Karlsruhe Institute of Technology

LOOPFEST 2026
BNL
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NATIONAL LABORATORY

Institute for Theoretical Physics

NLO analysis of the $\mathcal{O}_1 - \mathcal{O}_7$ interference in $\overline{B} \to X_s\gamma$ at subleading power

Based on JHEP 04 (2025) 066 and 26xx.xxxxx (in collaboration with Philipp Böer and Tobias Hurth)

LoopFest XXIV, BNL, 27th May 2026

Riccardo Bartocci - Karlsruhe Institute of Technology

Motivation: the $\mathcal{O}_1 - \mathcal{O}_7$ interference contribution

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In the **inclusive $\overline{B} \to X_s\gamma$ decay**, the CP-averaged photon-energy spectrum is given by

[1003.5012: Benzke, Lee, Neubert, Paz]

$$\frac{d\Gamma}{dE_\gamma} = \frac{G_F\alpha|V_{tb}V_{ts}^*|^2}{2\pi^4}\overline{m_b}^2(\mu)E_\gamma^3\Bigg[|H_\gamma(\mu)|^2\int d\omega m_b J(m_b(\omega+p_+);\mu)S(\omega;\mu)$$
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Resolved contributions (subleading power corrections)

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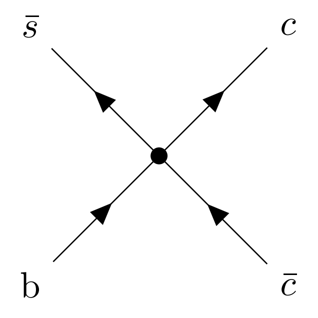
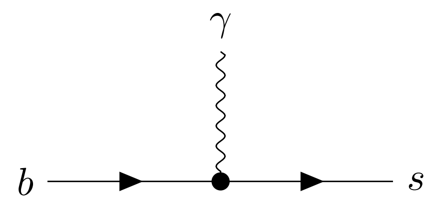
$$\mathcal{O}_8^{d_id_j} = g_s m_{d_i}\left(\bar{d}_j\sigma^{\mu\nu}T^A P_R d_i\right)G^A_{\mu\nu}$$





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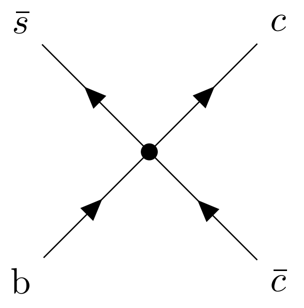

$$\mathcal{O}_2^{q,\,d_i d_j} = (\bar{d}_i^\alpha\gamma_\mu P_L q^\alpha)(\bar{q}^\beta\gamma^\mu P_L d_j^\beta)$$

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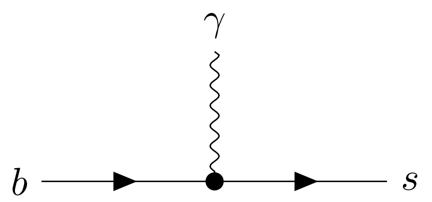

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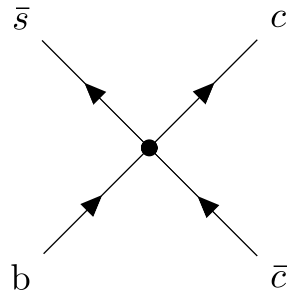

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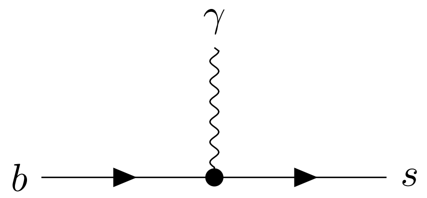

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Current uncertainties of **the $\mathcal{O}_1 - \mathcal{O}_7$ interference:**

- Estimated contribution $\approx [2.6\,\%\,, 8.9\%]$ (**largest uncertainty**)
- **Large scale ambiguity** $\approx 40\,\%$ (not included in the above estimates)

[2512.08902: Benzke, Hurth]

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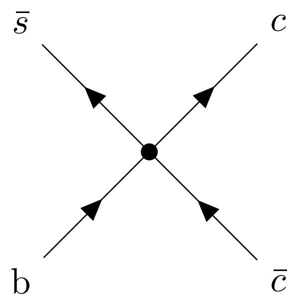

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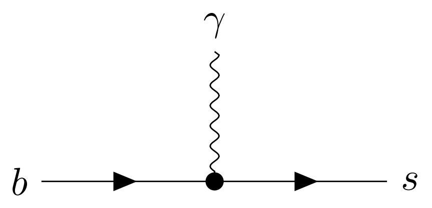

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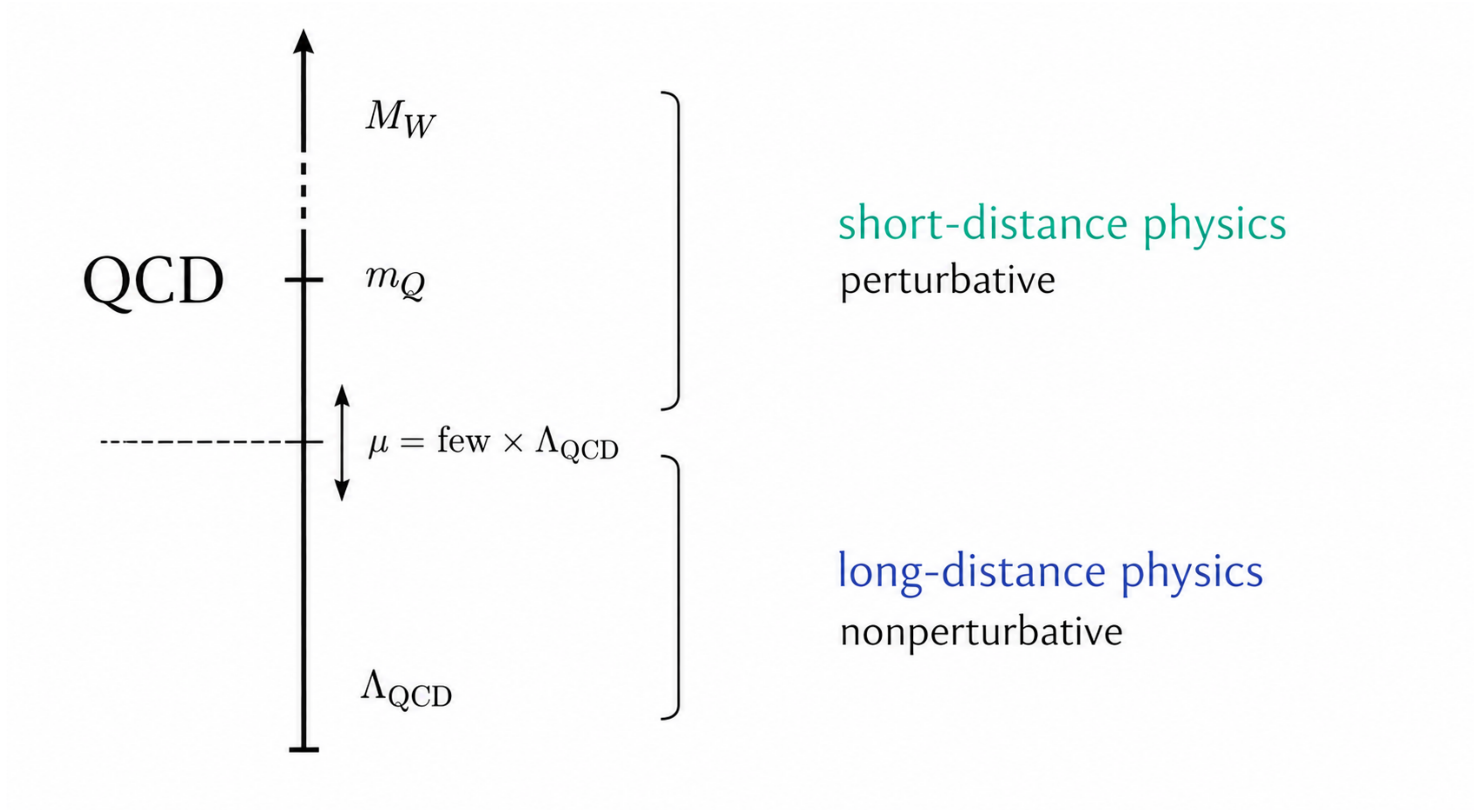
[2512.08902: Benzke, Hurth]

→ **NLO analysis and Renormalisation Group Evolution (RGE) will reduce this ambiguity**

Factorisation in Soft Collinear Effective Theory

Factorisation

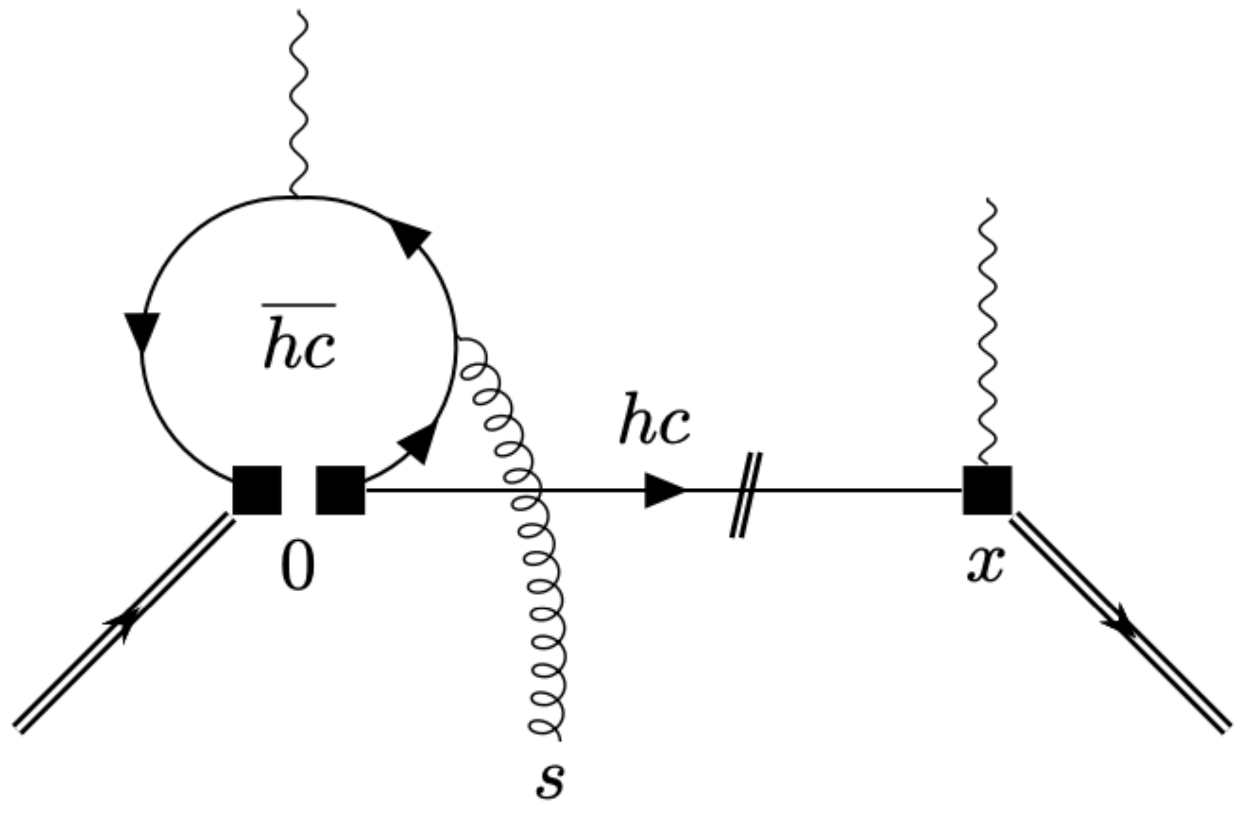
Factorisation ↔ Scale separation ↔ Perturbative and non-perturbative separation



Factorisation at LO

From the LO diagrams of the $\mathcal{O}_1 - \mathcal{O}_7$ interference:

[1003.5012: Benzke, Lee, Neubert, Paz]



we write:

$$d\Gamma(\bar{B} \to X_s\gamma) \propto \mathrm{Disc}_{\underline{\mathrm{restr.}}} \left[i \int d^4x \, \langle \bar{B} | \mathcal{H}^\dagger_{\mathrm{eff}}(x) \, \mathcal{H}_{\mathrm{eff}}(0) | \bar{B} \rangle \right]$$

Factorisation theorem:

$$d\Gamma(\bar{B} \to X_s\gamma) \sim H \cdot J \otimes g_{17} \otimes \bar{J}$$

Factorisation at LO

[1003.5012: Benzke, Lee, Neubert, Paz]

Factorisation formula at LO:

$$d\Gamma(\bar{B} \to X_s\gamma) \sim H \cdot J \otimes g_{17} \otimes \bar{J}$$

$$d\Gamma \sim Re \int_{-\infty}^{\bar{\Lambda}} d\omega\delta(\omega + p_+) \int_{-\infty}^{+\infty} d\omega_1 g_{17}(\omega, \omega_1; \mu) \frac{1}{\omega_1 + i\epsilon} \left[1 - F\left(\frac{m_c^2 - i\epsilon}{2E_\gamma\omega_1}\right)\right], \qquad F(x) = 4x \arctan^2\left(\frac{1}{\sqrt{4x-1}}\right)$$

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1. Hard function ($\mu^2 \sim m_b^2$) at LO ~ 1 (perturbative)

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2. Jet function ($\mu^2 \sim m_b \Lambda_{QCD}$) (perturbative)

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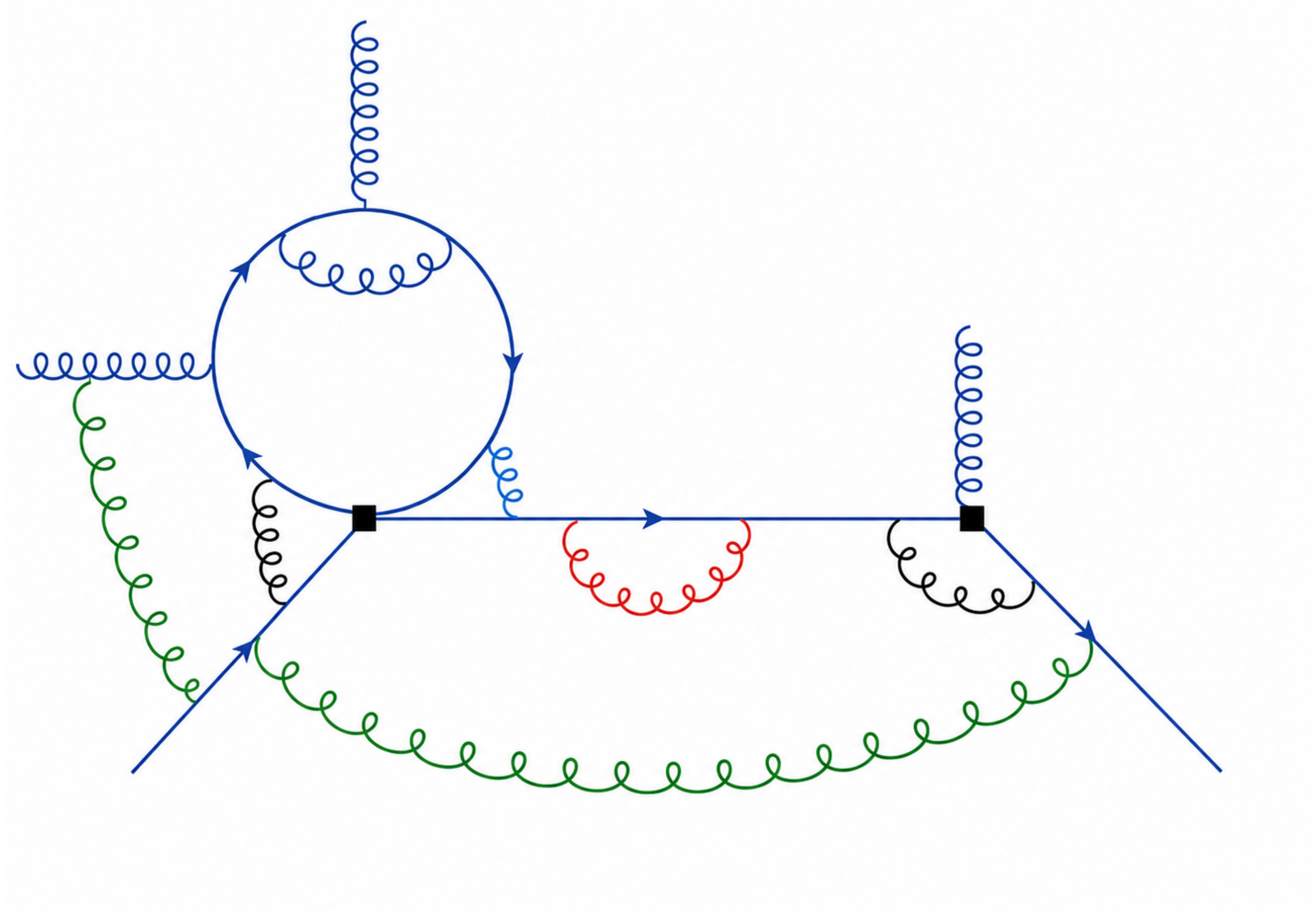
4. “Penguin”-jet function ($\mu^2 \sim m_b \Lambda_{QCD}$) (perturbative)

QCD radiative corrections (NLO)



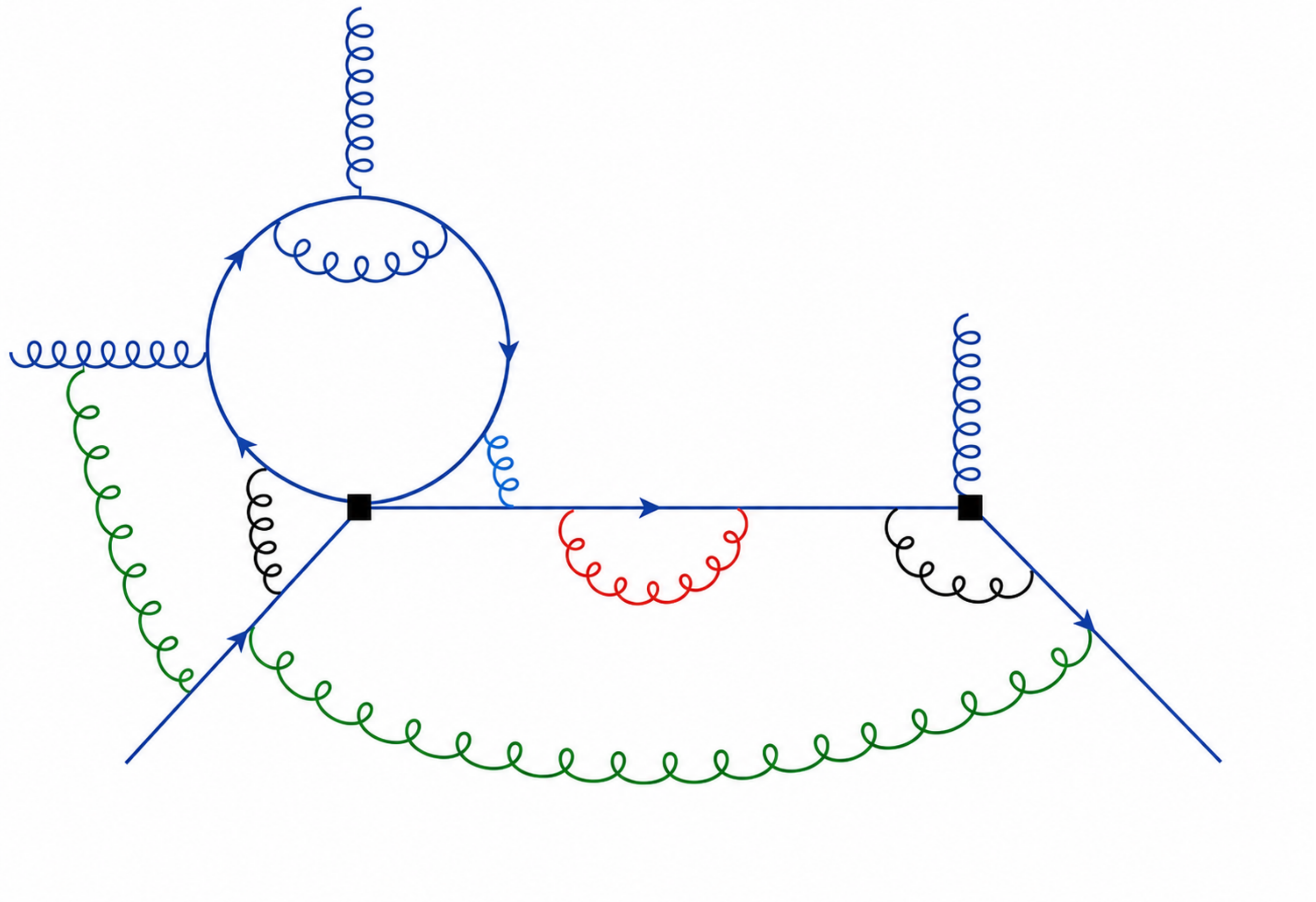
In SCET we can compute the different contributions to the factorisation theorem individually.

Factorisation formula at NLO:

$$d\Gamma(\bar{B}\to X_s\gamma) \sim \int_0^1 du\, \underbrace{H(u;\mu)\int_{-\infty}^{\infty} d\omega_1\, J^{(\bar{n})}(u}_{},\omega_1;\mu)\int_{-\infty}^{\bar{\Lambda}} d\omega\, J^{(n)}(\omega;\mu)\, g_{17}(\omega,\omega_1;\mu)$$

Additional convolution between hard and penguin jet function

QCD radiative corrections (NLO)



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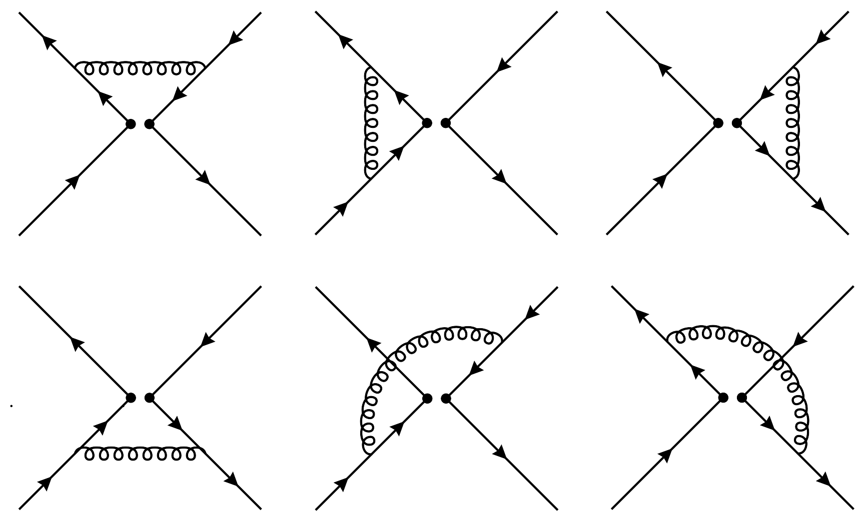
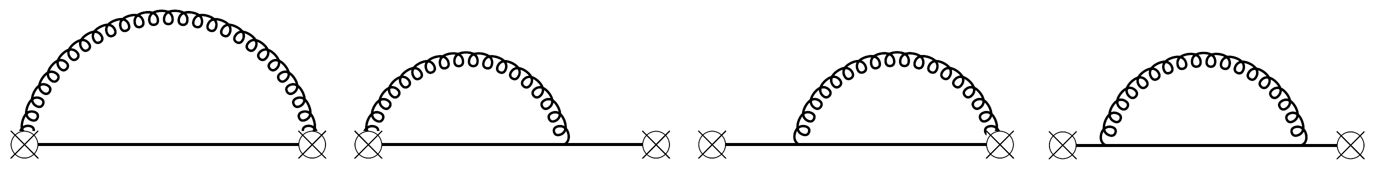
HARD FUNCTION

[2008.10615: Beneke et al.]



JET FUNCTION

[0603140: Becher, Neubert]



PENGUIN JET FUNCTION

[to appear: RB, Böer, Brune, Hurth]

THIS TALK

SHAPE FUNCTION g_{17}

[2411.16634: RB, Böer, Hurth]

THIS TALK

Renormalisation of the shape function g_{17}

The subleading shape function

The subleading shape function is given by the forward matrix element between two B-meson states

$$g_{17}(\omega, \omega_1; \mu) = \frac{1}{2M_B} \int \frac{dr}{2\pi}\, e^{-i\omega_1 r} \int \frac{dt}{2\pi}\, e^{-i\omega t} \, \langle \bar{B}_v | \mathcal{O}_{17}(t, r) | \bar{B}_v \rangle$$

where the operator is defined as

$$\mathcal{O}_{17}(t, r) \to (\bar{h}_v S_n)_-(tn)\, \not{\bar{n}} (S_n^\dagger S_{\bar{n}})_+(0)\, i\gamma_\alpha^\perp \bar{n}_\beta \, (S_{\bar{n}}^\dagger\, g_s G_s^{\alpha\beta} S_{\bar{n}})_+(r\bar{n})\, (S_{\bar{n}}^\dagger h_v)_+(0)$$

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Related to $\bar{n}$

Related to n

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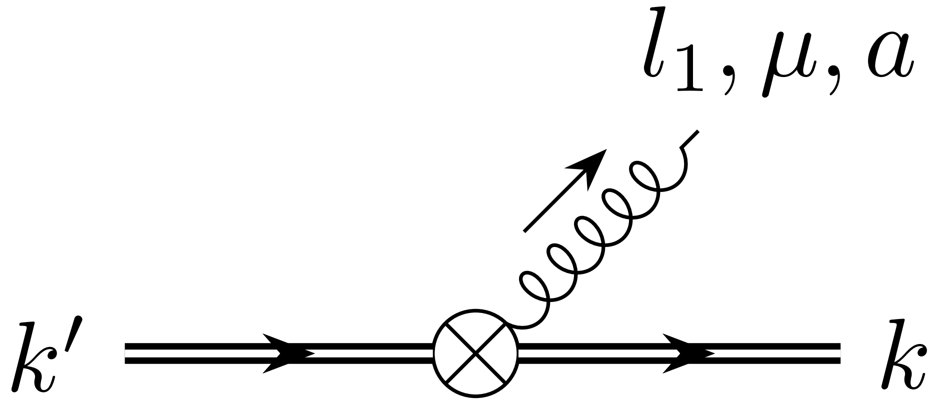
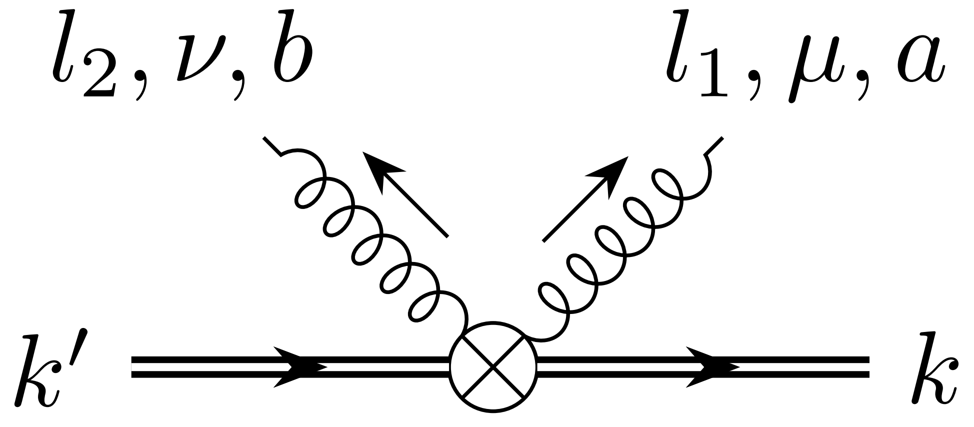
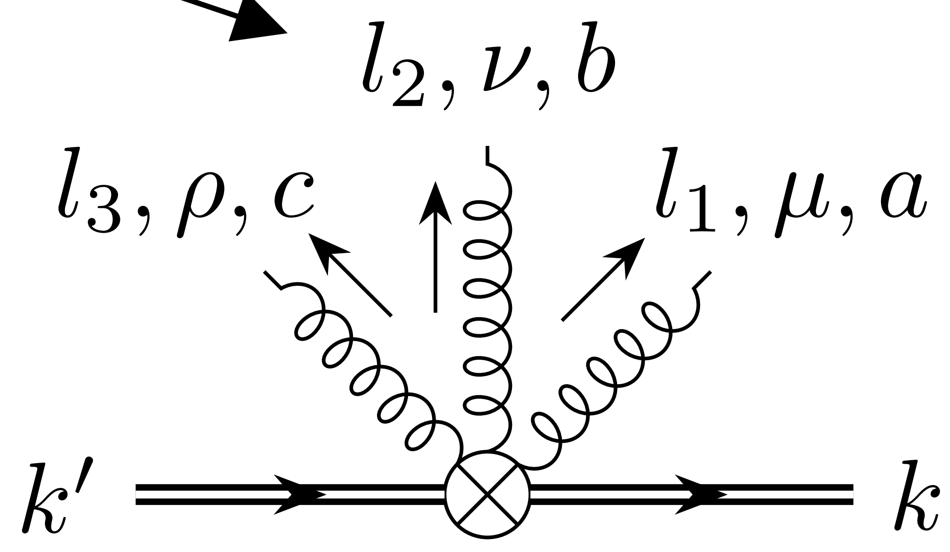


One gluon from $G_s^{\alpha\beta}$



- Two gluons from $G_s^{\alpha\beta}$
- One gluon from $G_s^{\alpha\beta}$ and one from $S_{n,\bar{n}}$



- Two gluons from $G_s^{\alpha\beta}$ and one from $S_{n,\bar{n}}$
- One gluon from $G_s^{\alpha\beta}$ and two from $S_{n,\bar{n}}$

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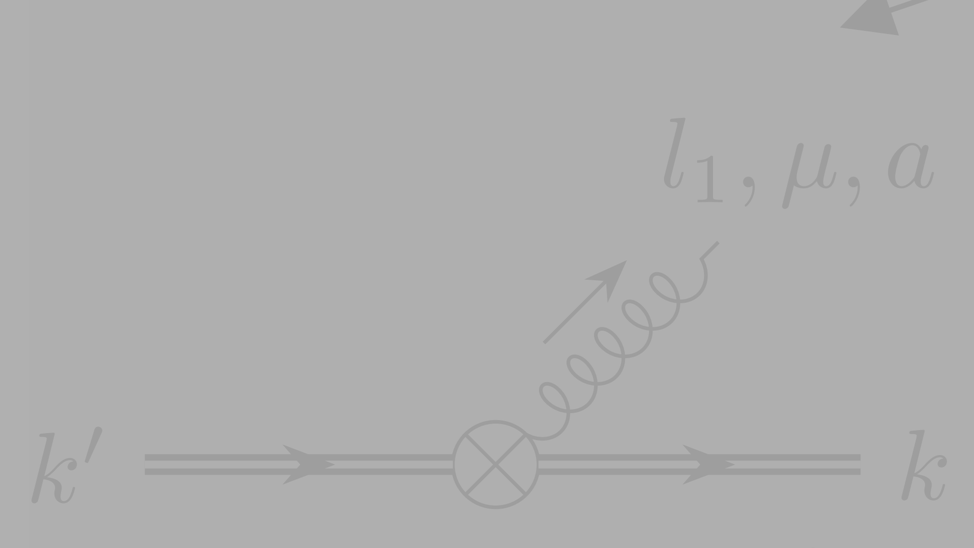
where the operator is defined as

$$\mathcal{O}_{17}(t, r) \to (\bar{h}_v S_n)$$

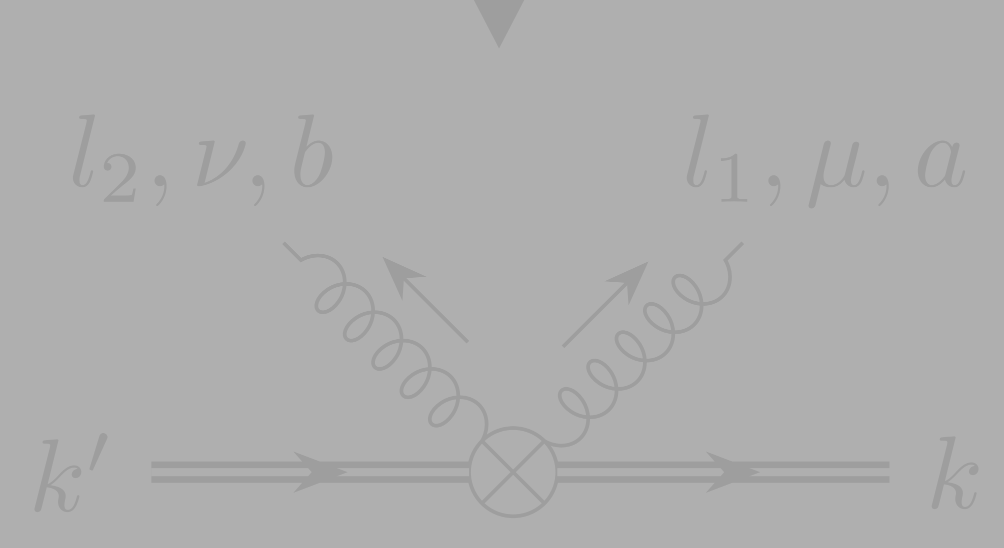
$$S_{\bar{n}})_+ (r\bar{n})\, (S^\dagger_{\bar{n}} h_v)_+ (0)$$

DISCLAIMER

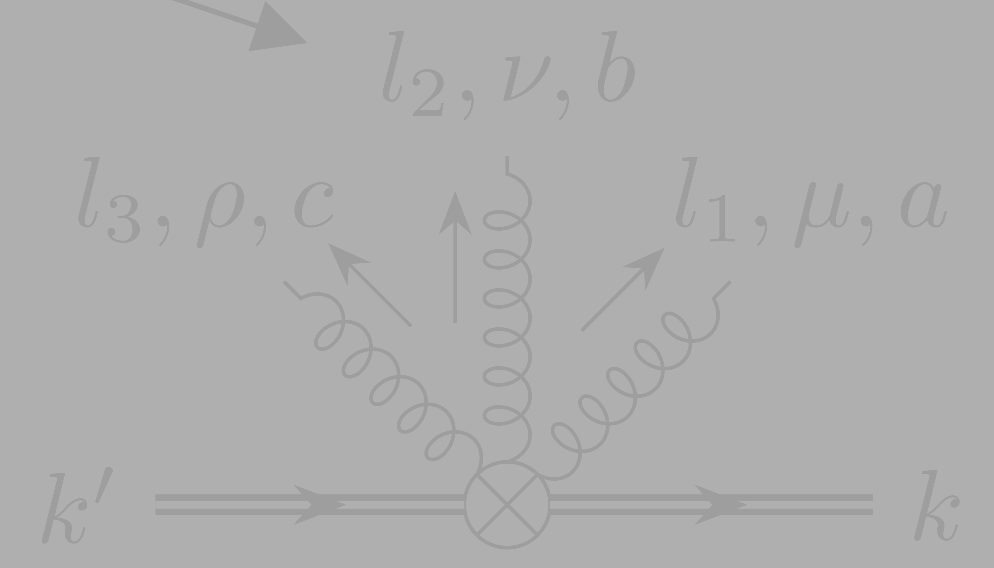
g_{17} must be interpreted as a distribution on the space of jet functions



One gluon from $G_s^{\alpha\beta}$



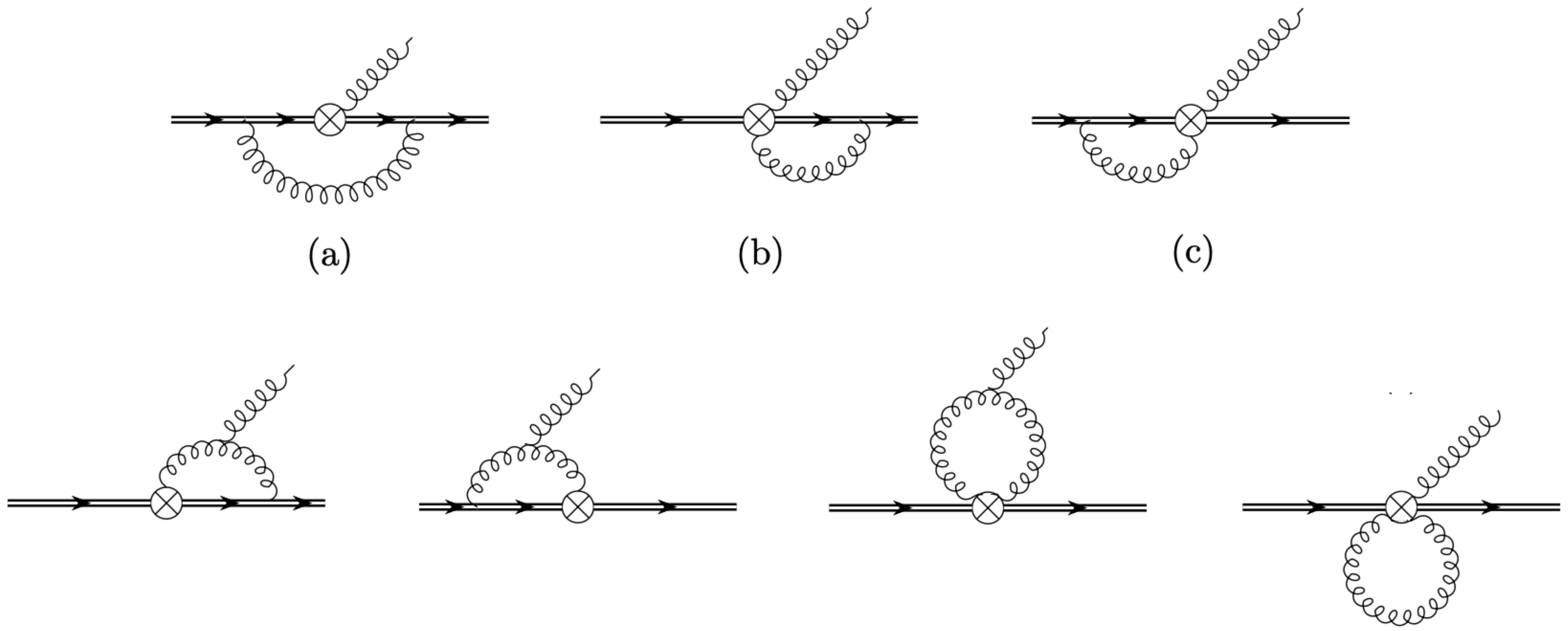
- Two gluons from $G_s^{\alpha\beta}$
- One gluon from $G_s^{\alpha\beta}$ and one from $S_{n,\bar{n}}$



- Two gluons from $G_s^{\alpha\beta}$ and one from $S_{n,\bar{n}}$
- One gluon from $G_s^{\alpha\beta}$ and two from $S_{n,\bar{n}}$

Renormalisation of the subleading shape function

We compute QCD corrections, employing the previous Feynman rules, calculating the following diagrams:


(a) (b) (c)

(d) (e) (f) (g)

[2411.16634: RB, Böer, Hurth]

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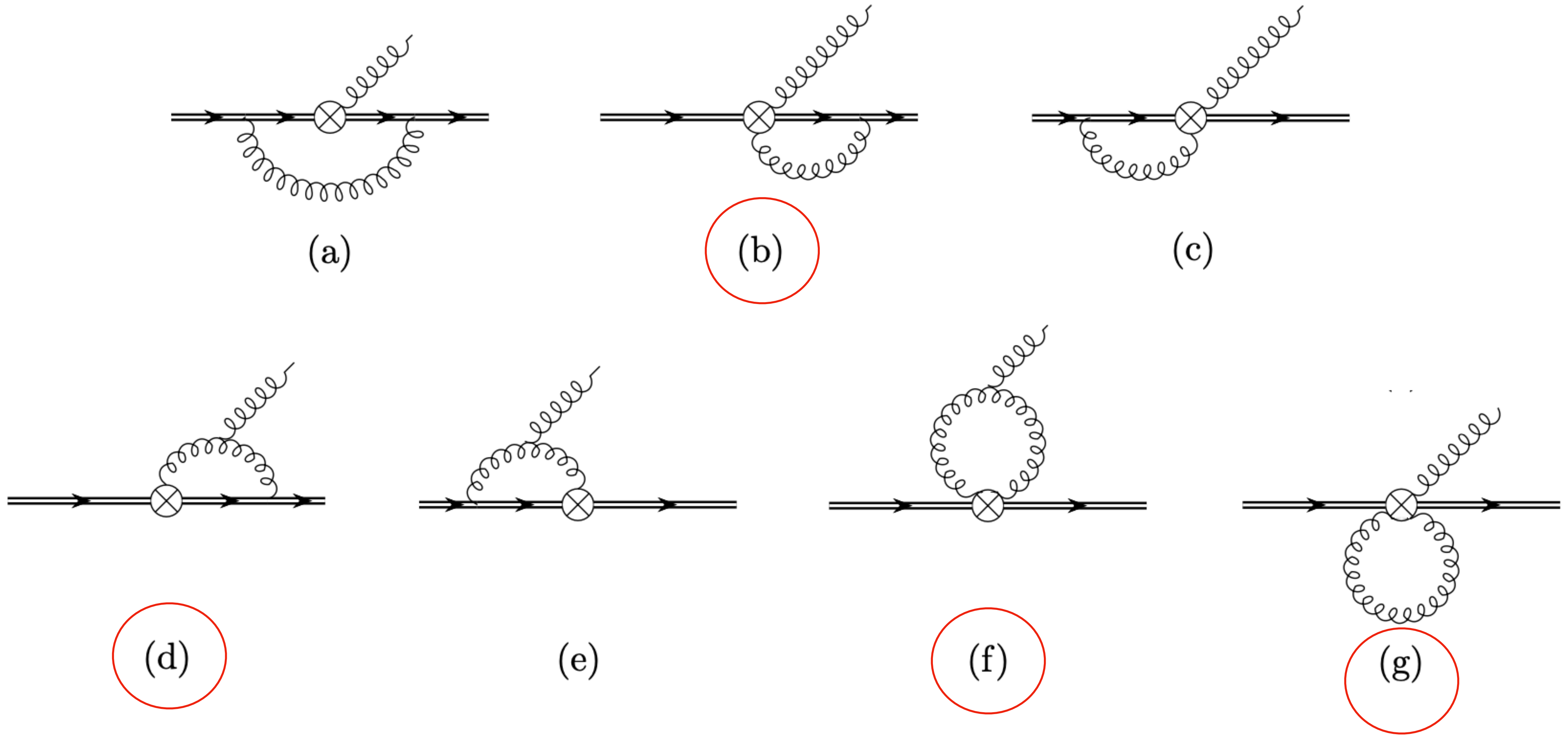
[2411.16634: RB, Böer, Hurth]

Possible mixing of ω and ω_1



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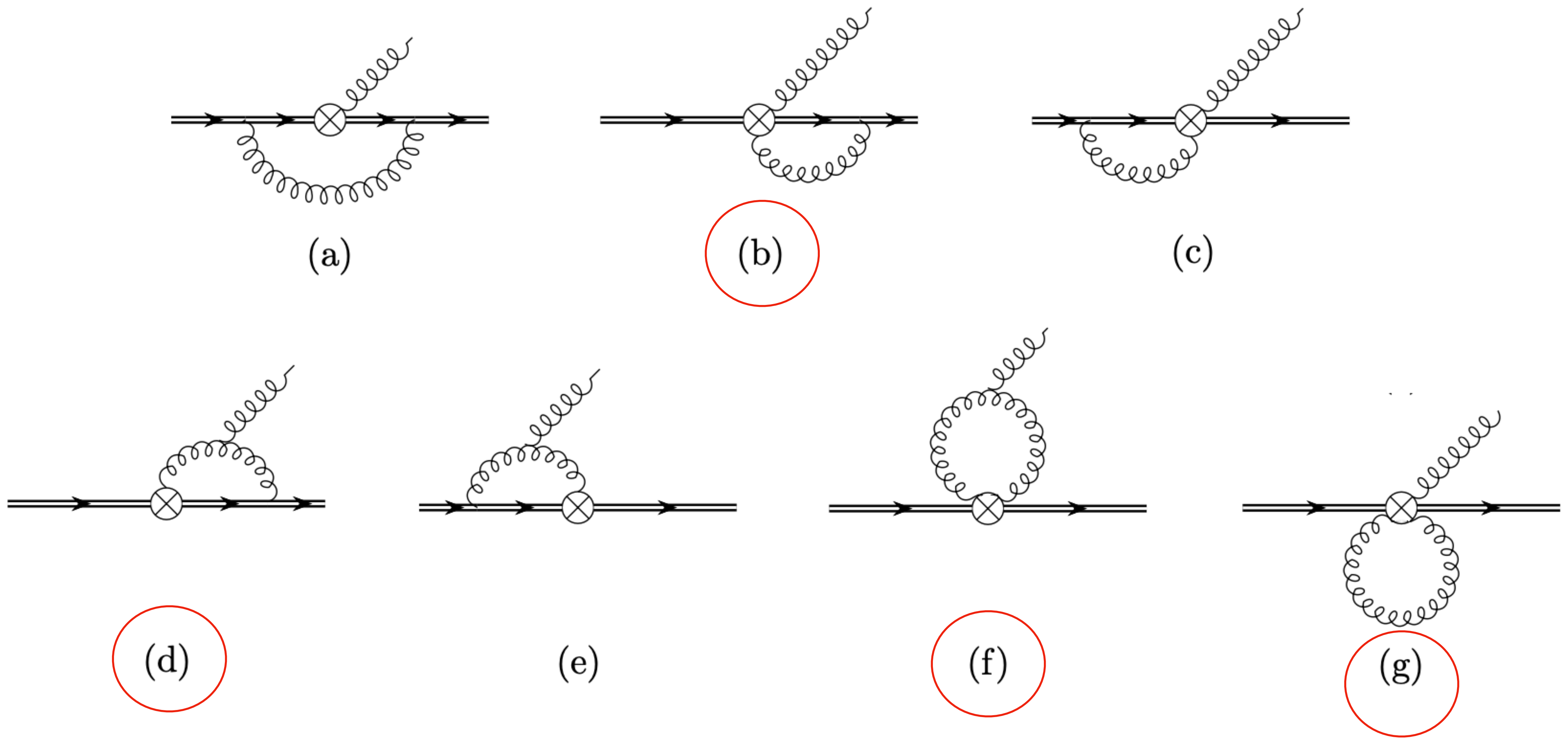

[2411.16634: RB, Böer, Hurth]

Possible mixing of ω and ω_1

For the anomalous dimension we get:

$$\gamma_{17}(\omega,\omega_1,\omega',\omega_1';\mu)=\frac{\alpha_s}{\pi}\Big\{C_F\,\delta(\omega_1-\omega_1')\gamma_n(\omega,\omega';\mu)+\frac{C_A}{2}\,\delta(\omega-\omega')\gamma_{\bar{n}}(\omega_1,\omega_1';\mu)\Big\}$$

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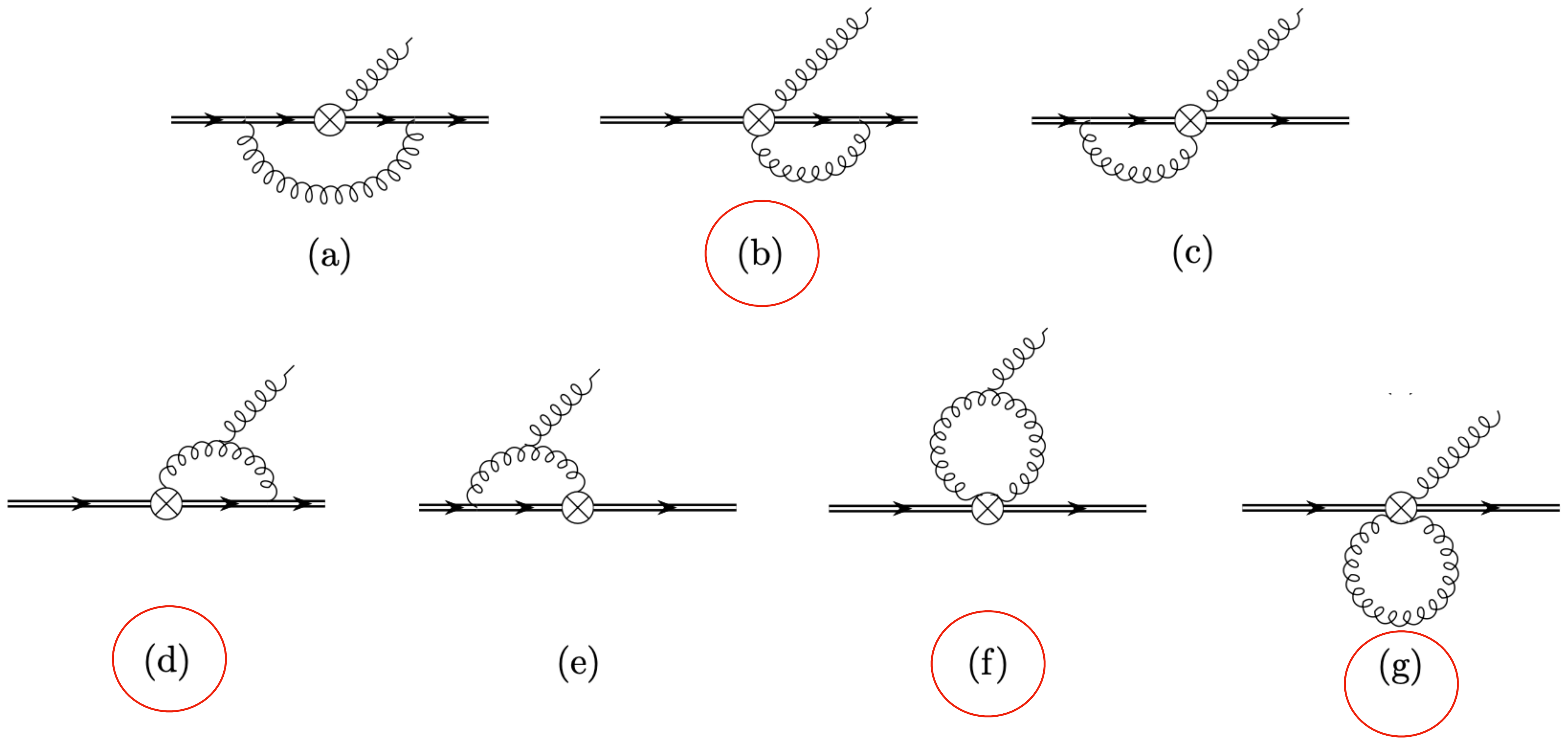

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$$\gamma_{\bar{n}}(\omega_1,\omega_1';\mu) = \ln\frac{\mu^2}{\omega_1^2}\,\delta(\omega_1-\omega_1') - \mathrm{Re}\,\mathbf{H}(\omega_1,\omega_1') + \frac{2\omega_1}{(\omega_1')^2}[\theta(\omega_1)\theta(\omega_1'-\omega_1)-\theta(-\omega_1)\theta(\omega_1-\omega_1')]$$

$$\gamma_n(\omega,\omega';\mu) = \left(\ln\frac{\mu^2}{\omega^2}-1\right)\delta(\omega-\omega') - 2\theta(\omega)\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}\right]_+\omega' - 2\theta(-\omega)\left[\frac{\theta(\omega'-\omega)}{\omega'-\omega}\right]_\ominus$$

DEFINITION

Plus-distribution:

$$\int d\omega'\,[\ldots]_+\,f(\omega') \equiv \int d\omega'\,[\ldots]\,(f(\omega')-f(\omega))$$

Modified distributions:

$$\int d\omega'\,[\ldots]_{\oplus/\ominus}\,f(\omega') \equiv \int d\omega'\,[\ldots]\,(f(\omega')-\theta(\pm\omega')f(\omega))$$

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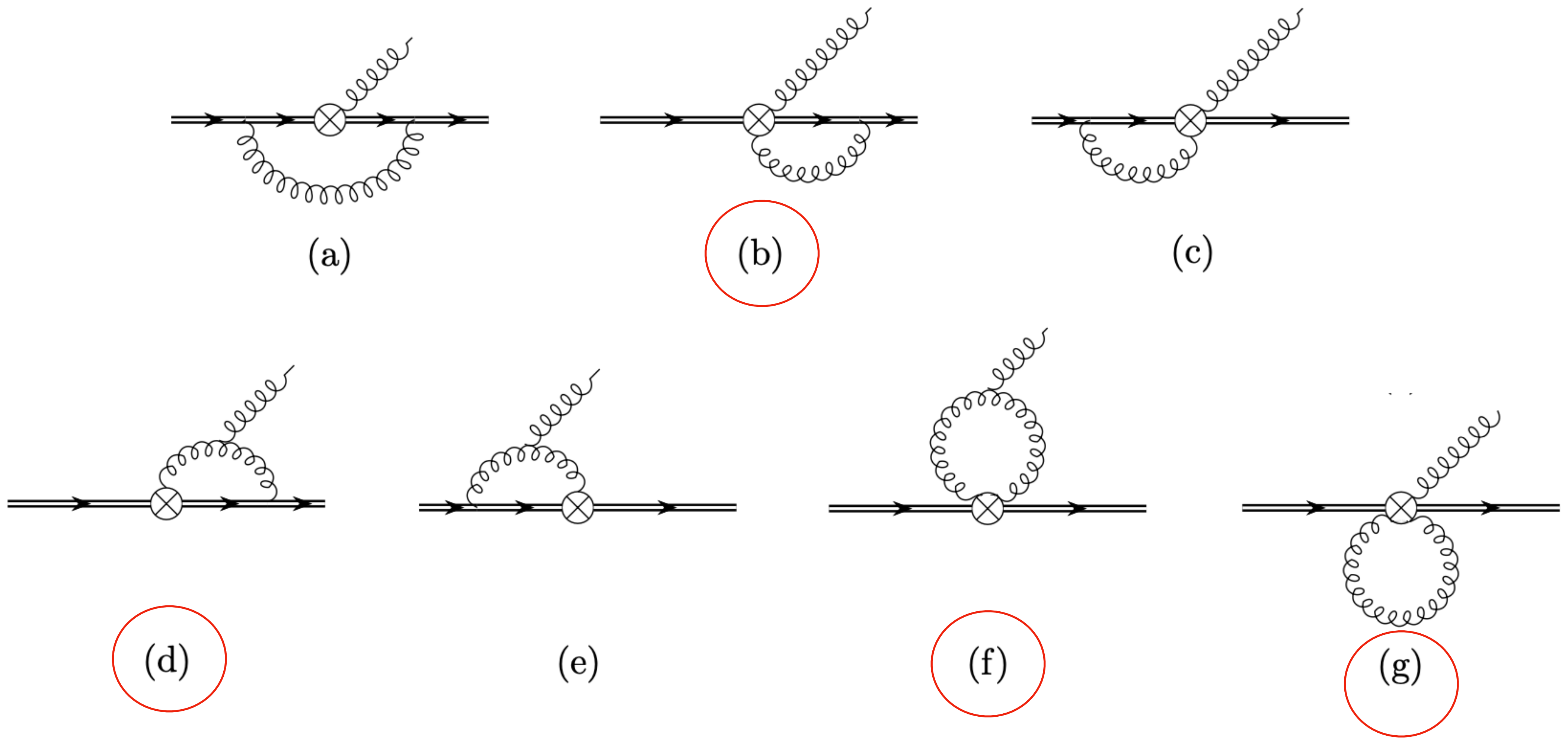

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RGE solution:

$$g_{17}(\omega,\omega_1;\mu) = \int_{\omega}^{\bar{\Lambda}} \frac{d\omega'}{\omega'-\omega}\, U_n^{(17)}(\omega,\omega';\mu,\mu_0) \int_{-\infty}^{\infty} \frac{d\omega_1'}{|\omega_1'|}\, U_{\bar{n}}^{(17)}(\omega_1,\omega_1';\mu,\mu_0)\, g_{17}(\omega',\omega_1';\mu_0)$$

“Factorisation” of two light-cones

The exclusive counterpart

In exclusive decays, such as $\bar{B}_{d,s} \to \gamma\gamma$, analogous soft functions appear

$$2\,\mathcal{F}_B(\mu)\,m_B \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \exp\left[-i(\omega_1\tau_1+\omega_2\tau_2)\right]\, \Phi_{\mathrm{G}}(\omega_1,\omega_2,\mu) \;=\; \langle 0|(\bar{q}_s S_n)(\tau_1 n)\,(S_n^\dagger\, S_{\bar{n}})(0)\,(S_{\bar{n}}^\dagger\, g_s\, G_{\mu\nu}\, S_{\bar{n}})(\tau_2\bar{n})\;\bar{n}^\nu\;\not{n}\gamma_\perp^\mu\gamma_5\,(S_{\bar{n}}^\dagger h_v)(0)|\bar{B}_v\rangle$$

[2312.15439: Huang et al.]

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With few modifications to the previous computation, we get the anomalous dimension

$$\Gamma_G(\omega,\omega_1,\omega',\omega_1';\mu) = \Gamma_G^n(\omega,\omega') + \Gamma_G^{\bar{n}}(\omega_1,\omega_1') + \frac{\alpha_s}{\pi}\frac{i}{4}\frac{C_A}{\pi}\Delta H(\omega,\omega')\Delta H(\omega_1,\omega_1')$$

“mixing”: spoil of factorisation theorem?

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“mixing”: spoil of factorisation theorem?

However, it must be interpreted as a distribution on the space of jet functions!

$$\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-i0}\int_{-\infty}^{+\infty}\frac{d\omega_1}{\omega_1+i0}\Delta H(\omega,\omega')\Delta H(\omega_1,\omega_1') = 0$$

Due to the location of the poles, this “mixing” term vanishes saving factorisation and making the evolution factorised again.

[2411.16634: RB, Böer, Hurth]

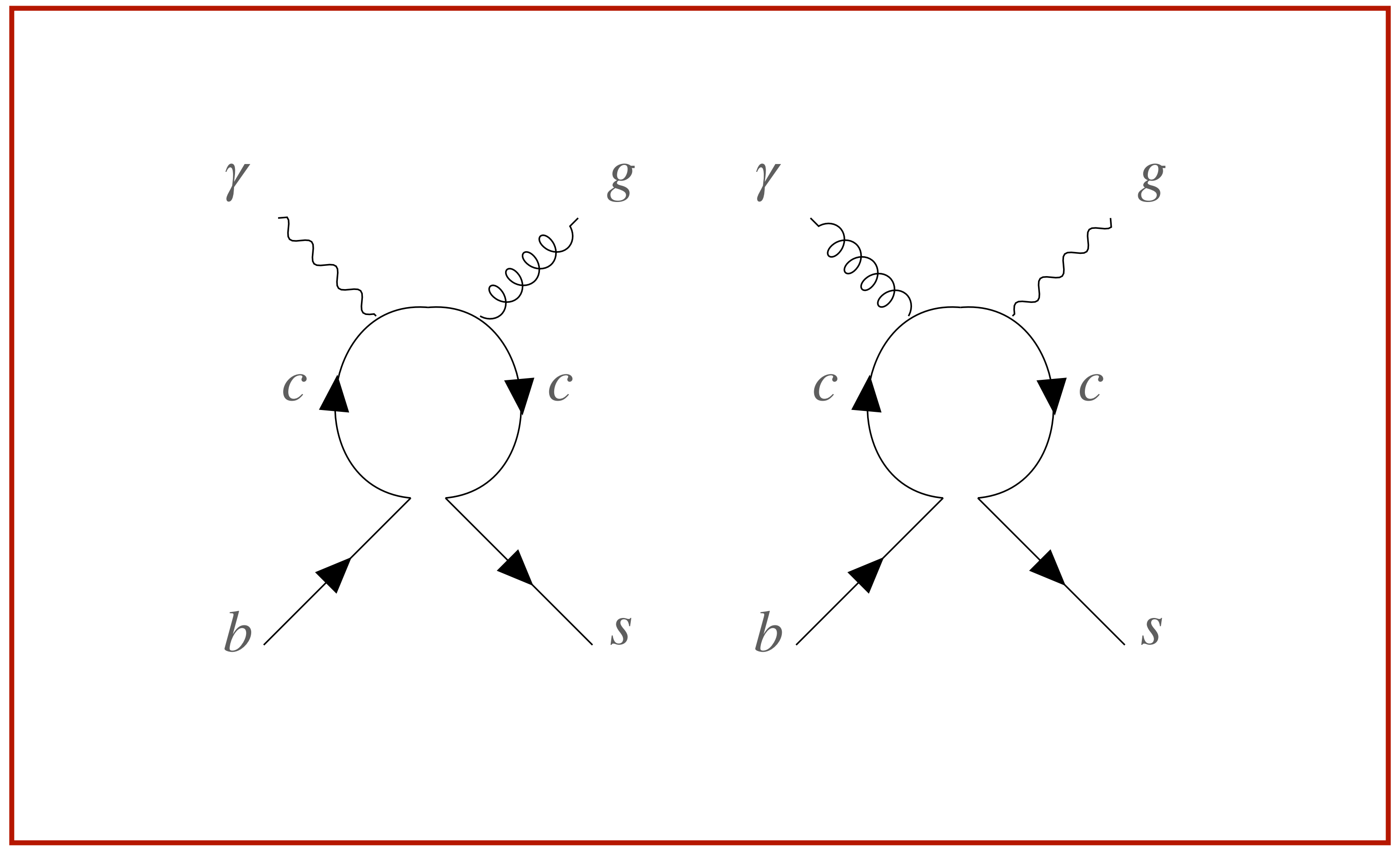
Two-loop “penguin”-jet function and pole cancellation

The “penguin”-jet function at NLO

The “penguin”-jet function is extracted from the anti-hard collinear region of $b \to s\gamma g$.

[to appear: RB, Böer, Hurth]

LO



LO contains charm-loop, therefore NLO is a two-loop amplitude

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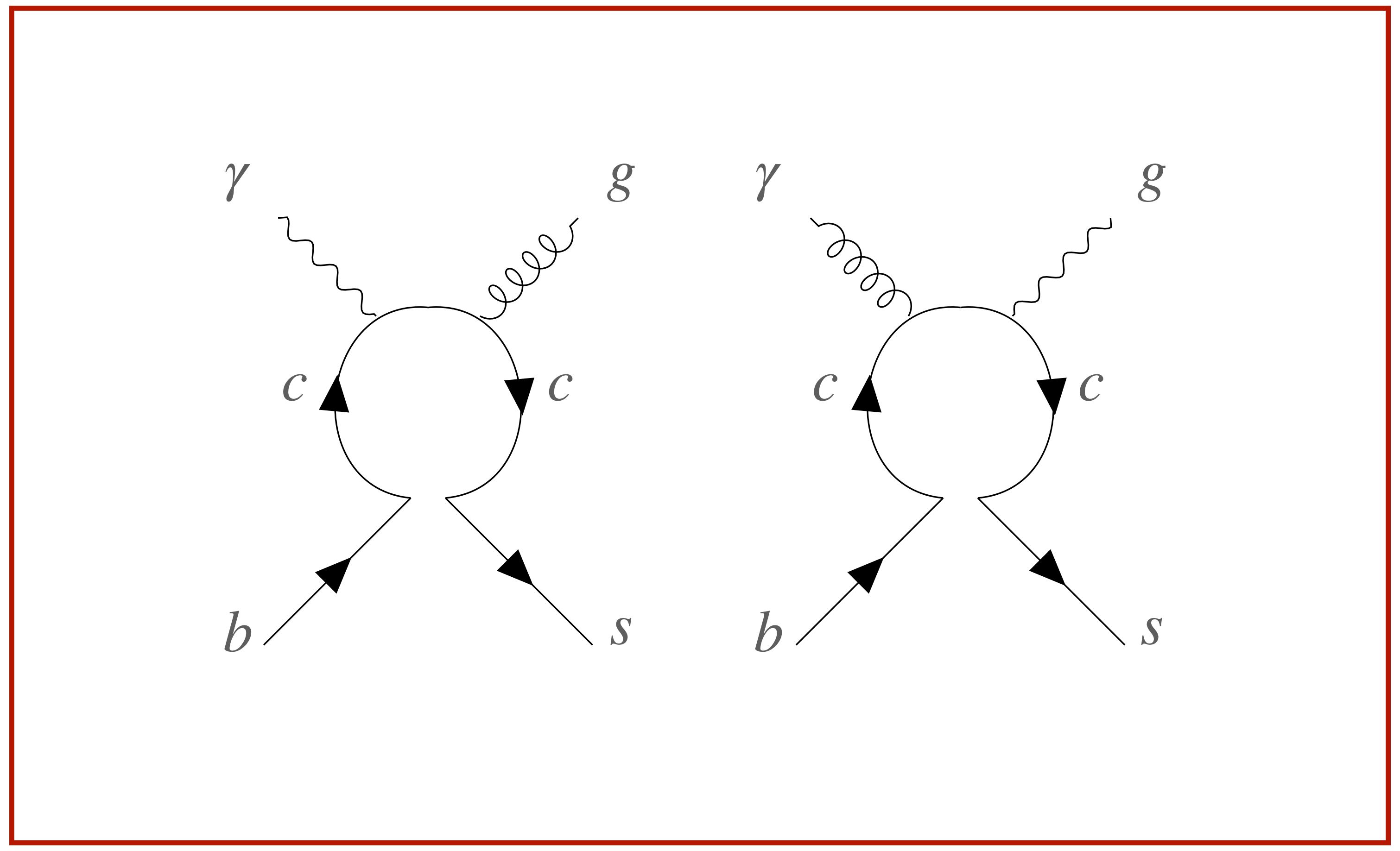

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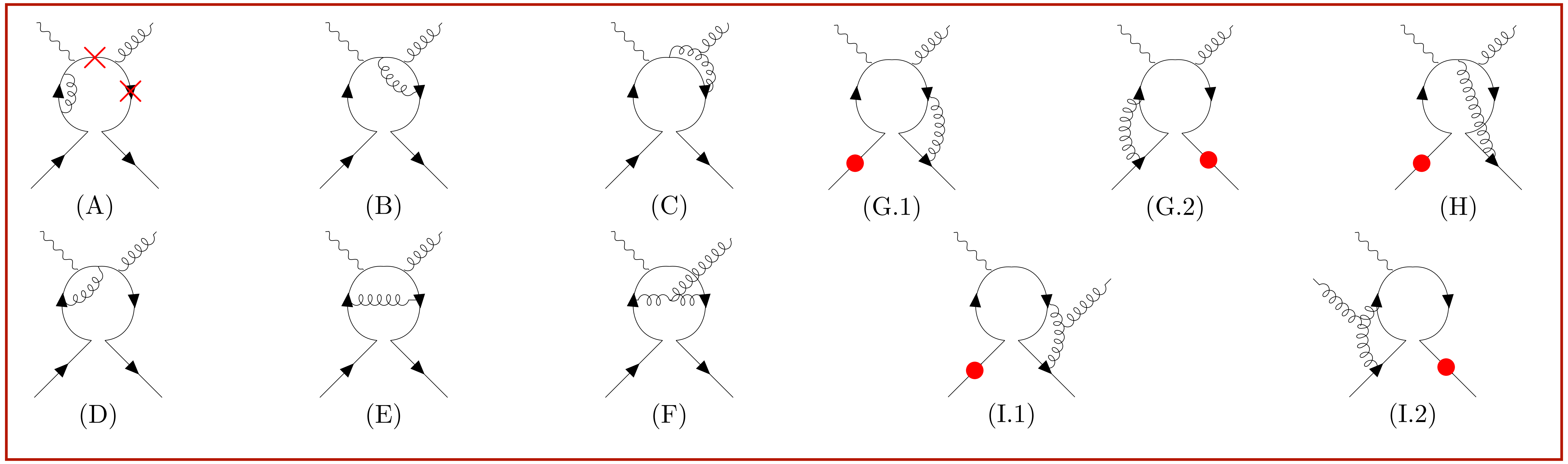

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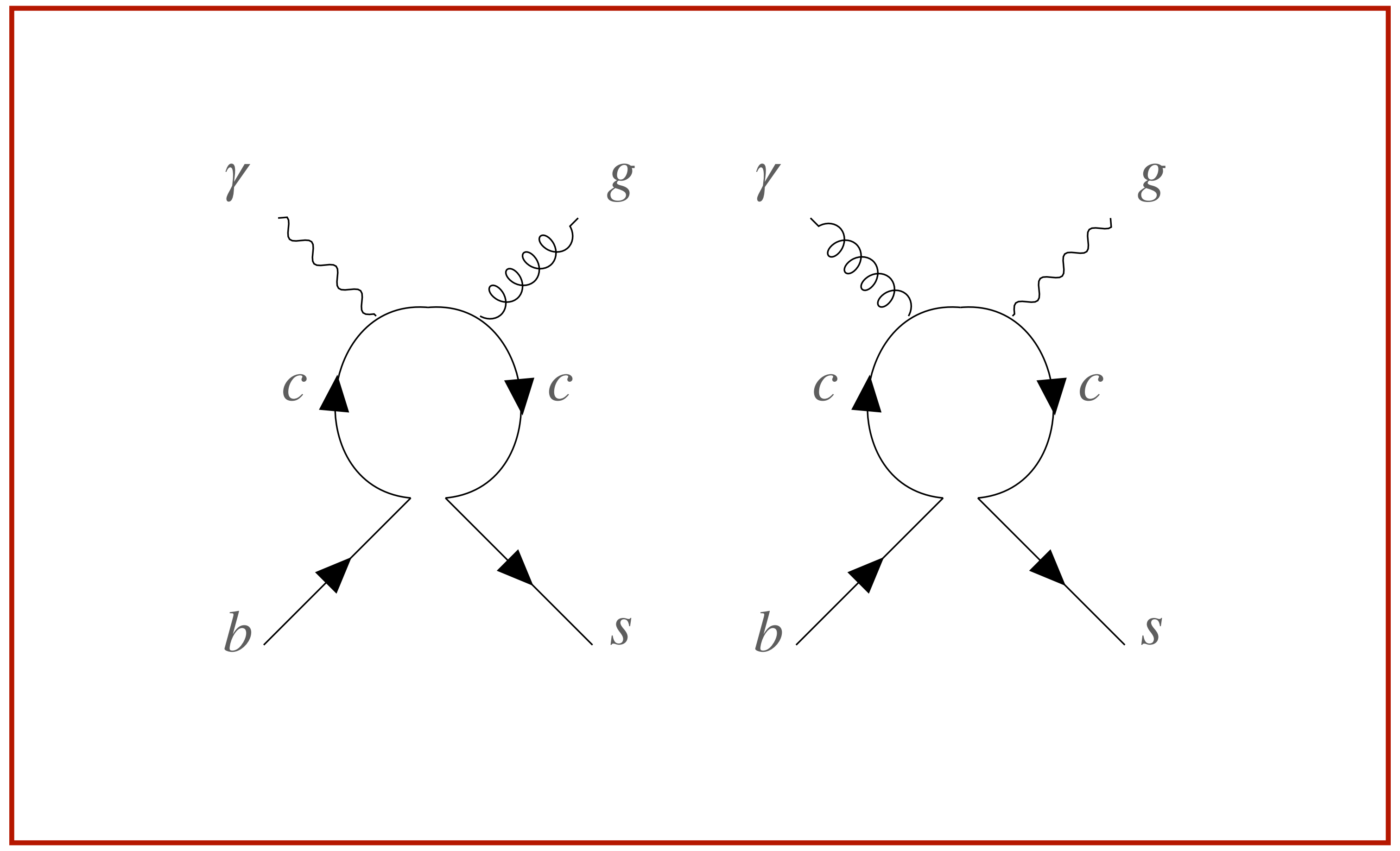

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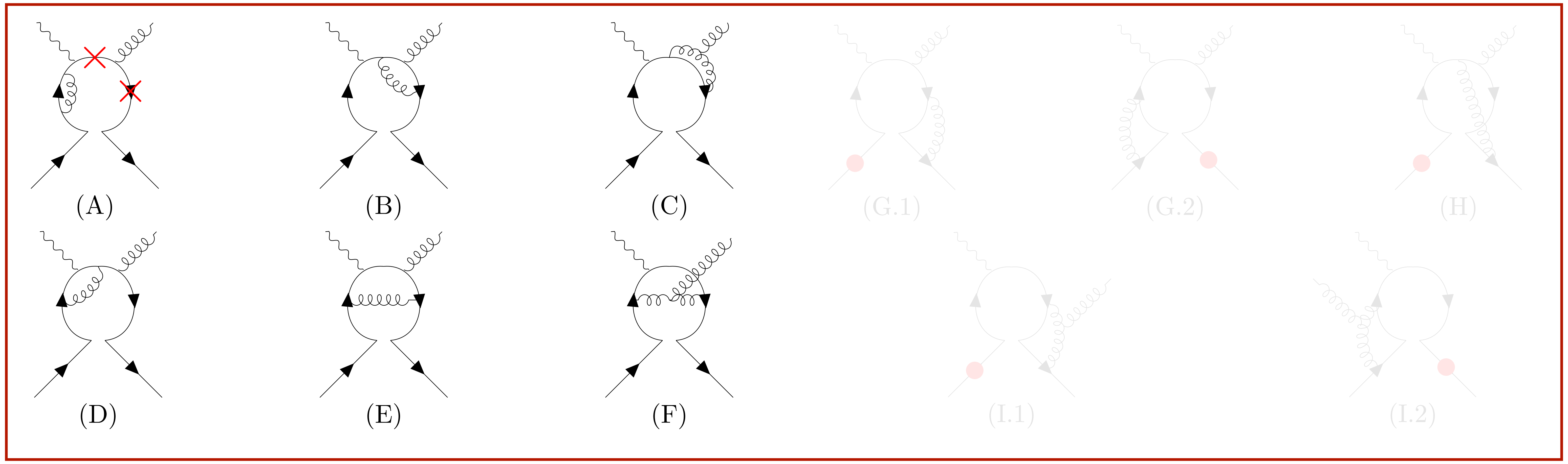

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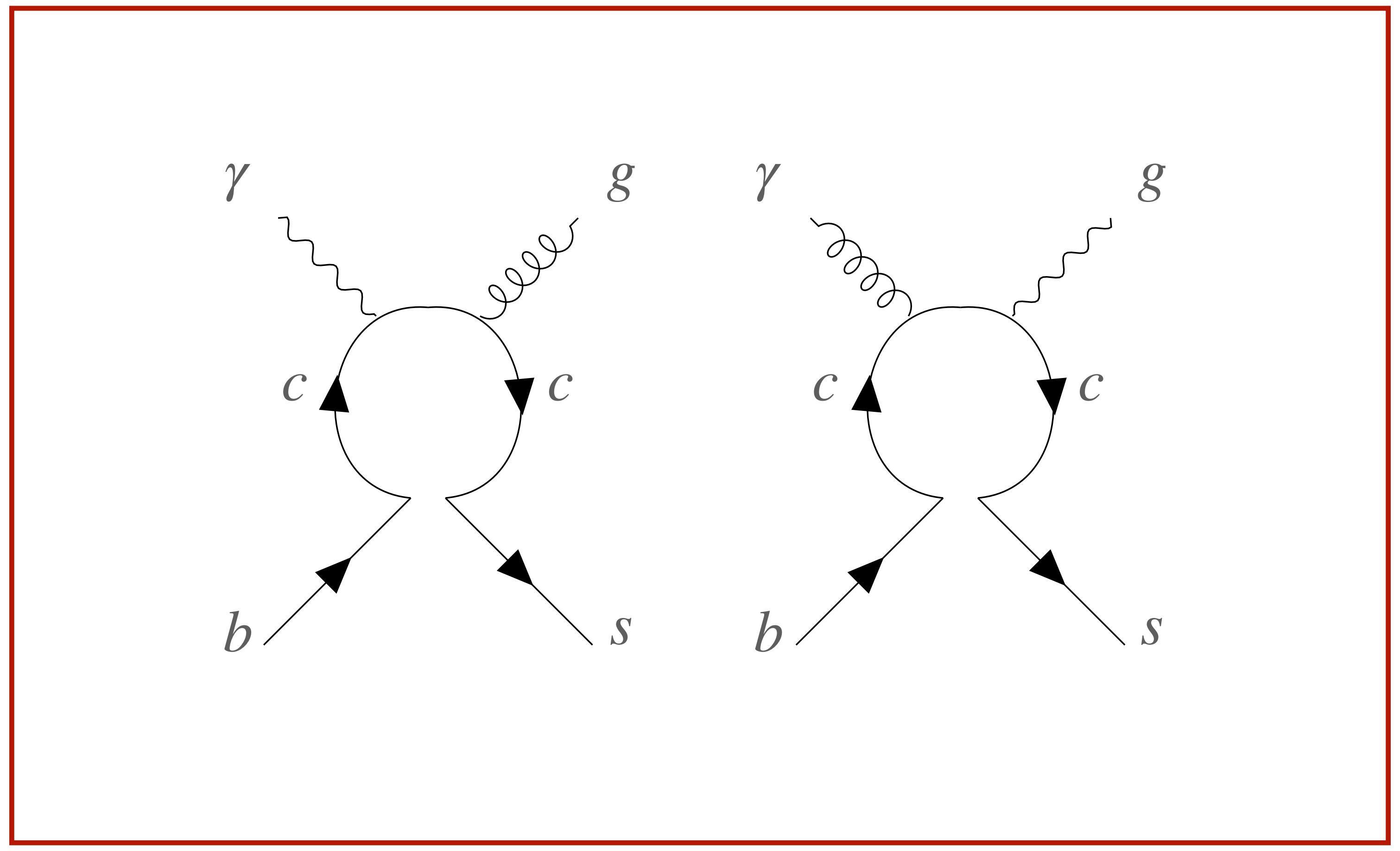

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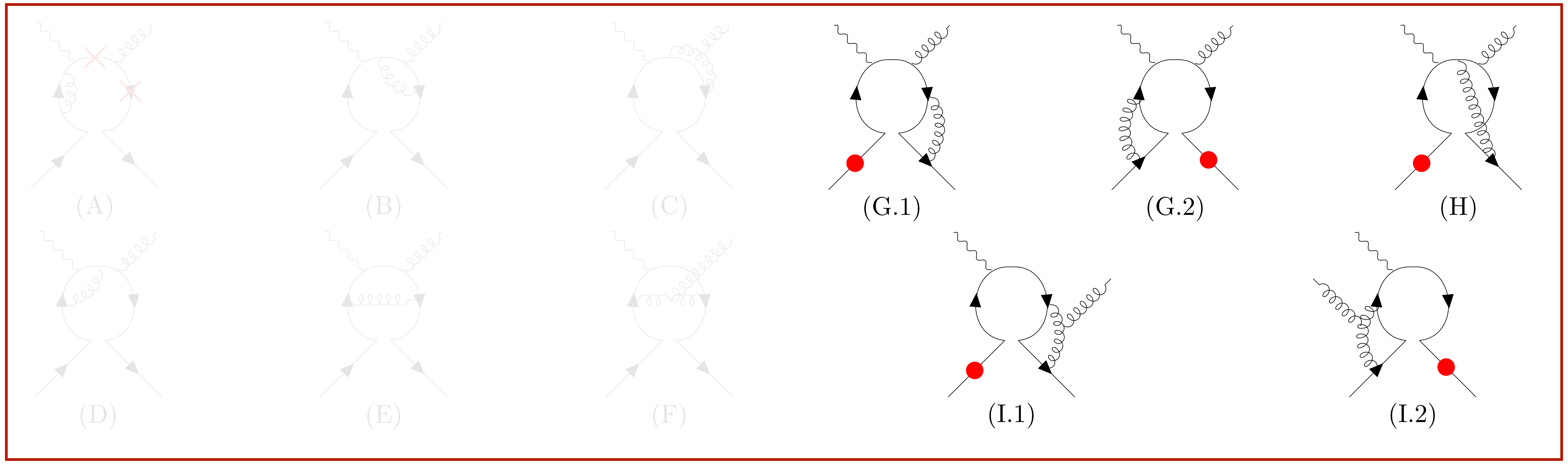

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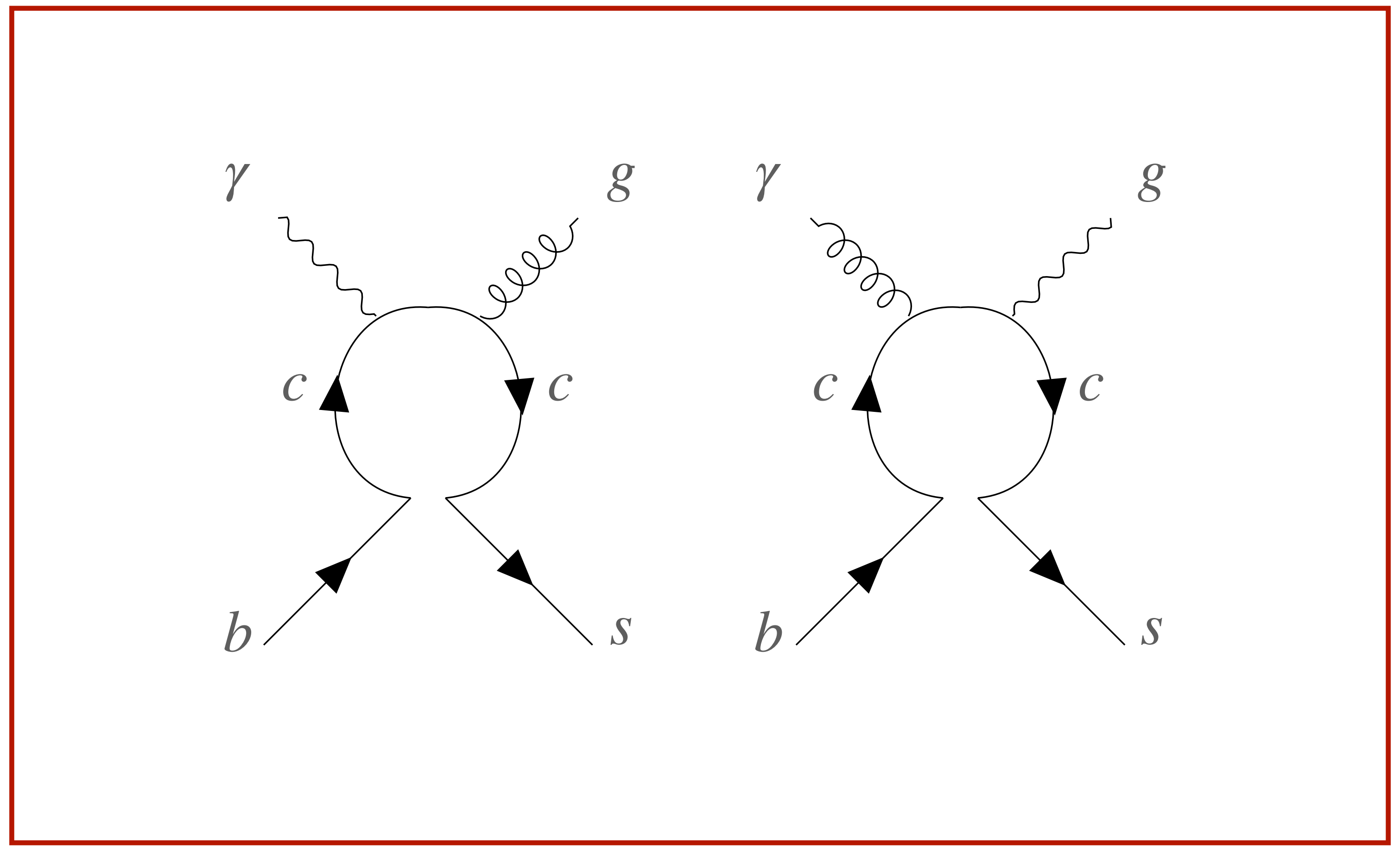

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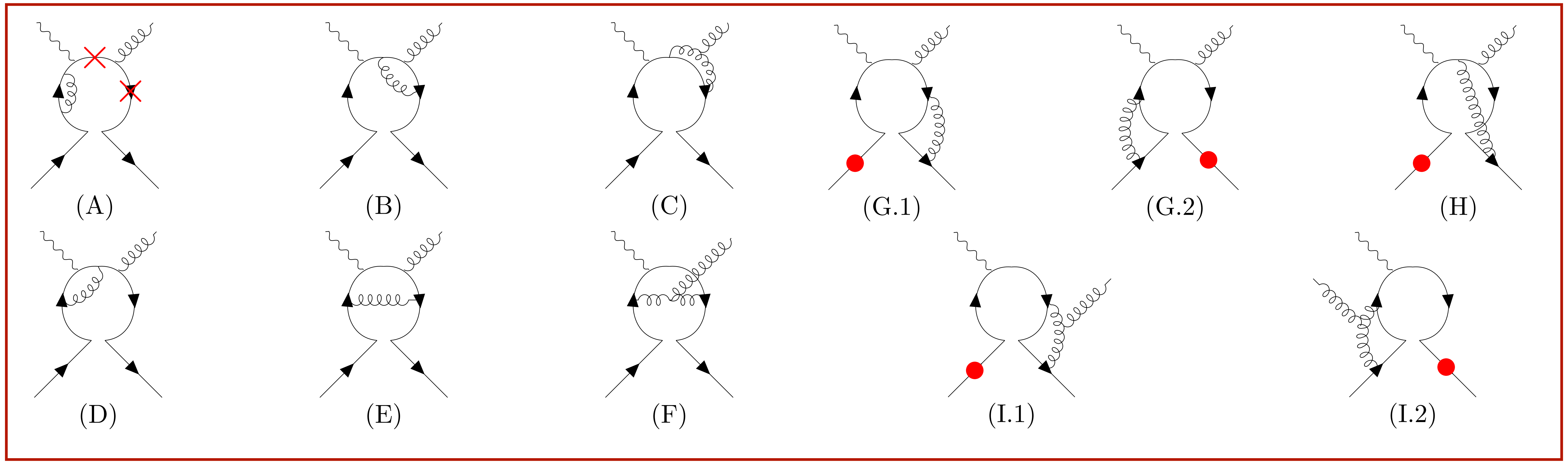

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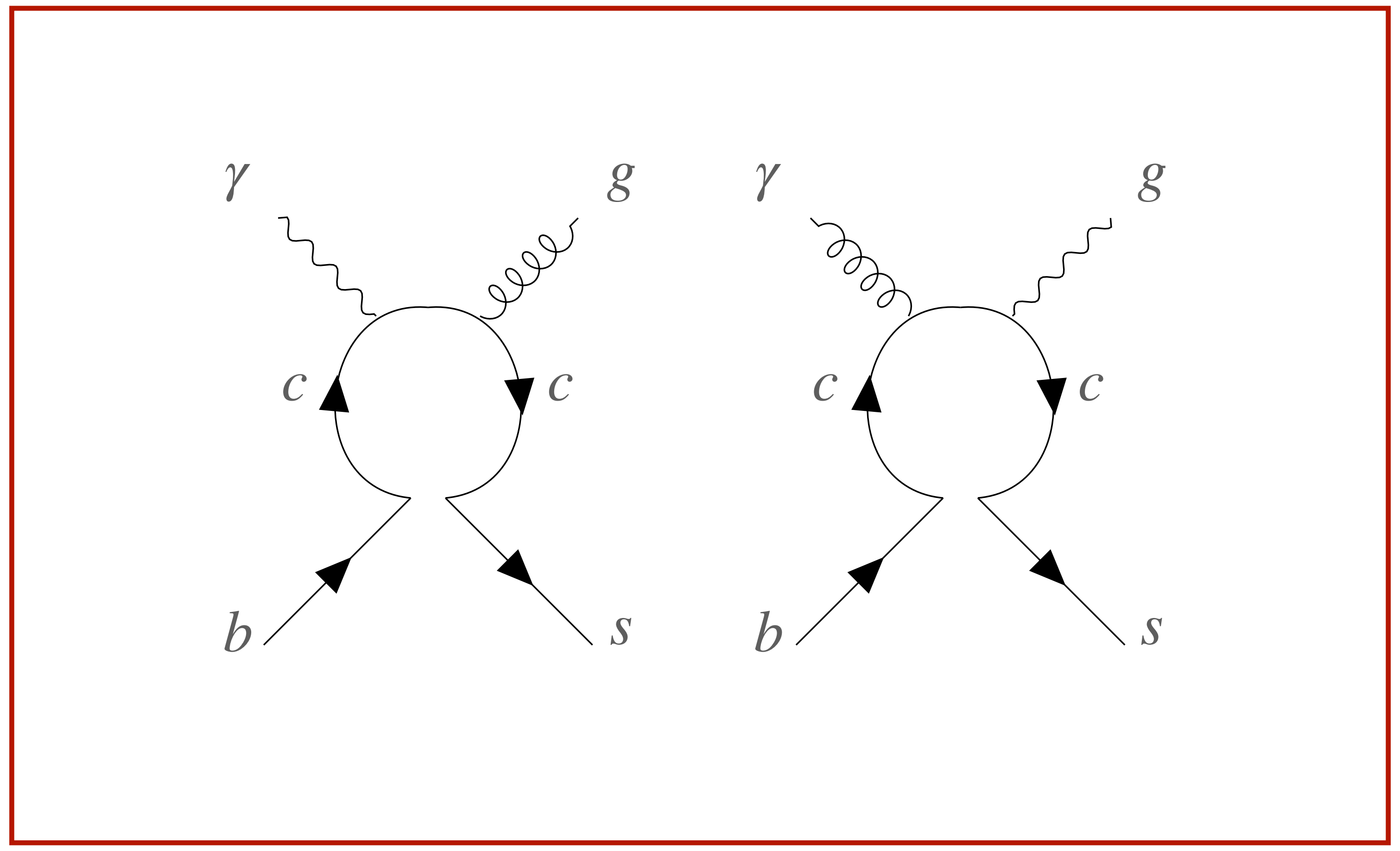

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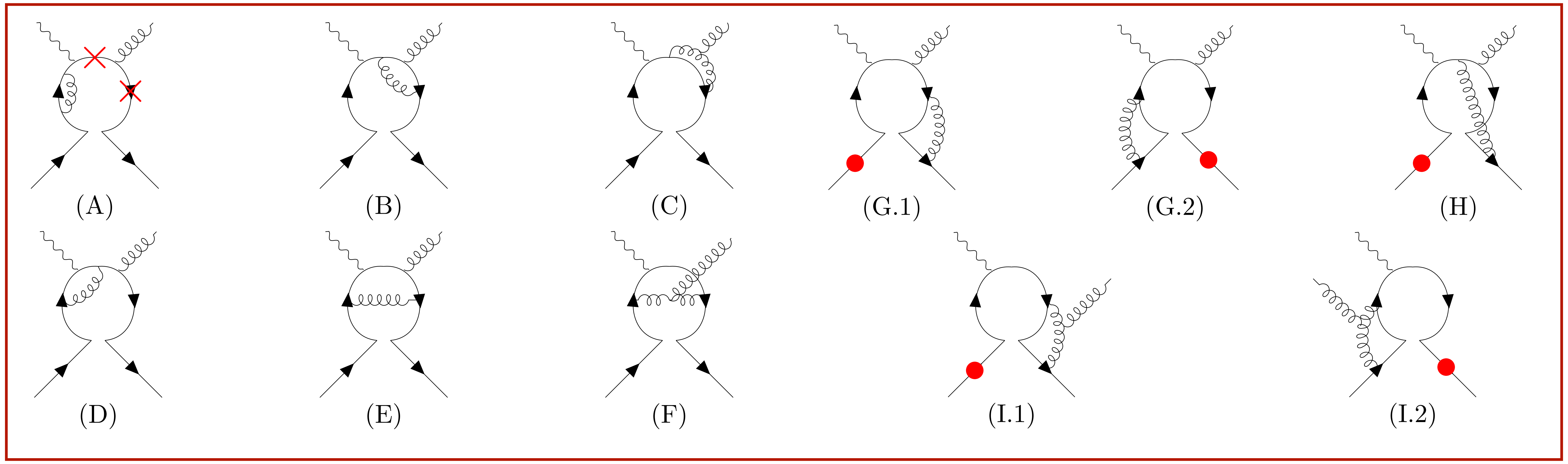

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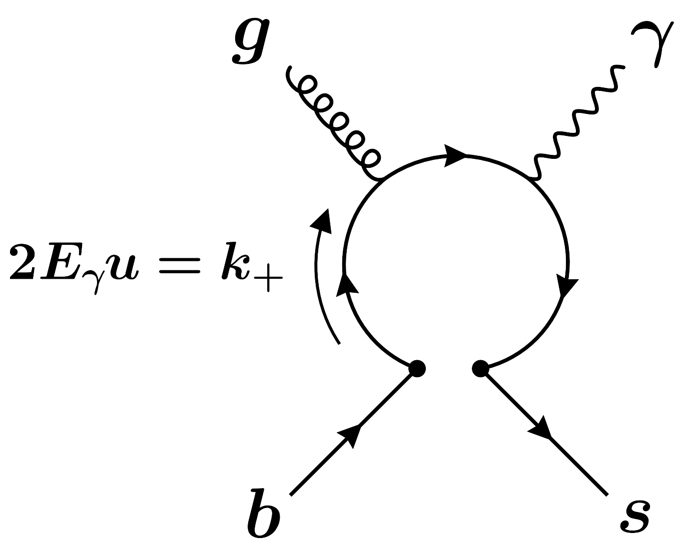
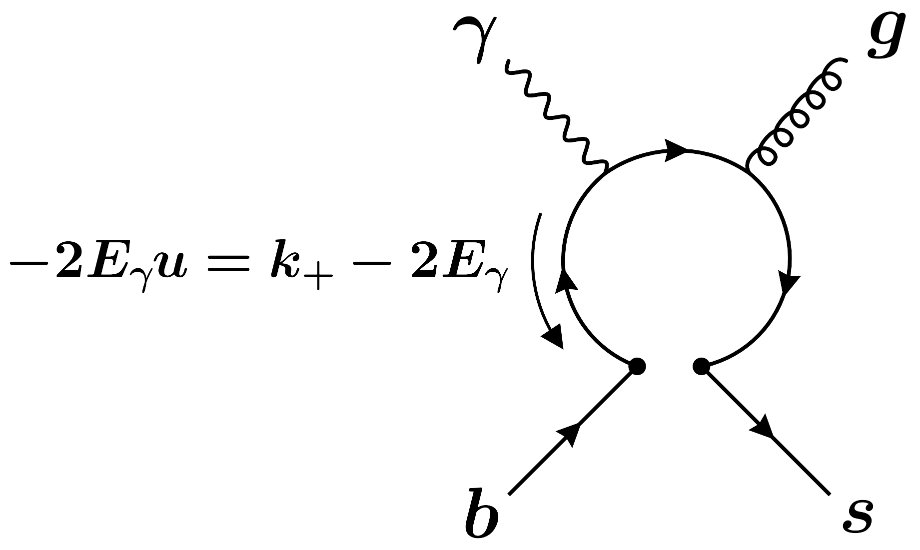
Additional complication:

$$d\Gamma(\bar{B} \to X_s\gamma) \sim \int_0^1 du\, H(u;\mu) \int_{-\infty}^{\infty} d\omega_1\, J^{(\bar{n})}(u,\omega_1;\mu) \int_{-\infty}^{\bar{\Lambda}} d\omega\, J^{(n)}(\omega;\mu)\, g_{17}(\omega,\omega_1;\mu)$$





Computation of the “penguin”-jet function at NLO

Initial amplitude

Anti-hard-collinear amplitude

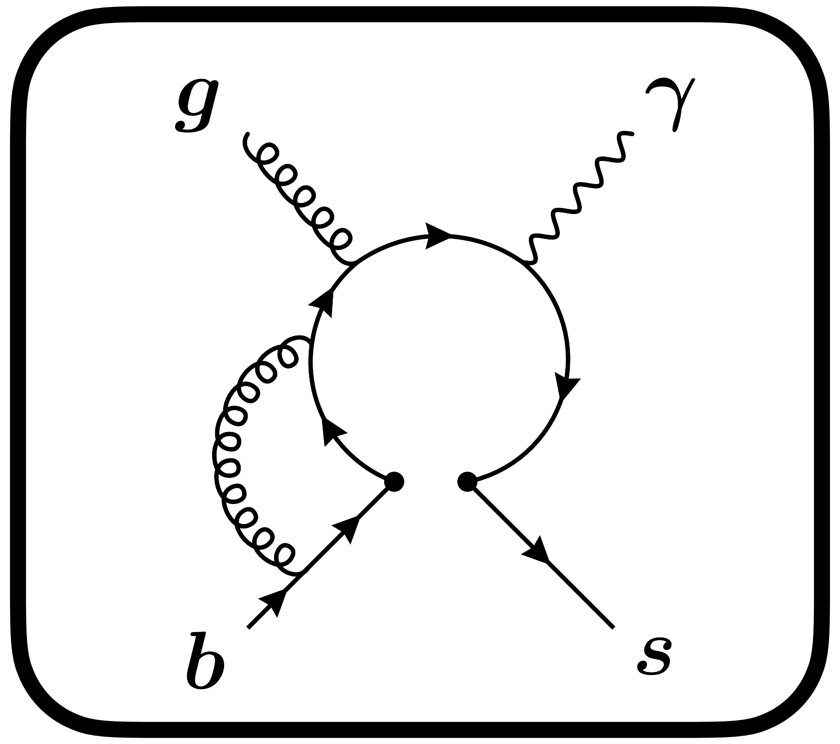
u-dependent delta function

IBP reduction

DE in canonical form

Set initial condition

Result



$$\tilde{\mu}^{4\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\bar{u}^{(s)}_{hc,L}\gamma^\mu i\frac{\not{k}-\not{k}_\gamma}{(k-k_\gamma)^2+i0}iQ_u\not{\epsilon}^*_\perp(k_\gamma)i\frac{\not{k}}{k^2+i0}$$
$$\times\, ig_s\not{\epsilon}^*_{s,\perp}(k_g)i\frac{\not{k}+\not{k}_g}{(k+k_g)^2+i0}ig_st^a\gamma^\alpha i\frac{\not{k}+\not{k}_g+\not{l}}{(k+k_g+l)^2+i0}$$
$$\times\,\gamma_\mu P_L i\frac{\not{k}_b+\not{l}-m_b}{(k_b+l)^2-m_b^2+i0}ig_st^a\gamma_\alpha\frac{-i}{l^2+i0}b_L\,,$$

Computation of the “penguin”-jet function at NLO

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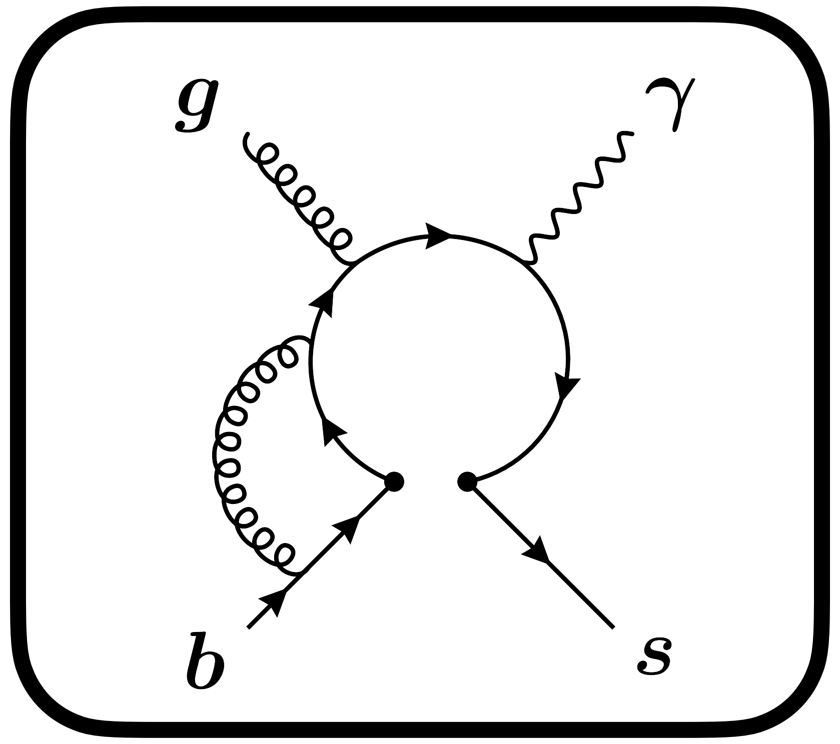
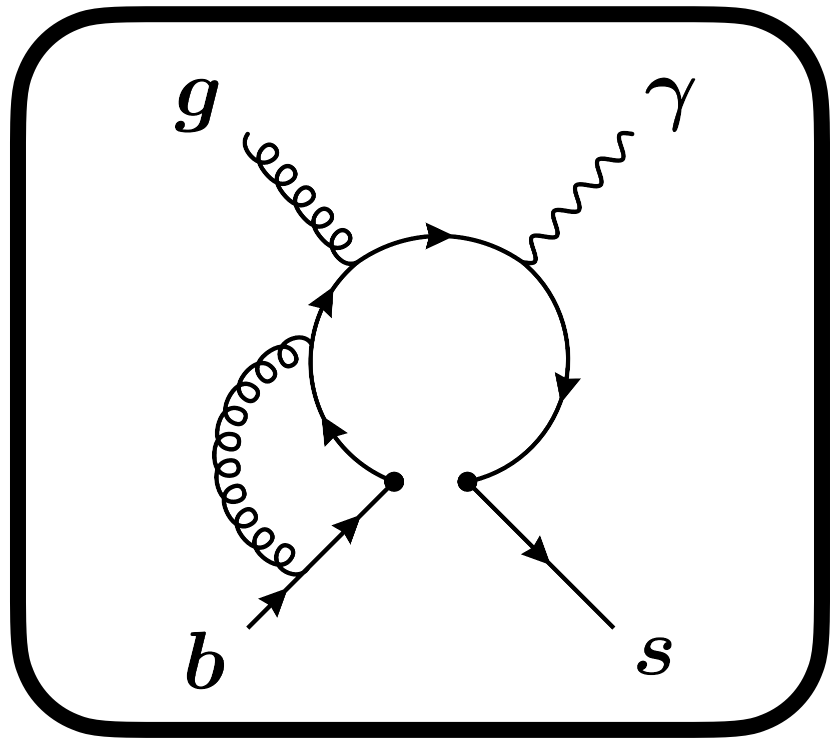
- **Result**



$$2g_s^3 Q_u C_F \tilde{\mu}^{4\varepsilon} \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \bar{u}^{(s)}_{hc,L} \gamma^\mu \frac{\not{k} - \not{k}_\gamma}{(k-k_\gamma)^2 + i0} \not{\epsilon}^*_\perp(k_\gamma) \frac{\not{k}}{k^2 + i0}$$

$$\times \not{\epsilon}^*_{s,\perp}(k_g) \frac{\not{k}}{(k+k_g)^2 + i0} \gamma_\mu \frac{1}{(k+k_g+l)^2 + i0} \frac{1}{l^2 + i0} \boxed{\frac{l_+ + k_+}{l_+ + i0}} b_L \,,$$

Computation of the “penguin”-jet function at NLO



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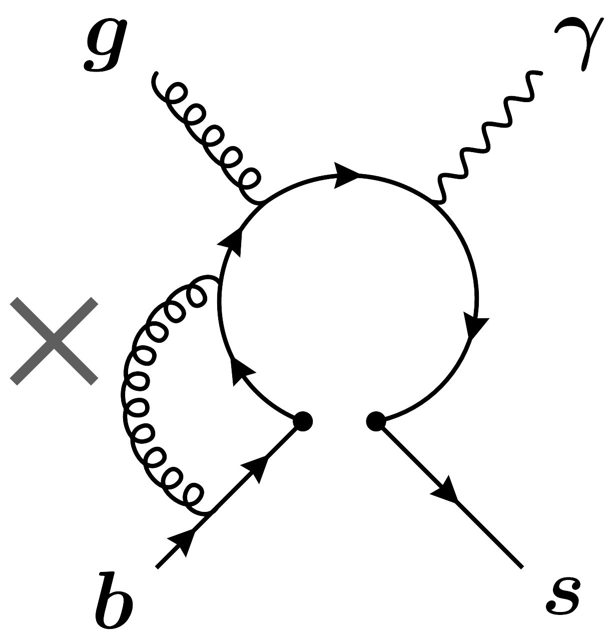
$$\delta\left(u-\frac{k_+}{2E_\gamma}\right)\times$$



$$\delta\left(u-\frac{k_+}{2E_\gamma}\right)=\frac{2E_\gamma}{2\pi i}\left(\frac{1}{k_+-2E_\gamma u-i0}-\frac{1}{k_+-2E_\gamma u+i0}\right)$$

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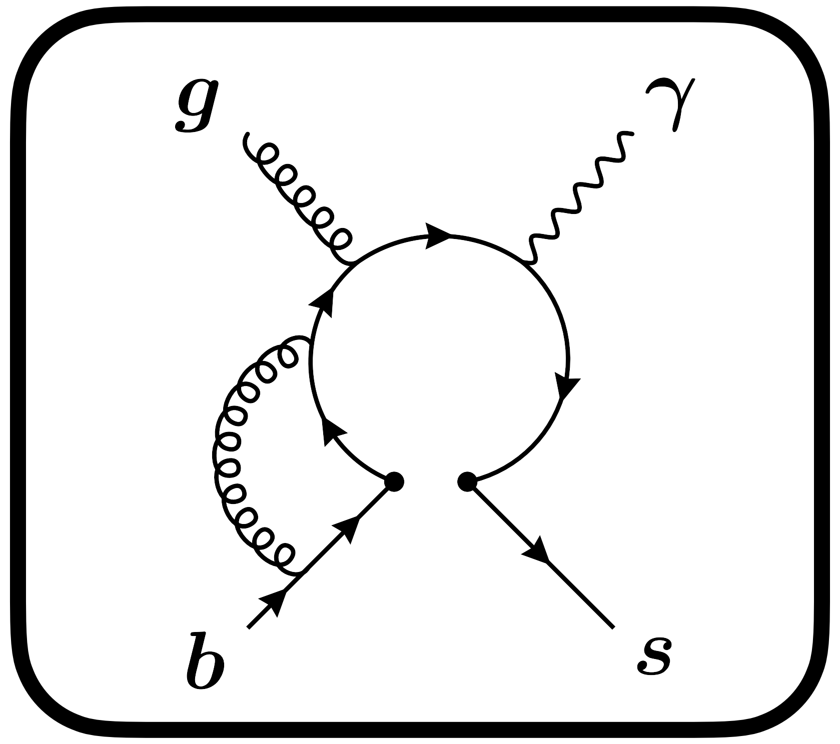

$$D = \left\{ \frac{1}{l^2+i0}, \frac{1}{(l+k_g)^2+i0}, \frac{1}{(k+k_g)^2+i0}, \frac{1}{(l-k_\gamma)^2+i0}, \frac{1}{(k-l)^2+i0}, \frac{1}{(k-k_\gamma)^2+i0}, \frac{1}{k^2+i0}, \frac{1}{n\cdot l+i0}, \delta\Big(k_+ - 2E_\gamma u(\bar{u})\Big) \right\}.$$

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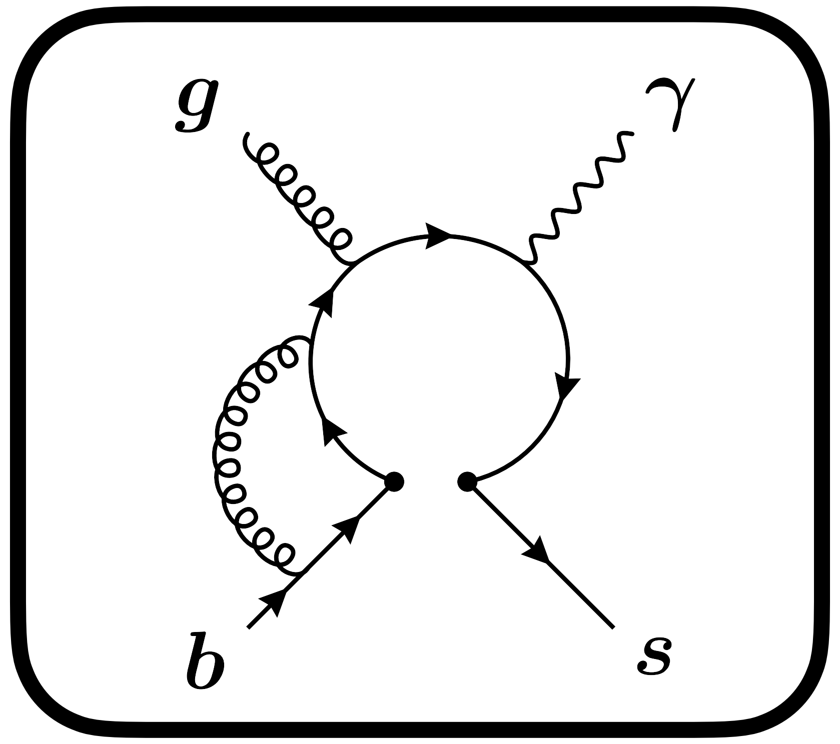

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$$v_c'(u) = \varepsilon A_c(u) v_c(u)$$

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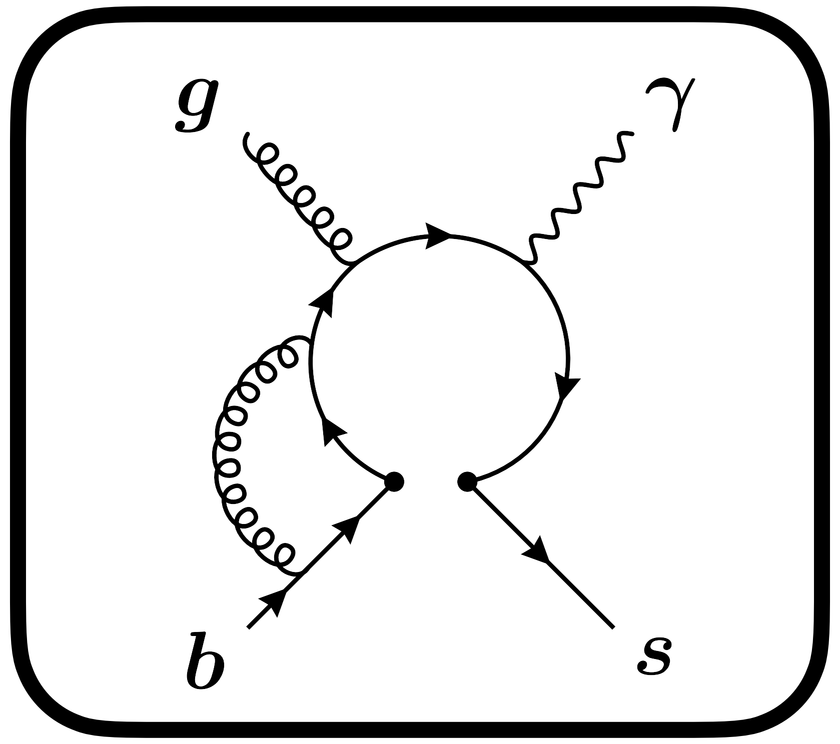
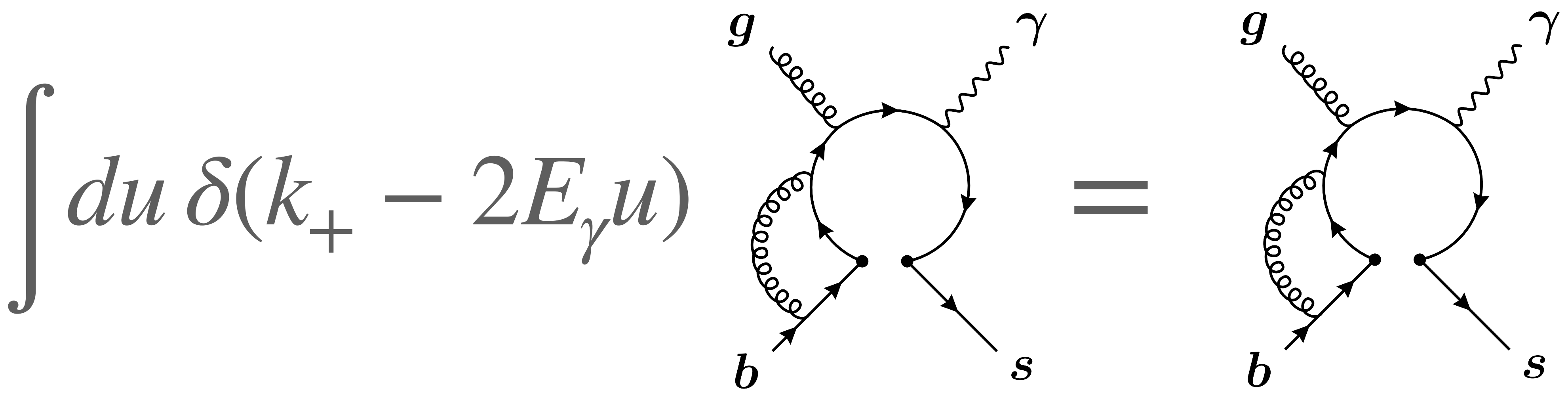
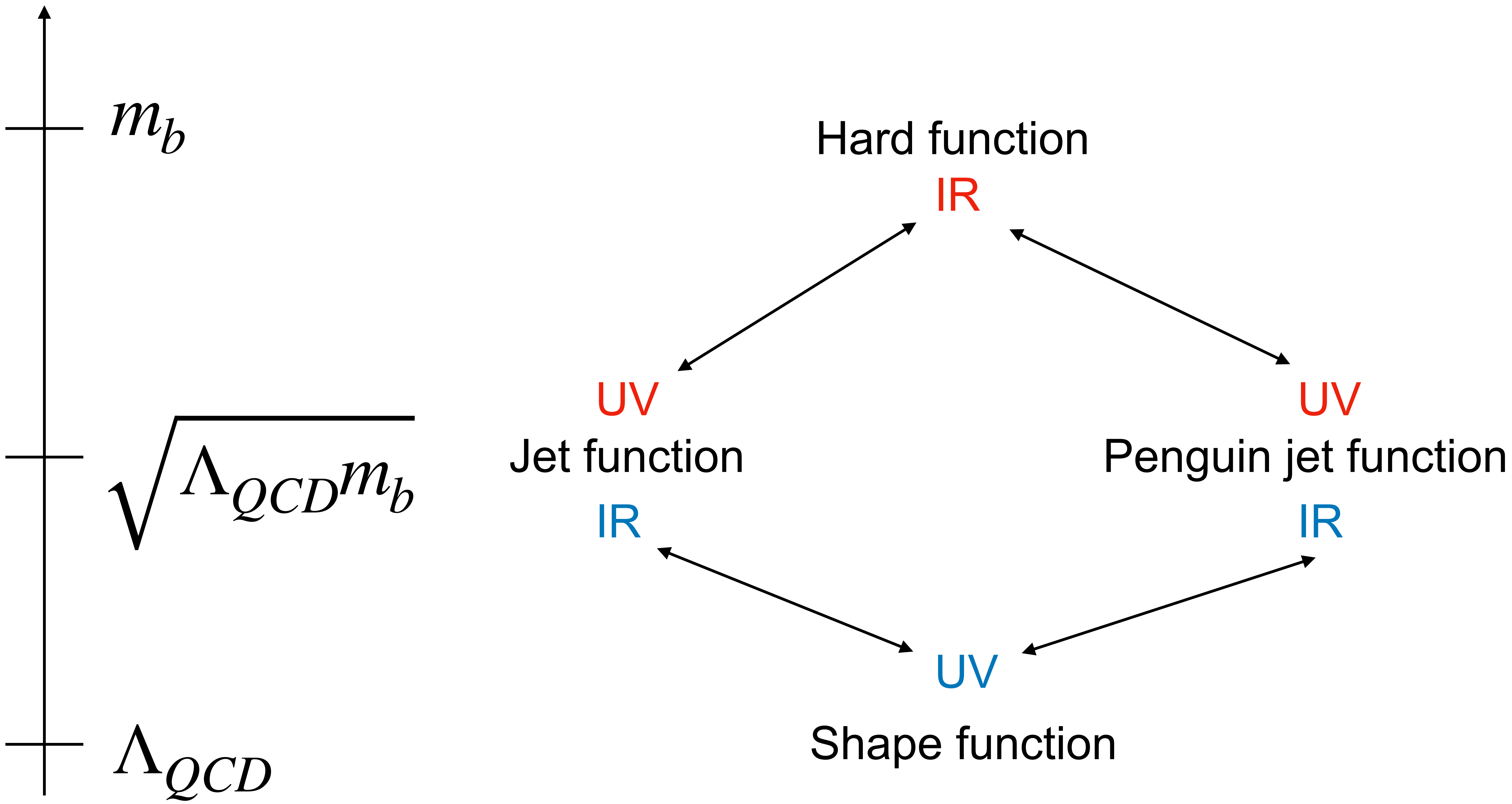



$$\int du\, \delta(k_+ - 2E_\gamma u) \quad = $$

Pole cancellation

All the poles cancel!

- Non-trivial crosscheck of the calculations.
- Reliability of factorisation framework.



Summary and outlook

Why NLO?

The $\mathcal{O}_1 - \mathcal{O}_7$ interference has the largest uncertainty.

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- Computed in the massless case proving pole cancellation
- Future work: Full pheno analysis

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Thank you for your attention!

Back up slides

RG evolution of the shape function

Given the previous anomalous dimension, the RG equation can be solved using the Mellin transform method.

$$g_{17}(\omega,\omega_1;\mu)=\int_{\omega}^{\bar{\Lambda}}\frac{d\omega'}{\omega'-\omega}\,U_n^{(17)}(\omega,\omega';\mu,\mu_0)\int_{-\infty}^{\infty}\frac{d\omega_1'}{|\omega_1'|}\,U_{\bar{n}}^{(17)}(\omega_1,\omega_1';\mu,\mu_0)\,g_{17}(\omega',\omega_1';\mu_0)$$

“Factorisation” of two light-cones

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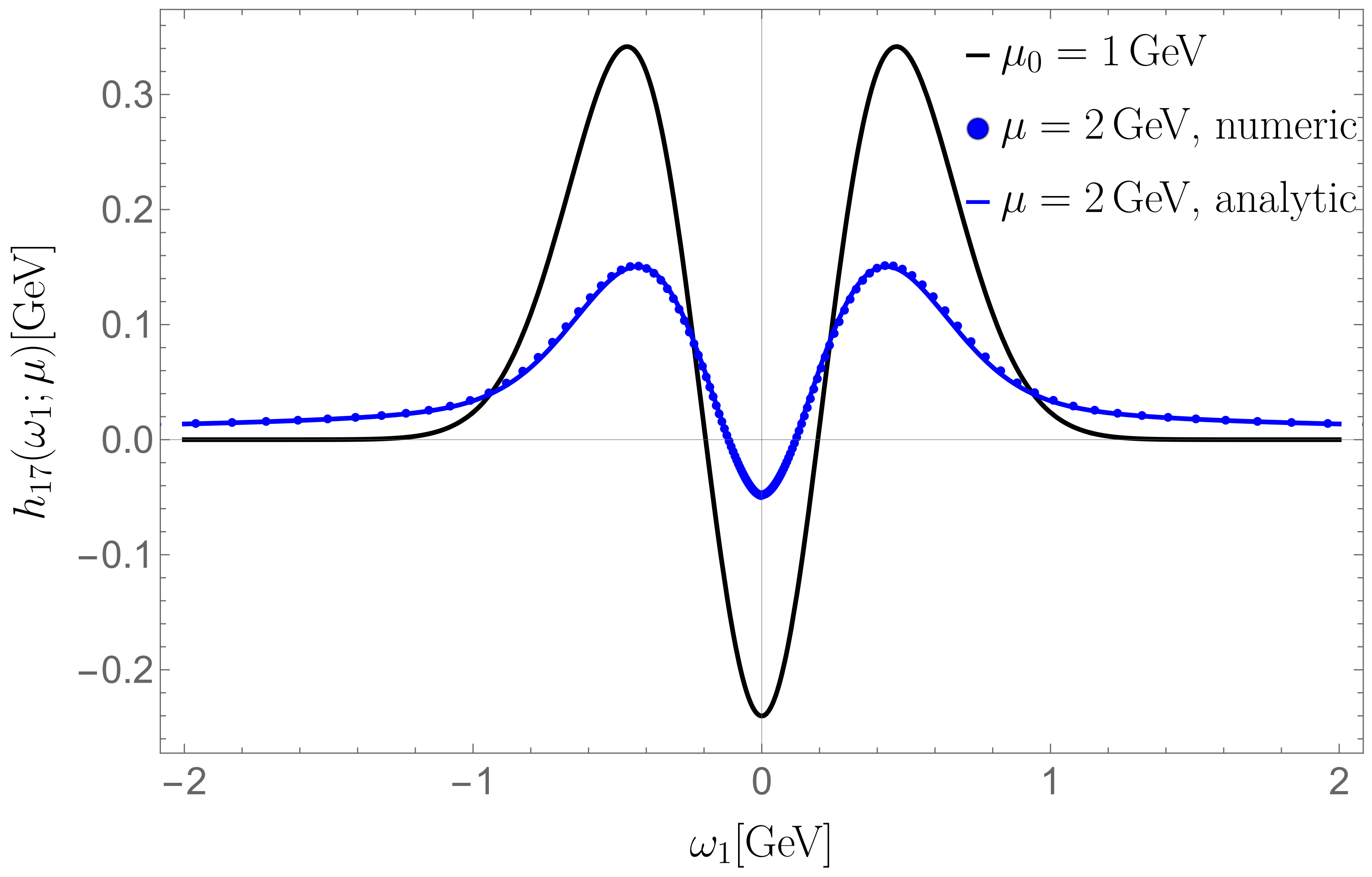
$$g_{17}(\omega,\omega_1;\mu) = \int_\omega^{\bar{\Lambda}} \frac{d\omega'}{\omega'-\omega}\, U_n^{(17)}(\omega,\omega';\mu,\mu_0) \int_{-\infty}^{\infty} \frac{d\omega_1'}{|\omega_1'|}\, U_{\bar{n}}^{(17)}(\omega_1,\omega_1';\mu,\mu_0)\, g_{17}(\omega',\omega_1';\mu_0)$$

“Factorisation” of two light-cones

All order properties of g_{17} are preserved in the RGE:

- g_{17} is real (from PT invariance)
- The function $h_{17} = \int d\omega\, g_{17}$ is even (from HQET trace formalism)

RGE for the function h_{17}



[2411.16634: RB, Böer, Hurth]

Distribution definition

Plus-distribution:

$$\int d\omega' \, [\ldots]_+ \, f(\omega') \equiv \int d\omega' \, [\ldots] \, (f(\omega') - f(\omega))$$

Modified distributions:

$$\int d\omega' \, [\ldots]_{\oplus/\ominus} \, f(\omega') \equiv \int d\omega' \, [\ldots] \, (f(\omega') - \theta(\pm\omega') f(\omega))$$

H-distributions:

$$\begin{aligned}
F^>(\omega_i, \omega_i') &= \left[\frac{\omega_i \, \theta(\omega_i' - \omega_i)}{\omega_i' \, (\omega_i' - \omega_i)}\right]_+ + \left[\frac{\theta(\omega_i - \omega_i')}{\omega_i - \omega_i'}\right]_\oplus \\
F^<(\omega_i, \omega_i') &= \left[\frac{\omega_i \, \theta(\omega_i - \omega_i')}{\omega_i' \, (\omega_i - \omega_i')}\right]_+ + \left[\frac{\theta(\omega_i' - \omega_i)}{\omega_i' - \omega_i}\right]_\ominus \\
G^>(\omega_i, \omega_i') &= (\omega_i + \omega_i') \left[\frac{\theta(\omega_i' - \omega_i)}{\omega_i' \, (\omega_i' - \omega_i)}\right]_+ - i\pi\delta(\omega_i - \omega_i') \\
G^<(\omega_i, \omega_i') &= (\omega_i + \omega_i') \left[\frac{\theta(\omega_i - \omega_i')}{\omega_i' \, (\omega_i - \omega_i')}\right]_+ + i\pi\delta(\omega_i - \omega_i')
\end{aligned}$$

$$H_\pm(\omega_i, \omega_i') = \theta(\pm\omega_i) F^{>(<)}(\omega_i, \omega_i') + \theta(\mp\omega_i) G^{<(>)}(\omega_i, \omega_i')$$

Basics for RGE

$$\widetilde{\mathcal{O}}_{17}^{(\mathrm{bare})}(\omega,\omega_1) = \int d\omega' \int d\omega_1'\, Z_{17}^{-1}(\omega,\omega_1,\omega',\omega_1';\mu)\, \widetilde{\mathcal{O}}_{17}^{(\mathrm{ren})}(\omega,\omega_1;\mu)$$

$$\frac{d}{d\ln\mu}\, g_{17}(\omega,\omega_1;\mu) = -\int d\omega' \int d\omega_1'\, \gamma_{17}(\omega,\omega_1,\omega',\omega_1';\mu)\, g_{17}(\omega',\omega_1';\mu)$$

$$\gamma_{17}(\omega,\omega_1,\omega',\omega_1';\mu) = -\int d\hat{\omega} \int d\hat{\omega}_1\, \frac{dZ_{17}(\omega,\omega_1,\hat{\omega},\hat{\omega}_1;\mu)}{d\ln\mu}\, Z_{17}^{-1}(\hat{\omega},\hat{\omega}_1,\omega',\omega_1';\mu)$$

Bottom-meson soft function renormalisation

$$
\begin{aligned}
\Gamma_{\mathrm{G}} \;&=\; \frac{\alpha_s}{\pi}\Bigg\{\left[C_F\left(\ln\frac{\mu}{\omega_1-i0}-\frac{1}{2}\right)+C_A\left(\ln\frac{\mu}{\omega_2-i0}+\frac{i}{2}\pi\right)\right]\delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')-C_F H_+(\omega_1,\omega_1')\delta(\omega_2-\omega_2') \\
&\quad -C_A H_+(\omega_2,\omega_2')\delta(\omega_1-\omega_1')+C_A\left(\frac{\omega_2}{{\omega_2'}^2}\right)\left[\theta(\omega_2)\theta(\omega_2'-\omega_2)-\theta(-\omega_2)\theta(\omega_2-\omega_2')\right]\delta(\omega_1-\omega_1')+\Delta\Gamma_{\mathrm{G}}\Bigg\} \\
\Delta\Gamma_{\mathrm{G}} \;&=\; \frac{i}{4}\frac{C_A}{\pi}\left[H_+(\omega_1,\omega_1')-H_-(\omega_1,\omega_1')-2i\pi\delta(\omega_1-\omega_1')\right]\left[H_+(\omega_2,\omega_2')-H_-(\omega_2,\omega_2')-2i\pi\delta(\omega_2-\omega_2')\right]
\end{aligned}
$$

Full evolution functions

The final result for the RGE reads:

$$g_{17}(\omega,\omega_1;\mu) = \int_{\omega}^{\bar{\Lambda}} \frac{d\omega'}{\omega'-\omega}\, U_n^{(17)}(\omega,\omega';\mu,\mu_0) \int_{-\infty}^{\infty} \frac{d\omega_1'}{|\omega_1'|}\, U_{\bar{n}}^{(17)}(\omega_1,\omega_1';\mu,\mu_0)\, g_{17}(\omega',\omega_1';\mu_0)$$

$$U_n^{(17)}(\omega,\omega';\mu,\mu_0) = \frac{e^{2V+2\gamma_E a}}{\Gamma(-2a)} \left(\frac{\mu_0}{\omega'-\omega}\right)^{2a}$$

$$\begin{aligned} U_{\bar{n}}^{(17)}(\omega_1,\omega_1';\mu,\mu_0) = &- e^{V_1+2\gamma_E a_1} \left(\frac{\mu_0}{|\omega_1'|}\right)^{a_1} \left\{\theta(\tau) G_{3,3}^{1,2}\left(\begin{matrix} -1, & 1, & a_1/2 \\ a_1+1, & a_1-1, & a_1/2 \end{matrix} \middle| \tau \right)\right. \\ &\left. + \frac{1}{2\pi}\sin\left(\frac{a_1\pi}{2}\right)\theta(-\tau)\Gamma(1+a_1)\Gamma(3+a_1)(-\tau)^{1+a_1}\,{}_2F_1(1+a_1,3+a_1,3;\tau)\right\} \end{aligned}$$

With the Meijer-G functions defined as:

$$G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = \int \frac{d\eta}{2\pi i} z^{\eta} \frac{\prod_{j=1}^{m}\Gamma(b_j-\eta)\prod_{j=1}^{n}\Gamma(1-a_j+\eta)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\eta)\prod_{j=n+1}^{p}\Gamma(a_j-\eta)}$$

Mellin space transform and equations

The Mellin transform reads:

$$g_{17}(\omega,\omega_1;\mu) = \theta(\omega)g_{17}^{>}(\omega,\omega_1;\mu) + \theta(-\omega)g_{17}^{<}(\omega,\omega_1;\mu)$$

$$\tilde{g}_{17}^{<}(\eta,\omega_1;\mu) = \int_0^\infty \frac{d\omega}{\omega}\left(\frac{\mu}{\omega}\right)^\eta g_{17}^{<}(-\omega,\omega_1;\mu)$$

$$\tilde{g}_{17}^{>}(\eta,\omega_1;\mu) = \int_0^\infty \frac{d\omega}{\omega}\left(\frac{\mu}{\omega}\right)^\eta g_{17}^{>}(\omega,\omega_1;\mu)$$

Applying this to the RG equations we get for the non-abelian part:

$$\left(\frac{d}{d\ln\mu} - \eta_1\right)\tilde{g}_{17}^{>}(\omega,\eta_1;\mu)$$

$$= -\frac{\alpha_s C_A}{2\pi}\Big\{\left[H_{-1-\eta_1} + H_{\eta_1} + 2H_{1-\eta_1} + 2\partial_{\eta_1}\right]\tilde{g}_{17}^{>}(\omega,\eta_1;\mu) - \Gamma(-\eta_1)\Gamma(1+\eta_1)\,\tilde{g}_{17}^{<}(\omega,\eta_1;\mu)\Big\}$$

and

$$\left(\frac{d}{d\ln\mu} - \eta_1\right)\tilde{g}_{17}^{<}(\omega,\eta_1;\mu)$$

$$= -\frac{\alpha_s C_A}{2\pi}\Big\{\left[H_{-1-\eta_1} + H_{\eta_1} + 2H_{1-\eta_1} + 2\partial_{\eta_1}\right]\tilde{g}_{17}^{<}(\omega,\eta_1;\mu) - \Gamma(-\eta_1)\Gamma(1+\eta_1)\,\tilde{g}_{17}^{>}(\omega,\eta_1;\mu)\Big\}$$