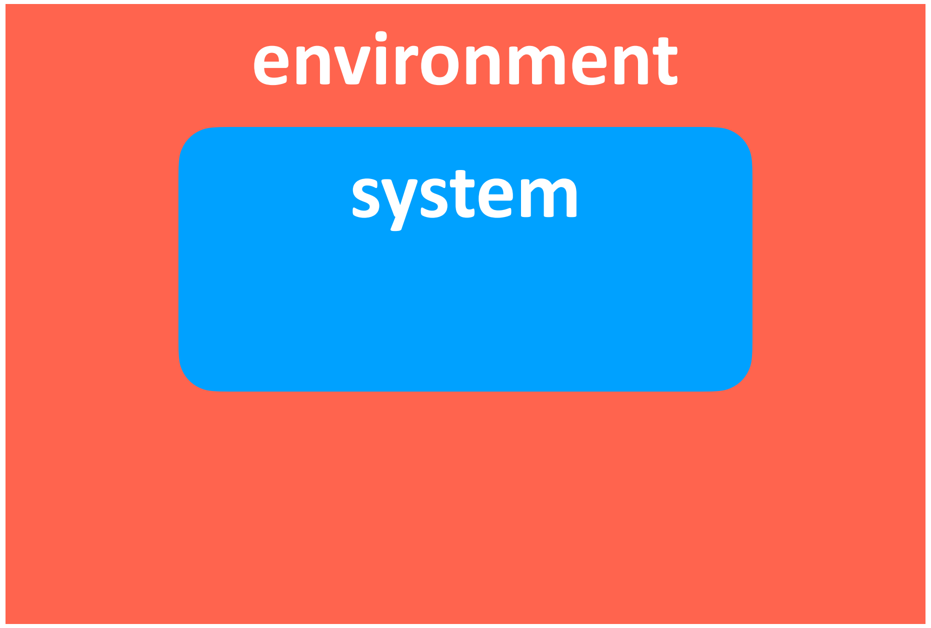
Assuming Markovian evolution.

Master equation:

$$\dot{\rho} = -i[H,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

Decoherence at colliders

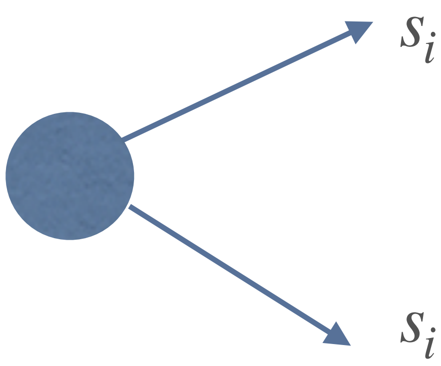




Radiation, leaking information from the “system”, decoherence in spin correction

Leading order, no radiation

Production a pair of particle:



Two qubit spin state can be characterized by

$$\hat{\rho} = \frac{1}{4}\Big(\hat{I}\otimes\hat{I}+P_i^+\,\hat{\sigma}_i\otimes\hat{I}+P_j^-\,\hat{I}\otimes\hat{\sigma}_j+\boxed{C_{ij}}\,\hat{\sigma}_i\otimes\hat{\sigma}_j\Big)$$

Spin correlation

In general, P and C are function of kinematics.

Entanglement measure

$$\hat{\rho} = \frac{1}{4}\Big(\hat{I}\otimes\hat{I} + P_i^{+}\,\hat{\sigma}_i \otimes \hat{I} + P_j^{-}\,\hat{I}\otimes\hat{\sigma}_j + C_{ij}\,\hat{\sigma}_i\otimes\hat{\sigma}_j\Big)$$

Concurrence:

$$\mathcal{C}(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

λ_i are eigenvalues of

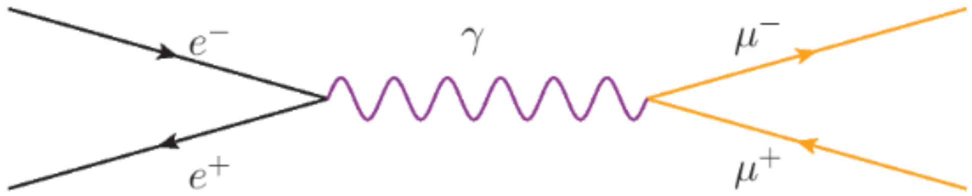
$$\tilde{\rho} = \sqrt{\rho}(\hat{\sigma}_y \otimes \hat{\sigma}_y)\,\rho^*\,(\hat{\sigma}_y \otimes \hat{\sigma}_y)\,\sqrt{\rho}.$$

Without radiation

Two qubit spin state can be characterized by

$$\hat{\rho} = \frac{1}{4}\Big(\hat{I}\otimes\hat{I} + P_i^+\,\hat{\sigma}_i \otimes \hat{I} + P_j^-\,\hat{I}\otimes\hat{\sigma}_j + C_{ij}\,\hat{\sigma}_i\otimes\hat{\sigma}_j\Big)$$

For example:



At leading order

$$\rho_{\rm LO} = \frac{1}{4}\left(\hat{I}_2\otimes\hat{I}_2 + \frac{\sin^2\theta}{1+\cos^2\theta}\hat{\sigma}_1\otimes\hat{\sigma}_1 + \frac{\sin^2\theta}{1+\cos^2\theta}\hat{\sigma}_2\otimes\hat{\sigma}_2 - \hat{\sigma}_3\otimes\hat{\sigma}_3\right)$$

Without radiation

At leading order

$$\rho_{\rm LO} = \frac{1}{4}\left(\hat{I}_2 \otimes \hat{I}_2 + \frac{\sin^2\theta}{1+\cos^2\theta}\hat{\sigma}_1 \otimes \hat{\sigma}_1 + \frac{\sin^2\theta}{1+\cos^2\theta}\hat{\sigma}_2 \otimes \hat{\sigma}_2 - \hat{\sigma}_3 \otimes \hat{\sigma}_3\right)$$

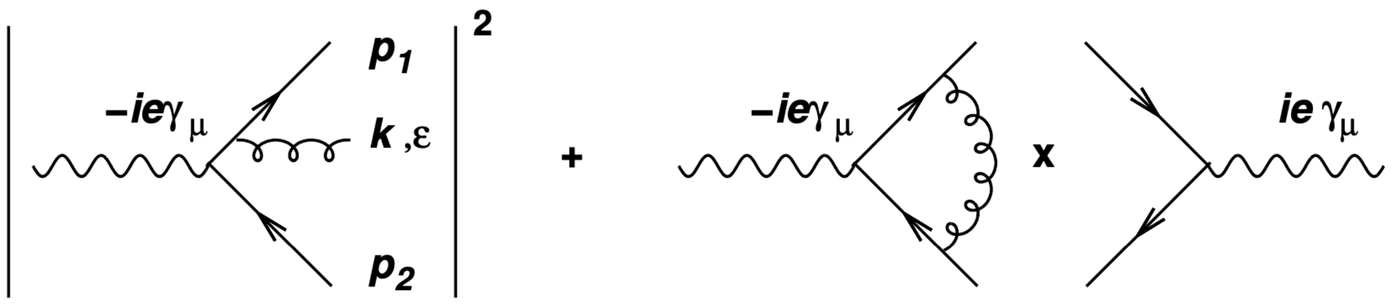
Maximal entanglement at $\cos\theta = 0$

$$\mathcal{C}[\rho_{\rm LO}] = 1$$

Corresponding to: $\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

At next leading order

Aoude, Barr, Maltoni, Satrioni ‘25



$$\rho^{\mathrm{red}}_{\mathrm{LO+NLO}} = \mathsf{p}_{\mathrm{LO}}\, \mathbb{1}\rho_{\mathrm{LO}}\mathbb{1} + \bar{\mathcal{E}}_{\mathrm{V}}[\rho_{\mathrm{LO}}] + \bar{\mathcal{E}}_{\mathrm{R}}[\rho_{\mathrm{LO}}]$$

$$\bar{\mathcal{E}}_{\mathrm{V}}[\rho_{\mathrm{LO}}] = \mathsf{p}_{\mathrm{V}}\mathbb{1}\rho_{\mathrm{LO}}\mathbb{1}$$

Virtual

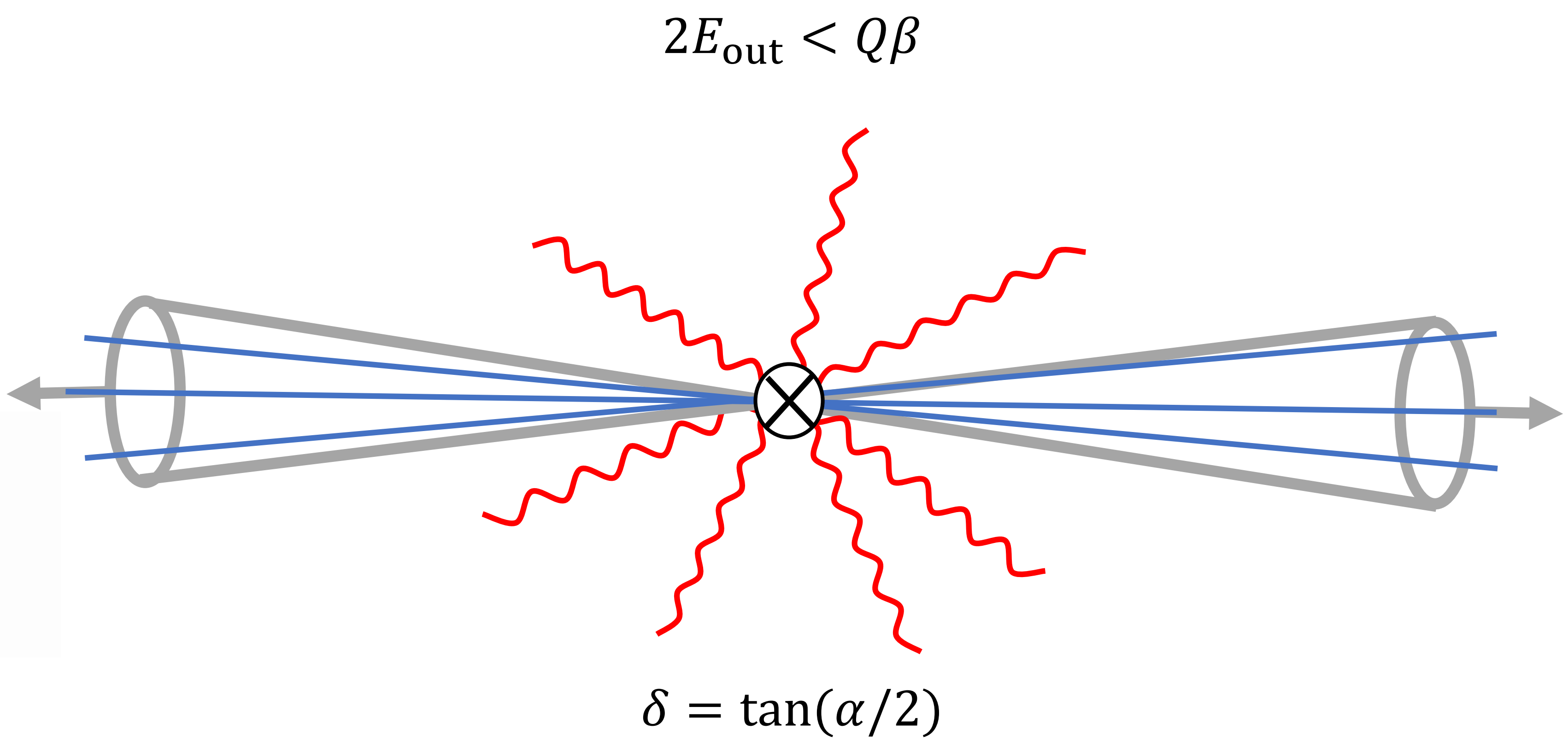
$$\bar{\mathcal{E}}_{\mathrm{R}}[\rho_{\mathrm{LO}}] = \sum_j K_j \rho_{\mathrm{LO}} K_j^\dagger$$

Real: hard, collinear, soft

More precisely

- We need
  - Precise definition of unobservable radiation with experimental resolution.
  - Potential large logs needs to be re-summed.
  - A framework for this purpose (Soft Collinear Effective Theory)
- A large body of knowledge in both quantum information and SCET.
  - Our work is a bridge between them.

Soft and collinear radiation



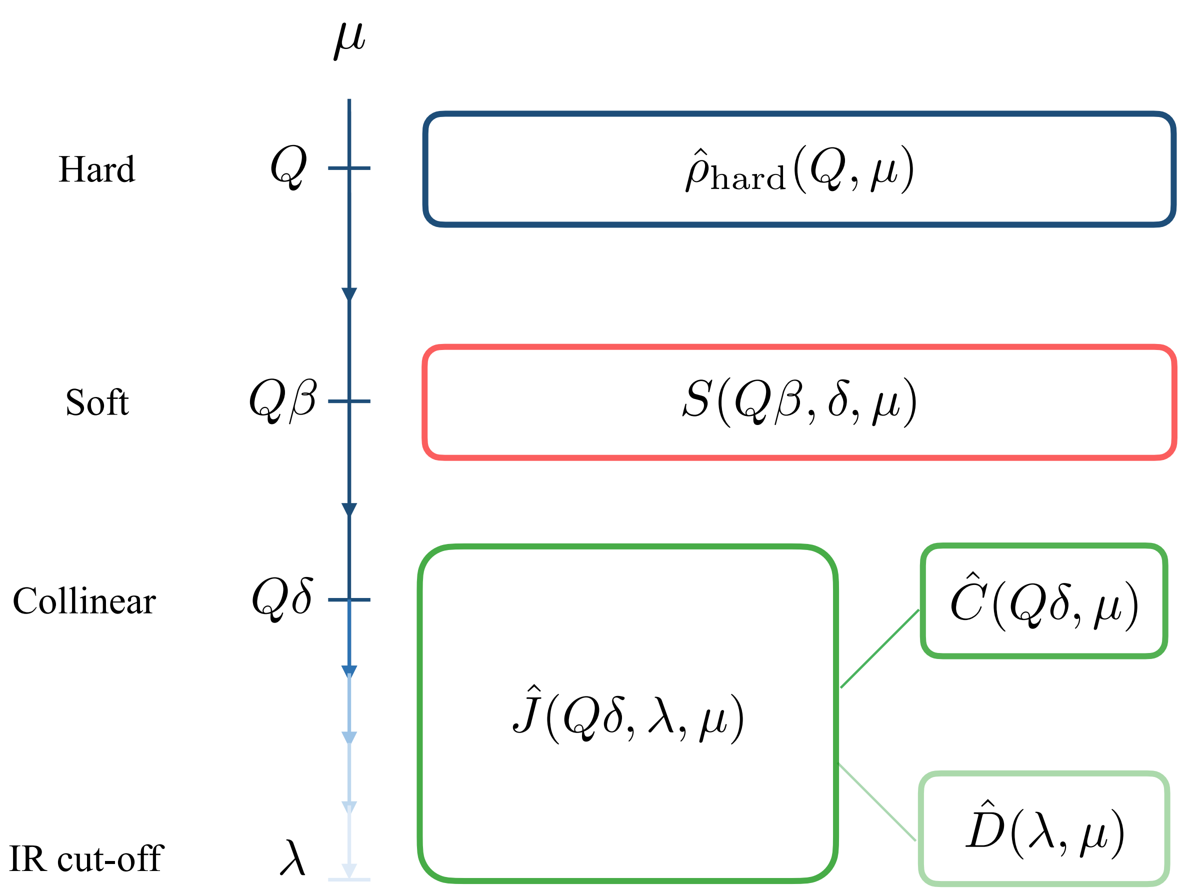
Sterman and Weinberg, 77

Q: hard scale of the scattering event.

A framework to tackle decoherence.

J. Gu, S.-J. Lin, D.-Y. Shao, S.-X. Yang, LTW, 2510.13591

The process of decoherence, the loss through unresolved radiation, can be systematically accounted for in an RGE/SCET framework.



Schematic drawing of the framework.
β, δ parameterize resolution

μ : renomalization scale

Factorization

- This jet definition allows us to apply the factorization theorems of SCET

$$\hat{\rho} = \underbrace{S(Q\beta, \delta, \mu)}_{\text{Soft function}} \underbrace{\hat{J}_f(Q\delta, \lambda, \mu)}_{\text{Jet function}} \overbrace{\hat{\rho}_{\mathrm{hard}}\ (Q, \mu)}^{\text{Hard function}} \underbrace{\hat{J}_{\bar{f}}(Q\delta, \lambda, \mu)}_{\text{Jet function}}$$

- Hard function:

$$\hat{\rho}_{\mathrm{hard}}(Q, \mu) = \frac{1}{4}\Big(\hat{I}\otimes\hat{I} + P_i^{+}\,\hat{\sigma}_i \otimes \hat{I} + P_j^{-}\,\hat{I}\otimes\hat{\sigma}_j + C_{ij}\,\hat{\sigma}_i\otimes\hat{\sigma}_j\Big)$$

Collinear radiation: jet function

- The fragmenting jet operators project the hard scattering state onto the Hilbert space of the observed particles. This effectively traces over unobserved collinear radiation, and induces decoherence

$$J_f \propto \quad \Xi_{\alpha\beta} = \sum_X \frac{\mathrm{d}z\,\mathrm{d}t}{2\pi z} e^{it\bar{n}\cdot P/z} \prod_i \theta\left(\delta^2 - \frac{n\cdot \hat{p}_{X_i}}{\bar{n}\cdot \hat{p}_{X_i}}\right) \langle 0|\, \chi_{n,\alpha}(t\bar{n})\, |f(PS)\, X\rangle \, \langle f(PS)\, X|\, \bar{\chi}_{n,\beta}(0)\, |0\rangle$$

Chen, Goldstein, Jaffe & Ji '94

- Spin decomposition

$$\hat{J}_f = \mathcal{J}_f^U \, \hat{I}\otimes\hat{I} + \mathcal{J}_f^L \hat{\sigma}_z\otimes\hat{\sigma}_z + \mathcal{J}_f^T (\hat{\sigma}_x\otimes\hat{\sigma}_x + \hat{\sigma}_y\otimes\hat{\sigma}_y)$$

- **J^U : unpolarized**
- **J^L : longitudinal polarized**
- **J^T : transverse polarized**

Jet evolution and decoherence

- Refactorization via an operator product expansion

$$\hat{J}(Q\delta,\lambda,\mu)=\hat{C}(Q\delta,\mu)\hat{D}(\lambda,\mu)$$

Fragmentation operator

- Define a scale-dependent effective production matrix

$$\hat{\rho}_{\rm eff}(\mu)\equiv S(Q\beta,\delta,\mu)\hat{C}_f(Q\delta,\mu)\hat{\rho}_{\rm hard}(Q,\mu)\hat{C}_{\bar{f}}(Q\delta,\mu)$$

Decoherence as RG flow

$$\hat{\rho}_{\rm eff}(t)=\hat{U}_f(t,0)\hat{\rho}_{\rm eff}(0)\hat{U}_{\bar{f}}(t,0)\qquad t\equiv\log(Q\delta/\mu).$$

Match to exp observable

We observe spin-dependent cross section

$$\mathrm{d}\sigma(\boldsymbol{S}_f, \boldsymbol{S}_{\bar{f}})$$

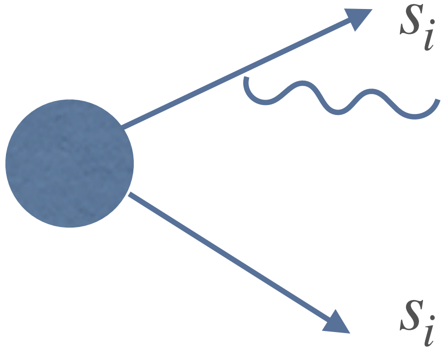
$S_{f,\bar{f}}$ spin of final states

Define projection operator

$$\hat{P}_{f(\bar{f})} = (\hat{I} + \boldsymbol{S}_{f(\bar{f})} \cdot \hat{\boldsymbol{\sigma}})/2$$

We have: $$\mathrm{d}\sigma(\boldsymbol{S}_f, \boldsymbol{S}_{\bar{f}}) \propto \mathrm{Tr}\left[\hat{P}_f \otimes \hat{P}_{\bar{f}}\, \hat{R}\right]$$

Example: QED-like theory



Radiation QED-like

Production: ρ_{hard}

Opposite helicity
(e.g.vector like production):

$$\hat{\rho}_{\text{OH}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{C}(\hat{\rho}_{\text{OH}}) = 2\,|\rho_{23}| = 2\,|\rho_{32}|$$

Same helicity
(such as Higgs decay)

$$\hat{\rho}_{\text{SH}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \qquad \mathcal{C}(\hat{\rho}_{\text{SH}}) = 2\,|\rho_{14}| = 2\,|\rho_{41}|$$

More general state:

$$\hat{\rho}_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}$$

$$\mathcal{C}(\hat{\rho}_X) = 2\max\left(0,\ |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}},\ |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\ \right)$$

Kraus operators in QED

- IR cut-off = lepton mass $\lambda = m$
- The phase-flip channel: randomly applies a "phase flip" to a qubit
- The Kraus operators in QED
  - phase-flip channel

$$\hat{K}_{(i,j)} = \hat{K}_i^{\ell^-} \otimes \hat{K}_j^{\ell^+} \qquad p = \sqrt{\frac{1}{2}\left[1 - \exp\left(-\frac{\alpha}{2\pi}t\right)\right]} \qquad t \equiv \log(Q\delta/\mu).$$

$$\hat{K}_0^{\ell^-} = \hat{K}_0^{\ell^+} = \sqrt{1-p^2}\,\mathbb{I}$$ **with probability 1 - *p*²**

$$\hat{K}_1^{\ell^-} = \hat{K}_1^{\ell^+} = p\,\hat{\sigma}_3$$ **with probability *p*²**

Master equation

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_{\text{eff}}}{\mathrm{d}t} = & -\frac{\alpha}{2\pi}\hat{\rho}_{\text{eff}} \\
& + \frac{\alpha}{4\pi}\left[(\hat{\sigma}_3 \otimes \hat{I})\,\hat{\rho}_{\text{eff}}\,(\hat{\sigma}_3 \otimes \hat{I}) + (\hat{I} \otimes \hat{\sigma}_3)\,\hat{\rho}_{\text{eff}}\,(\hat{I} \otimes \hat{\sigma}_3)\right]
\end{aligned}
$$

Corresponding to Lindblad operators

$$
\hat{L}_1 = \sqrt{\alpha/4\pi}\,\hat{\sigma}_3 \otimes \hat{I} \qquad \hat{L}_2 = \sqrt{\alpha/4\pi}\,\hat{I} \otimes \hat{\sigma}_3
$$

Decay of off-diagonal terms

$$
\frac{\hat{\rho}^{ij}_{\text{eff}}(t)}{\hat{\rho}^{ij}_{\text{eff}}(0)} = \begin{cases} 1 & i = j \quad (\text{diagonal})\,, \\ e^{-\frac{\alpha}{\pi}t} & ij = 14, 23, 32, 41 \quad (\text{anti-diagonal})\,, \\ e^{-\frac{\alpha}{2\pi}t} & \text{else}\,. \end{cases}
$$

Upper bound on decoherence

We derive a general upper bound for Concurrence decoherence

$$\mathcal{C}(t) \leq \mathcal{C}(0) e^{-\frac{\alpha}{\pi} t} \qquad t = \log(Q\delta/m)$$

$$\Rightarrow \quad \mathcal{C}_{\text{final}} \leq \mathcal{C}(0) \left(\frac{Q\delta}{m} \right)^{-\frac{\alpha}{\pi}}$$

Satisfied for example by:

$$\hat{\rho}_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}$$

$$\mathcal{C}(t) = 2 \max \left(0,\, e^{-\frac{\alpha}{\pi} t} \, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}},\, e^{-\frac{\alpha}{\pi} t} \, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}} \right)$$

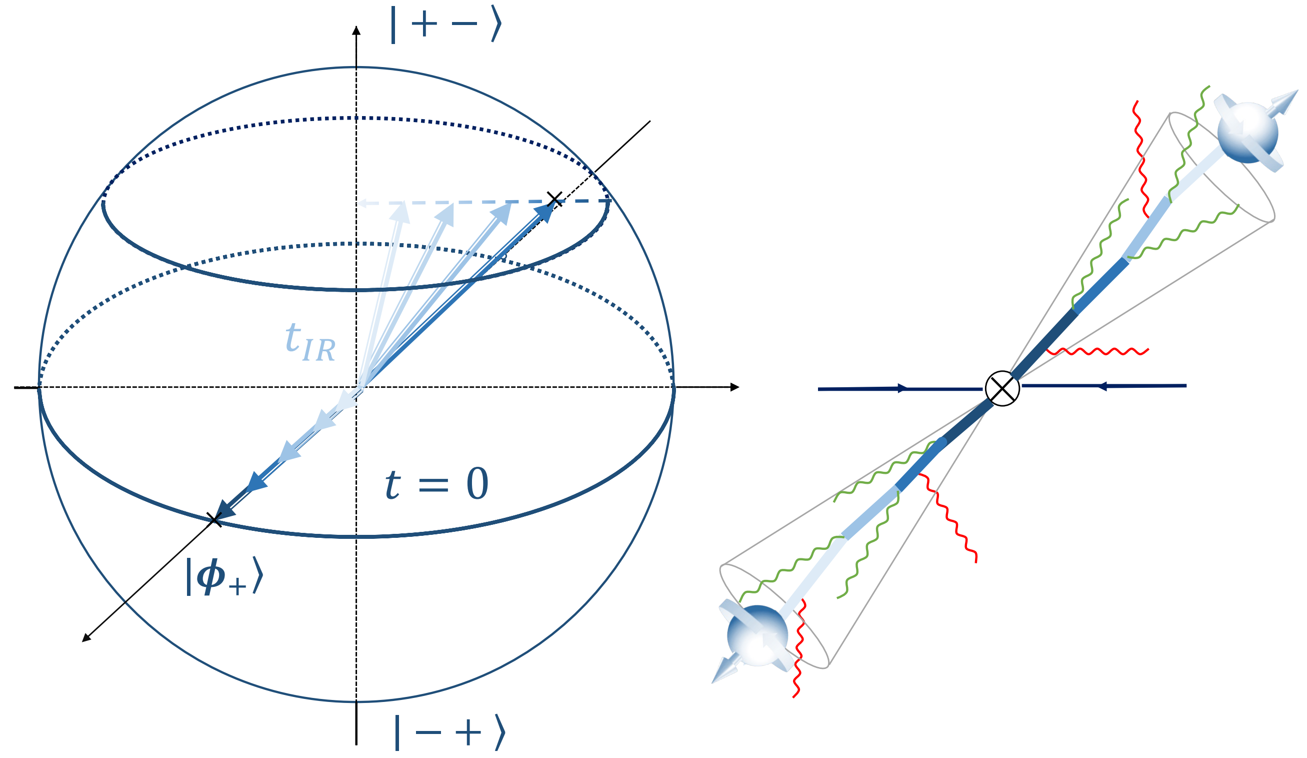
$$P_1^{\pm} = P_2^{\pm} = 0\,, \qquad C_{13} = C_{23} = C_{31} = C_{32} = 0$$

Inequality saturated for SH and OH states

Decoherence in action

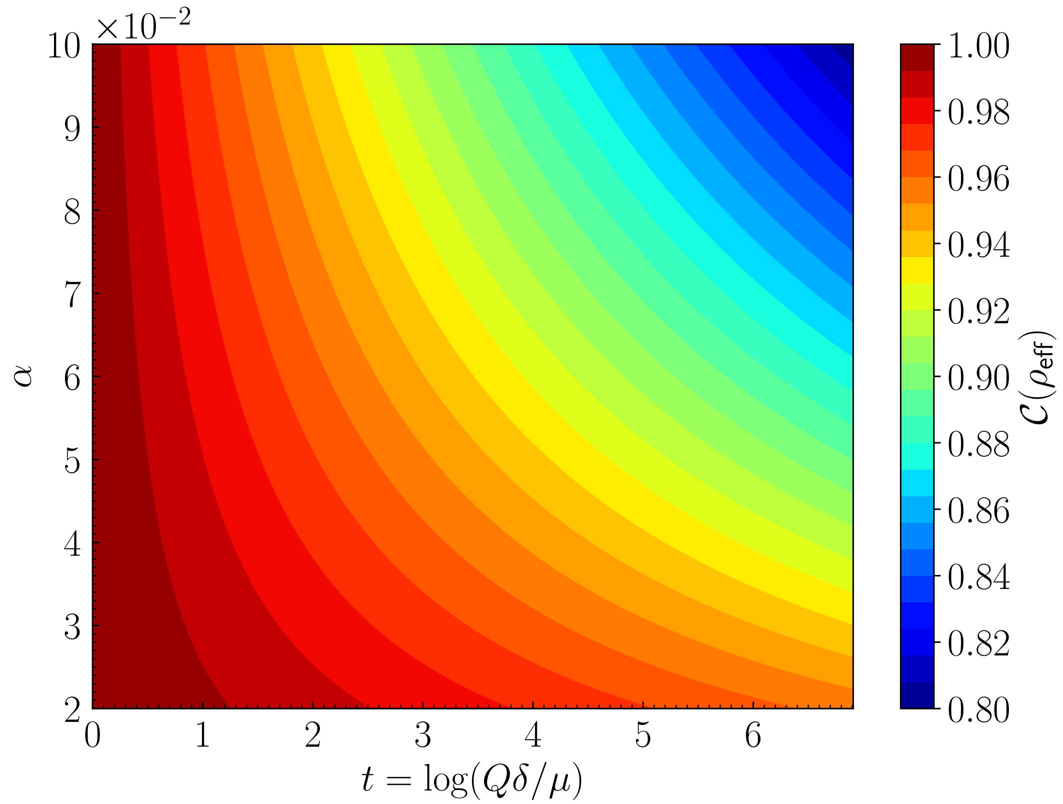
J. Gu, S.-J. Lin, D.-Y. Shao, S.-X. Yang, LTW, 2510.13591

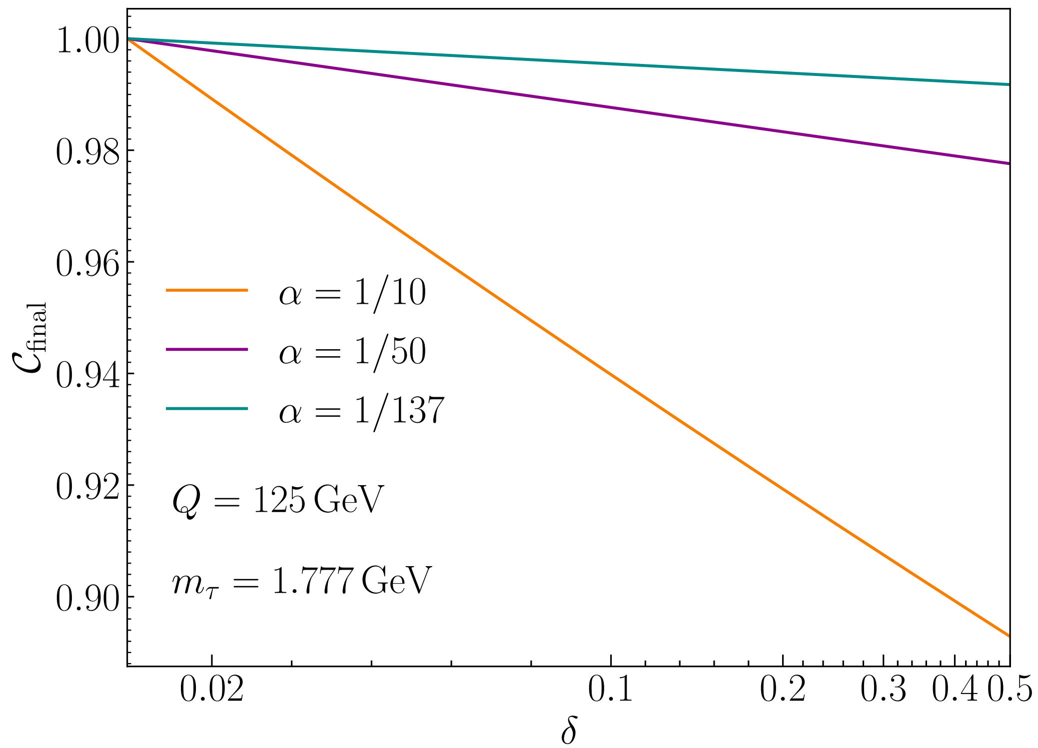
$|+-\rangle$ and $|-+\rangle$



RG evolution and decoherence, viewed through a Bloch sphere, QED-like theory

Decoherence in QED-like theory





$$\mathcal{C}(t) = \mathcal{C}(0)e^{-\frac{\alpha}{\pi}t}$$

$$\mathcal{C}_{\text{final}} \leq \mathcal{C}(0)\left(\frac{Q\delta}{m}\right)^{-\frac{\alpha}{\pi}}$$

Small decoherence in QED, as expected.
Resolution play an essential role.

Soft radiation decoherence.

Carney, Chaurette, Neuenfeld, Semenoff, 2017

Incoming: $|\alpha\rangle$ Outgoing: $|\text{out}\rangle = S|\alpha\rangle = \sum_{\beta,b} S_{\beta b,\alpha}|\beta b\rangle$ b: soft

Leading power factorization theorem: $S_{\beta b,\alpha} = S_{\beta,\alpha}\boxed{F_{\beta,\alpha}(\gamma)}$ Soft factor

Tracing out soft radiation:

$$\rho_{\beta\beta'} = S_{\beta,\alpha}S^*_{\beta',\alpha}\left(\frac{E}{\lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}}\left(\frac{\lambda}{Q}\right)^{A_{\beta,\alpha}/2+A_{\beta',\alpha}/2}$$

$\tilde{A}_{\beta\beta',\alpha}$ real emission $A_{\beta,\alpha}$ Virtual emission

$$\Delta A_{\beta\beta',\alpha} = \frac{A_{\beta,\alpha}}{2} + \frac{A_{\beta',\alpha}}{2} - \tilde{A}_{\beta\beta',\alpha}$$ positive.

Decoherence effect important as IR cutoff $\lambda \ll E < Q$

Applications to collider signal? Decohere momentum entanglement?

Future directions

- General question: for potential application of measuring entanglement, how important is the decoherence effect
  - Apply this to a broader range of observables: discord, etc.
  - More complicated (more relevant cases), such as QCD and hadronic collisions.
  - Decoherence beyond spin correlations: momentum, etc.

Future directions

- More general question: How does quantum entanglement help us?
  - Testing quantum mechanics? Probably not
  - Help with discovering new physics? Some SMEFT studies, not obvious.
  - Different from “pure kinematics”, should have something new to say. Need to think harder.

extra

Example: QED

$$\hat{R} = S(Q\beta, \delta, \mu)\hat{J}_f(Q\delta, \lambda, \mu)\hat{R}_{\text{hard}}(Q, \mu)\hat{J}_{\bar{f}}(Q\delta, \lambda, \mu)$$

$$\hat{\rho} = \hat{R}/\,\mathrm{Tr}[\hat{R}]$$

$$H(Q,\mu) = 1 + \frac{\alpha}{4\pi}\left[-8\log^2\left(\frac{Q}{\mu}\right) + 12\log\left(\frac{Q}{\mu}\right) - 16 + \frac{7\pi^2}{3}\right]$$

$$S(Q\beta,\delta,\mu) = 1 + \frac{\alpha}{4\pi}\left[-16\log\delta\log\left(\frac{Q\beta}{\mu}\right) - 8\log^2\delta - \frac{2\pi^2}{3}\right]$$

$$\Xi_{\alpha\beta} = \sum_X\!\!\!\!\!\!\int \frac{\mathrm{d}z\,\mathrm{d}t}{2\pi z} e^{it\bar{n}\cdot P/z} \prod_i \theta\left(\delta^2 - \frac{n\cdot\hat{p}_{X_i}}{\bar{n}\cdot\hat{p}_{X_i}}\right) \langle 0|\,\chi_{n,\alpha}(t\bar{n})\,|f(PS)\,X\rangle\,\langle f(PS)\,X|\,\bar{\chi}_{n,\beta}(0)\,|0\rangle$$

$$\mathcal{J}_f^U(Q\delta,m) = \mathrm{Tr}\left[\frac{\not{\bar{n}}}{2}\Xi\right],\quad \mathcal{J}_f^L(Q\delta,m) = \mathrm{Tr}\left[\frac{\not{\bar{n}}}{2}\gamma_5\Xi\right],\quad S_\perp^i\mathcal{J}_f^T(Q\delta,m) = \mathrm{Tr}\left[\frac{\not{\bar{n}}}{2}\gamma_\perp^i\gamma_5\Xi\right]$$

$$\mathcal{J}_f^{\mathcal{P}}(Q\delta,m) = 1 + \frac{\alpha}{4\pi}\left[4\log^2\left(\frac{Q\delta}{\mu}\right) - 6\log\left(\frac{Q\delta}{\mu}\right) - \frac{\pi^2}{6} + 7 + j^{\mathcal{P}}\right]$$

$$j^U = 1,\ j^L = -1,\ \text{and}\ j^T = 1 - 2\log(Q\delta/m)$$

QED

$$\hat{J}(Q\delta, \lambda, \mu) = \hat{C}(Q\delta, \mu)\hat{D}(\lambda, \mu)$$

$$\mathcal{D}_f^{\mathcal{P}} = 1 + \frac{\alpha}{4\pi} d^{\mathcal{P}}, \qquad \text{with} \quad d^U = 0, \quad d^L = -2, \quad d^T = -2\log(\mu/m)$$

$$\mathscr{C}_f^{\mathcal{P}(1)} = \mathcal{J}_f^{\mathcal{P}(1)} - \mathcal{D}_f^{\mathcal{P}(1)}$$

$$\mathscr{C}_f^U = \mathscr{C}_f^L = 1 + \frac{\alpha}{4\pi}\left[4\log^2\left(\frac{Q\delta}{\mu}\right) - 6\log\left(\frac{Q\delta}{\mu}\right) - \frac{\pi^2}{6} + 8\right]$$

$$\mathscr{C}_f^T = 1 + \frac{\alpha}{4\pi}\left[4\log^2\left(\frac{Q\delta}{\mu}\right) - 8\log\left(\frac{Q\delta}{\mu}\right) - \frac{\pi^2}{6} + 8\right].$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\mathcal{D}_f^{\mathcal{P}}(m,\mu) = \gamma^{\mathcal{P}}\,\mathcal{D}_f^{\mathcal{P}}(m,\mu)$$

$$\gamma^U = \gamma^L = 0, \qquad \gamma^T = -\frac{\alpha}{2\pi}$$

$$\hat{M}_{f(\bar{f})}(\boldsymbol{S}_{f(\bar{f})}) \equiv \hat{D}_{f(\bar{f})}\hat{P}_{f(\bar{f})}$$

$$\mathrm{d}\sigma(\boldsymbol{S}_f, \boldsymbol{S}_{\bar{f}}) \propto \mathrm{Tr}\left[\hat{M}_f(\boldsymbol{S}_f, t)\,\hat{R}_{\mathrm{eff}}(t)\,\hat{M}_{\bar{f}}(\boldsymbol{S}_{\bar{f}}, t)\right]$$

$$\hat{\rho}_{\rm eff}(t) = \hat{U}_f(t,0)\hat{\rho}_{\rm eff}(0)\hat{U}_{\bar{f}}(t,0)$$

$$U^{\mathcal{P}}(t,0) = \exp\left(\int_0^t {\rm d}t\,\gamma^{\mathcal{P}}\right) \qquad \gamma^{\mathcal{P}} \equiv \frac{\alpha}{\pi}P^{\mathcal{P}}_{ff}$$

1st Mellin moment of splitting function