MAX PLANCK INSTITUTE
FOR GRAVITATIONAL PHYSICS
(ALBERT EINSTEIN INSTITUTE)

MA∫CAMP

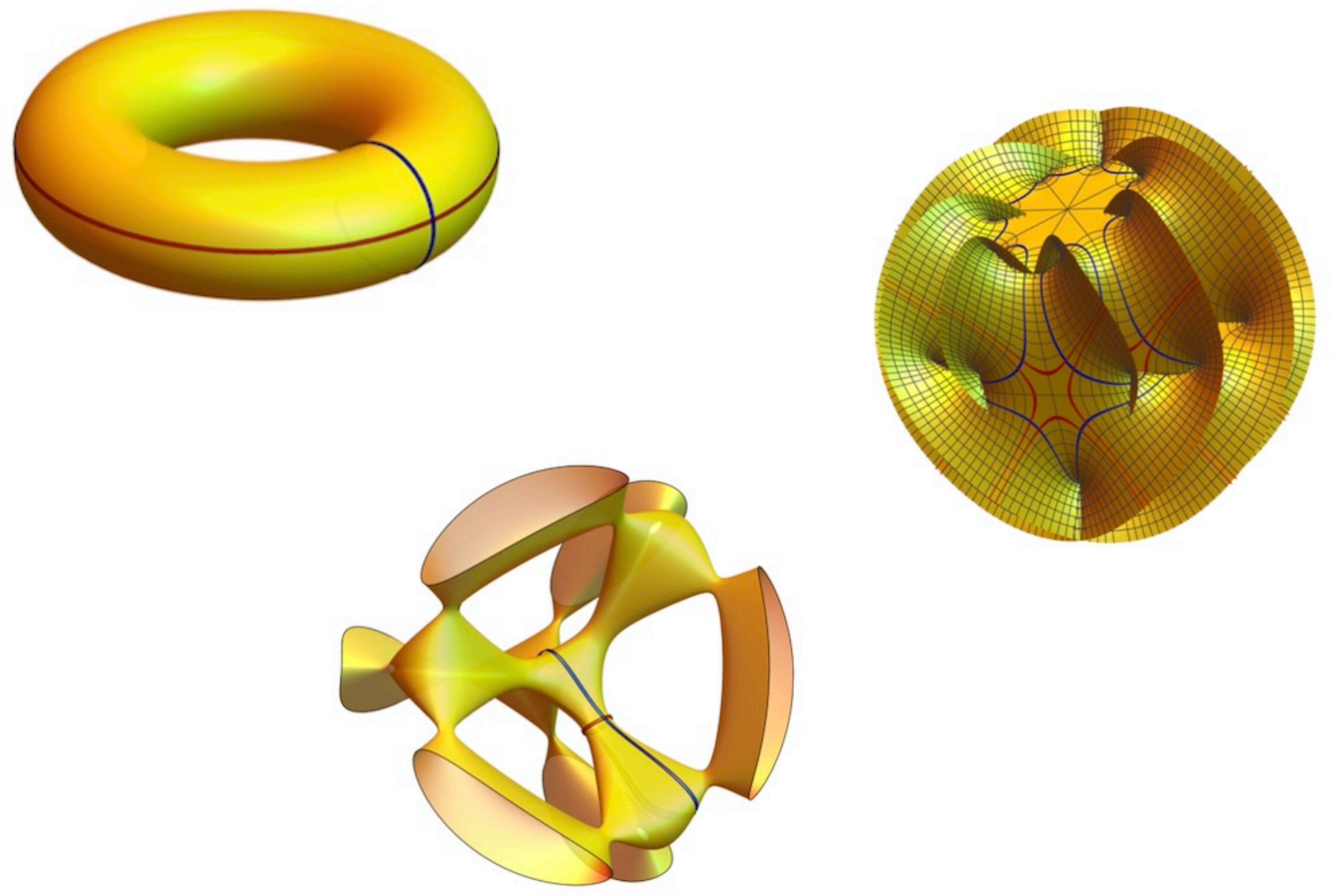
Funded by
the European Union

European Research Council
Established by the European Commission

Calabi-Yau Periods in Black Hole Scattering at the Fifth Post-Minkowskian Order

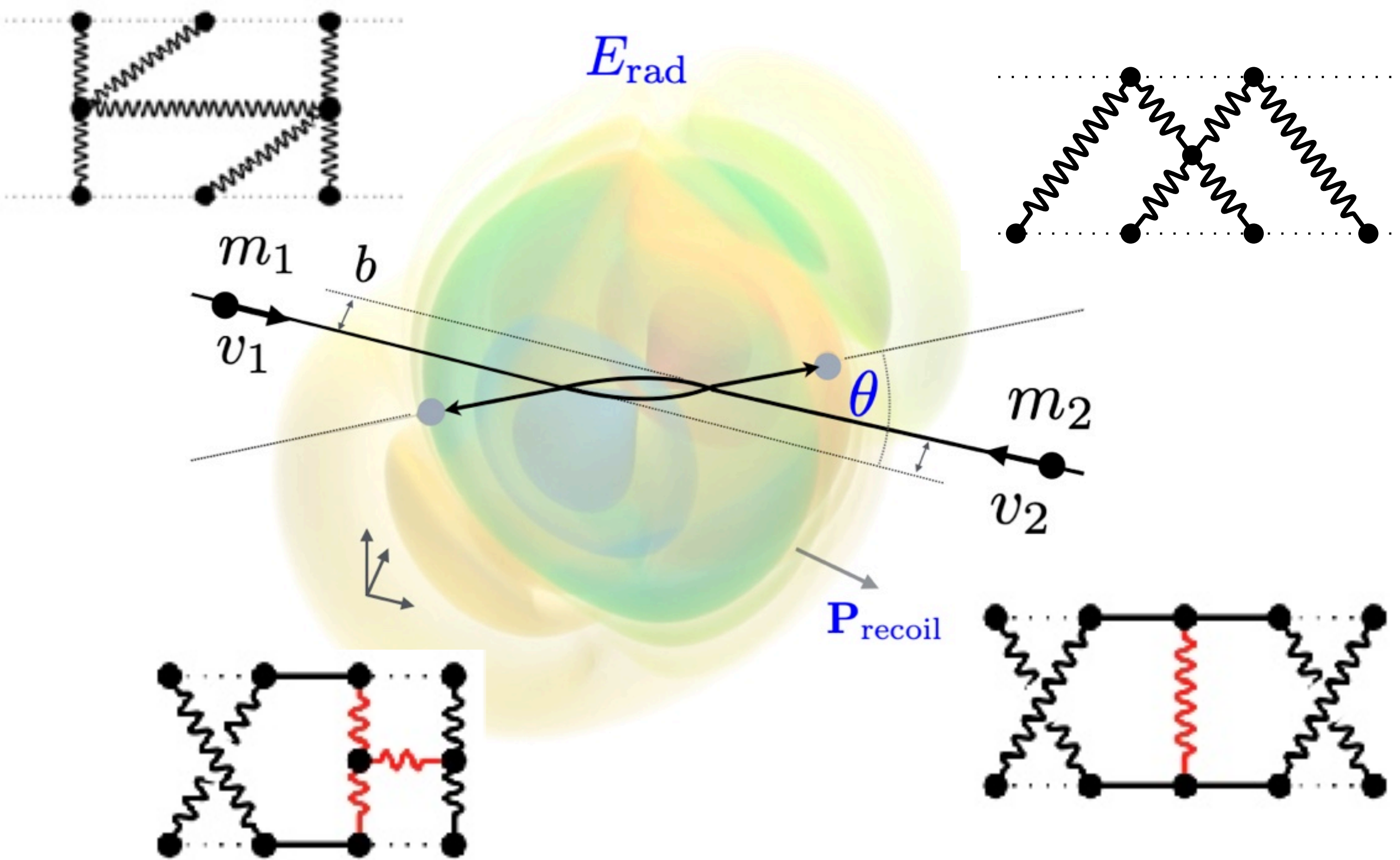
Christoph Nega

Calabi-Yau Periods



⟷

Black Hole Scattering



Collaborations

Joint work with:

Matthias Driesse, Felix Forner, Lennard Görges, Gustav Uhre Jakobsen, Albrecht Klemm, Cesare C. Mella, Gustav Mogull, Jan Plefka, Benjamin Sauer, Lorenzo Tancredi, Johann Usovitsch, and Fabian J. Wagner

Collaborations:

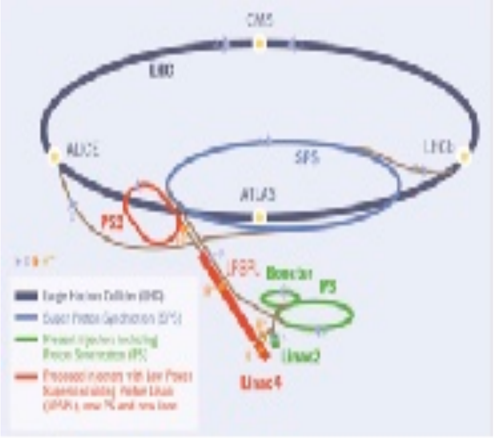
"Integrand Analysis, Leading Singularities and Canonical Bases beyond Polylogarithms" **[1]**,
"Aspects of canonical differential equations for Calabi-Yau geometries and beyond" **[2]**,
"On a procedure to derive ϵ-factorised differential equations beyond polylogarithms" **[3]**,

"Conservative Black Hole Scattering at Fifth Post-Minkowskian and Second Self-Force Order" **[4]**,
"Emergence of Calabi–Yau manifolds in high-precision black-hole scattering" **[5]**,
"Calabi-Yau periods for black hole scattering in classical general relativity" **[6]**

Motivation

- **Feynman integrals** are the backbone of many **perturbative computations,** even in **General Relativity** nowadays.
- **Current** and **future experiments** produce **highly precise data**. Thus, **high-precision predictions** are **essential** to compare this data with theory. This typically means **higher-loop calculations**.

Particle accelerators

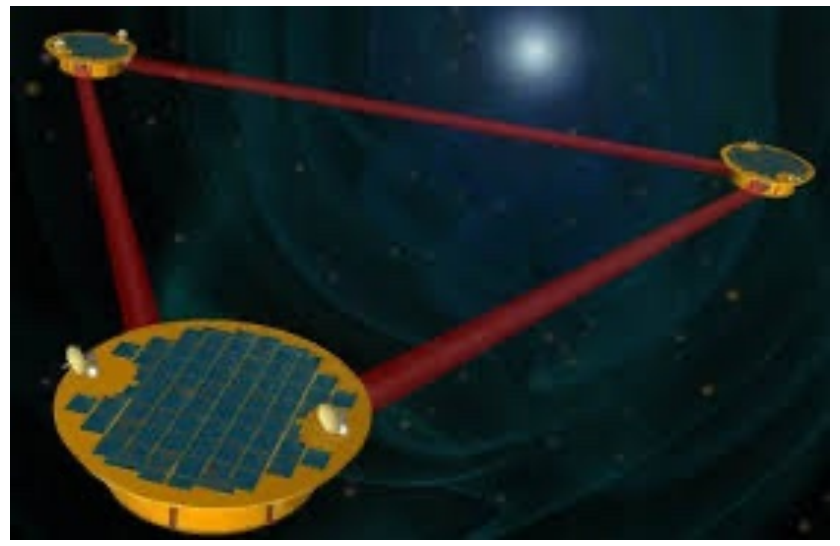


LHC©UNOSAT

FCC©dpa,SZ

Gravitational wave detectors

Virgo©Nicola Baldocchi/EGO



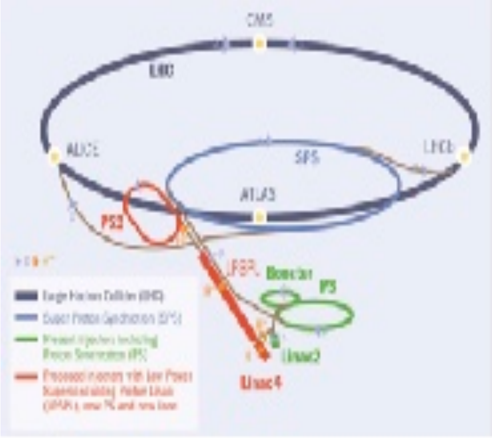
LISA©Nasa

Motivation

- **Feynman integrals** are the backbone of many **perturbative computations,** even in **General Relativity** nowadays.

- **Current** and **future experiments** produce **highly precise data**. Thus, **high-precision predictions** are **essential** to compare this data with theory. This typically means **higher-loop calculations**.

Particle accelerators

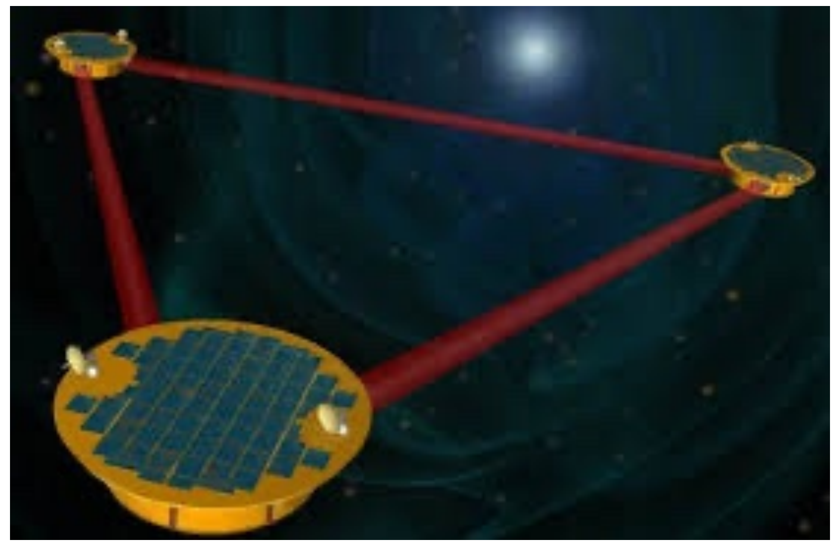


LHC©UNOSAT

FCC©dpa,SZ

Gravitational wave detectors

Virgo©Nicola Baldocchi/EGO



LISA©Nasa

- In recent years, we have discovered that **higher-loop computations** can reveal **new classes of functions beyond polylogarithms**. These functions are associated with **non-trivial geometries**, particularly **Calabi-Yau manifolds**.

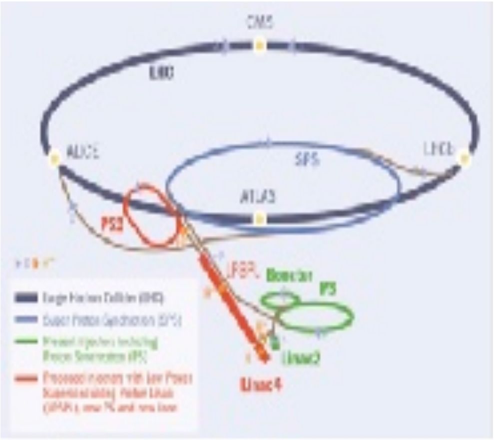
higher dimensions, growth in complexity, additional differential forms

sphere torus K3 Calabi-Yau

Motivation

- **Feynman integrals** are the backbone of many **perturbative computations,** even in **General Relativity** nowadays.
- **Current** and **future experiments** produce **highly precise data**. Thus, **high-precision predictions** are **essential** to compare this data with theory. This typically means **higher-loop calculations**.

Particle accelerators

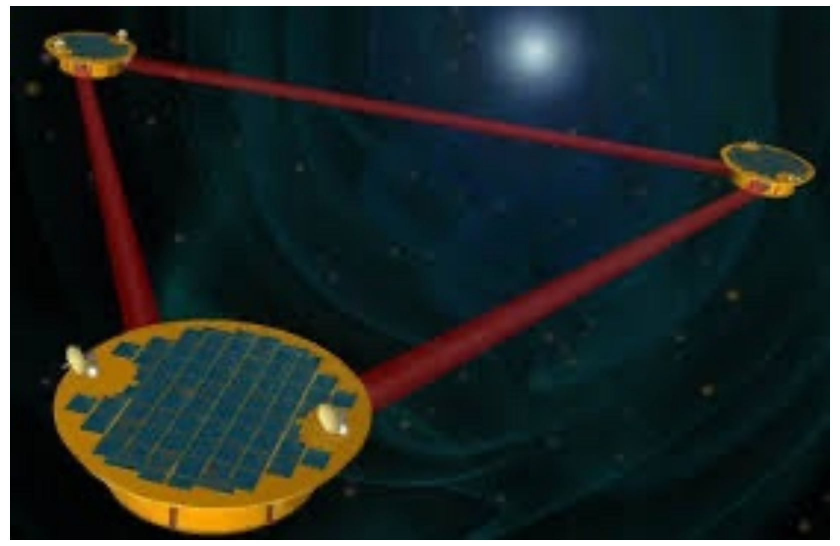


LHC©UNOSAT

FCC©dpa,SZ

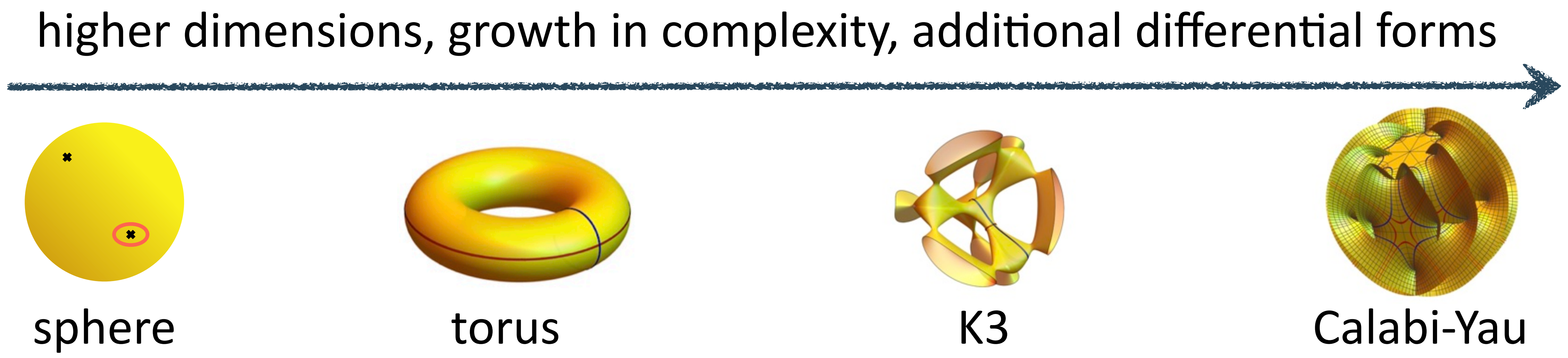
Gravitational wave detectors

Virgo©Nicola Baldocchi/EGO



LISA©Nasa

- In recent years, we have discovered that **higher-loop computations** can reveal **new classes of functions beyond polylogarithms**. These functions are associated with **non-trivial geometries**, particularly **Calabi-Yau manifolds**.



Today, we will discuss the scattering of two black holes at 5PM (4L) using Feynman integral techniques from particle physics and the mathematics of Calabi-Yau manifolds.

Table of Content

Calabi-Yau Manifolds

- **"Calabi-Yau geometries (CYs) are natural generalizations of elliptic curves to higher dimensions."**

Calabi-Yau Manifolds

- **"Calabi-Yau geometries (CYs) are natural generalizations of elliptic curves to higher dimensions."**

$$Y^2 = X(X-1)(X-\lambda)$$

elliptic curve/torus

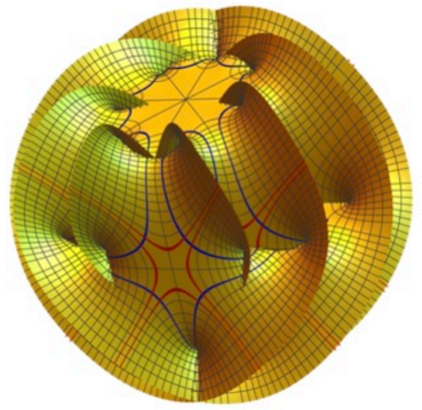
$$\int_{\Gamma_a} \frac{\mathrm{d}X}{Y}, \int_{\Gamma_b} \frac{\mathrm{d}X}{Y}$$

Elliptic integrals

$K(\lambda), K(1-\lambda)$

$\longrightarrow$

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 \\ - \psi X_0 X_1 X_2 X_3 X_4 = 0$$



CY n-fold

$$\varpi_i = \int_{\Gamma_i} \Omega$$

CY periods

Calabi-Yau Manifolds

- **"Calabi-Yau geometries (CYs) are natural generalizations of elliptic curves to higher dimensions."**

$$Y^2 = X(X-1)(X-\lambda)$$

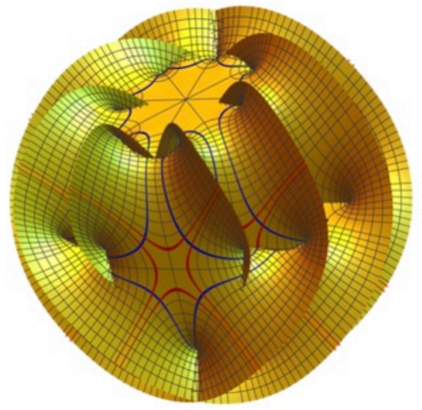
elliptic curve/torus

$$\int_{\Gamma_a} \frac{\mathrm{d}X}{Y}, \int_{\Gamma_b} \frac{\mathrm{d}X}{Y}$$

Elliptic integrals

$K(\lambda), K(1-\lambda)$

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 \\ - \psi X_0 X_1 X_2 X_3 X_4 = 0$$

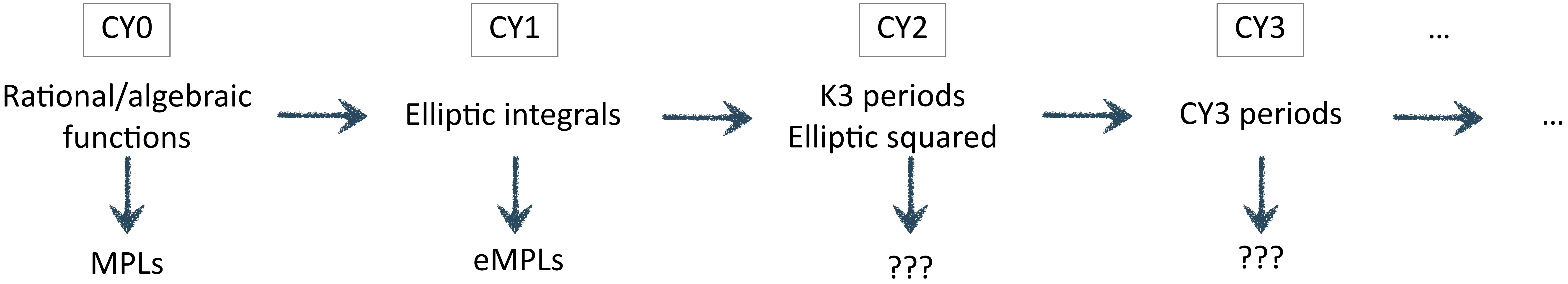


CY n-fold

$$\varpi_i = \int_{\Gamma_i} \Omega$$

CY periods

- **Calabi-Yau periods** generalize well-known classes of **special functions**.



CY0 | CY1 | CY2 | CY3 | ...

Rational/algebraic functions → Elliptic integrals → K3 periods Elliptic squared → CY3 periods → ...

Rational/algebraic functions ↓ MPLs

Elliptic integrals ↓ eMPLs

K3 periods Elliptic squared ↓ ???

CY3 periods ↓ ???

Calabi-Yau Manifolds

- **"Calabi-Yau geometries (CYs) are natural generalizations of elliptic curves to higher dimensions."**

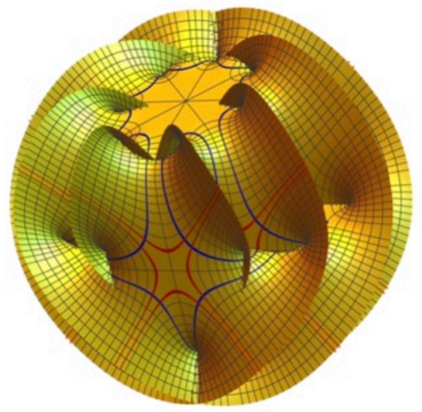
$$Y^2 = X(X-1)(X-\lambda)$$

elliptic curve/torus

$$\int_{\Gamma_a} \frac{\mathrm{d}X}{Y}, \int_{\Gamma_b} \frac{\mathrm{d}X}{Y}$$

Elliptic integrals
$K(\lambda), K(1-\lambda)$

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 \\ -\psi X_0 X_1 X_2 X_3 X_4 = 0$$

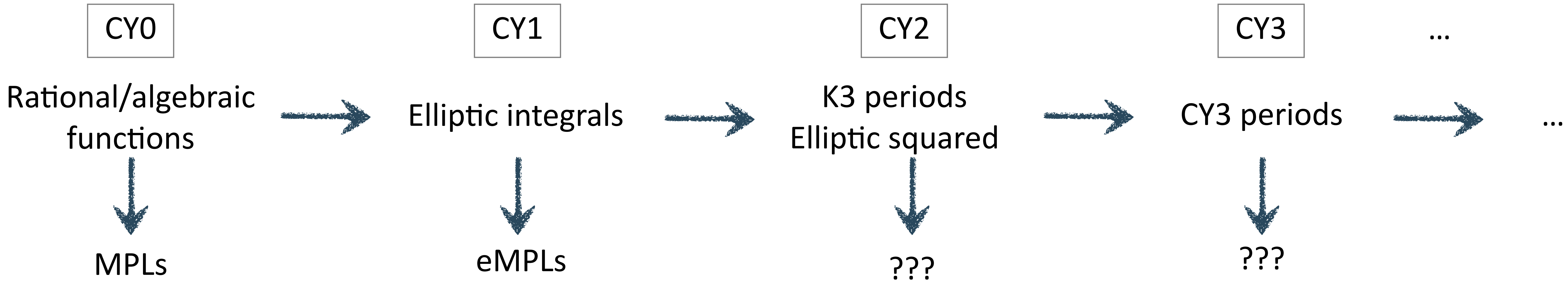


CY n-fold

$$\varpi_i = \int_{\Gamma_i} \Omega$$

CY periods

- **Calabi-Yau periods** generalize well-known classes of **special functions**.



CY0 | CY1 | CY2 | CY3 | ...

Rational/algebraic functions → Elliptic integrals → K3 periods Elliptic squared → CY3 periods → ...

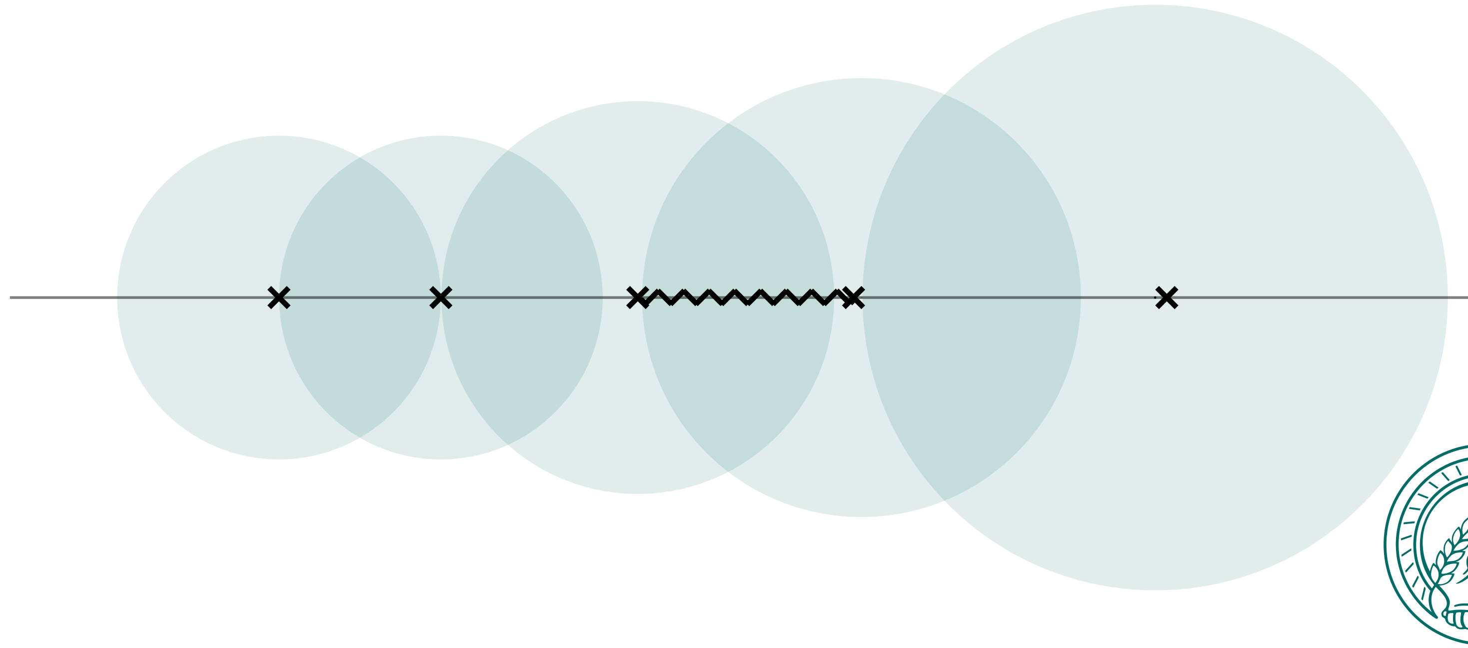
Rational/algebraic functions ↓ MPLs

Elliptic integrals ↓ eMPLs

K3 periods Elliptic squared ↓ ???

CY3 periods ↓ ???

- By using **differential equations** and **analytic continuation**, we can easily gain control over these new functions.



One only has to analytically continue $\log(z)$ and z^α.

Calabi-Yau Manifolds

- **"Calabi-Yau geometries (CYs) are natural generalizations of elliptic curves to higher dimensions."**

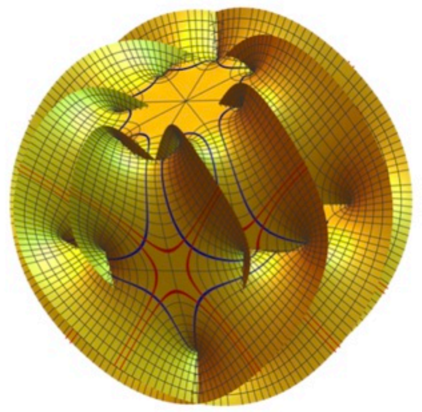
$$Y^2 = X(X-1)(X-\lambda)$$

elliptic curve/torus

$$\int_{\Gamma_a} \frac{\mathrm{d}X}{Y}, \int_{\Gamma_b} \frac{\mathrm{d}X}{Y}$$

Elliptic integrals
$K(\lambda), K(1-\lambda)$

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 \\ - \psi X_0 X_1 X_2 X_3 X_4 = 0$$

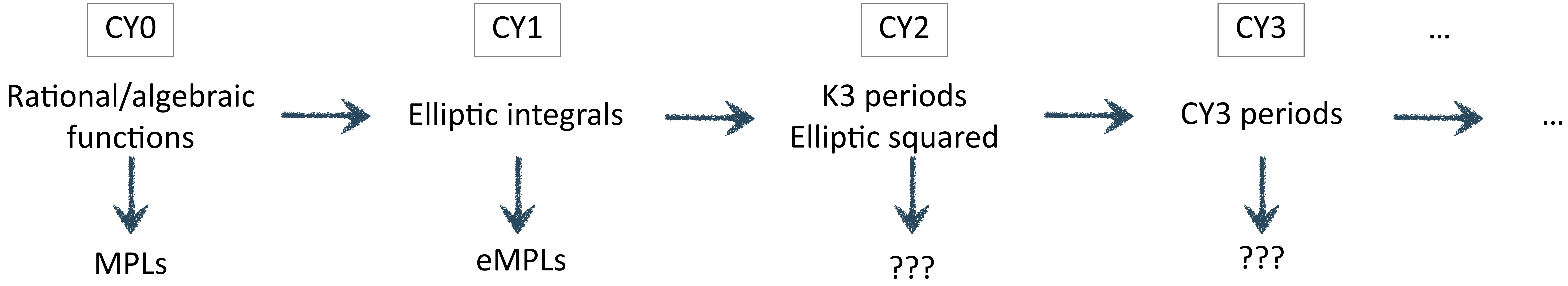


CY n-fold

$$\varpi_i = \int_{\Gamma_i} \Omega$$

CY periods

- **Calabi-Yau periods** generalize well-known classes of **special functions**.



CY0

Rational/algebraic functions

MPLs

CY1

Elliptic integrals

eMPLs

CY2

K3 periods
Elliptic squared

???

CY3

CY3 periods

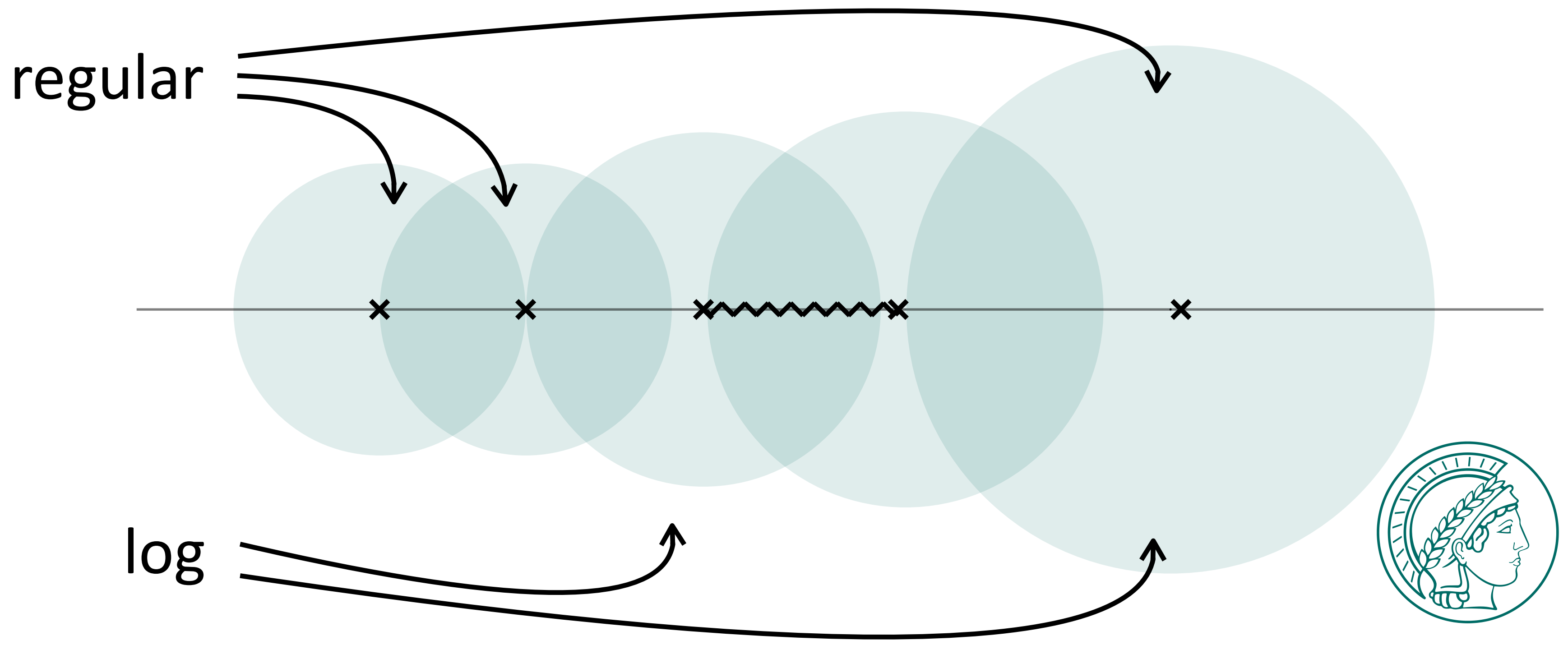
???

...

...

- By using **differential equations** and **analytic continuation**, we can easily gain control over these new functions.

One only has to analytically continue $\log(z)$ and z^α.



Canonical Integrals

- We aim to select **master integrals** that reveal the **analytic** and **transcendental structure** of **scattering amplitudes** while remaining **easy to compute**. To compute them, we use the **differential equations method**.

[Arkani-Hamed et al., Henn]

Canonical Integrals

- We aim to select **master integrals** that reveal the **analytic** and **transcendental structure** of **scattering amplitudes** while remaining **easy to compute**. To compute them, we use the **differential equations method**.

[Arkani-Hamed et al., Henn]

→ **Canonical Integrals**

Canonical Integrals

- We aim to select **master integrals** that reveal the **analytic** and **transcendental structure** of **scattering amplitudes** while remaining **easy to compute**. To compute them, we use the **differential equations method**. [Arkani-Hamed et al., Henn]

→ **Canonical Integrals**

- **Canonical integrals** satisfy ϵ**-factorized differential equations**.

ϵ**-factorized**

Uniform transcendental

$$\frac{\partial}{\partial z_i}\underline{I} = B(\underline{z};\epsilon)\underline{I} \xrightarrow[\text{integrals}]{\text{canonical}} \frac{\partial}{\partial z_i}\underline{J} = \epsilon A(\underline{z})\underline{J}$$

with solution

$$\underline{J} = \mathbb{P}\exp\left(\overbrace{\epsilon\int_\gamma A}\right)\underline{J}_0$$

[Kotikov, Gehrmann and Remiddi]

[Henn]

[Chen]

Iterated integrals

Related to **geometry**

Canonical Integrals

- We aim to select **master integrals** that reveal the **analytic** and **transcendental structure** of **scattering amplitudes** while remaining **easy to compute**. To compute them, we use the **differential equations method**.

[Arkani-Hamed et al., Henn]

→ **Canonical Integrals**

- **Canonical integrals** satisfy ϵ**-factorized differential equations**.

ϵ-**factorized**

Uniform transcendental

$$\frac{\partial}{\partial z_i}\underline{I} = B(\underline{z};\epsilon)\underline{I} \xrightarrow[\text{integrals}]{\text{canonical}} \frac{\partial}{\partial z_i}\underline{J} = \epsilon A(\underline{z})\underline{J}$$

with solution

$$\underline{J} = \mathbb{P}\exp\left(\overbrace{\epsilon\int_\gamma A}\right)\underline{J}_0$$

[Kotikov, Gehrmann and Remiddi]

[Henn]

[Chen]

Iterated integrals

Related to **geometry**

- To construct canonical integrals, it is essential to **identify integrals** related to the **three standard types of forms** given on a geometry. This is conveniently done by **integrand analysis**.

[3, Cesare's talk]

Canonical Integrals

- We aim to select **master integrals** that reveal the **analytic** and **transcendental structure** of **scattering amplitudes** while remaining **easy to compute**. To compute them, we use the **differential equations method**. [Arkani-Hamed et al., Henn]

→ **Canonical Integrals**

- **Canonical integrals** satisfy ϵ**-factorized differential equations**.

ϵ**-factorized**

Uniform transcendental

$$\frac{\partial}{\partial z_i}\underline{I} = B(\underline{z};\epsilon)\underline{I} \xrightarrow[\text{integrals}]{\text{canonical}} \frac{\partial}{\partial z_i}\underline{J} = \epsilon A(\underline{z})\underline{J}$$

with solution

$$\underline{J} = \mathbb{P}\exp\left(\overbrace{\epsilon\int_{\gamma} A}\right)\underline{J}_0$$

[Kotikov, Gehrmann and Remiddi]

[Henn]

[Chen]

Iterated integrals

Related to **geometry**

- To construct canonical integrals, it is essential to **identify integrals** related to the **three standard types of forms** given on a geometry. This is conveniently done by **integrand analysis**. [3, Cesare's talk]

1st-kind
(no pole)

$$\frac{\mathrm{d}X}{Y}$$

↓

periods

2nd-kind
(double pole)

$$\partial\frac{\mathrm{d}X}{Y}$$

↓

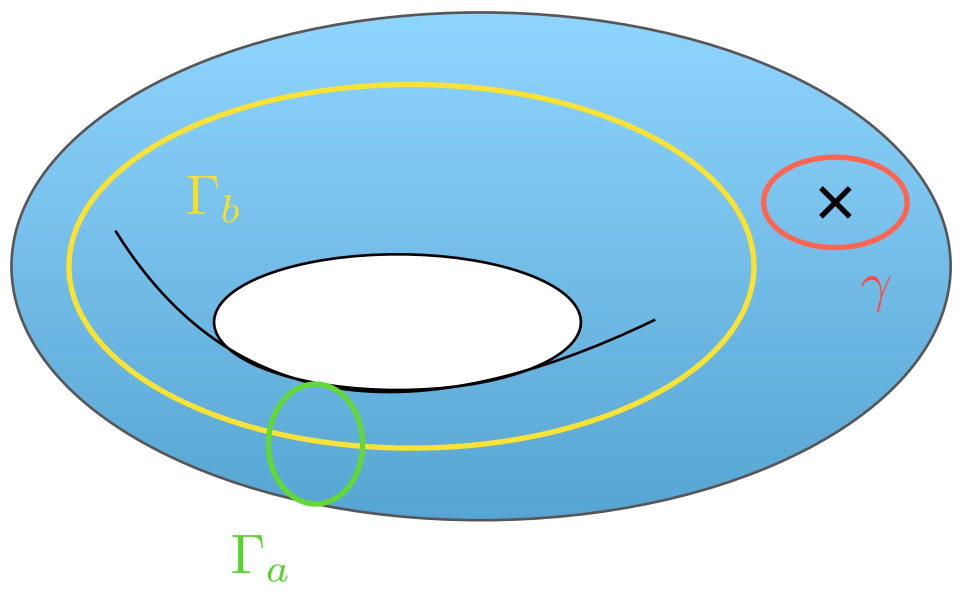
quasi-periods

3rd-kind
(single pole)

$$\frac{X\,\mathrm{d}X}{Y}$$

↓

third-kind periods,
G-functions



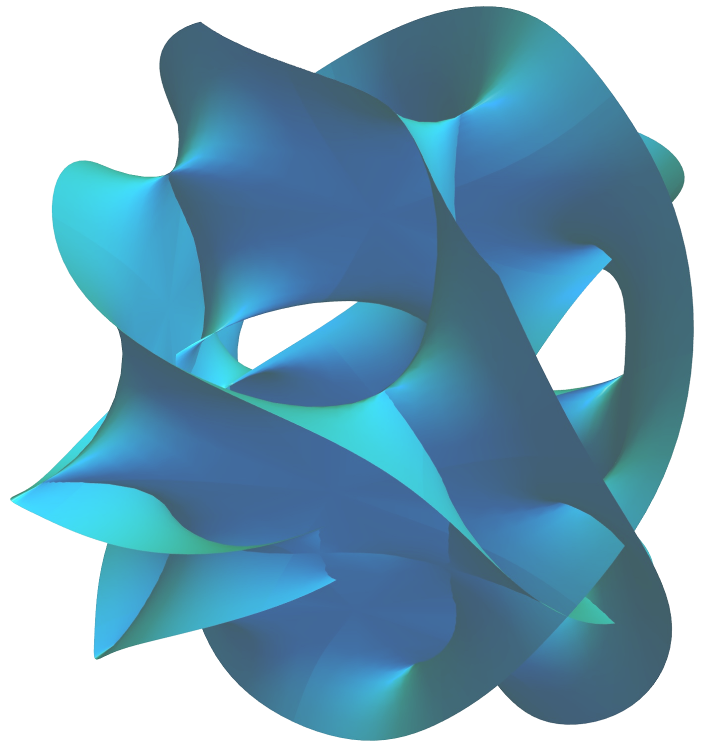
Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:

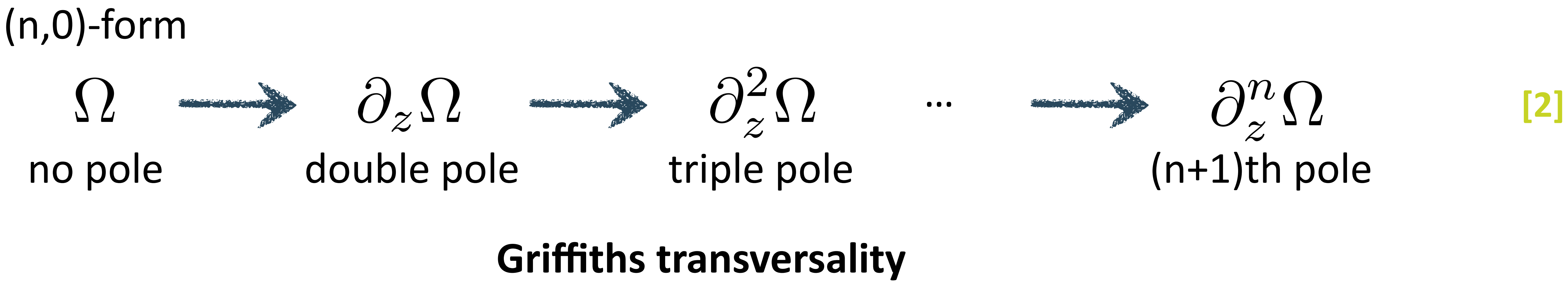
[2]

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.



(n,0)-form

$$\begin{tikzcd} \Omega \arrow[r] & \partial_z \Omega \arrow[r] & \partial_z^2 \Omega & \cdots \arrow[r] & \partial_z^n \Omega \end{tikzcd}$$

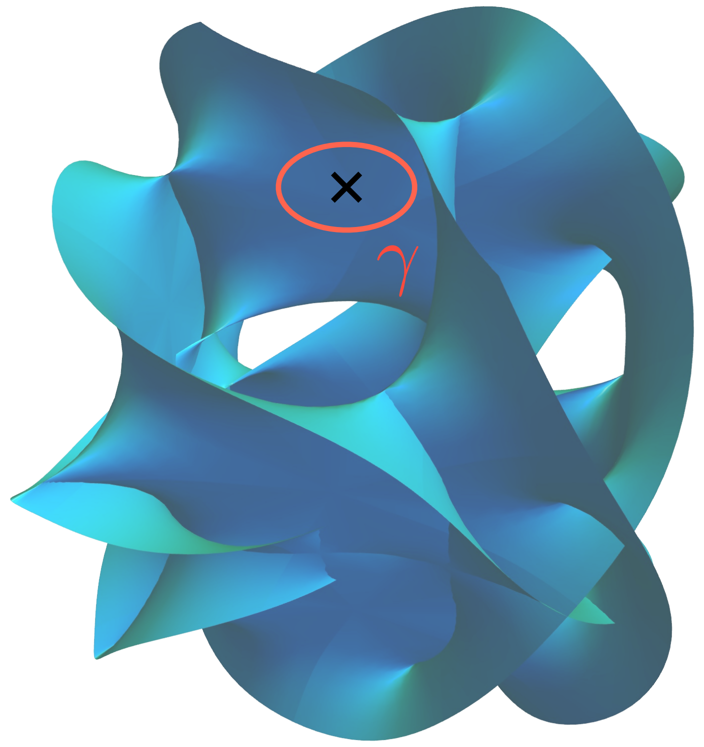
no pole — double pole — triple pole — (n+1)th pole

[2]

Griffiths transversality

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.

(n,0)-form

$$\underset{\text{no pole}}{\Omega} \longrightarrow \underset{\text{double pole}}{\partial_z \Omega} \longrightarrow \underset{\text{triple pole}}{\partial_z^2 \Omega} \quad \cdots \quad \longrightarrow \underset{\text{(n+1)th pole}}{\partial_z^n \Omega}$$

[2]

Griffiths transversality

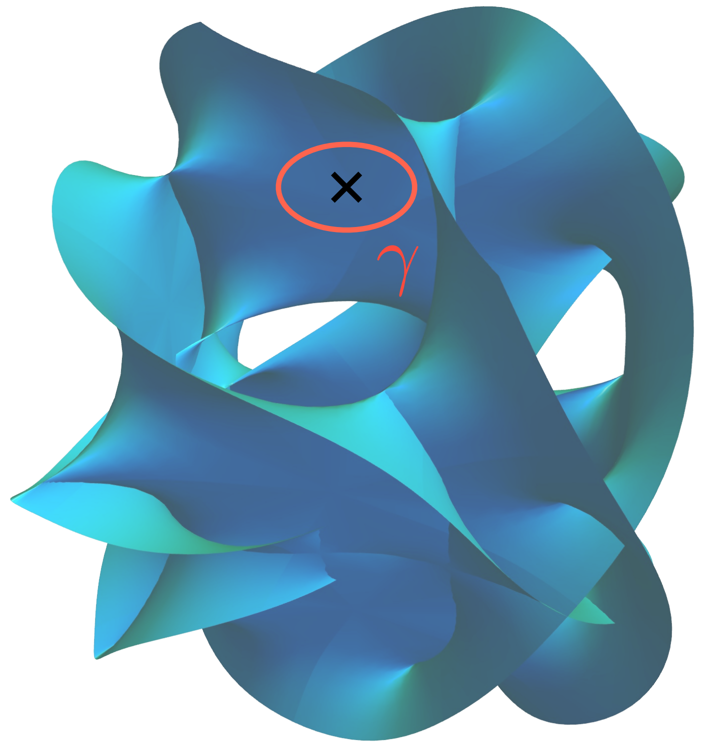
- The generalization of differential forms with **additional residues** (third kind forms) is very complicated and **not fully understood** on a CY n-fold.

$\dfrac{\Omega}{X - x_0}$ just a single pole

λ single <u>and</u> higher poles

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.

(n,0)-form

$$\underset{\text{no pole}}{\Omega} \longrightarrow \underset{\text{double pole}}{\partial_z\Omega} \longrightarrow \underset{\text{triple pole}}{\partial_z^2\Omega} \quad \cdots \longrightarrow \underset{\text{(n+1)th pole}}{\partial_z^n\Omega}$$

[2]

Griffiths transversality

- The generalization of differential forms with **additional residues** (third kind forms) is very complicated and **not fully understood** on a CY n-fold.

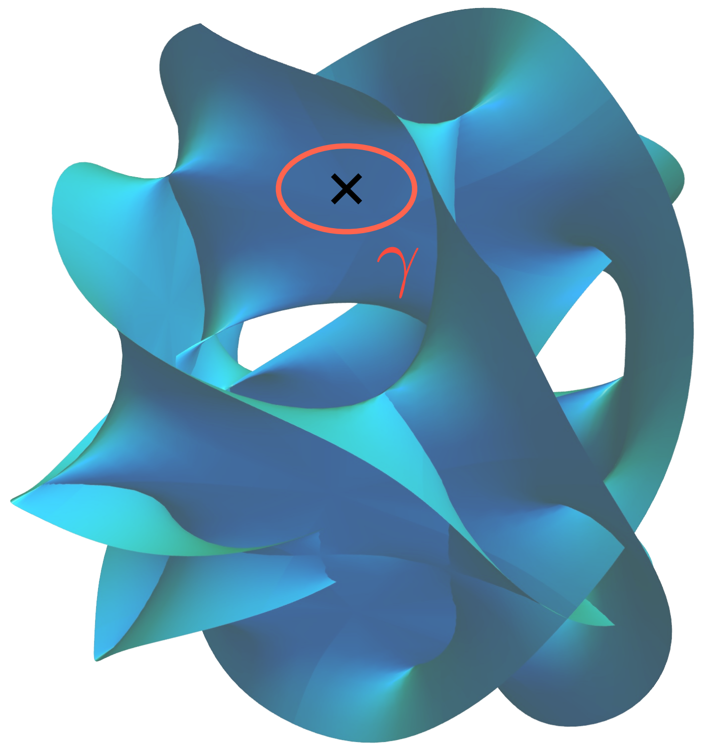
$\frac{\Omega}{X - x_0}$ just a single pole

λ single <u>and</u> higher poles

- Some **general comments** about **our method** [1-3] to compute canonical integrals:

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.

(n,0)-form

Ω	$\longrightarrow$	$\partial_z\Omega$	$\longrightarrow$	$\partial_z^2\Omega$	…	$\longrightarrow$	$\partial_z^n\Omega$	[2]
no pole		double pole		triple pole			(n+1)th pole	

Griffiths transversality

- The generalization of differential forms with **additional residues** (third kind forms) is very complicated and **not fully understood** on a CY n-fold.

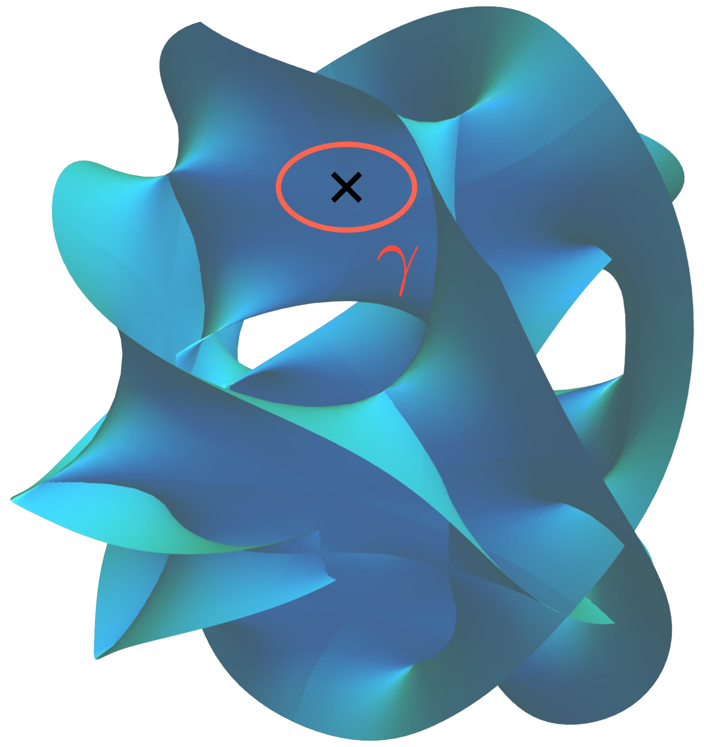
$\frac{\Omega}{X-x_0}$ just a single pole

λ single and higher poles

- Some **general comments** about **our method** [1-3] to compute canonical integrals:
  - Our approach is applicable to **any geometry**. Through integrand analysis, we find **representatives** for the **three different types of forms**. Therefore, we say that our **basis reflects** the **underlying geometry** (mixed Hodge structure).

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.

(n,0)-form

$$\Omega \longrightarrow \partial_z\Omega \longrightarrow \partial_z^2\Omega \quad \cdots \longrightarrow \partial_z^n\Omega$$

no pole, double pole, triple pole, (n+1)th pole [2]

Griffiths transversality

- The generalization of differential forms with **additional residues** (third kind forms) is very complicated and **not fully understood** on a CY n-fold.

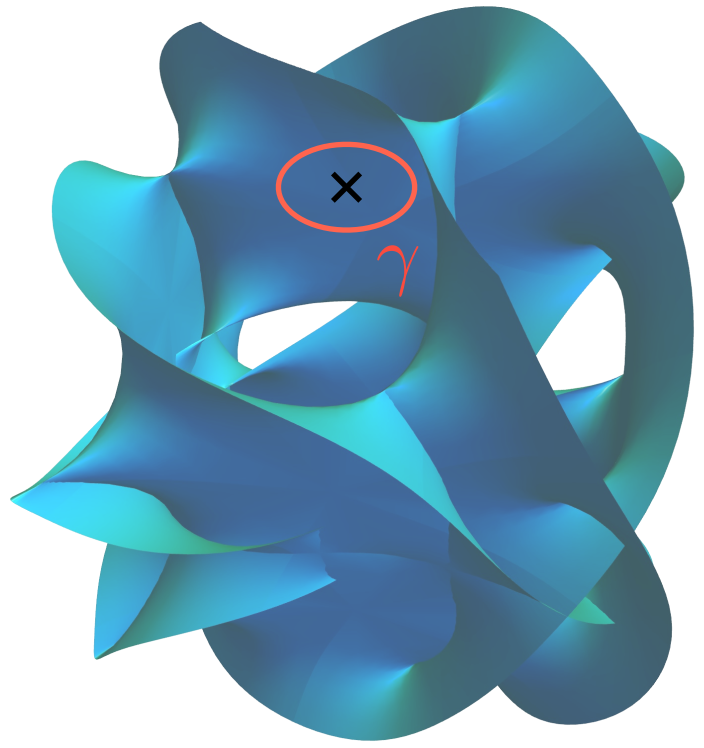
$\frac{\Omega}{X - x_0}$ just a single pole

λ single <u>and</u> higher poles

- Some **general comments** about **our method** [1-3] to compute canonical integrals:
  - Our approach is applicable to **any geometry**. Through integrand analysis, we find **representatives** for the **three different types of forms**. Therefore, we say that our **basis reflects** the **underlying geometry** (mixed Hodge structure).
  - Due to the appearance of **forms with higher poles**, we have to extend the **integrand analysis beyond** $\epsilon = 0$. Taking derivatives of forms with at most single poles gives us control over the **maximal number of corrections** needed to make them canonical.

Canonical Integrals and Calabi-Yaus

- **Simple generalization** for **Calabi-Yau manifolds**:



- There is **no natural representation** for other **n-forms** on the Calabi-Yau. But we can **use derivatives** to generate the whole cohomology group.

(n,0)-form

$$\Omega \longrightarrow \partial_z\Omega \longrightarrow \partial_z^2\Omega \quad \cdots \quad \longrightarrow \partial_z^n\Omega$$

no pole — double pole — triple pole — (n+1)th pole

[2]

Griffiths transversality

- The generalization of differential forms with **additional residues** (third kind forms) is very complicated and **not fully understood** on a CY n-fold.

$\frac{\Omega}{X - x_0}$ just a single pole

λ single and higher poles

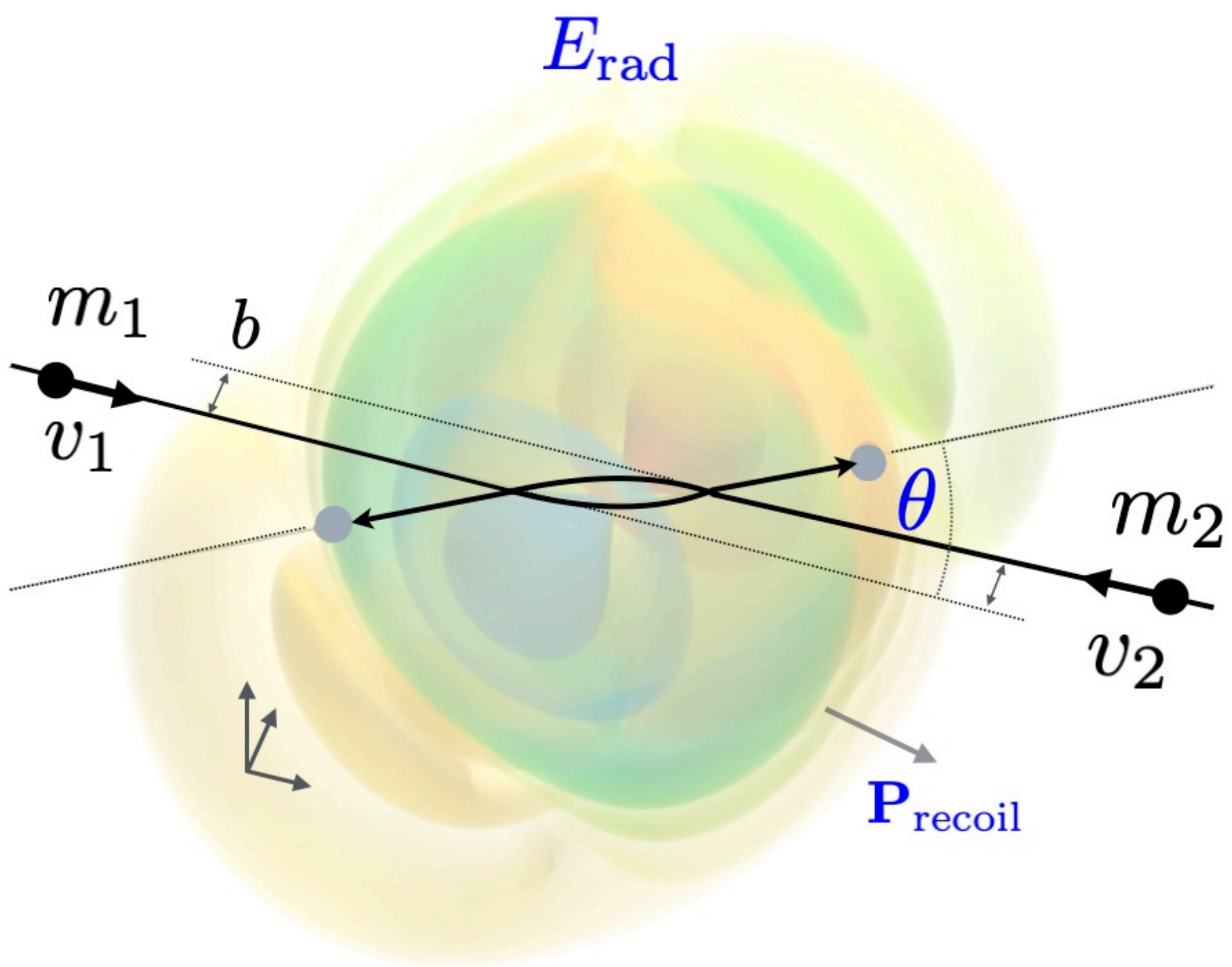
- Some **general comments** about **our method** [1-3] to compute canonical integrals:
  - Our approach is applicable to **any geometry**. Through integrand analysis, we find **representatives** for the **three different types of forms**. Therefore, we say that our **basis reflects** the **underlying geometry** (mixed Hodge structure).
  - Due to the appearance of **forms with higher poles**, we have to extend the **integrand analysis beyond** $\epsilon = 0$. Taking derivatives of forms with at most single poles gives us control over the **maximal number of corrections** needed to make them canonical.
  - **New transcendental functions** that we call **G-functions** are just **additional leading singularities** that have to be **removed** to obtain a canonical integral. If they appear already for $\epsilon = 0$, they are **generalized third-kind integrals**. Otherwise, they describe LSs at higher orders in ϵ.

Black Hole Scattering

- Through the **Worldline QFT** approach, we can model the scattering of **black holes**.

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^{\mu} \dot{x}_i^{\nu}\right] - \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R(g) + S_{\mathrm{g.f.}}$$

[Mogull, Plefka, Steinhoff]

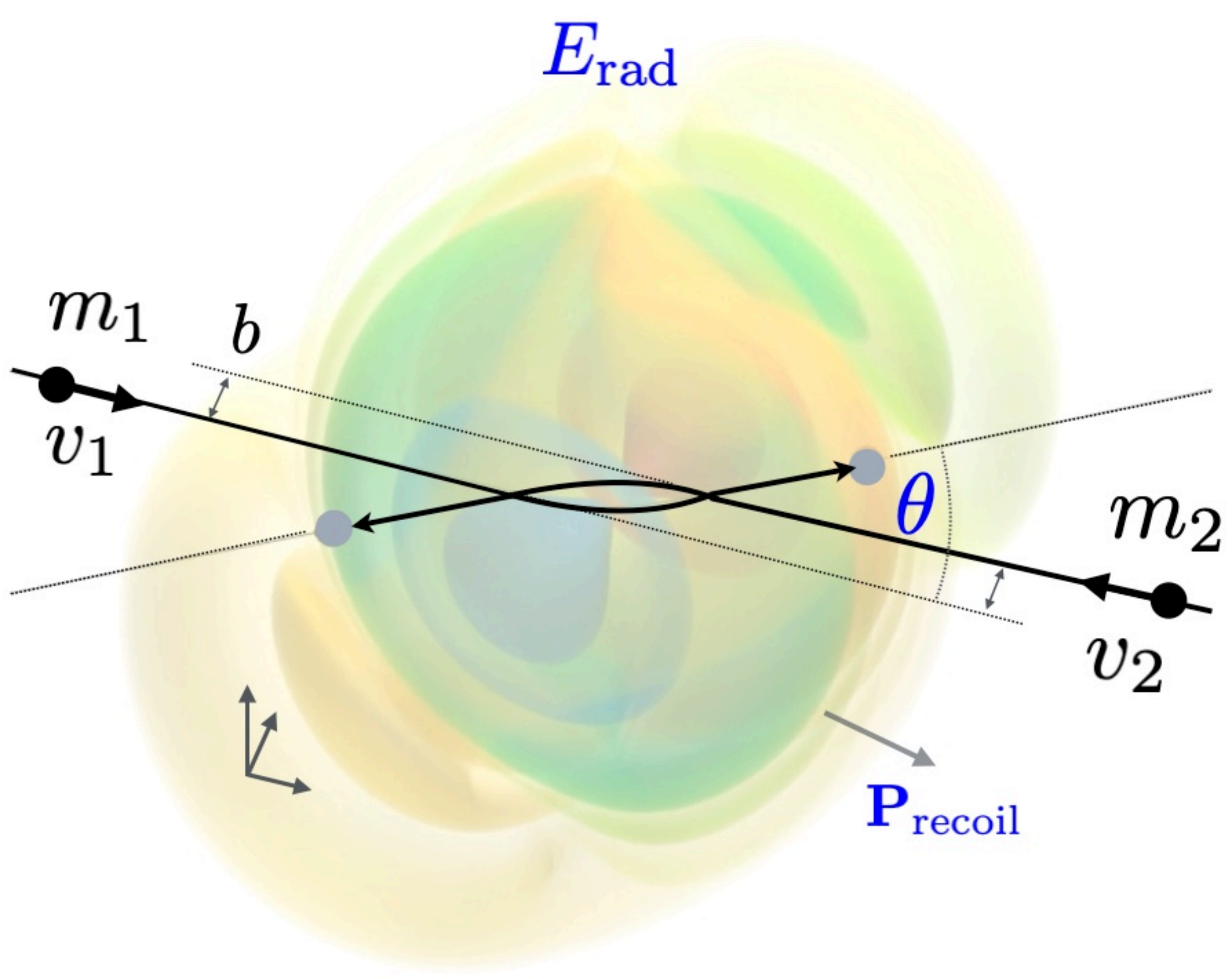


Black Hole Scattering

- Through the **Worldline QFT** approach, we can model the scattering of **black holes**.

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu\right] - \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R(g) + S_{\text{g.f.}}$$

[Mogull, Plefka, Steinhoff]



- **Post-Minkoskian** approximation: We expand in Newton's constant G.

deflection

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu$$

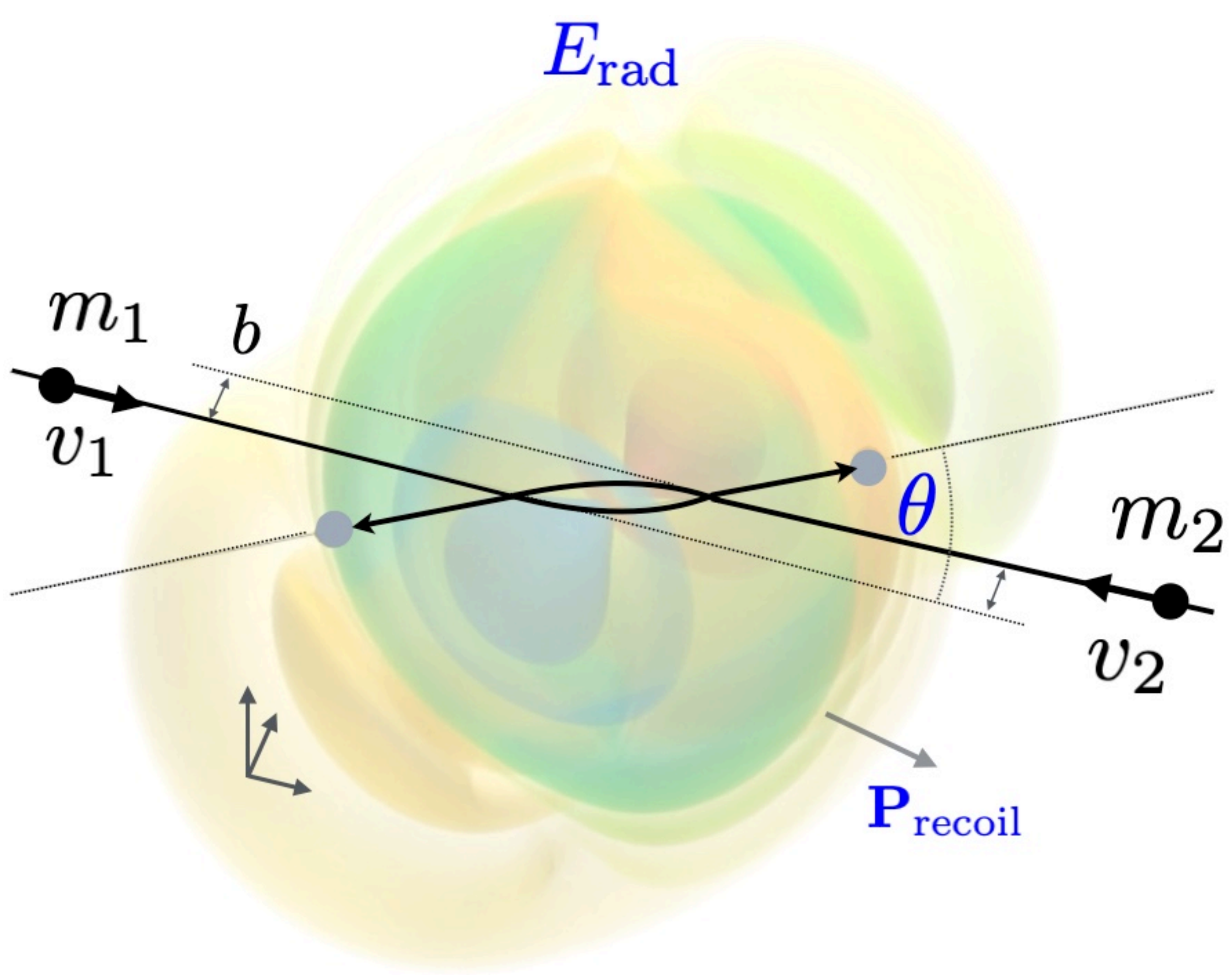
deflection angle θ ← → radiated energy E_{rad}

Black Hole Scattering

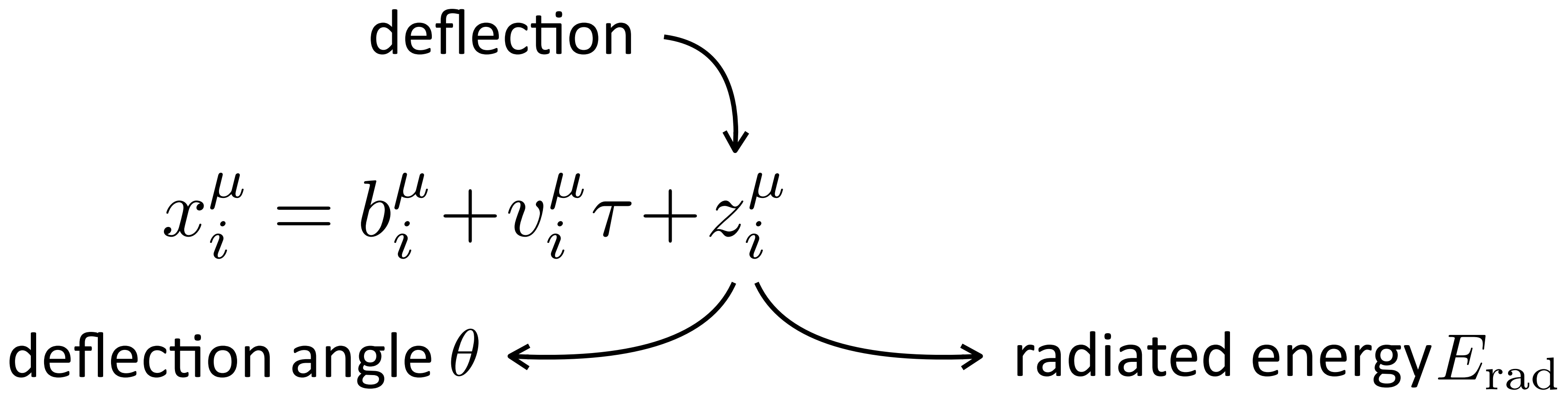
- Through the **Worldline QFT** approach, we can model the scattering of **black holes**.

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu\right] - \frac{1}{16\pi G}\int \mathrm{d}^4x \sqrt{-g}R(g) + S_{\text{g.f.}}$$

[Mogull, Plefka, Steinhoff]



- **Post-Minkoskian** approximation: We expand in Newton's constant G.



deflection

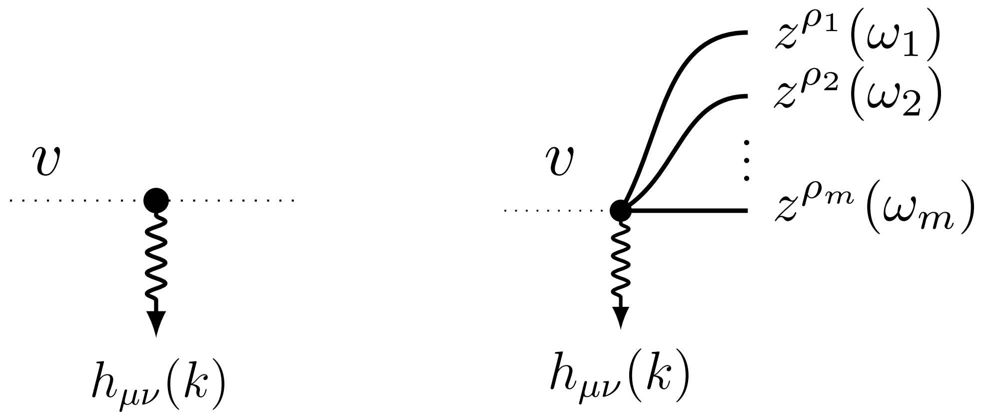
$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu$$

deflection angle θ

radiated energy E_{rad}

  - The advantage of the **WQFT** approach is that **tree-level one-point functions solve the classical e.o.m.**
  - Due to the **WQFT Feynman rules tree-level** functions still contain **"loop"-integrals**.

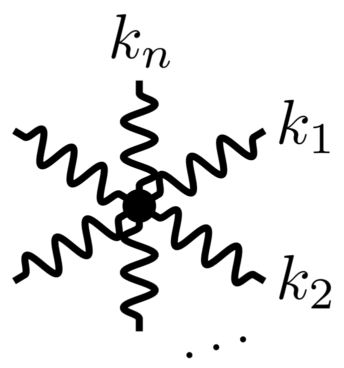
world line vertices



only energy conservation

$$\sim \delta\Big(v \cdot k + \sum_i \omega_i\Big)$$

graviton vertices



full four-momentum conservation

$$\sim \delta^{(4)}\Big(\sum_i k_i\Big)$$

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^{\mu} &= p_i^{\mu}(\tau \to +\infty) - p_i^{\mu}(\tau \to -\infty)\\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} +\end{aligned}$$

$$\Delta p_1^{(5)\mu} = m_1 m_2\Big(m_2^4\Delta p^{(5)\mu}_{0\mathrm{SF}} + m_1 m_2^3\Delta p^{(5)\mu}_{1\mathrm{SF}} + m_1^2 m_2^2\Delta p^{(5)\mu}_{2\mathrm{SF}} + m_1^3 m_2\Delta p^{(5)\mu}_{\overline{1\mathrm{SF}}} + m_1^4\Delta p^{(5)\mu}_{\overline{0\mathrm{SF}}}\Big)$$

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty) \\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} + \end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2\Big(m_2^4\Delta p_{0\text{SF}}^{(5)\mu} + m_1 m_2^3\Delta p_{1\text{SF}}^{(5)\mu} + m_1^2 m_2^2\Delta p_{2\text{SF}}^{(5)\mu} + m_1^3 m_2\Delta p_{\overline{1\text{SF}}}^{(5)\mu} + m_1^4\Delta p_{\overline{0\text{SF}}}^{(5)\mu}\Big)$$

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty)\\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} + \end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2 \Big(m_2^4 \Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3 \Delta p_{1\mathrm{SF}}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2 \Delta p_{\overline{1\mathrm{SF}}}^{(5)\mu} + m_1^4 \Delta p_{\overline{0\mathrm{SF}}}^{(5)\mu} \Big)$$

0SF: 63 diagrams
(simple)

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty)\\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} +\end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2\Big(m_2^4\Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3\Delta p_{1\mathrm{SF}}^{(5)\mu} + m_1^2 m_2^2\Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2\Delta p_{\overline{1\mathrm{SF}}}^{(5)\mu} + m_1^4\Delta p_{\overline{0\mathrm{SF}}}^{(5)\mu}\Big)$$

0SF: 63 diagrams
(simple)

1SF: 363 diagrams

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty) \\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} + \end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2\Big(m_2^4\Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3\Delta p_{1\mathrm{SF}}^{(5)\mu} + m_1^2 m_2^2\Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2\Delta p_{\overline{1\mathrm{SF}}}^{(5)\mu} + m_1^4\Delta p_{\overline{0\mathrm{SF}}}^{(5)\mu}\Big)$$

0SF: 63 diagrams
(simple)

1SF: 363 diagrams

2SF: 641 diagrams

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty)\\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} +\end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2\Big(m_2^4\Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3\Delta p_{1\mathrm{SF}}^{(5)\mu} + m_1^2 m_2^2\Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2\Delta p_{\overline{1\mathrm{SF}}}^{(5)\mu} + m_1^4\Delta p_{\overline{0\mathrm{SF}}}^{(5)\mu}\Big)$$

0SF: 63 diagrams
(simple)

1SF: 363 diagrams

2SF: 641 diagrams

- The computation can be further divided into two types: **dissipative** and **conservative**.

Dissipative:

$$\Delta p_1^\mu + \Delta p_2^\mu = P_{\mathrm{rad}}^\mu \neq 0$$

retarded worldline propagators (in-in)
retarded graviton propagators (in-in)

physical result

Conservative:

$$\Delta p_1^\mu + \Delta p_2^\mu = 0$$

retarded worldline propagators (in-in)
Feynman graviton propagators (in-out)
only take the real and even velocity part

ongoing debates on how to properly
define the conservative part

Black Hole Scattering

- We want to compute the **change of momentum** at 5PM ($\mathcal{O}(G^5)$).

$$\begin{aligned}\Delta p_i^\mu &= p_i^\mu(\tau \to +\infty) - p_i^\mu(\tau \to -\infty)\\ &= G\Delta p_i^{(1)\mu} + G^2\Delta p_i^{(2)\mu} + \cdots + G^5\Delta p_i^{(5)\mu} +\end{aligned}$$

self-force expansion

$$\Delta p_1^{(5)\mu} = m_1 m_2\left(m_2^4 \Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3 \Delta p_{1\mathrm{SF}}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2 \Delta p_{\overline{1\mathrm{SF}}}^{(5)\mu} + m_1^4 \Delta p_{\overline{0\mathrm{SF}}}^{(5)\mu}\right)$$

0SF: 63 diagrams
(simple)

1SF: 363 diagrams

2SF: 641 diagrams

- The computation can be further divided into two types: **dissipative** and **conservative**.

Dissipative:

$$\Delta p_1^\mu + \Delta p_2^\mu = P_{\mathrm{rad}}^\mu \neq 0$$

retarded worldline propagators (in-in)
retarded graviton propagators (in-in)

physical result

Conservative:

$$\Delta p_1^\mu + \Delta p_2^\mu = 0$$

retarded worldline propagators (in-in)
Feynman graviton propagators (in-out)
only take the real and even velocity part

ongoing debates on how to properly
define the conservative part

We discuss today:
5PM 1SF dissipative [5]
5PM 2SF conservative [4]

Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

$$v_1 \cdot v_2 = \gamma = (x + 1/x)/2,$$
$$\theta = x\,\partial_x$$

1SF

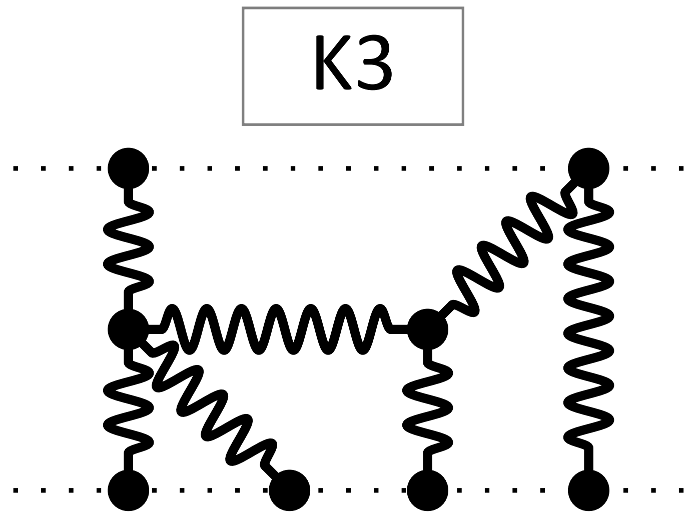
2SF

Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

$$v_1 \cdot v_2 = \gamma = (x + 1/x)/2,$$
$$\theta = x\,\partial_x$$

1SF

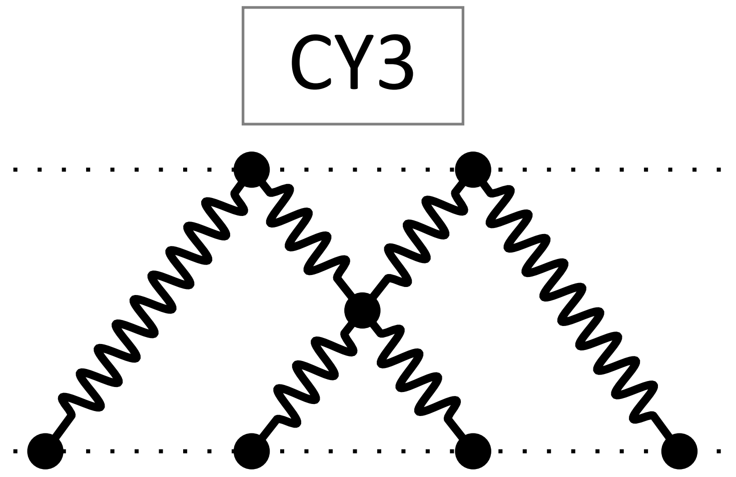


$$\mathcal{L}_{\mathrm{K3}} = \theta^3 - 2x^2\left(2 + 4\theta + 3\theta^2 + \theta^3\right) + x^4(2+\theta)^3$$

Symmetric square:

$$\mathcal{L}_{\mathrm{Leg}} = \theta^2 - x(\theta + 1/2)^2$$

$$\mathcal{L}_{\mathrm{K3}} = \mathrm{Sym}^2(\mathcal{L}_{\mathrm{Leg}})$$



$$\mathcal{L}_{\mathrm{CY3}} = (\theta - 1)^4 - x^4(\theta + 1)^4$$

Hadamard product: [AESZ]

$$\mathcal{L}_{\mathrm{CY3}} = \mathcal{L}_{\mathrm{Leg}} * \mathcal{L}_{\mathrm{Leg}}$$

2SF

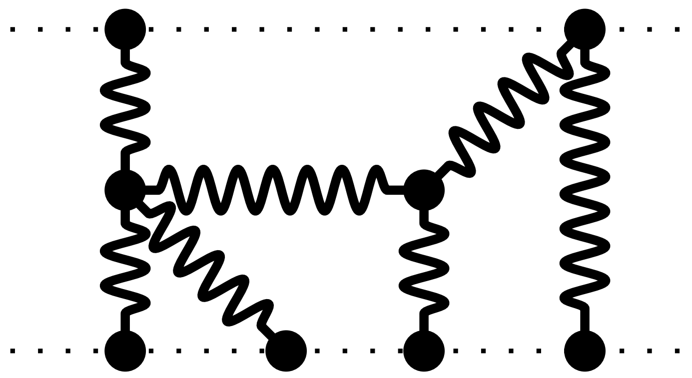
Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

$$v_1 \cdot v_2 = \gamma = (x + 1/x)/2,$$
$$\theta = x\,\partial_x$$

1SF

K3



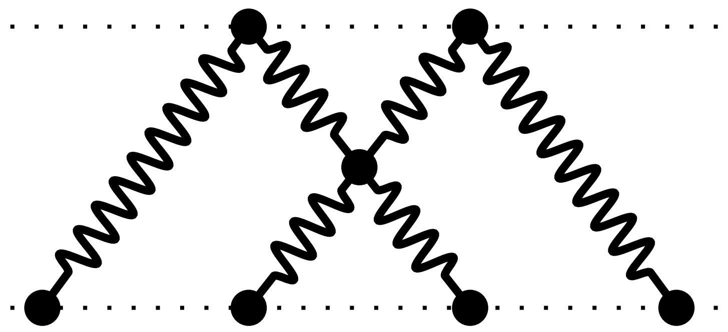
$$\mathcal{L}_{\mathrm{K3}} = \theta^3 - 2x^2\left(2 + 4\theta + 3\theta^2 + \theta^3\right) + x^4(2+\theta)^3$$

Symmetric square:

$$\mathcal{L}_{\mathrm{Leg}} = \theta^2 - x(\theta + 1/2)^2$$

$$\mathcal{L}_{\mathrm{K3}} = \mathrm{Sym}^2(\mathcal{L}_{\mathrm{Leg}})$$

CY3



$$\mathcal{L}_{\mathrm{CY3}} = (\theta - 1)^4 - x^4(\theta + 1)^4$$

Hadamard product: [AESZ]

$$\mathcal{L}_{\mathrm{CY3}} = \mathcal{L}_{\mathrm{Leg}} * \mathcal{L}_{\mathrm{Leg}}$$

simple, no third-kind

2SF

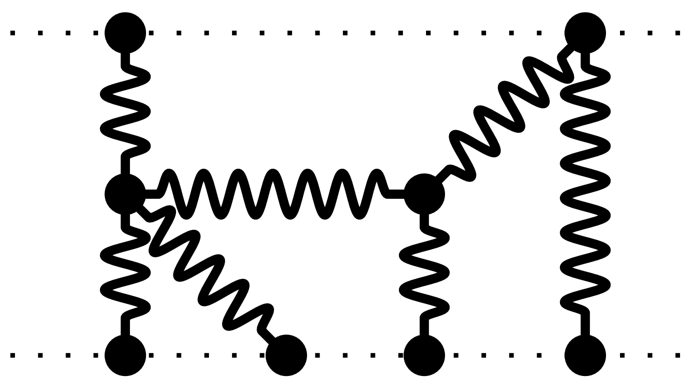
Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

$$v_1 \cdot v_2 = \gamma = (x + 1/x)/2, \qquad \theta = x\,\partial_x$$

1SF

K3



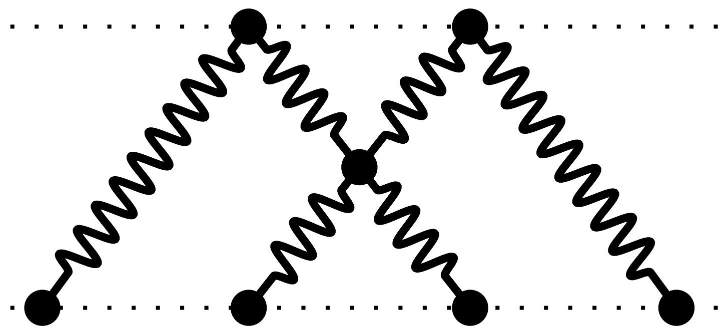
$$\mathcal{L}_{\mathrm{K3}} = \theta^3 - 2x^2\left(2 + 4\theta + 3\theta^2 + \theta^3\right) + x^4(2+\theta)^3$$

Symmetric square:

$$\mathcal{L}_{\mathrm{Leg}} = \theta^2 - x(\theta + 1/2)^2$$

$$\mathcal{L}_{\mathrm{K3}} = \mathrm{Sym}^2(\mathcal{L}_{\mathrm{Leg}})$$

CY3



$$\mathcal{L}_{\mathrm{CY3}} = (\theta - 1)^4 - x^4(\theta + 1)^4$$

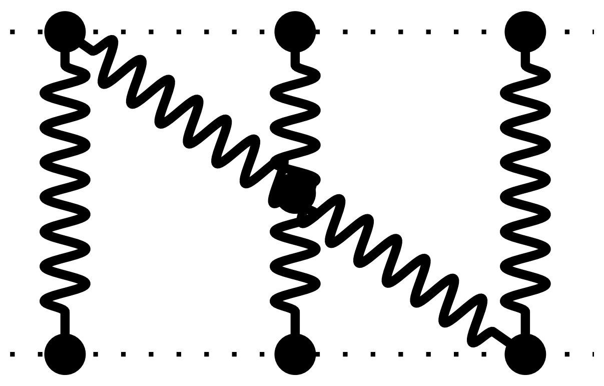
Hadamard product: [AESZ]

$$\mathcal{L}_{\mathrm{CY3}} = \mathcal{L}_{\mathrm{Leg}} * \mathcal{L}_{\mathrm{Leg}}$$

simple, no third-kind

2SF

K3

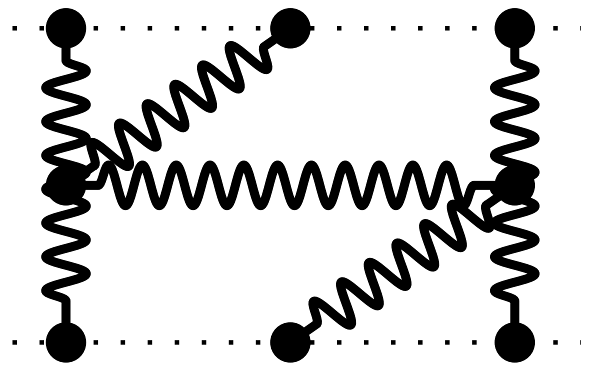


$$\mathcal{L}_{\mathrm{K3}} = \left(1 - 34x^2 + x^4\right)\theta^3 - 6x^2\left(17 - x^2\right)\theta^2 - 12(3-x)(3+x)x^2\theta - 8\left(5 - x^2\right)x^2$$

Apery operator [Apery, Zagier]

⟶ singularity structure crucial

CY3



$$\mathcal{L}_2^{(4)} = \theta^4 - 2^{30}z^3(\theta + \tfrac{1}{2})^4 - 2^4 z(192\theta^4 + 128\theta^3 + 112\theta^2 + 48\theta + 7) + 2^{14}z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7)$$

again Hadamard product [AESZ, van Straten]

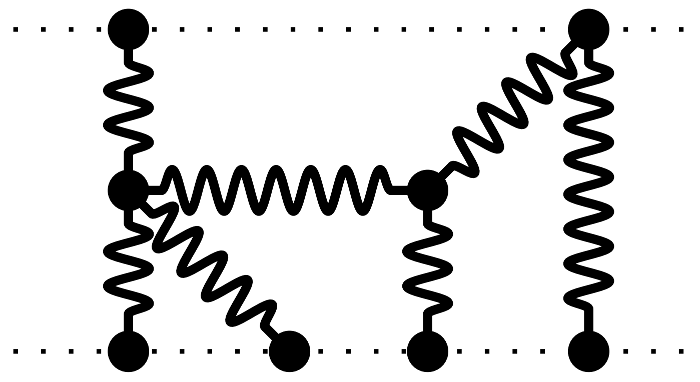
Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

$$v_1 \cdot v_2 = \gamma = (x + 1/x)/2,$$
$$\theta = x\,\partial_x$$

1SF

K3



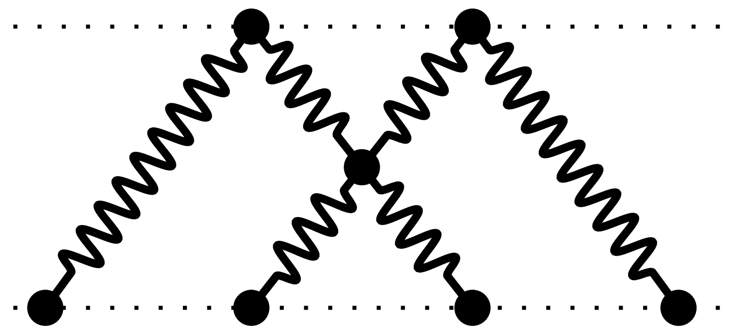
$$\mathcal{L}_{\mathrm{K3}} = \theta^3 - 2x^2\left(2 + 4\theta + 3\theta^2 + \theta^3\right) + x^4(2+\theta)^3$$

Symmetric square:

$$\mathcal{L}_{\mathrm{Leg}} = \theta^2 - x(\theta + 1/2)^2$$

$$\mathcal{L}_{\mathrm{K3}} = \mathrm{Sym}^2(\mathcal{L}_{\mathrm{Leg}})$$

CY3



$$\mathcal{L}_{\mathrm{CY3}} = (\theta - 1)^4 - x^4(\theta + 1)^4$$

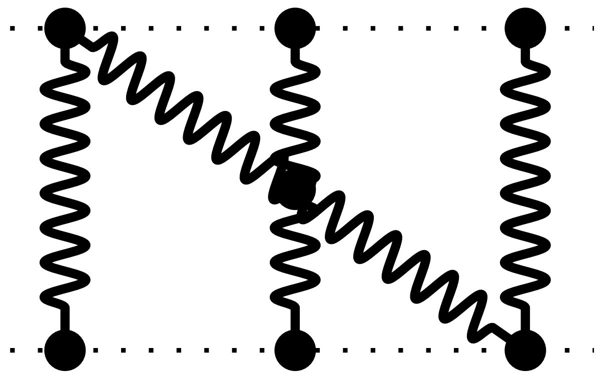
Hadamard product: [AESZ]

$$\mathcal{L}_{\mathrm{CY3}} = \mathcal{L}_{\mathrm{Leg}} * \mathcal{L}_{\mathrm{Leg}}$$

simple, no third-kind

2SF

K3

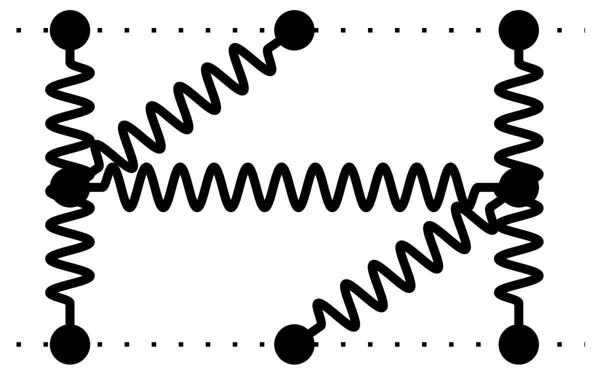


$$\mathcal{L}_{\mathrm{K3}} = \left(1 - 34x^2 + x^4\right)\theta^3 - 6x^2\left(17 - x^2\right)\theta^2 - 12(3-x)(3+x)x^2\theta - 8\left(5 - x^2\right)x^2$$

Apery operator [Apery, Zagier]

→ singularity structure crucial

CY3



$$\mathcal{L}_2^{(4)} = \theta^4 - 2^{30}z^3(\theta + \tfrac{1}{2})^4 - 2^4 z(192\theta^4 + 128\theta^3 + 112\theta^2 + 48\theta + 7) + 2^{14}z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7)$$

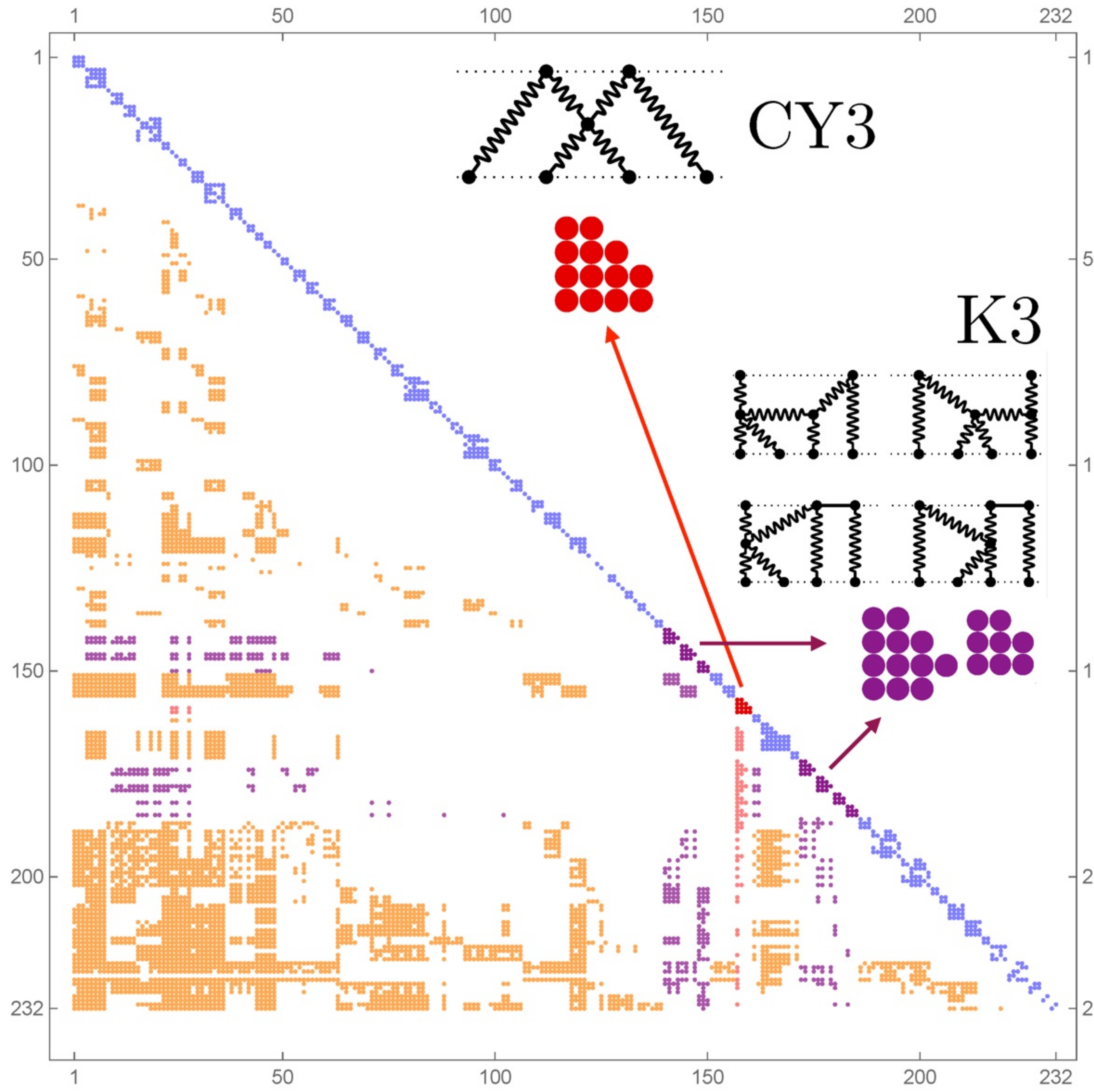
again Hadamard product [AESZ, van Straten]

complicated, many third-kinds on max cut, K3 7x7, CY3 5x5

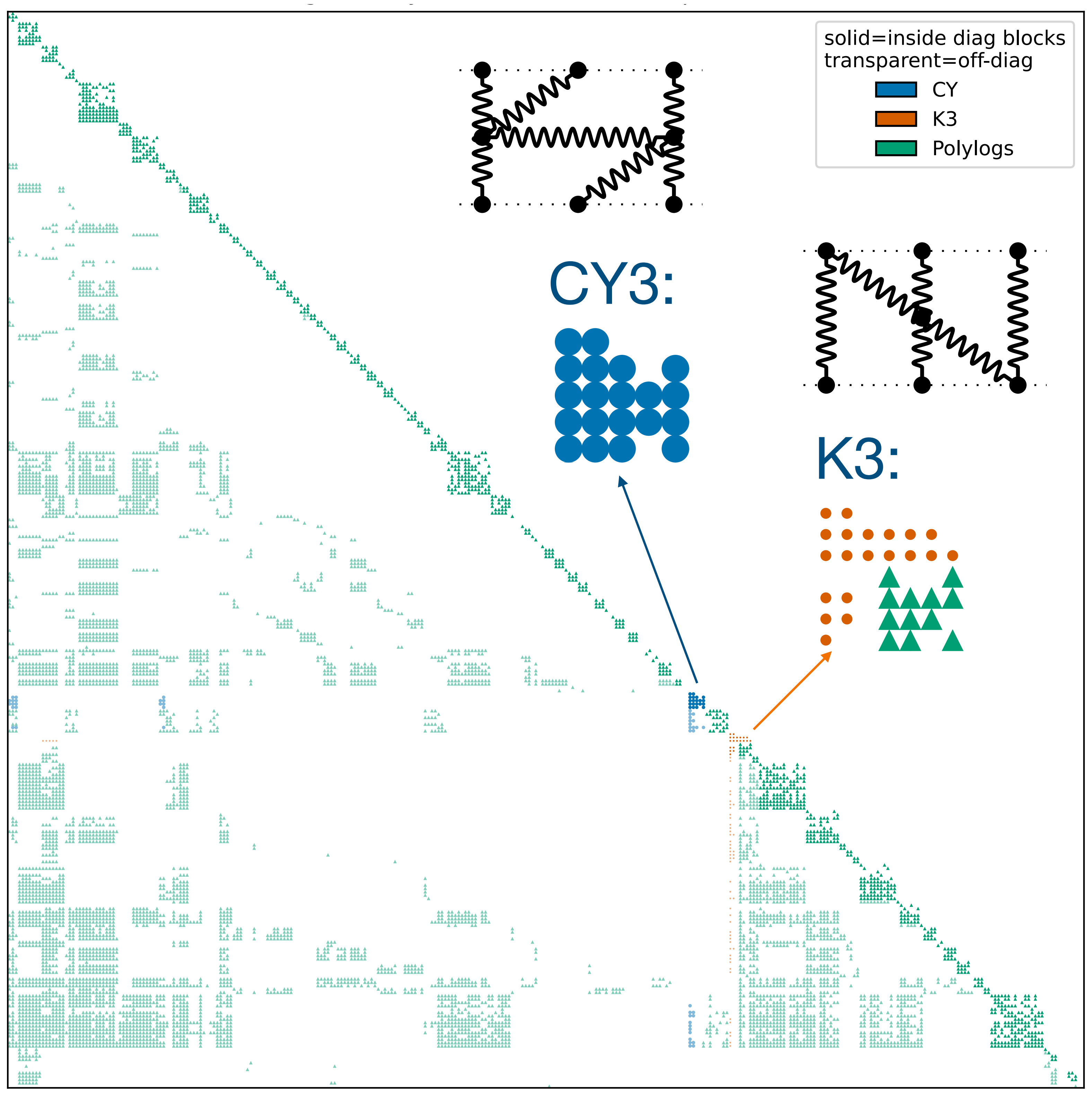
Calabi-Yaus @ 5PM

- In total, at **5PM**, we found **two interesting families containing** the **Calabi-Yau integrals**. At **1SF**, this family contained **232 masters**, and at **2SF**, **321 masters**. We managed to bring **both systems** into **canonical form**. Here, it was **crucial** to use **our canonicalization method** in **combination** with **integrand analysis**.

1SF



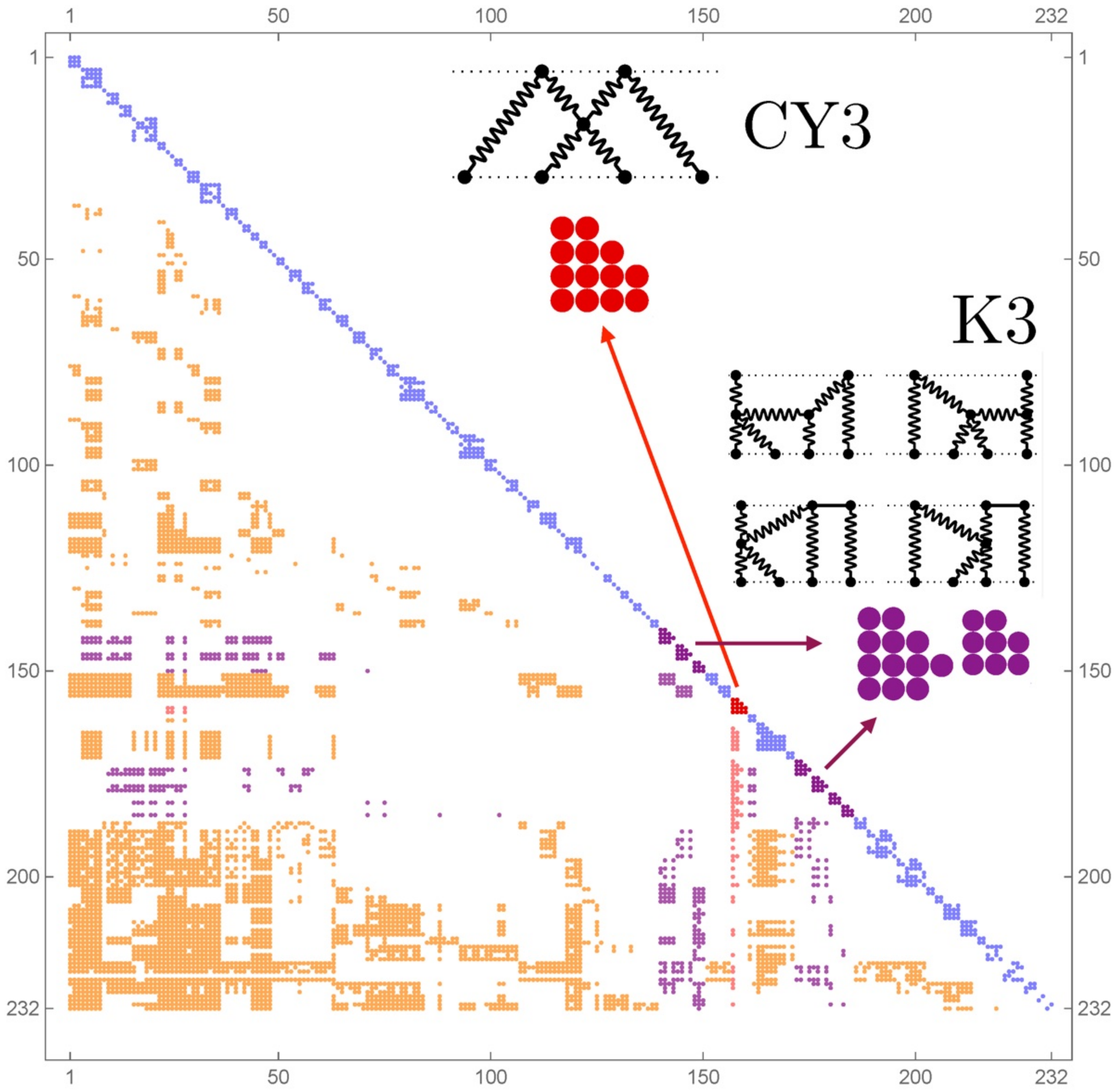
2SF



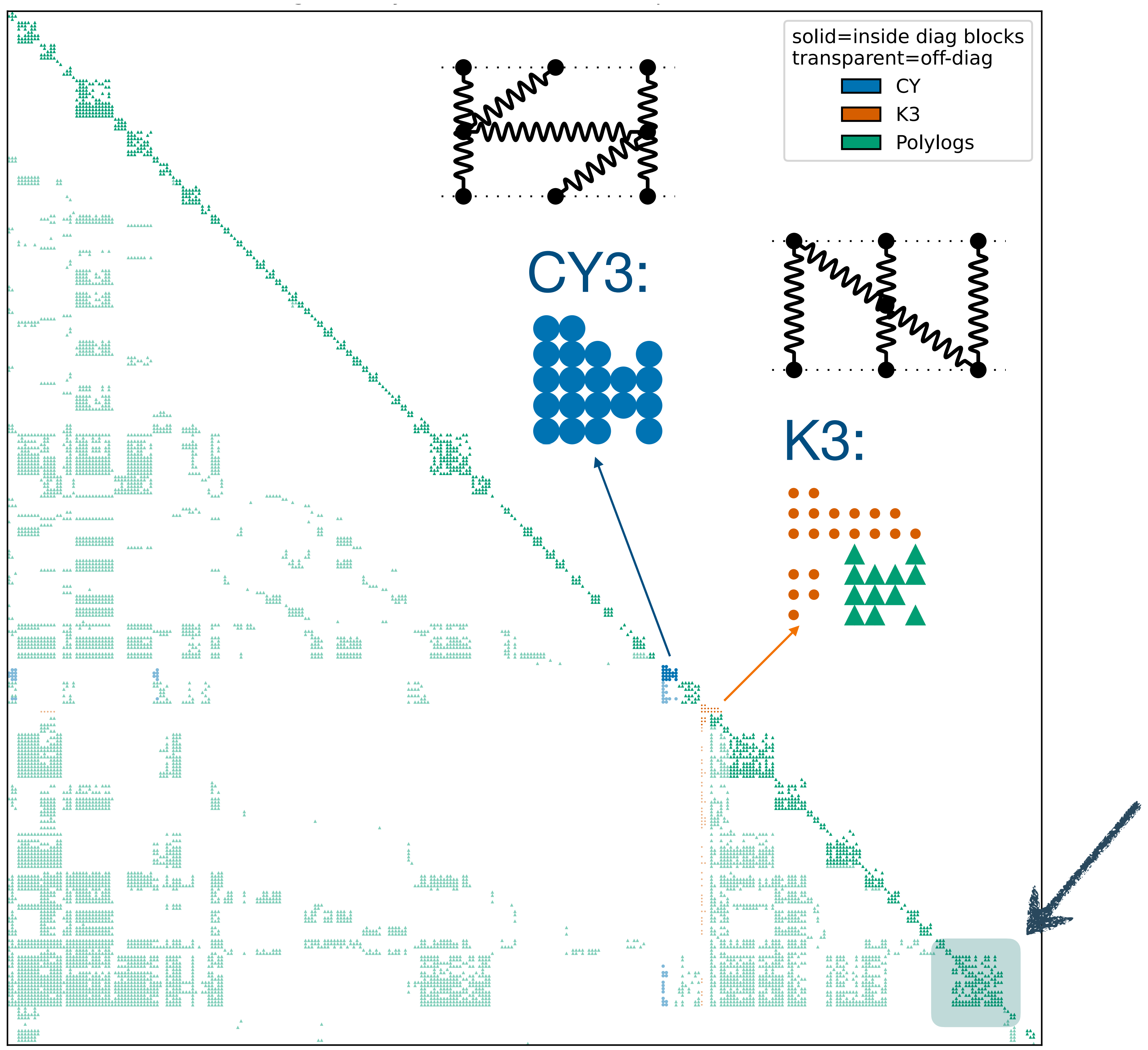
Calabi-Yaus @ 5PM

- In total, at **5PM**, we found **two interesting families containing** the **Calabi-Yau integrals**. At **1SF**, this family contained **232 masters**, and at **2SF**, **321 masters**. We managed to bring **both systems** into **canonical form**. Here, it was **crucial** to use **our canonicalization method** in **combination** with **integrand analysis**.

1SF



2SF



16x16 polylog sector

Comments on Results @ 5PM

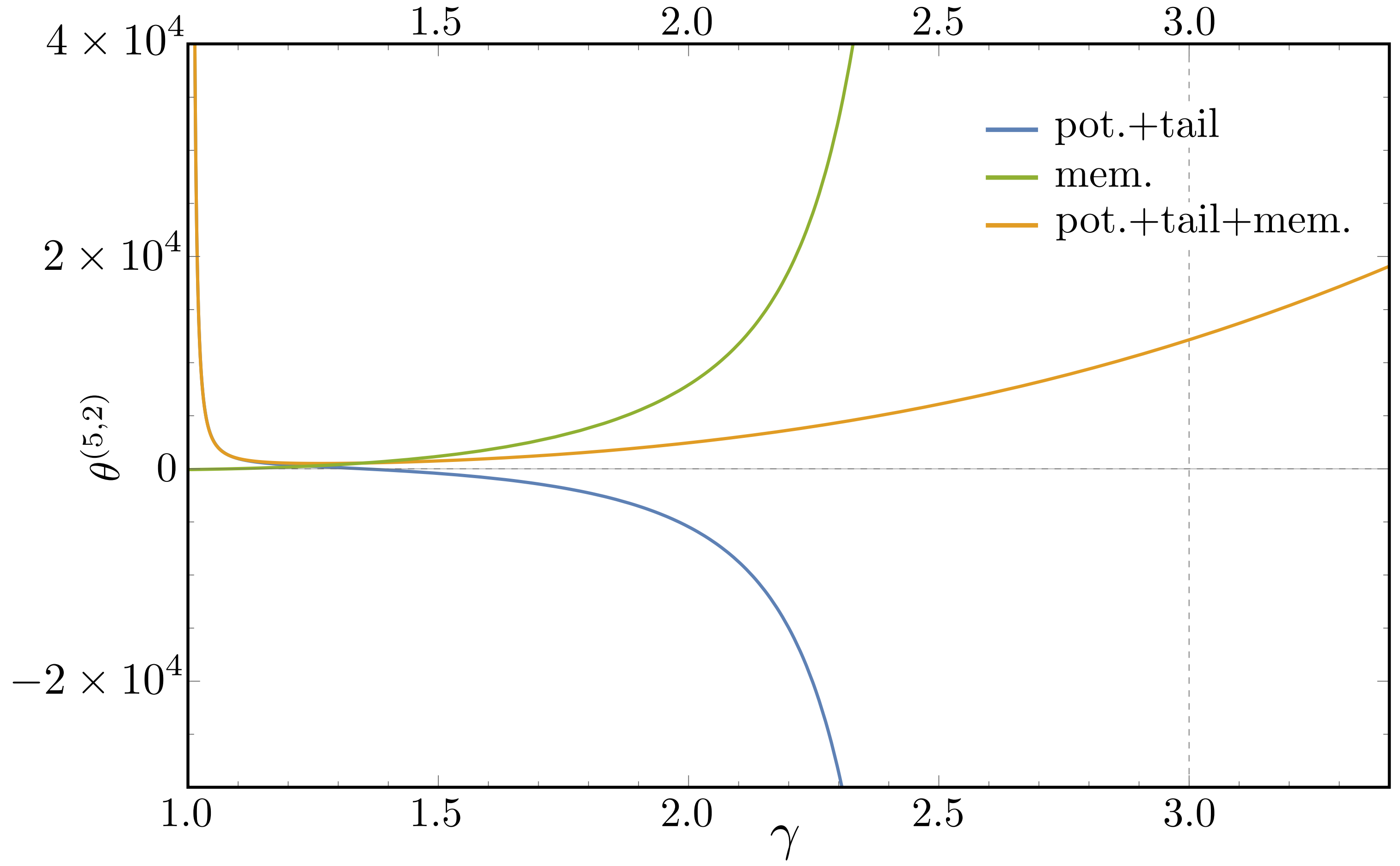
- **Further comments** on the two computations:
    - For the **1SF dissipative**, we computed all **boundary values analytically**.
    - For the **2SF conservative**, we had to **modify** the **boundary prescription** (still ongoing debate).

Comments on Results @ 5PM

- **Further comments** on the two computations:
  - For the **1SF dissipative**, we computed all **boundary values analytically**.
  - For the **2SF conservative**, we had to **modify** the **boundary prescription** (still ongoing debate).

We fixed bounadry values by **imposing cancellations** of ϵ**- and kinematical singularities**. Due to the **analytic form** of our result, it was quite **easy** to **analyze** the **kinematical singularity** at $\gamma = 3$.

$\longrightarrow$ $(\gamma - 3)$**-prescription**

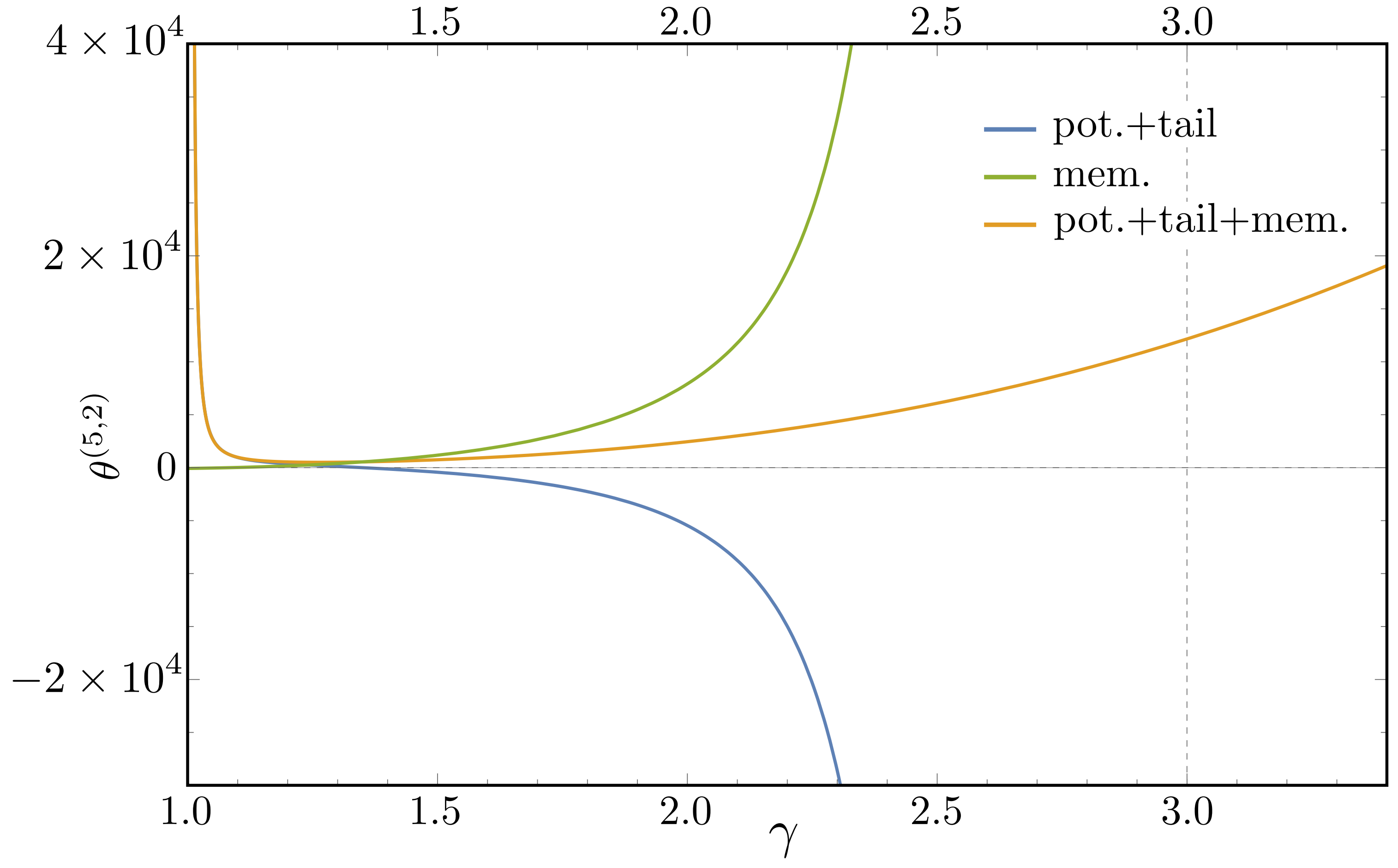


Comments on Results @ 5PM

- **Further comments** on the two computations:
  - For the **1SF dissipative**, we computed all **boundary values analytically**.
  - For the **2SF conservative**, we had to **modify** the **boundary prescription** (still ongoing debate).

We fixed bounadry values by **imposing cancellations** of ϵ**- and kinematical singularities**. Due to the **analytic form** of our result, it was quite **easy** to **analyze** the **kinematical singularity** at $\gamma = 3$.

$\longrightarrow$ $(\gamma - 3)$**-prescription**



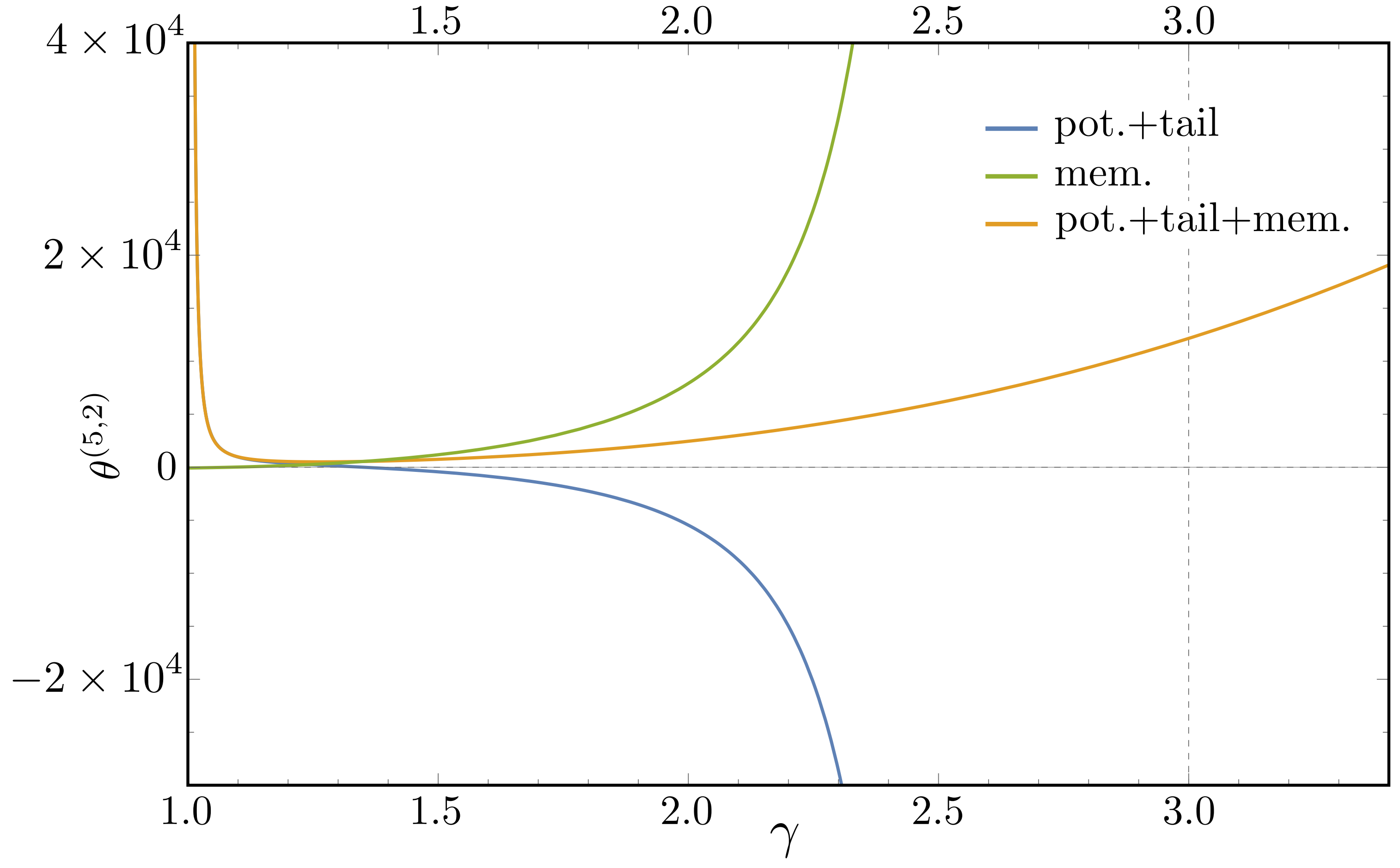
- At **1SF dissipative**, the **Calabi-Yau three-fold periods** survive in the final **result** for the **radiated energy**. [5]

Comments on Results @ 5PM

- **Further comments** on the two computations:
  - For the **1SF dissipative**, we computed all **boundary values analytically**.
  - For the **2SF conservative**, we had to **modify** the **boundary prescription** (still ongoing debate).

We fixed bounadry values by **imposing cancellations** of ϵ**- and kinematical singularities**. Due to the **analytic form** of our result, it was quite **easy** to **analyze** the **kinematical singularity** at $\gamma = 3$.

$\longrightarrow$ $(\gamma - 3)$**-prescription**



- At **1SF dissipative**, the **Calabi-Yau three-fold periods** survive in the final **result** for the **radiated energy**. [5]

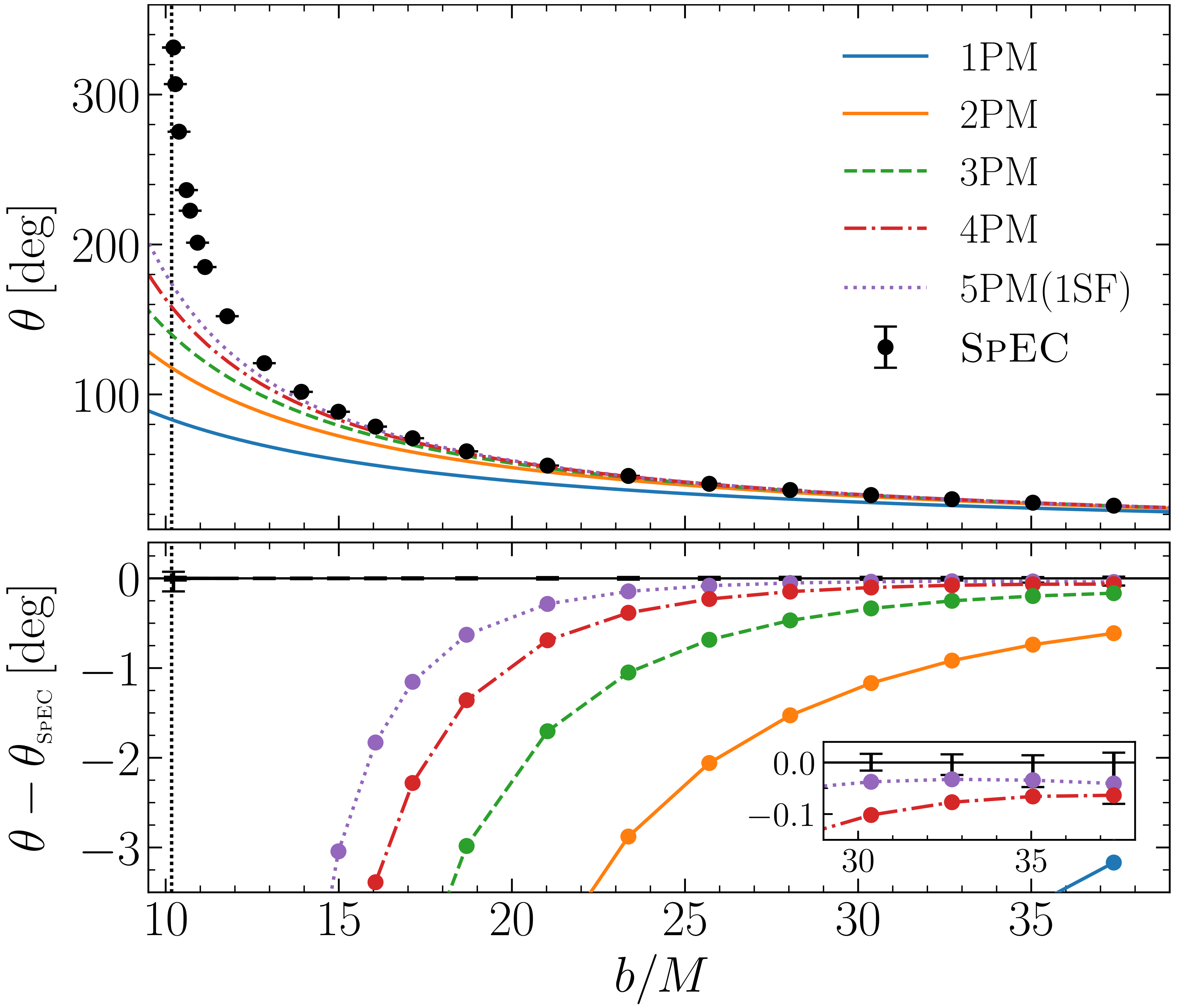
- At **2SF conservative**, only the K3 periods appear in the scattering angle. [4, Bern et al.]

Comments on Results @ 5PM

- There have already been several **checks/comparisons** on **our results**, particularly at 1SF:

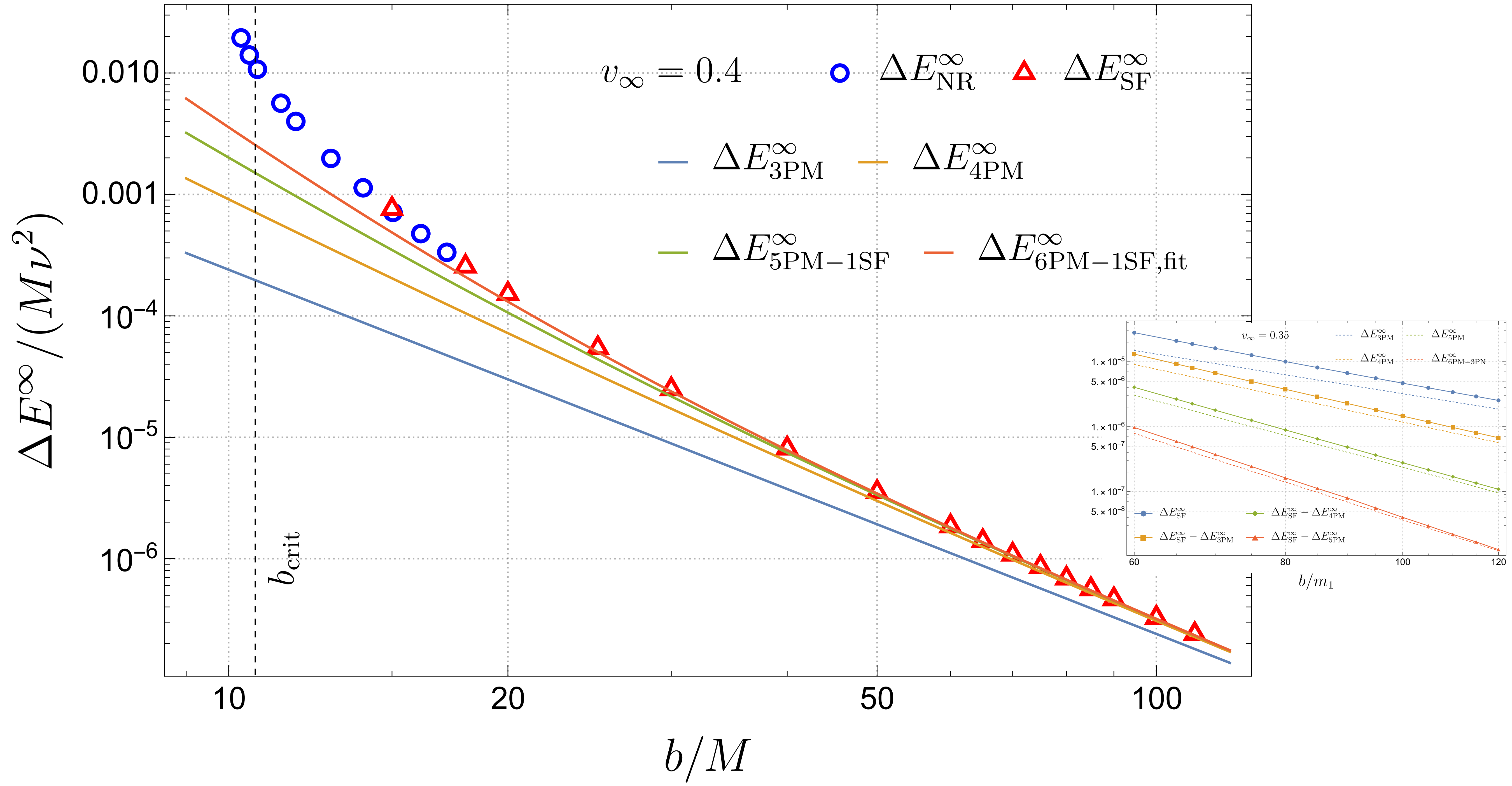
[Long, Pfeiffer, Kidder, Scheel]

Scattering angle



[Warbuton]

Radiated energy

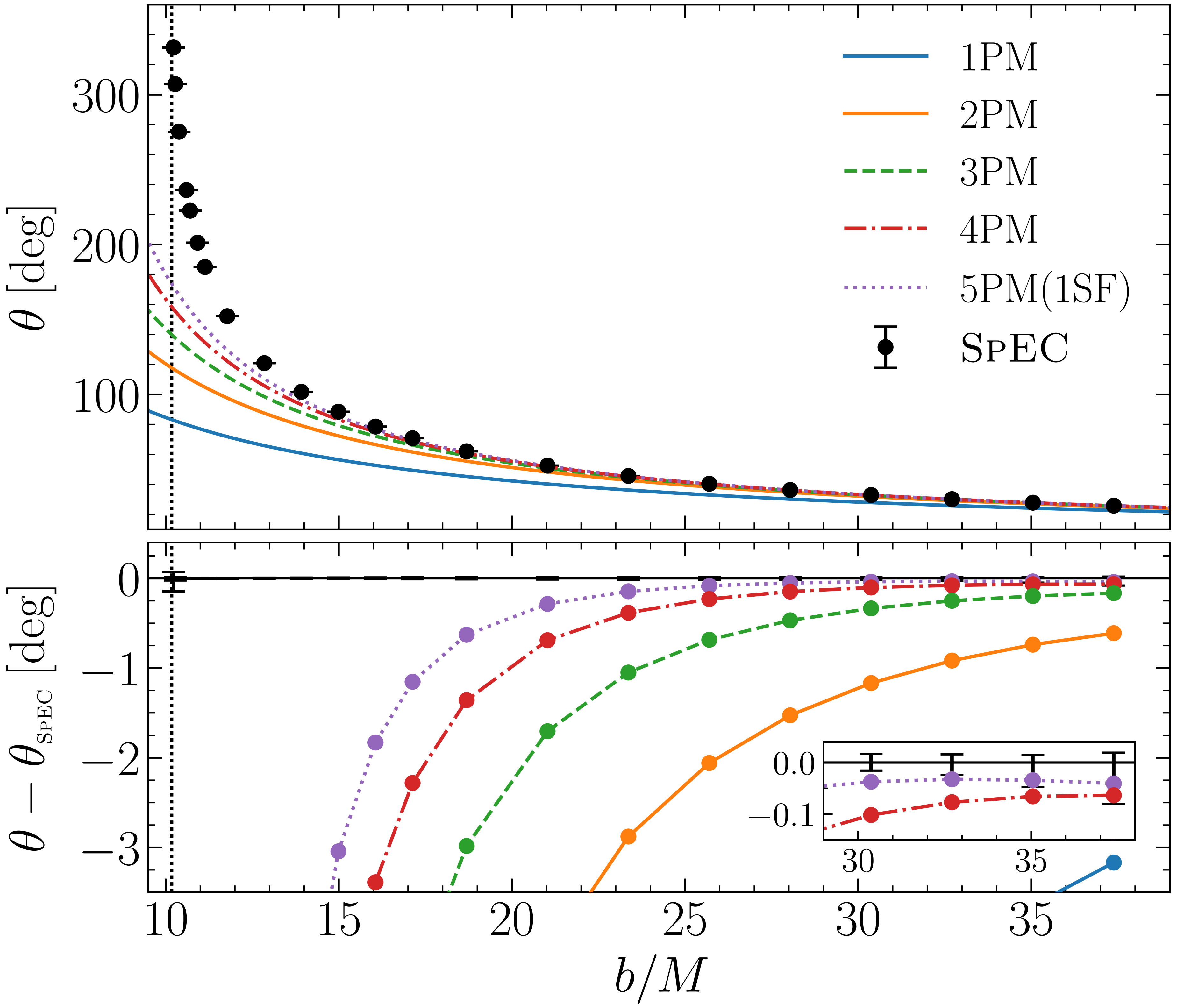


Comments on Results @ 5PM

- There have already been several **checks/comparisons** on **our results**, particularly at 1SF:

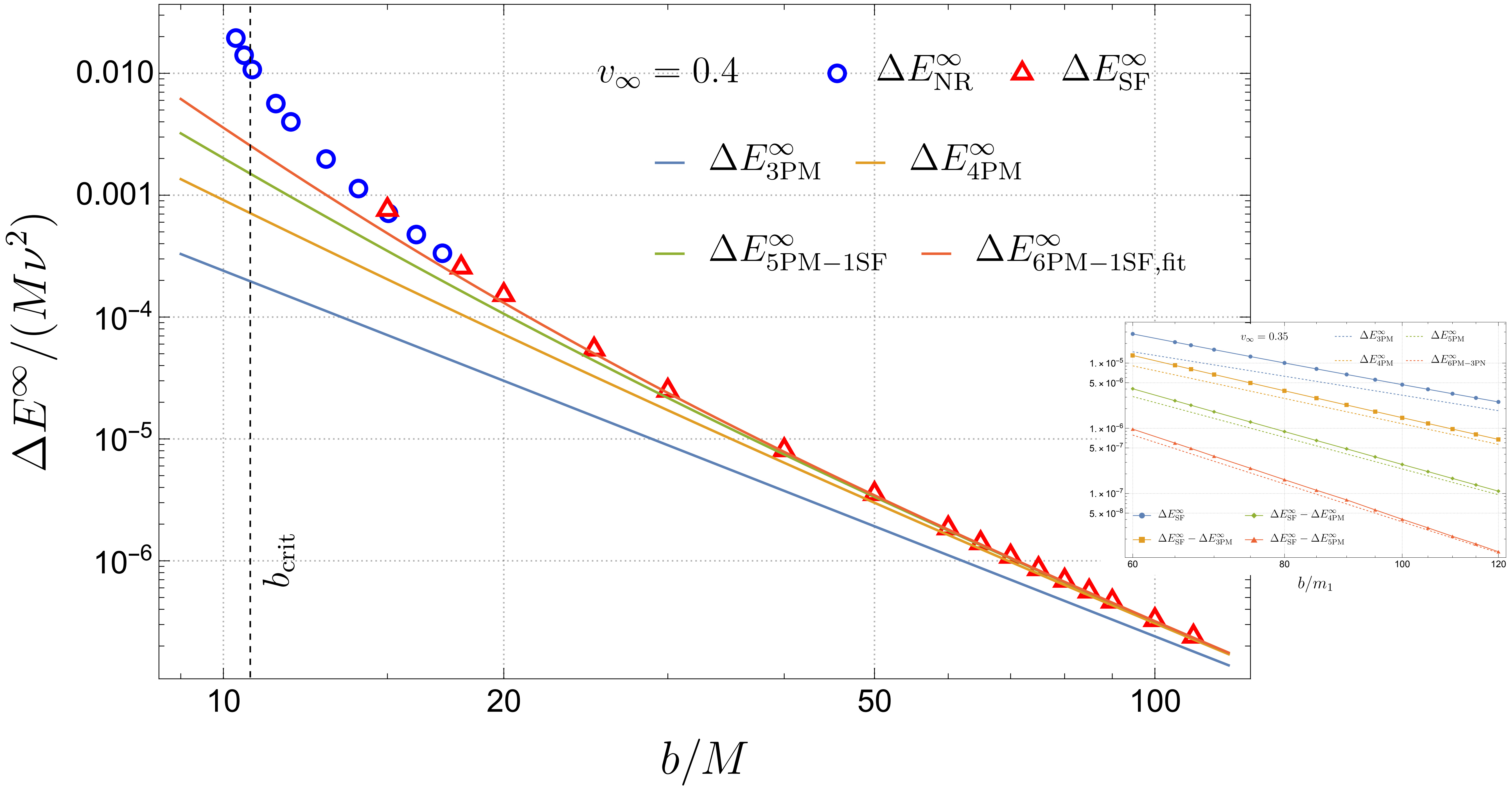
[Long, Pfeiffer, Kidder, Scheel]

Scattering angle



[Warbuton]

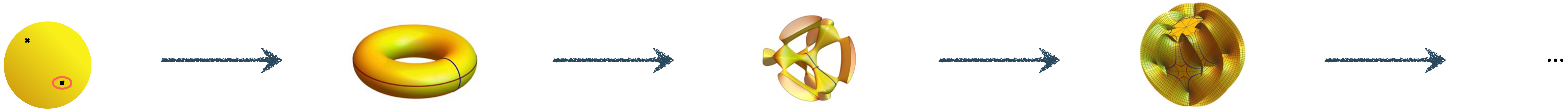
Radiated energy



Check on our result containg Calabi-Yau periods!

Conclusions

- **Calabi-Yau periods** are a **natural generalization** of **elliptic functions** to higher dimensions.



0 dim. → 1 dim. → 2 dim. → 3 dim. → ...

sphere

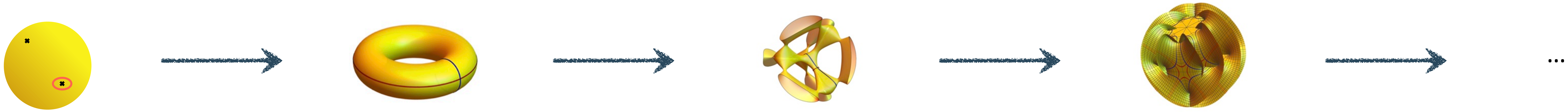
torus

K3

Calabi-Yau

Conclusions

- **Calabi-Yau periods** are a **natural generalization** of **elliptic functions** to higher dimensions.



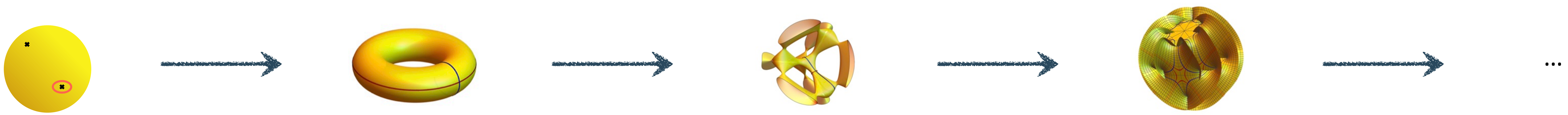
0 dim.	1 dim.	2 dim.	3 dim.	
sphere	torus	K3	Calabi-Yau	...

- The **black hole scattering** example was the **most complicated situation** in which our canonicalization procedure was applied yet. We are currently working on even **more general cases**. Nevertheless, the crucial point is to make the **geometry manifest** (mixed Hodge structure) using **integrand analysis**.

1st-kind	**2nd-kind**	**3rd-kind**
(no pole)	(double pole)	(single pole)
periods	quasi-periods	third-kind periods, G-functions

Conclusions

- **Calabi-Yau periods** are a **natural generalization** of **elliptic functions** to higher dimensions.



0 dim.	1 dim.	2 dim.	3 dim.	
sphere	torus	K3	Calabi-Yau	...

- The **black hole scattering** example was the **most complicated situation** in which our canonicalization procedure was applied yet. We are currently working on even **more general cases**. Nevertheless, the crucial point is to make the **geometry manifest** (mixed Hodge structure) using **integrand analysis**.

1st-kind	**2nd-kind**	**3rd-kind**
(no pole)	(double pole)	(single pole)
periods	quasi-periods	third-kind periods, G-functions

- Using **state-of-the-art perturbation theory techniques** originally developed for particle physics, we have computed the **most accurate predictions** for **black hole scattering** (5PM 1SF and 2SF). This was only possible due to the **interplay** of **advanced IBP programs**, the theory of **canonical integrals beyond polylogarithms**, **Calabi-Yau mathematics**, and **huge computer resources**.

Thank you for your attention