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NNLO QCD with Scalar Radiators and pure splittings

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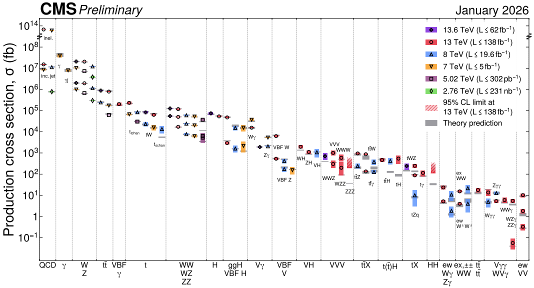
Based on:

J. Campbell, S. Höche, MK, C. Preuss, D. Reichelt arXiv:2505.10408

J. Campbell, S. Höche, MK arXiv:2606.XXXXX (in preparation)

Motivation

- HL-LHC: $3000\,\mathrm{fb}^{-1}$ → sub-percent experimental precision
- FCC-ee Tera-Z: 10^{12} Z bosons → need extreme theoretical precision
- Theory uncertainty becomes limiting factor
- Need **fully differential** predictions including realistic hadronic final states



Central goal of this program:

NNLO + NNLL matching for general-purpose MC generators

(Sherpa, Pepper, …): fully differential, shower-matched, percent-level accurate

The NNLO challenge

- At NNLO: double-real, real-virtual, double-virtual
  $\rightsquigarrow$ individually divergent; cancellation must be local
- **Overlapping soft and collinear singularities**
- Standard NNLO schemes **sectorize phase space**
  $\rightsquigarrow$ non-trivial cuts; hard analytic integration
- **But: PS evolution covers the full phase space**
  $\rightsquigarrow$ **sectorized subtraction $\not\Leftrightarrow$ PS matching**
  $\rightsquigarrow$ ambiguous assignment of emissions to branchings
- **Goal: non-sectorized**, locally finite NNLO subtraction with NLL-safe PS recoil

$$\begin{aligned}\mathcal{M}_{RR} = \mathcal{M}_{n+2} &- \sum \mathcal{M}_n \otimes S_2 \\ &- \sum \left[\mathcal{M}_{n+1} - \mathcal{M}_n \otimes S_1\right] \otimes S_1\end{aligned}$$

- S_1: NLO subtraction terms
- S_2: NNLO subtraction terms
- Key: S_1, S_2 locally finite over the **full phase space**

The idea: scalar radiators

[Campbell, Höche, MK, Preuss, Reichelt, 2505.10408]

- Quark-gluon vertex and quark propagator decompose as

$$\frac{\not{p}+\not{q}}{(p+q)^2}\,T^a_{ij}\gamma^\mu = T^a_{ij}\left[S^\mu(p,q)+\frac{i\sigma^{\nu\mu}q_\nu}{(p+q)^2}-\frac{\gamma^\mu\not{p}}{(p+q)^2}\right]$$

- Dominant behavior (leading singular) given by **scalar current**

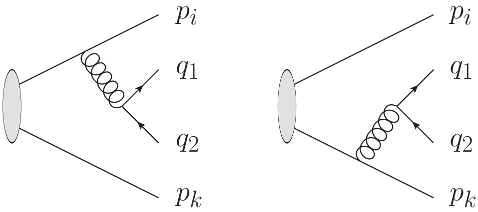
$$S^\mu(p,q)=\frac{(2p+q)^\mu}{(p+q)^2}$$

- Magnetic term $\sigma^{\nu\mu}$ (spin) and $\gamma^\mu\not{p}$ are subleading in all singular limits
- Similarly: decompose triple gluon vertex

$$\frac{d^\mu_\sigma(p_{12})}{p_{12}^2}f^{abc}\Gamma^{\sigma\nu\rho}(p_1,p_2)=i d^\mu_\sigma(p_{12})\mathrm{Tr}(T^c[T^b,T^a])\left[S^\nu(p_1,p_2)g^{\rho\sigma}-\frac{(p_1-p_2)^\rho}{2p_{12}^2}g^{\sigma\nu}\right]+\text{swap}$$

- **All 1→3 splitting functions = scalar radiator + subleading spin remainder**
  $\rightsquigarrow$ Remainder subleading in (almost) all singular limits
  $\rightsquigarrow$ No sectorization needed; single global subtraction term

Scalar radiators: $q\bar{q}$ emission



- Replace soft current J^μ by scalar current S^μ

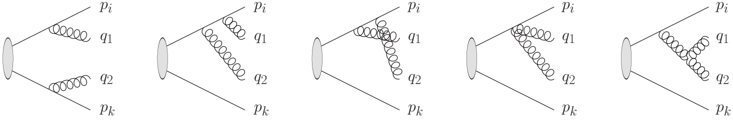
$$J^\mu(q) = \sum_i \hat{\mathbf{T}}_i \frac{p_i^\mu}{p_i \cdot q} \quad \rightarrow \quad \sum_i \hat{\mathbf{T}}_i S^\mu(p_i, q)$$

- Scalar radiator for $q\bar{q}$ pair with color structure $\hat{\mathbf{T}}_i \hat{\mathbf{T}}_k T_R$:

$$\mathcal{S}^{(q\bar{q})}_{i;k}(q_1, q_2; \bar{n}) = \frac{2}{s_{i12} s_{k12}} \left[\frac{2}{s_{12}} \left(\frac{z_i s_{k12} + z_k s_{i12}}{z_1 + z_2} - s_{ik} \right) + 1 - \frac{t_{12,i} t_{12,k}}{s_{12}^2} \right]$$

- **Coincides with Campbell-Glover in the double-soft limit**
- **Purely scalar:** the $q\bar{q}$ radiator has no splitting remainder
  $\rightsquigarrow$ fully captured by the scalar current in Feynman gauge, no sectorization

Scalar radiators: gg emission



Abelian + non-abelian components: **complete** & validated

Simplified abelian radiators (charge conservation):

$$\mathcal{S}^{(\mathrm{ab})}_{i,k;l,m} = \tfrac{1}{4}\,\mathcal{S}_{i;l}(q_1)\mathcal{S}_{k;m}(q_2) = \frac{s_{il}\,s_{km}}{s_{i1}\,s_{l1}\,s_{k2}\,s_{m2}}$$

$$\mathcal{S}^{(\mathrm{ab})}_{i,k;l} = \frac{s_{il}s_{kl}}{s_{l12}\,s_{i1}s_{k2}}\left(\frac{s_{k1}}{s_{kl}s_{l1}} + \frac{s_{i2}}{s_{il}s_{l2}} - \frac{s_{ik}}{s_{il}s_{kl}} - \frac{s_{12}}{s_{l1}s_{l2}}\right)$$

$$\mathcal{S}^{(\mathrm{ab})}_{i;l} = \frac{\lambda^2(s_{i1}s_{l2},\, s_{i2}s_{l1},\, s_{12}s_{il})}{s_{i12}s_{l12}s_{i1}s_{l1}s_{i2}s_{l2}} + \frac{2(1-\varepsilon)}{s_{i12}s_{l12}}$$

$\lambda^2(a,b,c) = (a{-}b{-}c)^2 - 4bc$ Non-abelian: see next slide

- **Leading double-soft** $\mathcal{S}^{(\mathrm{ab})}_{i,k;l,m}$: squared one-gluon current
- **Sub-leading** $\mathcal{S}^{(\mathrm{ab})}_{i,k;l}$: no double-collinear singularity, purely soft + triple-collinear
- **Sub-sub-leading** $\mathcal{S}^{(\mathrm{ab})}_{i;l}$: azimuthal correlation in dipole rest frame

Non-abelian radiator in covariant gauge

- Strongly-ordered limit reproduces Catani-Grazzini [Catani, Grazzini (2000)], Eq. (110):

$$\mathcal{S}_{i;k}^{(\mathrm{nab,\,s.o.})}(q_1,q_2) = 4\left(\frac{s_{ik}}{s_{i1}s_{12}s_{k2}} + \frac{s_{ik}}{s_{k1}s_{12}s_{i2}} - \frac{s_{ik}^2}{s_{i1}s_{k1}s_{i2}s_{k2}}\right)$$

- **Full non-abelian radiator** (covariant gauge, charge conservation applied) differences to Catani-Grazzini highlighted in red:

$$\begin{aligned}\mathcal{S}_{i;k}^{(\mathrm{nab})}(q_1,q_2) = \mathcal{S}_{i;k}^{(\mathrm{nab,\,s.o.})}(q_1,q_2) \;+\; &\frac{4(1-\varepsilon)}{s_{i12}s_{k12}}\left(\frac{s_{i1}s_{k2}+s_{i2}s_{k1}}{s_{12}^2}-\frac{1}{2}\right) - \frac{4}{s_{i12}s_{k12}}\left(\frac{2s_{ik}}{s_{12}}-1\right)\\ &- \left[\frac{s_{i1}s_{k2}}{s_{i12}s_{k12}}\left(1-\frac{s_{i2}+s_{k1}}{s_{ik}}-\frac{s_{12}}{2s_{ik}}\right)+\frac{s_{12}}{2s_{i12}}\left(1-\frac{s_{12}}{2s_{k12}}\right)+(i\leftrightarrow k)\right]\frac{1}{2}\,\mathcal{S}_{i;k}^{(\mathrm{nab,\,s.o.})}\end{aligned}$$

- Red corrections are **sub-leading in the double-soft limit** (a remainder survives in the triple-collinear limit) ⇝ small numerical impact, but required for correct ε-poles upon integration
- Strongly-ordered $(1-\varepsilon)$ contribution vanishes in soft-gluon limit by color conservation

Overlap removal via partial fractioning [Campbell, Höche, MK, 2606.XXXXX]

- **NLO:** single-gluon radiator $\mathcal{S}_{i;k}(q_1) = -4\,p_ip_k/(p_{i1}^2 p_{k1}^2)$ singular for $q_1 \parallel p_i$ *and* $q_1 \parallel p_k$

$$\mathcal{S}_{i;k}(q_1;n) = \mathcal{S}_{i;k}^{(i)} + \mathcal{S}_{i;k}^{(k)}\ , \quad \text{where} \quad \mathcal{S}_{i;k}^{(i)} = \frac{\eta_{k1}}{\eta_{k1}+\eta_{i1}}\,\mathcal{S}_{i;k}(q_1)\ , \quad \eta_{ij} = \frac{2p_ip_j}{\zeta_i\zeta_j n^2}$$

  ⇝ each term has a **unique emitter** ⇒ well-defined kinematic mapping

- **NNLO abelian** radiator $\mathcal{S}_{i,k;l,m}^{(\mathrm{ab})} = \frac{1}{4}\mathcal{S}_{i;l}(q_1)\mathcal{S}_{k;m}(q_2)$ is **fully factorized**
  ⇒ partial fraction each factor as at NLO, no ordering needed
  Non-abelian $\mathcal{S}_{i;k}^{(\mathrm{nab})}$: angular partial fractioning + energy ordering $\zeta_1/(\zeta_1+\zeta_2)$

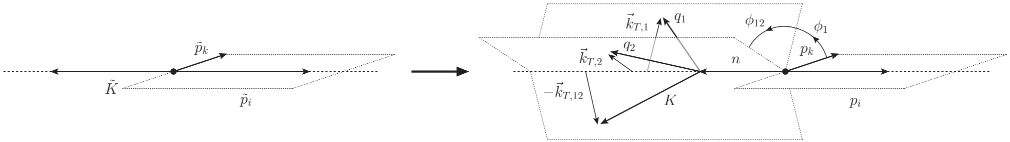
- $[\mathcal{M}_3 - \mathcal{M}_2 \otimes S_1] \otimes S_1$ **counterterms:** first **order the two emissions** by transverse momentum [Dulat, et al. (2018)]

$$(S_1 \otimes S_1)_{i,k;l,m}^{(1)} = \frac{s_{k2}\,s_{m2}}{s_{k2}\,s_{m2} + s_{i1}\,s_{l1}}\,(S_1 \otimes S_1)_{i,k;l,m}$$

  then partial fraction each ordered term as at NLO

- ⇒ **Unique branching topology for every counterterm** — no sectorization
  ⇝ each counterterm ↔ a single parton-shower branching: basis for NNLO+PS matching

Kinematics: iterated NLO mappings



- Apply identified-particle mapping [Catani, Seymour (1997)] twice: $(\tilde{p}_i, q_1, q_2, \tilde{K}) \to (p_i, q_{12}, K) \to (p_{i12}, K')$
- Each step: Sudakov decomposition along $\tilde{p}_i$, N^μ $p_i^\mu = z\tilde{p}_i^\mu, \quad N^\mu = \tilde{K}^\mu + (1-z)\tilde{p}_i^\mu$
- On-shell conditions preserved at every step
- **Crucial for PS matching:** parton showers build emissions through the **same iterated** $1 \to 2$ **kinematics** ⇝ recoil matches NLL-safe PS algorithms [Herren, et al. (2023); Assi, Höche (2023)]
- **Natural path toward NNLO+PS matching** [Campbell, et al. (2023)]

Phase-space factorization

- Iterated 2-body decomposition of the 4-body phase space [Byckling, Kajantie (1969)]

$$\mathrm{d}\Phi_4(-K;p_i,q_1,q_2,Q)=\mathrm{d}\Phi_2(-K;q_2,-M)\,\frac{\mathrm{d}M^2}{2\pi}\,\mathrm{d}\Phi_2(-M;q_1,-N)\,\frac{\mathrm{d}N^2}{2\pi}\,\mathrm{d}\Phi_2(-N;p_i,Q)$$

- Born remap relates $\Phi_2(-N;p_i,Q)$ to the underlying Born phase space [Assi, Höche (2023)]:
  $\dfrac{\mathrm{d}\Phi_2(-N;p_i,Q)}{\mathrm{d}\Phi_2(-\tilde{K};\tilde{p}_i,\tilde{Q})}=z^{1-2\varepsilon}$
- Two-emission phase-space element ($\boldsymbol{\zeta=z_1/(1-z)}$):

$$\mathrm{d}\Phi_{+2}^{(a)}(-\tilde{K},\tilde{p}_i;q_1,q_2)=\left(\frac{2\tilde{p}_i\tilde{K}}{16\pi^2}\right)^{2-2\varepsilon}\frac{(1-z)^{3-4\varepsilon}\big(z\zeta(1-\zeta)\big)^{1-2\varepsilon}}{\big((1-z)(1-\zeta\,\eta_{12})+\kappa\big)^{2-2\varepsilon}}\,\mathrm{d}z\,\mathrm{d}\zeta\,\frac{\mathrm{d}\Omega_{1,N}^{2-2\varepsilon}}{4\pi}\,\frac{\mathrm{d}\Omega_{2,N}^{2-2\varepsilon}}{4\pi}$$

- Singular limits cleanly identified in these variables:
  double-soft $\leftrightarrow z\to 1$, **single-soft** $\leftrightarrow \zeta\to 0$ or $\zeta\to 1$
- Phase-space volume in back-to-back kinematics ($\kappa=-1$) reproduces [Gehrmann-De Ridder, et al. (2004)]

Integrability of subtraction terms

[work in progress]

- After partial fractioning, the NLO single-gluon radiator becomes

$$-(\tilde{p}_i\tilde{K})\,\mathcal{S}^{(i)}_{i;k}(q_1;N) = \frac{1-z+\kappa}{(1-z)^2\,\eta_1}\,\frac{\eta_k}{\eta_{k1}+\eta_1}$$

Discontinuity at $\eta_k, \eta_1 \to 0$ is unphysical (hard partons i, k have $\eta_k > 0$)
⇝ Integrable, matches known analytic results [van Neerven (1986); Höche, et al. (2025)]

- Iterated partial fractioning at NNLO yields ($i = k$ case)

$$(\tilde{p}_i\tilde{K})^2\,\mathcal{S}_{i,i;l,m}(q_1,q_2;N) = \frac{\big((1-z)(1-\zeta\,\eta_{12})+\kappa\big)^2}{(1-z)^4\zeta^2(1-\zeta)^2\,\eta_1\,\eta_2}\,\mathcal{F}_{i,i;l,m}(q_1,q_2;N)$$

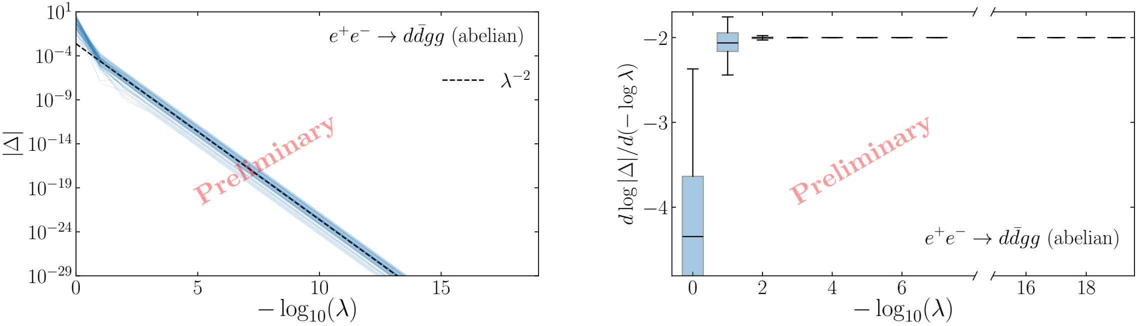
where $\mathcal{F}_{i,i;l,m} = \frac{\eta_l}{\eta_{l1}+\eta_1}\,\frac{\eta_m}{\eta_{m2}+\eta_2}$ is continuous over phase space
both emissions partial-fractioned to emitter i, *no* p_T ordering (that enters only the $S_1 \otimes S_1$ terms)

- Singular structure **fully factorized** into the denominator
⇝ Laurent series expansion in z, ζ + numerical Ω integration
- Same technique applies to non-abelian strongly-ordered, double-soft, and sub-leading radiators

Numerical tests

$e^+e^- \to 2$ jets

Numerical validation: convergence



- Relative remainder $\Delta = |\mathcal{M}_{RR}|/\mathcal{M}_4$ scaled toward double-soft limit
  Left: trajectories of $|\Delta|$, each $\to 0$ as $\lambda \to 0$ (along λ^2) Right: distribution of slopes over 10^5 points, stabilizing at -2
- **Slopes stabilize at -2:** the abelian $e^+e^- \to q\bar{q}gg$ double-soft limit is one of the few with λ^2 suppression (generic limits give λ^1, slope -1)

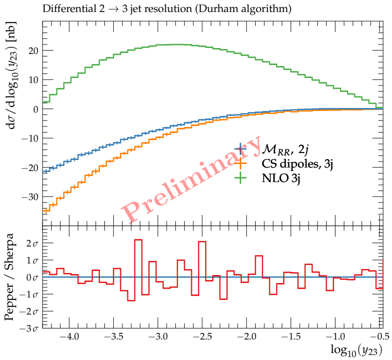
Numerical validation: all limits

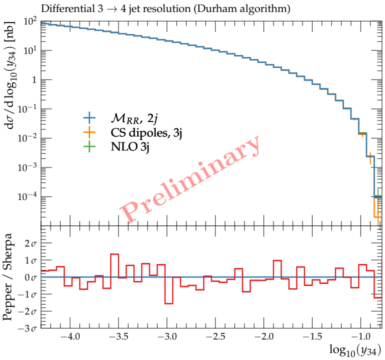
Slope of $\log|\Delta|$ vs. $\log(1/\lambda)$ at $-\log_{10}\lambda = 20$ for all limits [Bothmann, et al. (2024)]:

Limit	$q\bar{q}Q\bar{Q}$	$q\bar{q}q\bar{q}$	$q\bar{q}gg$ (ab)	$q\bar{q}gg$ (nab)
Single Soft	–	–	-1 ± 10^{-15}	-1 ± 10^{-16}
Double Collinear	-1 ± 10^{-14}	-1 ± 10^{-15}	-1 ± 10^{-14}	-1 ± 10^{-13}
Double Soft	-1 ± 10^{-15}	-1 ± 10^{-16}	-2 ± 10^{-15}	-1 ± 10^{-16}
Triple Collinear	-1 ± 10^{-12}	-1 ± 10^{-16}	-1 ± 10^{-14}	-1 ± 10^{-14}
Soft×Collinear	–	–	-2 ± 10^{-15}	–
Double Collinear×2	–	–	-2 ± 10^{-16}	–

- **Slope $= -1$ in (almost) all limits** (correct leading-power subtraction); -2 **only in a few abelian double-unresolved limits**, with $\lesssim 10^{-12}$ variation
- Independently implemented and cross-checked in Sherpa and Pepper [Bothmann, et al. (2024)]

First physics: $e^+e^- \to 2$ jets at NNLO (double-real)





- Numerically integrated double-real contribution to $e^+e^- \to q\bar{q}$ at NNLO:
  4.1×10^8 phase-space points $\Rightarrow$ 1σ-uncertainty of 372 fb, $< 10^{-5}$ relative to Born
- Durham jet rates $\log_{10}(y_{23})$ and $\log_{10}(y_{34})$
- Double-real building block in place (real, real-virtual and double-virtual to come):
  $\rightsquigarrow$ Demonstrates numerical performance; relevant for Tera-Z at FCC-ee

Conclusions & Outlook

Goal: NNLO + NNLL matching
for general-purpose MC generators

- **All 1→3 splitting functions decomposed into scalar radiators and pure remainders**
  - ⇝ Computed as full amplitudes; no kinematic limits required
  - ⇝ Complete: $q\bar{q}$, gg-abelian and gg-non-abelian topologies
- **Partial fractioning resolves overlapping singularities without sectorization**
  - ⇝ Unique branching topology for every counterterm
  - ⇝ Iterated NLO kinematics; natural path toward NNLO+PS matching
- **Implemented and validated in Sherpa and Pepper independently**
  - ⇝ All singular limits confirmed; first differential results for Durham jet rates
- **Outlook:**
  - ⇝ Integrated counterterms (analytic phase-space integration)
  - ⇝ Virtual corrections (real-virtual, double-virtual)
  - ⇝ Hadron-collider processes; NNLO+PS matching