

# The spectrum of Feynman integral geometries at two loops

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[**PB**, Yang [2408.06325](#)]

[**PB**, Yang [2503.16299](#)]

[**PB**, Frellesvig, Marzucca,  
Morales, Seefeld, Wilhelm,  
Yang [2512.13794](#)]

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LoopFest XXIV

Brookhaven National Laboratory



European Research Council

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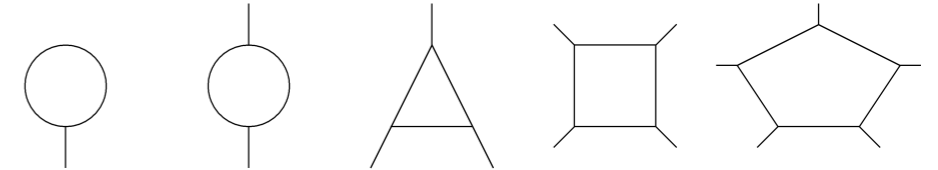
# Presentation plan

- Integrand **reduction** to a minimal basis of topologies
- Functionally distinct **basis** of all 2-loop Master Integrals
- Spectrum of all 2-loop Feynman integral **geometries**

# The finite basis problem

1-loop n-point : linearly related to subsectors of 5-point

$$\mathcal{F}_{1,n} = \sum_{i_1=1}^n \frac{\mathcal{N}_{\text{tadpole},i_1}}{\mathcal{D}_{i_1}} + \sum_{\substack{i_1, i_2=1 \\ i_1 \neq i_2}}^n \frac{\mathcal{N}_{\text{bubble},i_1,i_2}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \sum_{\substack{i_1, i_2, i_3=1 \\ i_j \neq i_m}}^n \frac{\mathcal{N}_{\text{triangle},i_1,i_2,i_3}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{\substack{i_1, i_2, i_3, i_4=1 \\ i_j \neq i_m}}^n \frac{\mathcal{N}_{\text{box},i_1,i_2,i_3,i_4}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}} + \sum_{\substack{i_1, i_2, i_3, i_4, i_5=1 \\ i_j \neq i_m}}^n \frac{\mathcal{N}_{\text{pentagon},i_1,i_2,i_3,i_4,i_5}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}\mathcal{D}_{i_5}},$$



beyond 1-loop : new integrals with each new leg ?

- 1-loop : “no” i.e. up to 5-point [Passarino, Veltman, Ossola, Papadopoulos, Pittau]
- 2-loop massless planar in  $d=4-2\epsilon$  : “also no” i.e. up to 11 denominators [Gluza, Kajda, Kosower [1009.0472](#)]
- 2-loop in  $d=d_0-2\epsilon$  : “also no” [Kleiss, Malamos, Papadopoulos, Verheyen [1206.4180](#)]
- 2-loop in  $d=4$  : “also no” i.e. up to 8 denominators [Feng, Huang [1209.3747](#)]
- L-loop in  $d=4$  : “also no” [Bourjaily, Herrmann, Langer, Trnka [2007.13905](#)]
- **L-loop in  $d=d_0-2\epsilon$  : “also no”** **[PB, Tong-Zhi Yang [2408.06325](#)]**  
(for integer  $d_0$  & propagator powers)

# Central idea


't Hooft - Veltman dimensional regularization scheme :

loop momenta  $k_l$  live in  $d = 4 - 2\epsilon$  dimensions

while external momenta  $p_n$  live in 4 dimensions

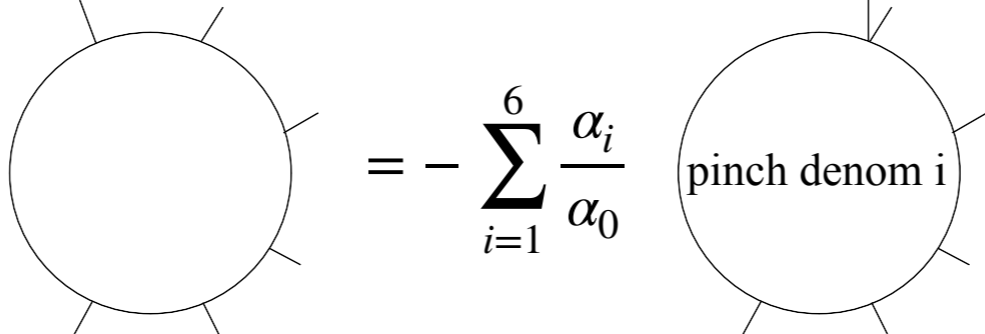
$$p_n = \sum_{i=1}^4 \beta_{ni} p_i$$

basis of 4  
independent vectors



- Standard Model renormalizability : 't Hooft - Veltman, 1972
- physical tensor and spinor projectors : Tancredi - Peraro
- finite number of independent Feynman integrals per loop : here

# 1-loop example

- CDR integral  $\mathcal{F}_{1,6} = \frac{1}{k^2(k+p_1)^2(k+p_{12})^2(k+p_{123})^2(k+p_{1234})^2(k+p_{12345})^2}$
- only 4 independent external momenta  $p_5 = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4$
- tHV integral  $\mathcal{F}_{1,6} = \frac{1}{k^2(k+p_1)^2(k+p_{12})^2(k+p_{123})^2(k+p_{1234})^2(k+\sum_{i=1}^4 z_i p_i)^2}$
- # independent scalar products = # independent propagators
- $\{k_1^2, p_1 \cdot k_1, \dots, p_4 \cdot k_1\} \Rightarrow$  **at most 5** linearly independent propagators at 1 loop  
 $(k + \sum_{i=1}^4 z_i p_i)^2 = \alpha_0 + \alpha_1 k^2 + \alpha_2 (k + p_1)^2 + \alpha_3 (k + p_{12})^2 + \alpha_4 (k + p_{123})^2 + \alpha_5 (k + p_{1234})^2$
- therefore, can partial fraction 

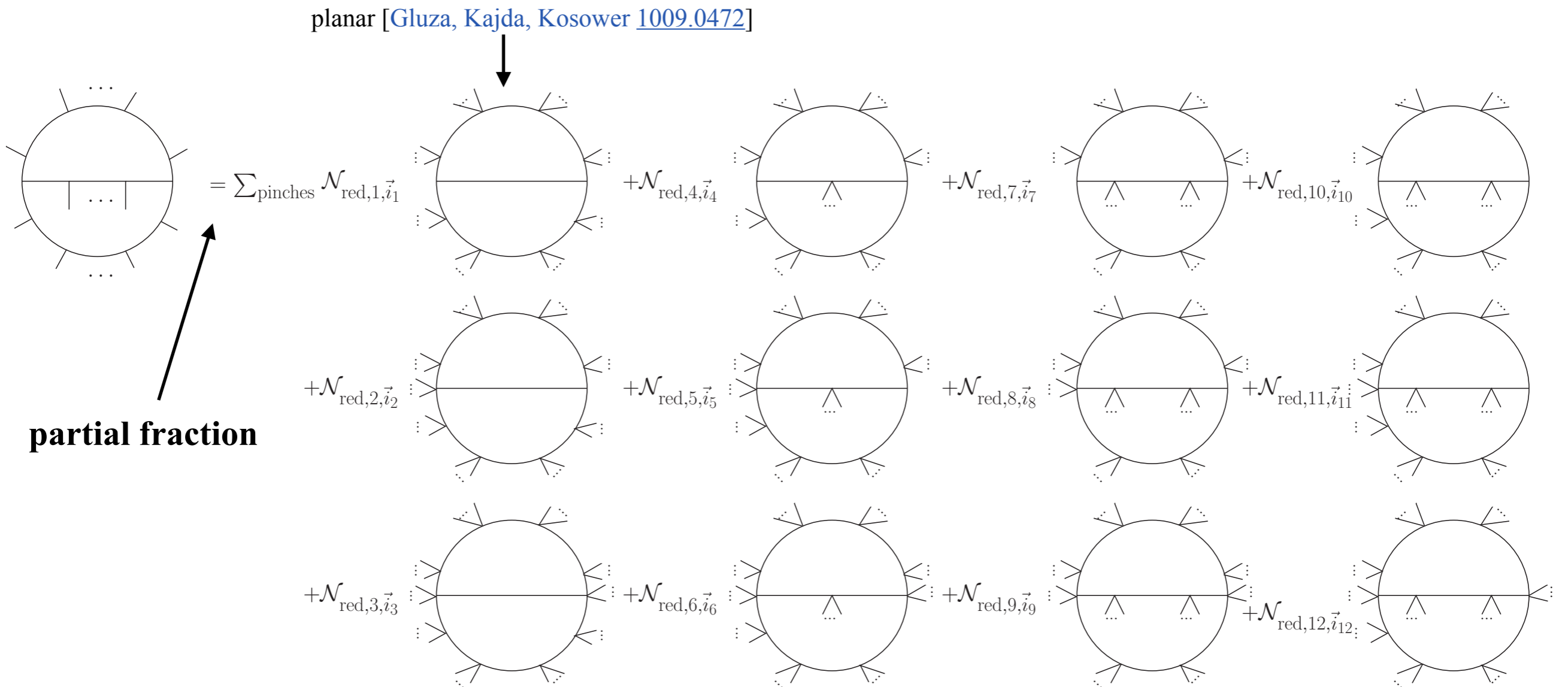
# 2-loop point-by-point

- (2 loop)\*(4 external) scalar products  $\{p_1 \cdot k_1, \dots, p_4 \cdot k_2\}$
- + 3 (loop\*loop) scalar products  $\{k_1^2, k_2^2, k_1 \cdot k_2\}$
- $\Rightarrow$  **at most 11** linearly independent propagators in tHV

kinematics	denominators	CDR ISPs	tHV ISPs	generalized CDR propagators	generalized tHV propagators
2-loop 4-point	7	2	2	9	9
2-loop 5-point	8	3	3	11	<b>11</b>
2-loop 6-point	9	4	2	13	<b>11</b>
2-loop 7-point	10	5	1	15	<b>11</b>
<b>2-loop 8-point</b>	<b>11</b>	6	<b>0</b>	17	<b>11</b>
2-loop 9-point	12	7	<b>0</b>	19	<b>11</b>

only 11 independent  $\Rightarrow$  **partial fraction**

# Finite basis topologies at 2 loops

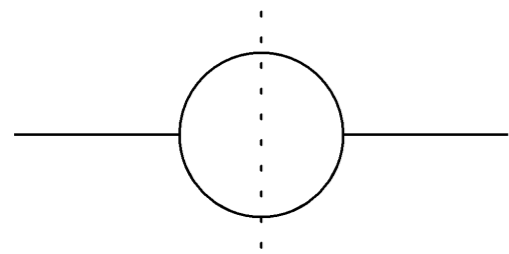


lower dimensions : less denominators e.g. 7 in  $d=2-2\epsilon$

higher loops : 12p@3L, 17p@4L, ... (loop-by-loop approach analogous but many more topologies)

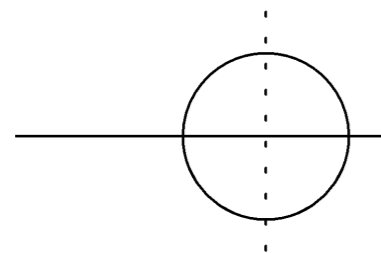


# Integral geometries from Baikov



$$= \int \frac{d^d k}{(2\pi)^d} (2\pi i) \delta^+(k^2 - m_1^2) (2\pi i) \delta^+((k-p)^2 - m_2^2) \stackrel{d \rightarrow 2}{\sim} \int dx \frac{1}{\sqrt{(x-x_1)(x-x_2)}}$$

$\int_0^x \frac{dx_1}{x_1 - a_1} \int_0^{x_1} \frac{dx_2}{x_2 - a_2} \dots$  Polylogarithm defined on a **sphere**  
 well-established

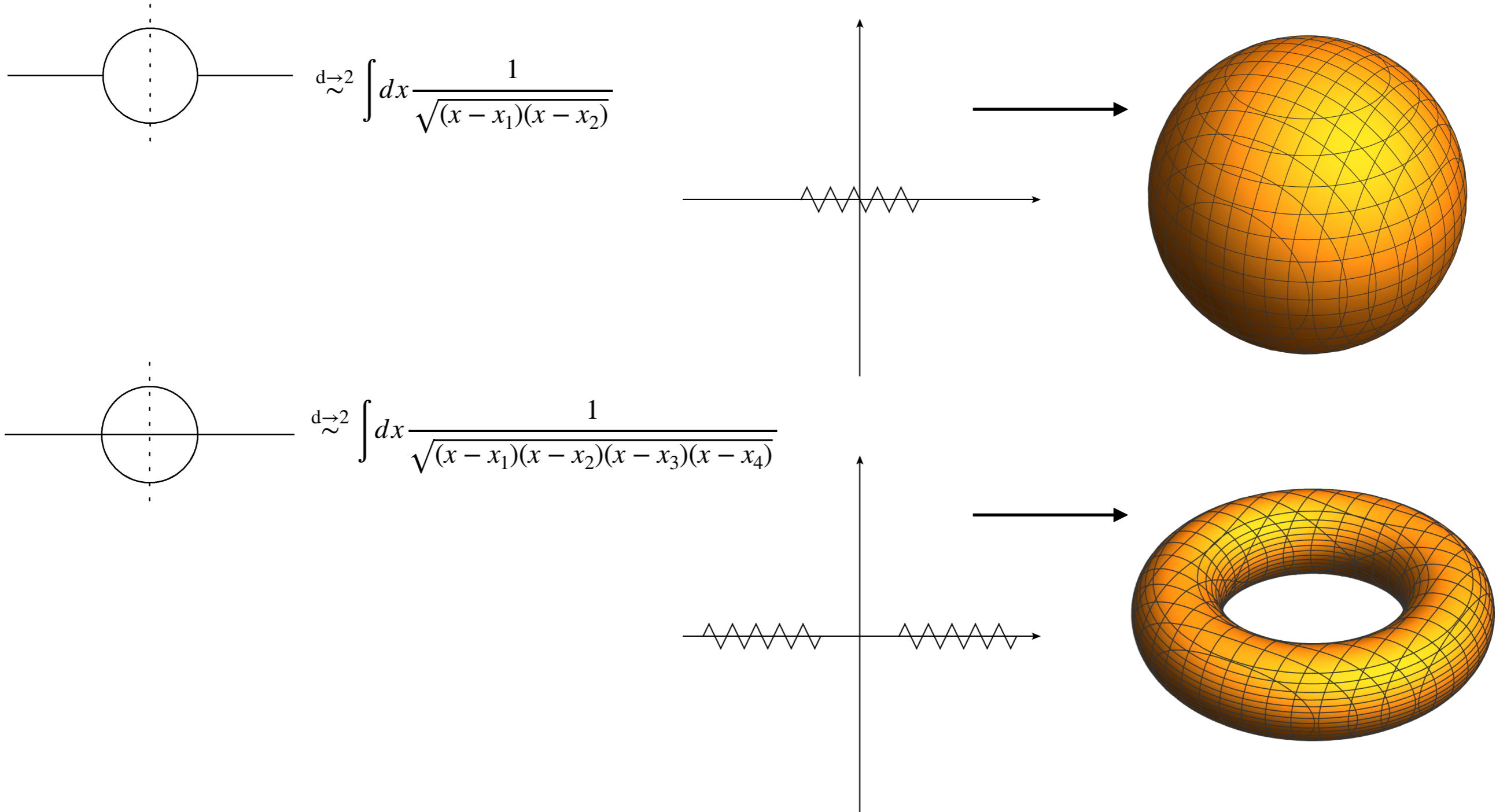


$$\sim \int dz_1 dz_2 \text{Baikov}_{\max \text{ cut}}^{\beta'(d)}(z_1, z_2) \stackrel{d \rightarrow 2}{\sim} \int dx \frac{1}{\sqrt{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}}$$

[Sabry 1962] Elliptic Polylogarithm defined on a **torus** [Broedel, Duhr, Dulat, Tancredi 1712.07089]  
 recently well-understood

[Baikov 9611449] [Frellesvig, Papadopoulos 1701.07356]

# Where dLog forms are naturally defined



# Leading singularities

fully reduced

$$\text{LS} \sim \text{max cut} \sim \int d\vec{z} \frac{1}{\sqrt{P_{\text{degree}}(\vec{z})}}$$

Geometry	Characteristic equation
Elliptic curve	$P_4(z)$
Hyperelliptic curve (genus $g$ )	$P_{2g+2}(z)$
Del Pezzo surface of degree 2	$P_4(z_1, z_2)$
K3 surface	$P_6(z_1, z_2)$
Calabi–Yau $n$ -fold	$P_{2n+2}(z_1, \dots, z_n)$

functions recently defined

functions unknown

[Snowmass review [2203.07088](#)]

# Integral geometries from differential equations

[see talks by Christoph and Cesare]

canonical form [Henn 1304.1806]

$$dM_i(\epsilon, \vec{x}) = \epsilon dA_{ij}(\vec{x}) M_j(\epsilon, \vec{x})$$

$$\partial^n + a(\epsilon, \vec{x})\partial^{n-1} + \dots$$

Picard - Fuchs form [Adams, Chaubey, Weinzierl 1702.04279]

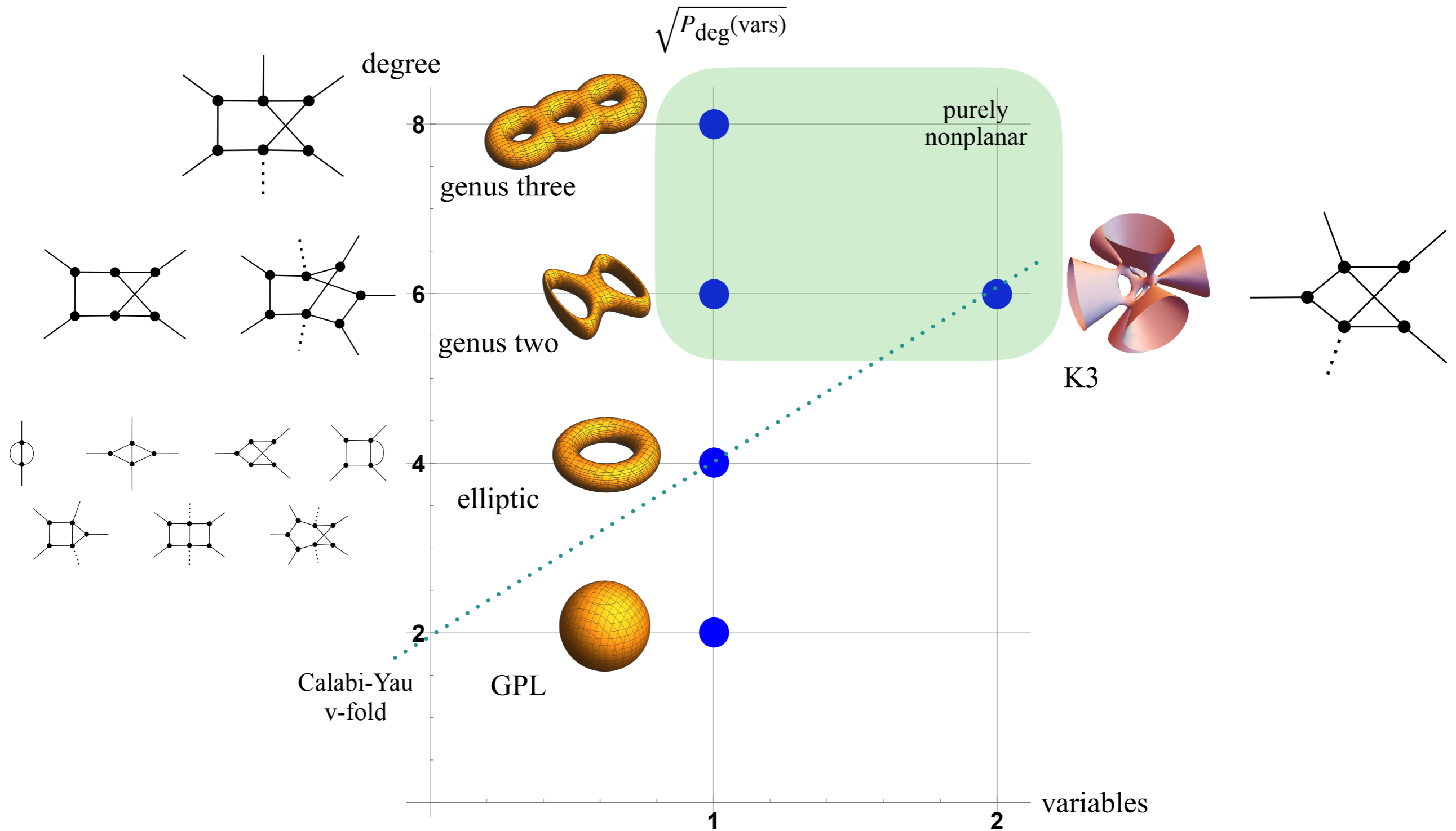
$$\prod_n \mathcal{L}_n(\partial, \epsilon, \vec{x}) M_1(\epsilon, \vec{x}) = 0$$

fully factorized

Geometry	Max PF degree
sphere	1
torus	2
genus 2	4
genus 3	6
K3	11

[Doran, Harder, Vanhove 2302.14840]

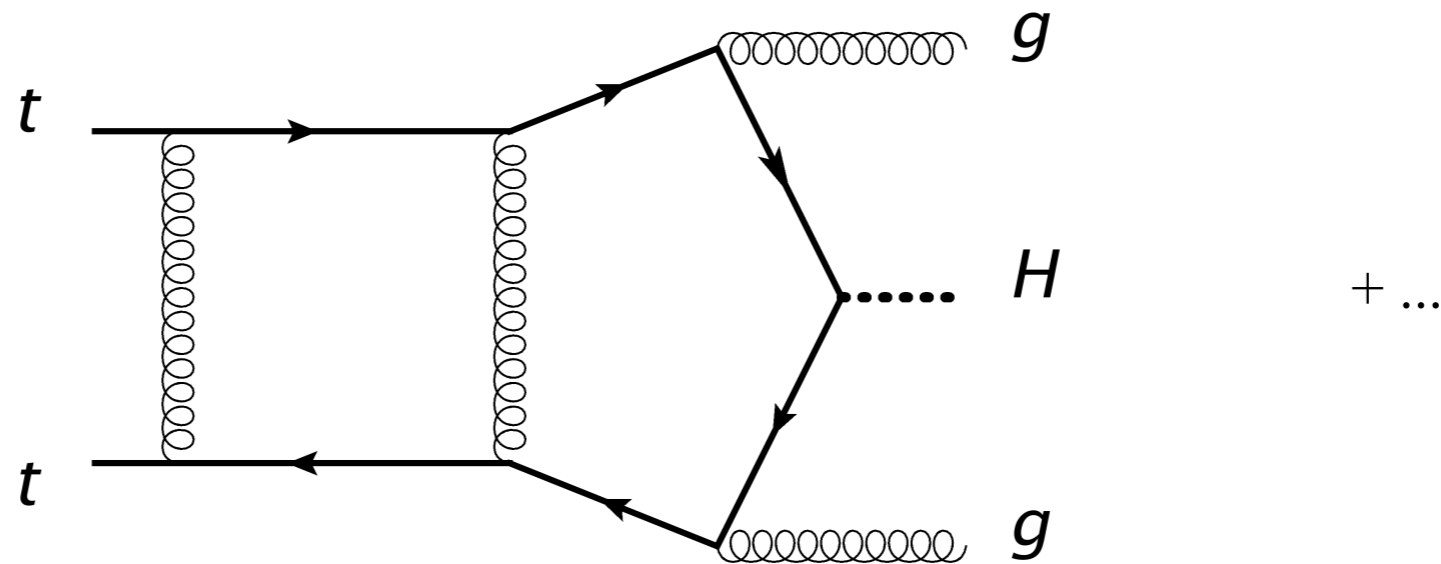
# Spectrum of 2-loop Feynman integral geometries



# Example amplitude : $gg \rightarrow t\bar{t}H$ @ 2-loop QCD

ongoing:  
Les Houches wishlist

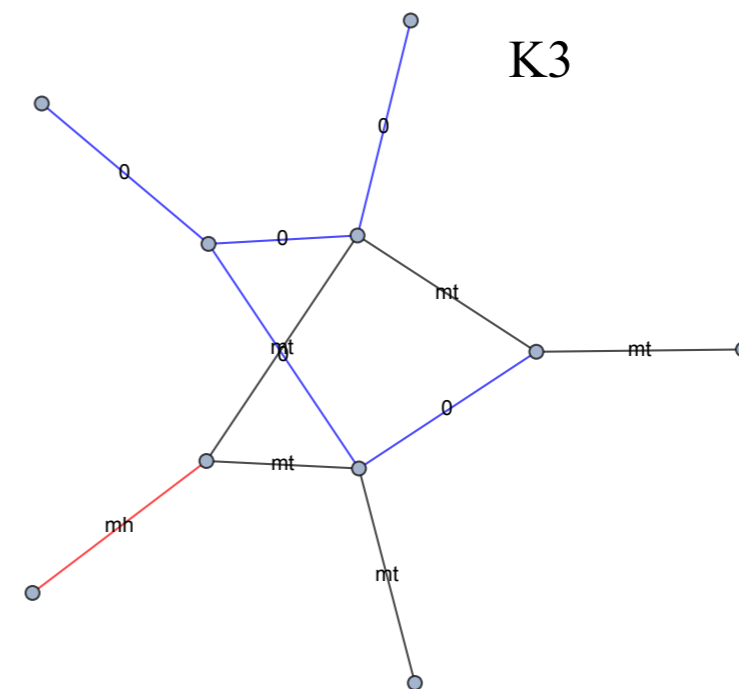
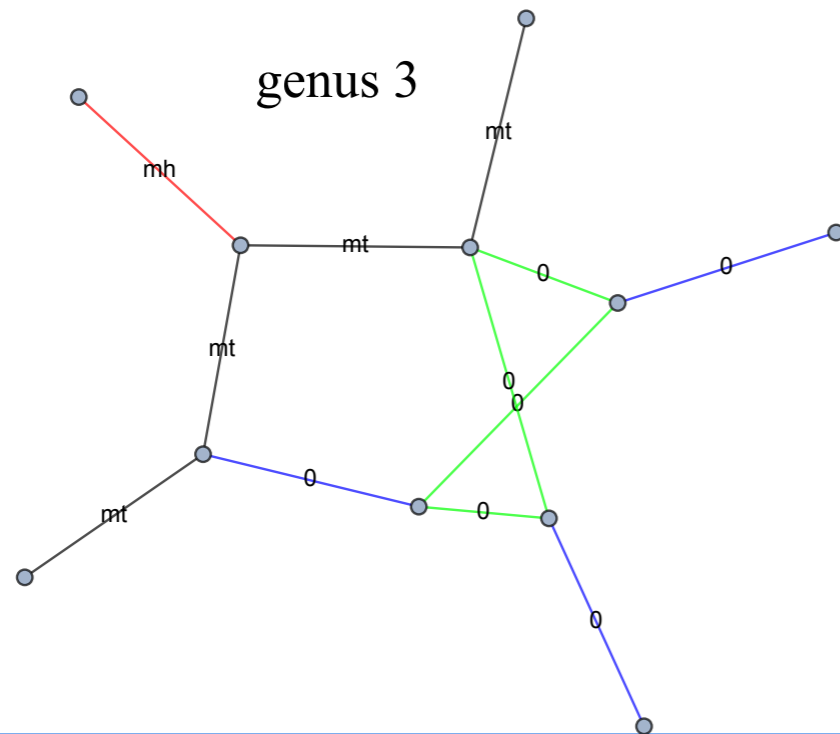
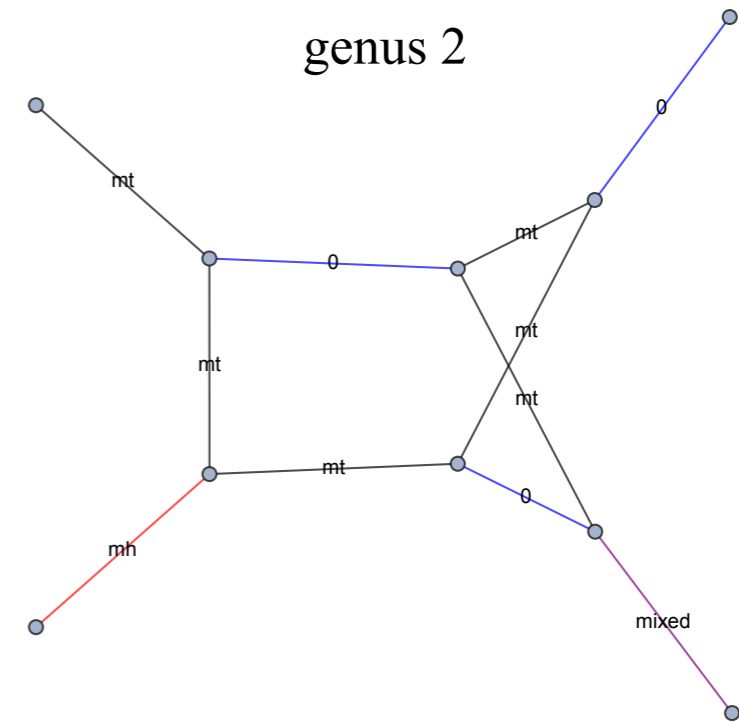
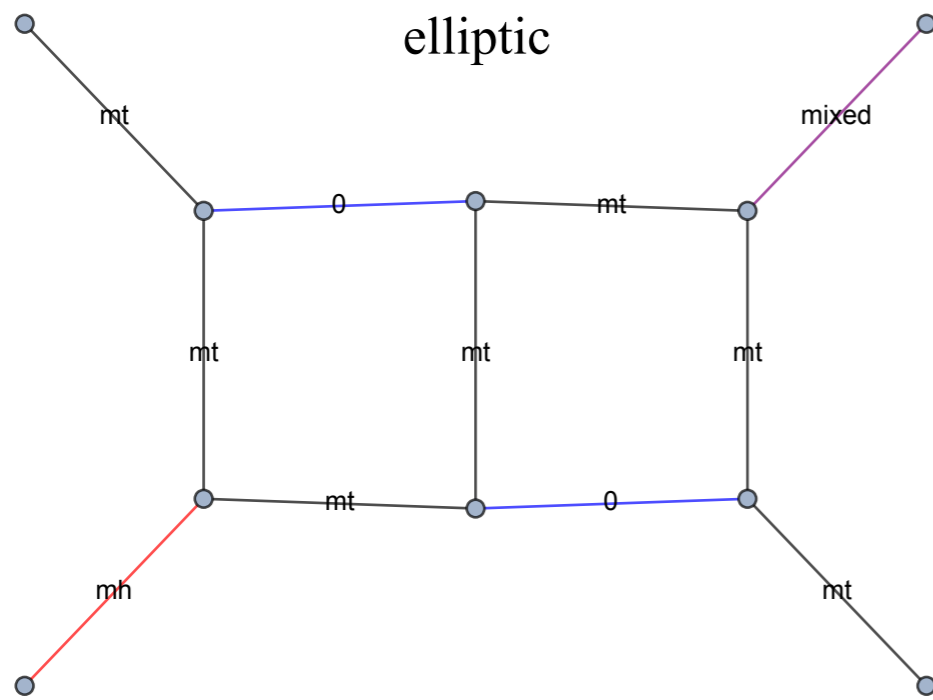
Amplitude =  $\Sigma$  Feynman diagrams



many subsectors

degenerate kinematics

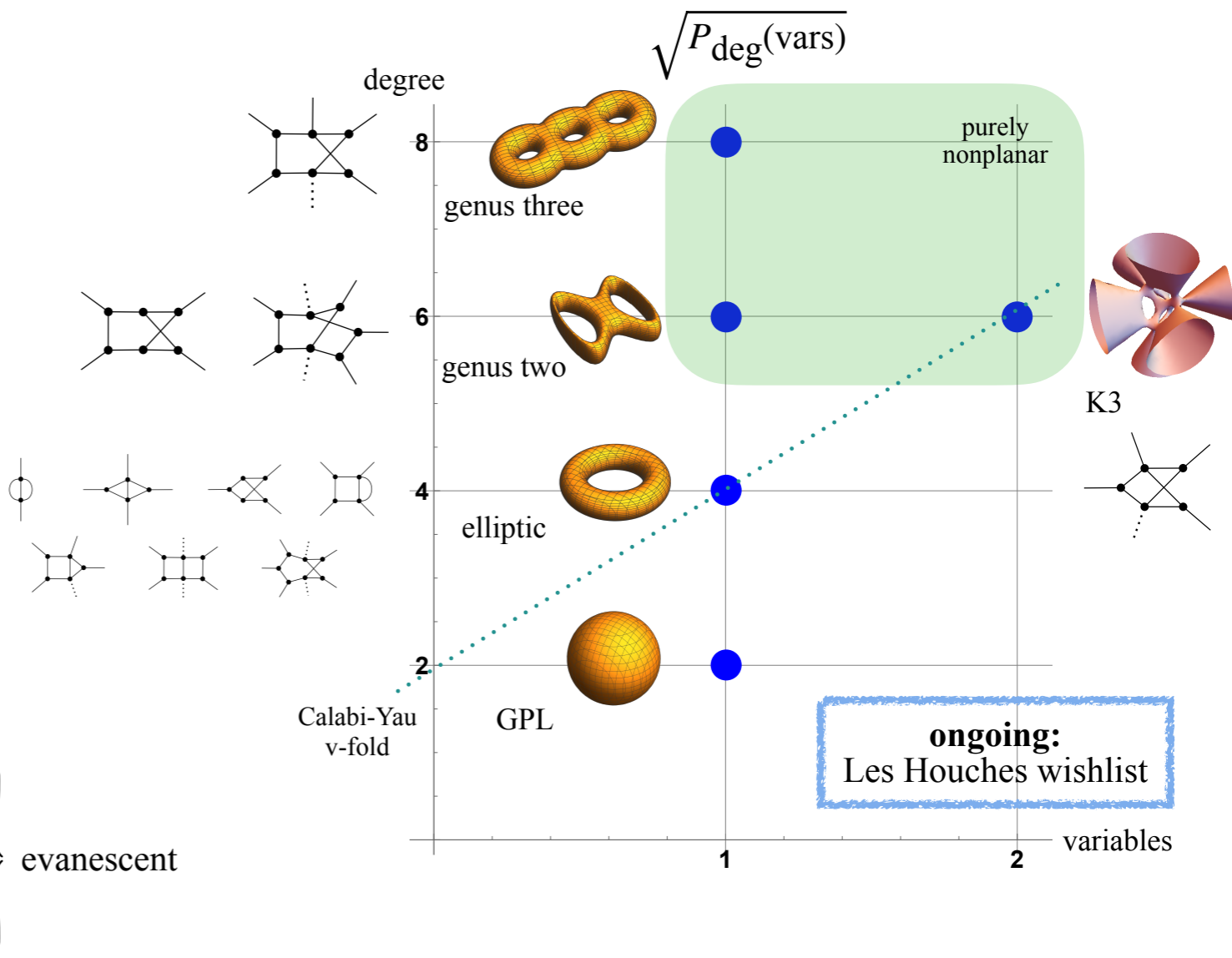
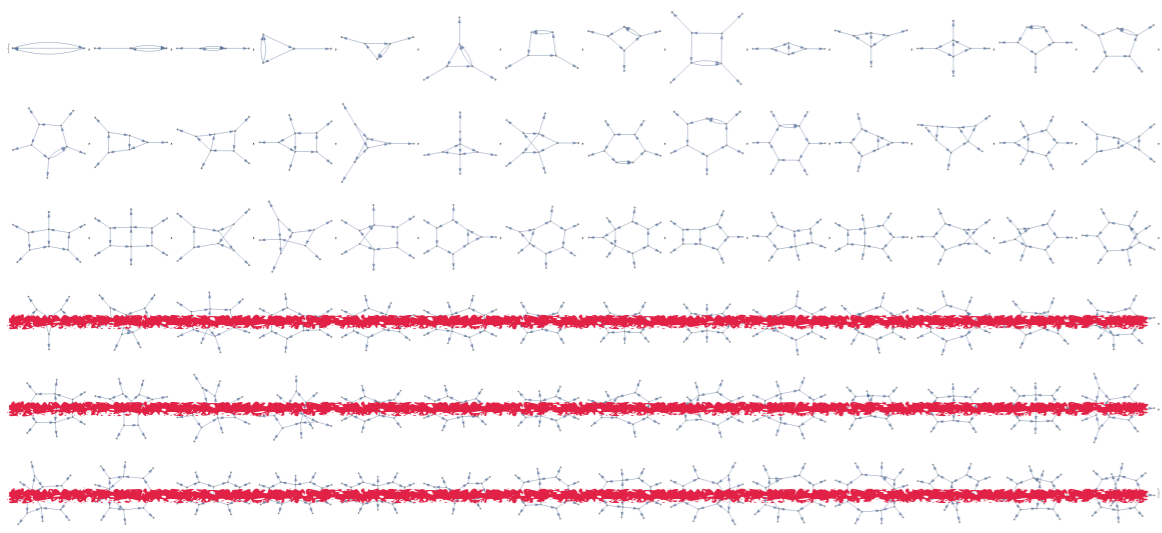
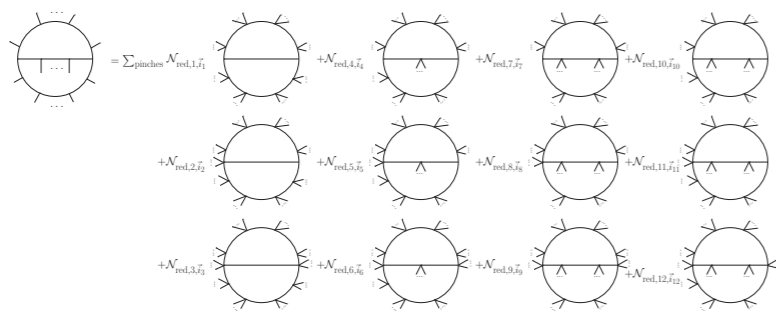
# Preliminary results : $gg \rightarrow t\bar{t}H$ @ 2-loop QCD



# Conclusions

- **motivation** : use 't Hooft-Veltman scheme in Feynman integrals
- **result** : {finite basis topologies, finite integral basis, bound on special functions} per loop

$$P_{n,\text{ext}} = \sum_{q_i \in \text{basis}} \beta_{n,i} q_i$$



THANK YOU

# Appendix

# Explicit reduction coefficients

$$\mathcal{F}_{L,n} = \prod_{i=1}^{D(n,L)} \frac{1}{\mathcal{D}_i} = \sum_{\substack{i_1, \dots, i_A = 1 \\ i_j \neq i_m}}^{D(n,L)} \frac{c_{i_1, \dots, i_A}}{\mathcal{D}_1 \cdots \hat{\mathcal{D}}_{i_1} \hat{\mathcal{D}}_{i_2} \cdots \hat{\mathcal{D}}_{i_A} \cdots \mathcal{D}_{D(n,L)}}$$

with

$$c_{i_1, \dots, i_A} = \frac{(-B_{i_1, \dots, i_A})^A}{B_{0, \dots, i_A} B_{i_1, 0, \dots, i_A} \cdots B_{i_1, \dots, 0}}$$

$$A = D(n, L) - D(N(L, d_0), L)$$

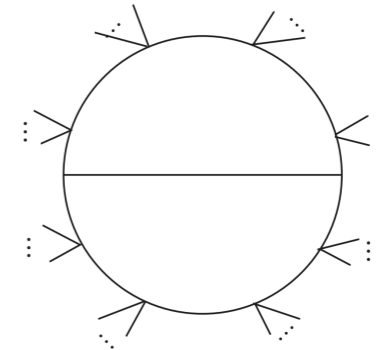
$$B_{i_1, \dots, i_A} = \begin{vmatrix} \alpha_{D(N(L, d_0), L)+1, i_1} & \cdots & \alpha_{D(N(L, d_0), L)+1, i_A} \\ \alpha_{D(N(L, d_0), L)+2, i_1} & \cdots & \alpha_{D(N(L, d_0), L)+2, i_A} \\ \vdots & \vdots & \vdots \\ \alpha_{D(n, L), i_1} & \cdots & \alpha_{D(n, L), i_A} \end{vmatrix}$$

$$\begin{aligned} \mathcal{D}_{i > D(N(L, d_0), L)} &= \alpha_{i,0} + \sum_{j=1}^{D(N(L, d_0), L)} \alpha_{i,j} \mathcal{D}_j \\ \alpha_{i,i} &= -1 \\ \alpha_{i, j > D(N(L, d_0), L)} &= 0 \end{aligned}$$

partial fractions implemented in Mathematica package **Apart** [[Feng 1204.2314](#)]

# Improving the IBP efficiency beyond 5-point

- explicit example : consider the 2-loop 8-point topology



- it has 17 CDR generalized propagators

$$\{k_1 - k_2, k_1, k_2, k_1 + p_1, k_1 + p_{12}, k_1 + p_{123}, k_1 + p_{1234}, k_2 + p_{1234}, k_2 + p_{12345}, k_2 + p_{123456}, k_2 + p_{1234567}, k_1 - p_5, k_1 - p_6, k_1 - p_7, k_2 - p_1, k_2 - p_2, k_2 - p_3\}$$

- apply tHV momentum decomposition  $p_{j>4} = \sum_{i=1}^4 p_i \frac{\det(\{p_1, \dots, \hat{p}_i, p_j, \dots, p_4\} \cdot \{p_1, \dots, p_4\})}{\det(\{p_1, \dots, p_4\} \cdot \{p_1, \dots, p_4\})}$

- then, only **11** independent tHV generalized propagators

$$\{k_1 - k_2, k_1, k_2, k_1 + p_1, k_1 + p_{12}, k_1 + p_{123}, k_1 + p_{1234}, k_2 + p_{1234}, k_2 + \sum_{j=1}^4 p_j z_{9,j}, k_2 + \sum_{j=1}^4 p_j z_{10,j}, k_2 + \sum_{j=1}^4 p_j z_{11,j}\}$$

	CDR	tHV	$R_t$
D2	31.4m/3126MI/13.9G	6.8m/2368MI/1.88G	4.6
D3	51.5m/3302MI/18.19G	10.1m/2368MI/2.9G	5.1
N3	115.2m/4497MI/25.7G	5.7m/2358MI/2.59G	20.2
N4	321.3m/6742MI/56.4G	7.6m/2368MI/4.3G	42.3
N5	908.9m/9779MI/137G	12.5m/2368MI/7.35G	72.7
N6	-	20.1m/2368MI/10.34G	-

- 1-off-shell all-internally-massless
- no cut propagators
- one numerical probe, one finite field
- with FIRE6 [Smirnov, Chukharev 1901.07808]

# Improving the numerical evaluation efficiency

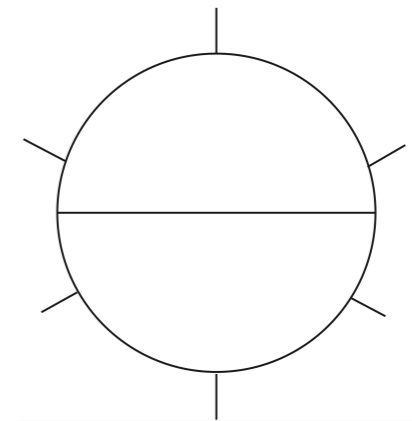
## Auxiliary Mass Flow (AMFlow)

[Liu, Ma [2201.11669](#)]

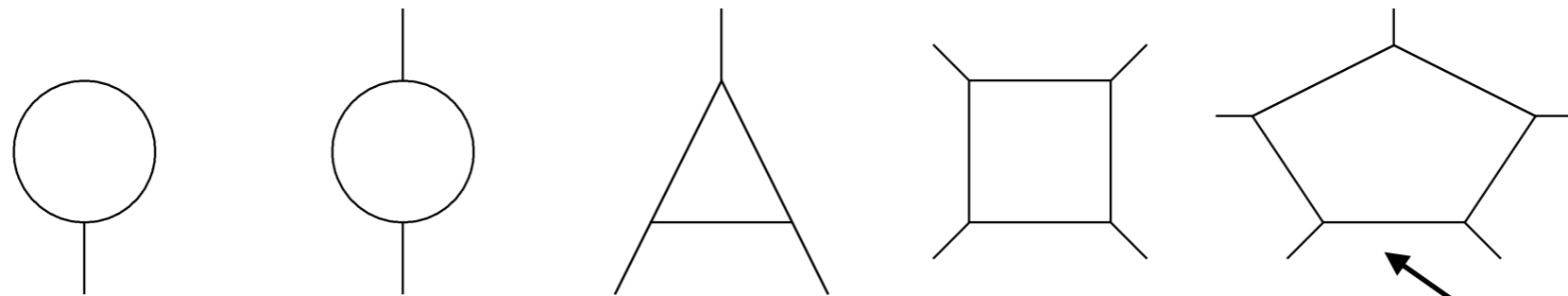
- numerical evaluator of Feynman integrals
- provides very high precision
- requires multiple internal IBP reductions

## tHV for IBP

- decompose external legs in 4 dimensions
- beyond 5-point, IBP performance improved
- explicit example : 2-loop 6-point 2-off-shell all-internally-massless
- AMFlow setup : no cut, 1-digit numerical sample, 20-digit precision, 5 leading  $\epsilon^n$  orders, 16 cores, Kira backend [[Klappert, Lange, Maierhöfer, Usovitsch 2008.06494](#)]
- tHV : **30h**
- CDR : **didn't finish** in 1 week



# 1-loop evanescent sector



evanescent sector

[Bern, Dixon, Kosower 9306240]

$$I_5^{(4D, \text{sing})} = \epsilon a^{(\text{fin})} I_5^{(6D, \text{fin})} + \text{subsectors}^{(\text{sing})}$$

dimension shift relations

[Baikov, Tarasov 1996]

no IR singularities beyond 4 dimensions  
& no UV singularities SDoD < 0

**amplitude** finite part : no contribution from Master Integrals with more than **4 denominators**

$$\mathcal{A} = a_i M_{\leq 4 \text{den}, i} + \mathcal{O}(\epsilon)$$

# 2-loop evanescent sectors

- shown evanescence of sectors with more than 8 denominators by a direct IBP reduction

recently shown for 2-loop 6-point planar massless

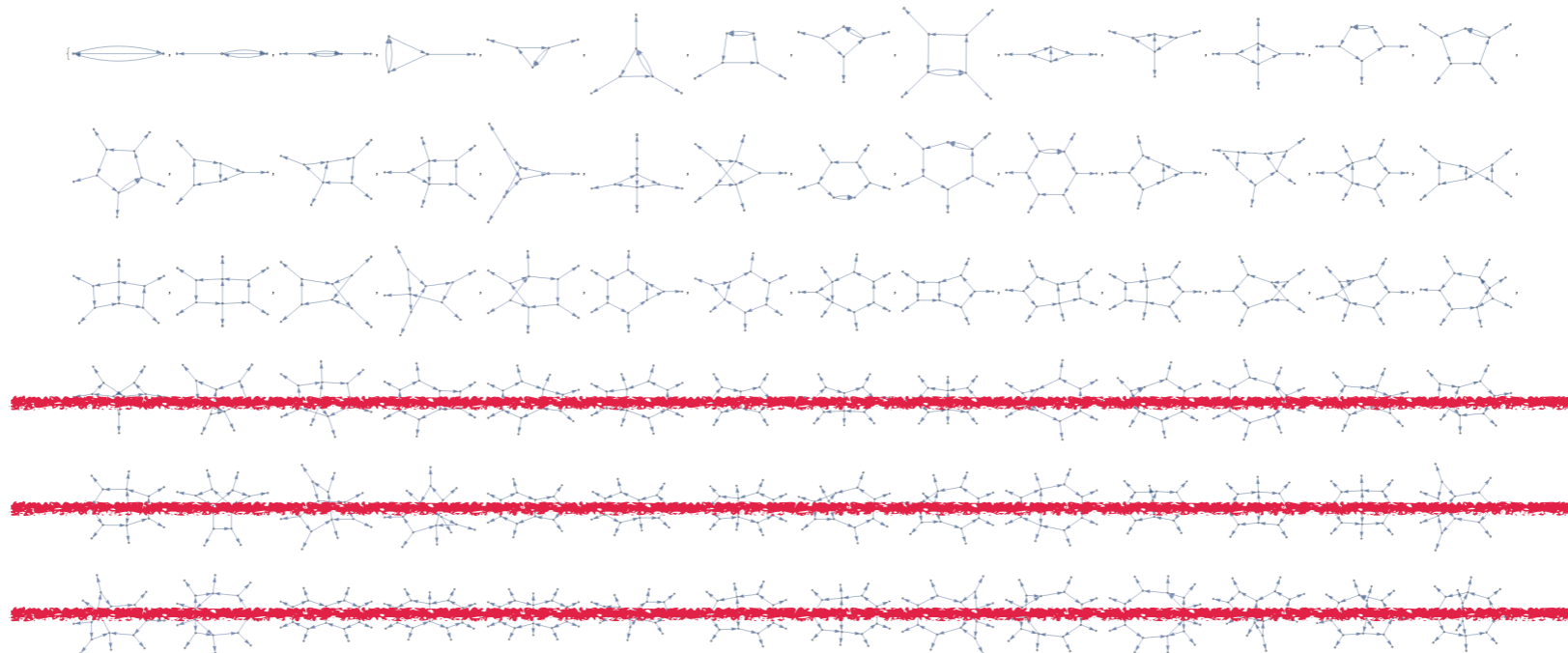
[Abreu, Monni, Page, Usovitsch [2412.19884](#)]

$$I^{(4D, \text{sing})} = \epsilon a_i^{(\text{fin})} M_i^{(6D, \text{fin})} + \text{subsectors}^{(\text{sing})}$$



- **amplitude** finite part : no contribution from Master Integrals with more than **8 denominators**

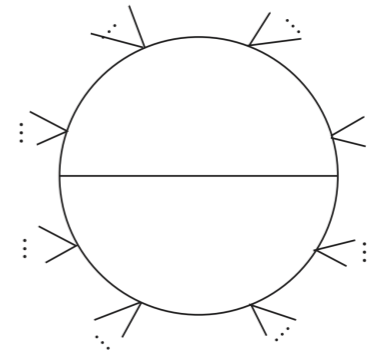
$$\mathcal{A} = a_i M_{\leq 8 \text{den}, i} + \mathcal{O}(\epsilon)$$



# Example applications beyond 5-point

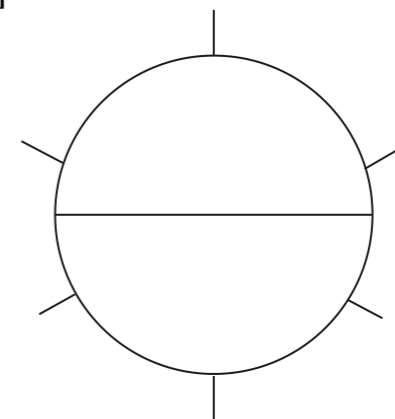
efficient IBP reduction (FIRE) [[Smirnov, Chukharev 1901.07808](#)]

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efficient numerical evaluation (AMFlow) [[Liu, Ma 2201.11669](#)]

CDR	tHV
>1week	30h



already applied for 2-loop 6-point integrals e.g. in [[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang 2501.01847](#)]