NNLO Matching for Meson Distribution Amplitudes from Euclidean Correlators

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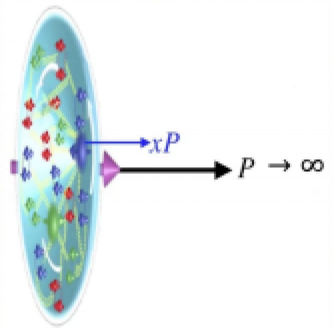
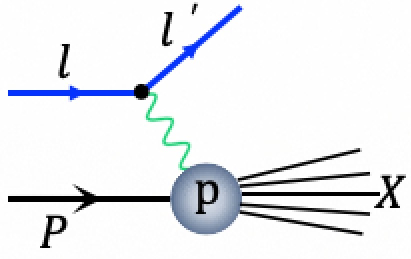
Based on Y. Ji, **F. Yao**, and J. H. Zhang, arXiv:2504.09367

PDFs vs. DAs

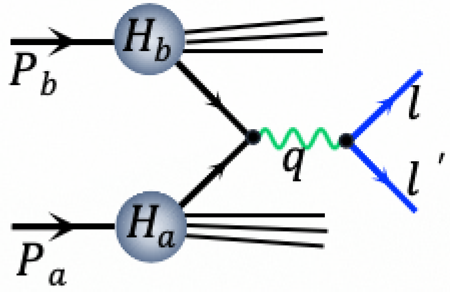
- Collinear PDFs: the probability distribution of a single parton within a hadron **(inclusive processes)**



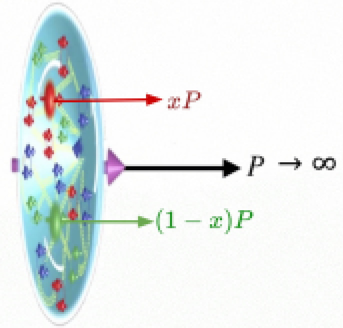
DIS



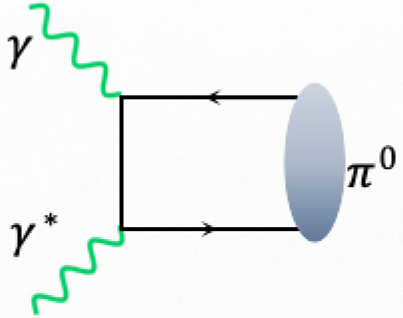
Drell-Yan



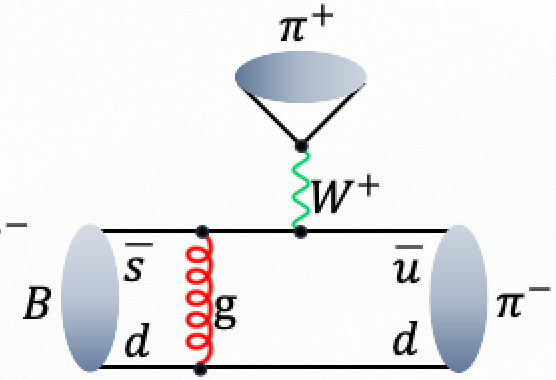
- Distribution Amplitudes (DAs): the probability distribution amplitude of partons during the creation or annihilation of a hadron **(Exclusive process)**



$\gamma\gamma^* \to \pi$



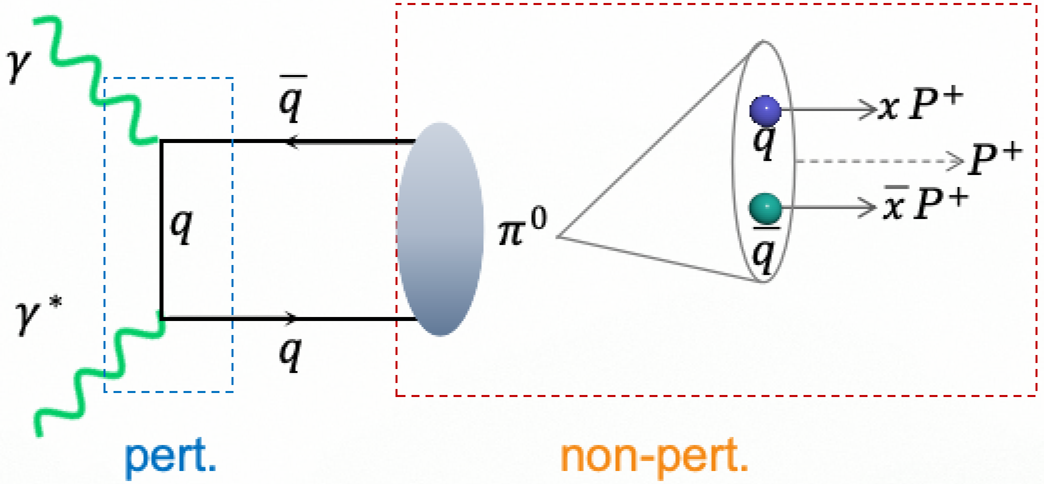
$B \to \pi^+\pi^-$



From Exclusive Processes to Meson DAs

- QCD factorization: a simple example

$\gamma\gamma^* \to \pi$



$\bar{x} = 1 - x$

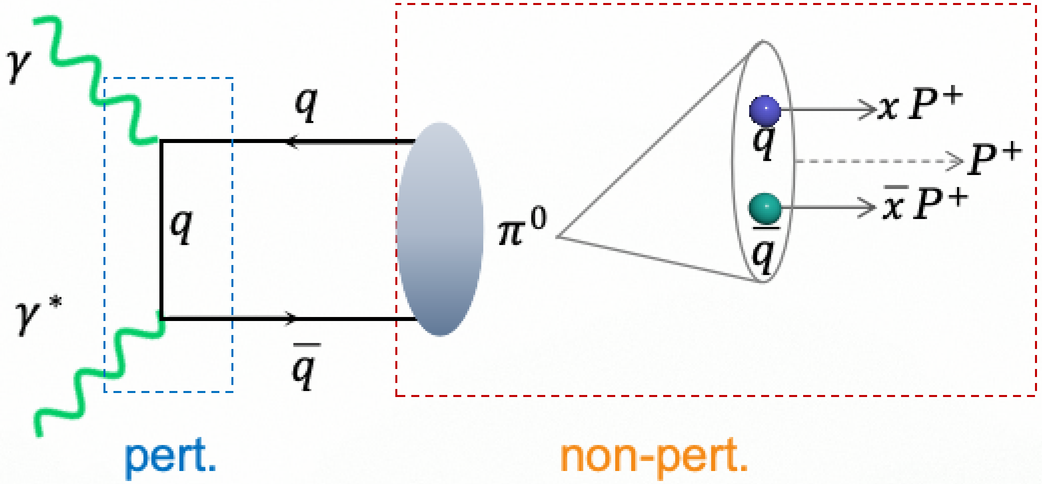
$F_{\gamma\gamma^*\to\pi} \approx$ hard scattering kernels $\otimes$ light-cone distribution amplitudes

- Experiment: limited clean processes → model-dependent extraction
- Lattice QCD: first-principles complementary input

From Exclusive Processes to Meson DAs

- QCD factorization: a simple example

$\gamma\gamma^* \to \pi$



$\bar{x} = 1 - x$

F_γ

- Experi
- Lattice

The definition of light-cone DA

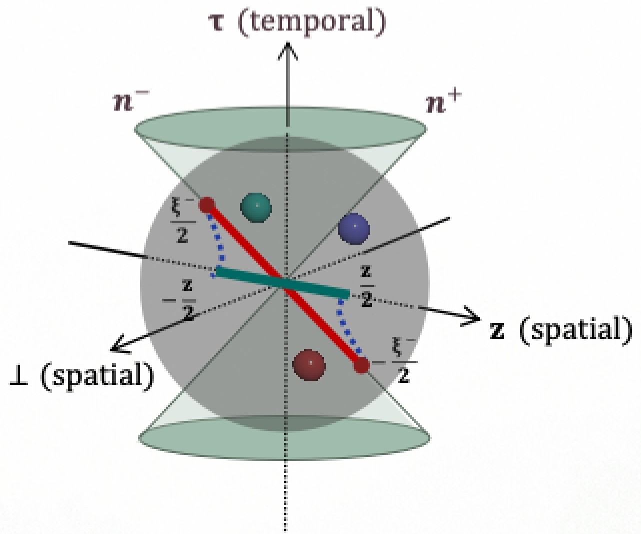
$$\mathrm{i} f_M(P^+)\,\phi_M(x) = \int \frac{d\xi^-}{2\pi}\, e^{ixP^+\xi^-} \langle 0|\bar{\psi}_1(0)\gamma^+\gamma_5 W(0,\xi^-)\psi_2(\xi^-)|M(P)\rangle.$$

Not directly calculable on a Euclidean lattice!

Perturbative matching in LaMET

- **Large momentum effective theory (LaMET):** X.D. Ji, PRL 110 (2013); X.D. Ji et al., Rev.Mod.Phys. 93 (2021)

equal-time correlation $\tilde{h}$ $\xrightarrow{\text{Lorentz boost}}$ light-cone correlation h



- **Perturbative matching:**

Quasi−LF correlation / LF correlation

$$\tilde{h}(z,\lambda) = \int_0^1 d\alpha d\beta\ C(\alpha,\beta,\mu^2 z^2)\ h(\alpha,\beta,\lambda,\mu) + \mathcal{O}(\Lambda_{QCD}^2 z^2, M^2 z^2);\quad \lambda = z P_z$$

matching coefficient / Power correction

$$\tilde{\phi}_\pi(x,P_z) = \int_0^1 dy\ \mathbb{C}(x,y,\mu,P_z)\ \phi_\pi(x,\mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{(xP_z)^2},\frac{\Lambda_{QCD}^2}{((1-x)P_z)^2},\frac{M^2}{(xP_z)^2},\frac{M^2}{((1-x)P_z)^2}\right)$$

Quasi−DA / LCDA

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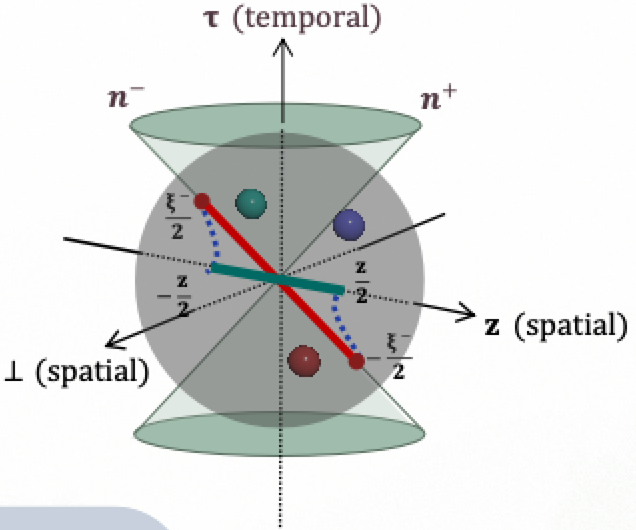


- **Perturbative matching:**

Quasi−LF correlation   LF correlation

$\tilde{h}(z, \lambda) =$

$\tilde{\phi}_\pi(x, P_z)$

Quasi−DA

- Precise extractions of LCDA: control of lattice artifacts and improvement of perturbative calculations.
- **Higher-order studies mainly focus on PDFs.**

See Tobias's talk
Chen, Wang, Zhu, PRL 126 (2021);
Li, Ma, Qiu, PRL 126 (2021);
Cheng et al., PRL 134 (2025).

Workflow of this work

$$\tilde{h}^R(z,\lambda,\mu) = \int_0^1 d\alpha d\beta\, C(\alpha,\beta,z^2,\mu) h(\alpha,\beta,\lambda,\mu) + h.t.$$

1. Correlation function in coordinate space

$$\tilde{h}(z,\lambda) = \langle 0|\bar{\psi}_1(0)\gamma^z\gamma_5\mathcal{W}(0,z)\psi_2(z)|\pi(P)\rangle$$

2. Feynman diagrams at NNLO

3. Results in coordinate space

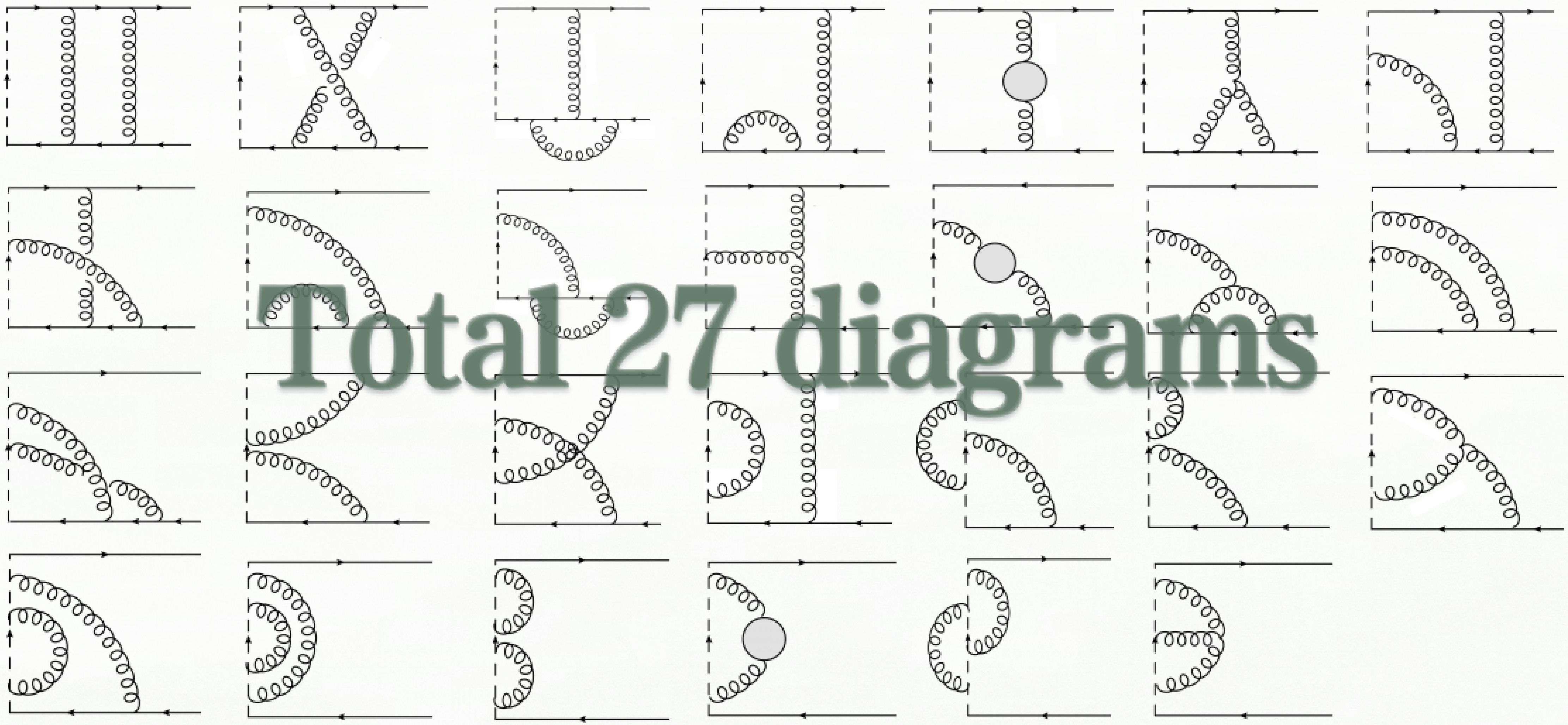
4. Fourier transform into momentum space

$$\tilde{\phi}(x,\mu) = \int_0^1 dy\, \mathbb{C}(x,y,\mu)\phi(y,\mu) + h.t.$$

5. Matching in the state-of-the-art scheme

6. Numerical test

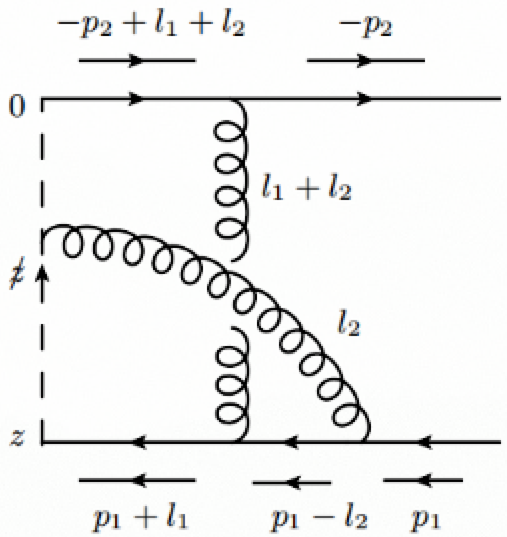
Feynman diagrams at NNLO



Total 27 diagrams

The conjugate diagrams are not shown.

Calculation details: an example



- **Feynman amplitude**

$$-ig^4C_F^2\left(C_F-\frac{C_A}{2}\right)\int_0^1 du\int\frac{d^dl_1d^dl_2}{(2\pi)^{2d}}\,e^{-i(p_1+l_1+ul_2)\cdot z}\gamma_\rho\frac{-\not{p}_2+\not{l}_{12}}{(-p_2+l_{12})^2}\not{z}\frac{\not{p}_1+\not{l}_1}{(p_1+l_1)^2}\gamma^\rho\frac{\not{p}_1-\not{l}_2}{(p_1-l_2)^2}\not{z}\frac{1}{l_2^2(l_1+l_2)^2}$$

- **Making a substitution**: $l_1^\alpha\to -i\,\partial_{z_1}^\alpha,\ l_2^\alpha\to -i\,\partial_{z_2}^\alpha$

$$-ig^4C_F^2\left(C_F-\frac{C_A}{2}\right)e^{-ip_1\cdot z}\gamma_\rho\gamma_\sigma\not{z}\gamma_\beta\gamma^\rho\gamma_\alpha\not{z}\int_0^1 du\Big\{(-p_2^\sigma-i\partial_{z_1}^\sigma-i\partial_{z_2}^\sigma)(p_1^\beta-i\partial_{z_1}^\beta)(p_1^\alpha+i\partial_{z_2}^\alpha)\underbrace{\int\frac{d^dl_1d^dl_2}{(2\pi)^{2d}}\frac{e^{i(l_1\cdot z_1+l_2\cdot z_2)}}{(p_1-l_2)^2(p_1+l_1)^2(-p_2+l_{12})^2l_2^2(l_1+l_2)^2}}_{J_{12}}\Big\}_{z_1=-z,z_2=-uz},$$

typical integral ⇓

$$J_{12}\equiv\frac{\Gamma(-1-2\epsilon)}{4^{1+2\epsilon}(4\pi)^d}\int_0^1\frac{d\alpha d\beta d\gamma d\tau\,\alpha\tau^\epsilon\bar{\tau}}{(\alpha\bar{\alpha})^{1+\epsilon}}e^{-i(\bar{\alpha}p_1-\alpha\beta p_2)\cdot z_1}e^{i[(\tau+\bar{\tau}\gamma)p_1+\beta\tau p_2]\cdot(z_2-\alpha z_1)}\left[-z_1^2\alpha\bar{\alpha}/\tau-(z_2-\alpha z_1)^2\right]^{1+2\epsilon}$$

Calculation details: an example

Useful formulas for J_{12}

$$\int \frac{d^d l}{(2\pi)^d} \frac{e^{-il\cdot z}}{[l^2]^n} = \frac{(-1)^{-n} i}{4^{n-d/2}(4\pi)^{d/2}} \frac{\Gamma(d/2-n)}{\Gamma(n)} (-z^2)^{n-d/2}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{e^{-il\cdot z}}{[l^2]^n} = \frac{(-1)^{-n} i \Gamma(n-d/2+1/2)}{(4\pi)^{\frac{d-1}{2}}\Gamma(n)} \int \frac{dl^z}{2\pi} e^{-il^z z} \left(\frac{1}{l_z^2}\right)^{n-\frac{d-1}{2}}$$

$$\int \frac{dl^z}{2\pi} e^{-il^z z} \left(\frac{1}{l_z^2}\right)^a = \frac{\Gamma(1/2-a)}{4^a \sqrt{\pi}\Gamma(a)} (z^2)^{a-1/2}$$

$$\int \frac{dl_1^z dl_2^z}{(2\pi)^2} \frac{e^{-ia_1 l_1^z + ia_2 l_2^z}}{((l_1^z)^2 + (l_2^z)^2)^{n-d}} = \frac{\Gamma(-n+d+1)}{4^{n-d}\pi\Gamma(n-d)} [a_1^2 + a_2^2]^{n-d-1}$$

- Then applying the partial derivative to the typical integral J_{12}.

Full matching coefficient

- Appropriate **change of variables** and integrating over the **remaining Feynman parameters**:

MB and MBcreate, HypExp

$$\int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \mathrm{d}\tau \mathrm{d}u \{\ldots\}\, e^{i\ [\ldots]}$$

$$\to \int_0^1 d\alpha d\beta\, e^{-i(\bar{\alpha}p_1+\beta p_2)\cdot z} \{\theta(1-\alpha-\beta)\, f_1(\alpha,\beta) + \theta(\alpha+\beta-1)\, f_2(\alpha,\beta)\}$$

- Conjugate diagram: $C(\alpha \leftrightarrow \beta, z, \epsilon)\, e^{-i(\bar{\alpha}p_1+\beta p_2)\cdot z}$

All diagrams added:

$$C_B(\alpha,\beta,z,\epsilon) = \delta(\alpha)\delta(\beta) + a_s^B\left(\frac{1}{\epsilon}\mathcal{C}_{11}^{(1)} + \mathcal{C}_{10}^{(1)} + \epsilon\, \mathcal{C}_{1E}^{(1)}\right) + {a_s^B}^2 \left(\frac{1}{\epsilon^2}\mathcal{C}_{22}^{(2)} + \frac{1}{\epsilon}\mathcal{C}_{21}^{(2)} + \mathcal{C}_{20}^{(2)}\right)$$

$$a_s = \frac{\alpha_s}{4\pi}$$

$$C_R(\alpha,\beta,z) = Z_{\alpha_s} Z_{UV}\,(Z_O \otimes C_B(\alpha,\beta,z,\epsilon))$$

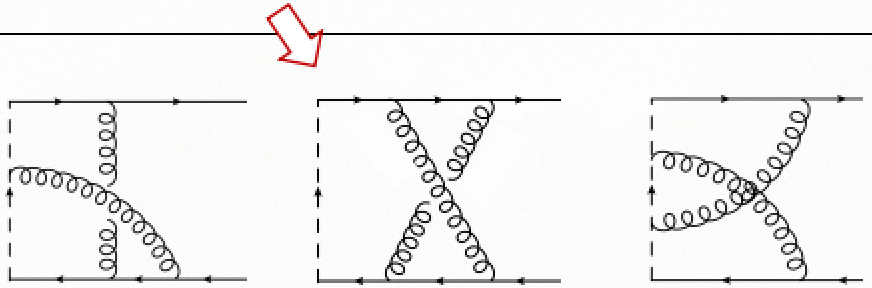
V. M. Braun et. al., JHEP 07 (2020)
V. M. Braun et. al., JHEP 03 (2016)

Full matching coefficient

$$C_{\overline{MS}}(\alpha,\beta,\mu^2 z^2) = \delta(\alpha)\delta(\beta) + a_s(\mu)\,(C_{11}L + C_{10}) + a_s^2(\mu)\,(C_{22}L^2 + C_{21}L + C_{20}) \qquad L = \ln[\mu^2 z^2]$$

F. Yao, Y. Ji and J. H. Zhang, JHEP 11 (2023)

• **PolyLog structures:** $\mathrm{Li}_{n\_}[\alpha]$, $\mathrm{Li}_{n\_}[\beta]$, $\mathrm{Li}_{n\_}[\bar{\alpha}]$, $\mathrm{Li}_{n\_}[\bar{\beta}]$, $\mathrm{Li}_{n\_}[\bar{\alpha}-\beta]$, $\mathrm{Li}_{n\_}\left[\frac{\alpha\,\beta}{\bar{\alpha}\,\bar{\beta}}\right]$, … $n = 1,2,3$
• **Three color structures:** $C_F C_A$, $C_F{}^2$, $C_F\,\beta_0$
• **Displayed in a plus function:** $C1^{(2)}[\alpha,\beta] + C2^{(2)}[\alpha]_+\delta(\beta) + C3^{(2)}[\beta]_+\delta(\alpha) + C4^{(2)}\delta(\alpha)\delta(\beta)$
• **Two physical regions:** $\theta(1-\alpha-\beta)$, $\theta(\alpha+\beta-1)$



Full matching coefficient

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Cross check I	Cross check II
Recursive relation: • $C_{22} = \frac{1}{2}C_{11}\otimes(C_{11}-\beta_0)$; • $\mathcal{C}_{21}^{(2)} = C_{10}\otimes(C_{11}-\beta_0) - (Z_{21} - Z_{21}^{UV})$ Poles are completely cancelled which is a highly non−trivial check.	Reduction to helicity PDF case (forward limit): $C_{\overline{MS}}(\alpha',\mu^2z^2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \delta(\alpha'-\alpha-\beta) C_{\overline{MS}}(\alpha,\beta,\mu^2z^2)$ which is consistent with Ref. [Z. Y. Li, Y. Q. Ma and J. W. Qiu, PRL 126 (2021)].

😀 It is directly applicable to nonsinglet quark unpolarized and helicity GPDs as well.

Momentum space matching coefficient

Taking $p_1 = x\,P, p_2 = (1-x)\,P$

$$\mathbb{C}^{(2)}(\mathrm{x},\mathrm{y},\mu,P_z) = \int_0^1 d\tau \int \frac{d\lambda}{2\pi} e^{i\,(x-\tau)\lambda}\ \mathfrak{C}^{(2)}\left(\tau,\mathrm{y},\frac{\mu^2\lambda^2}{{P_z}^2}\right)$$

$$= \mathrm{a}_s^2(\mu)\begin{cases} h_1(x,y,\mu/P_z) & x<0<y \\ h_2(x,y,\mu/P_z) & 0<x<y \\ h_2(\bar{x},\bar{y},\mu/P_z) & y<x<1 \\ h_1(\bar{x},\bar{y},\mu/P_z) & y<1<x \end{cases} + \mathrm{a}_s^2(\mu)\begin{cases} h_3(x,y,\mu/P_z) & x<0<\bar{y} \\ h_4(x,y,\mu/P_z) & 0<x<\bar{y} \\ h_4(\bar{x},\bar{y},\mu/P_z) & \bar{y}<x<1 \\ h_3(\bar{x},\bar{y},\mu/P_z) & \bar{y}<1<x \end{cases}$$

Quasi-LF correlation $\tilde{h}_{\mathrm{R}}$ $\mathrm{C}(\alpha,\beta,\mu^2 z^2)$

Step I →

Radyushkin, PRD 100 (2019)

Pseudo-DA $\mathfrak{C}(\tau,y,\mu^2 z^2)$

Step II →

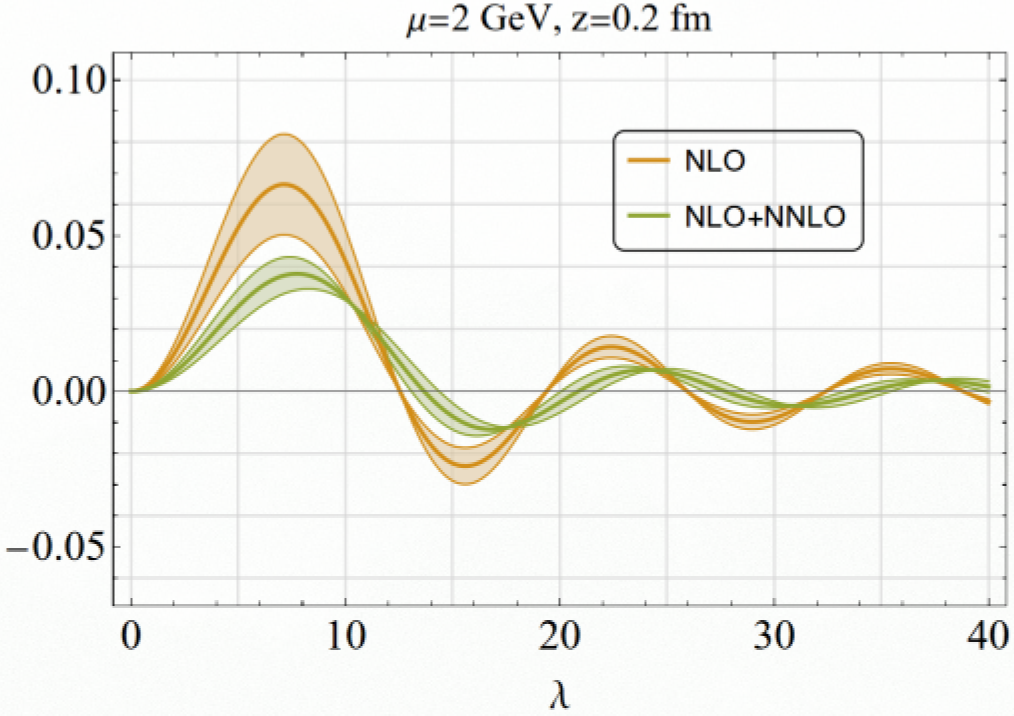
Quasi-DA $\mathbb{C}(x,y,\mu/P_z)$

$$\mathfrak{C}^{(2)}(\tau,\mathrm{y},\mu^2 \mathrm{z}^2) = \int \frac{d\lambda}{2\pi} e^{-i\,\tau\lambda} \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, e^{i\,(\bar{\alpha}y+\beta\bar{y})\lambda}\ \mathrm{C}^{(2)}(\alpha,\beta,\mu^2 \mathrm{z}^2)$$

$$= \mathrm{a}_s^2(\mu)\begin{cases} f_1(\tau,y,\mu^2 \mathrm{z}^2) & y<\tau<1 \\ f_1(\bar{\tau},\bar{y},\mu^2 \mathrm{z}^2) & 0<\tau<y \end{cases} + \mathrm{a}_s^2(\mu)\begin{cases} f_2(\tau,y,\mu^2 \mathrm{z}^2) & \bar{y}<\tau<1 \\ f_2(\bar{\tau},\bar{y},\mu^2 \mathrm{z}^2) & 0<\tau<\bar{y} \end{cases}$$

Numerical test in coordinate space

- Coordinate−space kernel: $C_r(\alpha, \beta, \mu^2 z^2)$ ⟹ **small-z region**
- Asymptotic model: $\phi(x) = 6\,x\,(1 - x)$
- Light−cone correlation: $h^{l.t.}(\alpha, \beta, \lambda) = \int_0^1 dx\, e^{-i\,(1-\alpha-\beta)x\,\lambda + i\,(\frac{1}{2}-\beta)\lambda}\, \phi(x)$



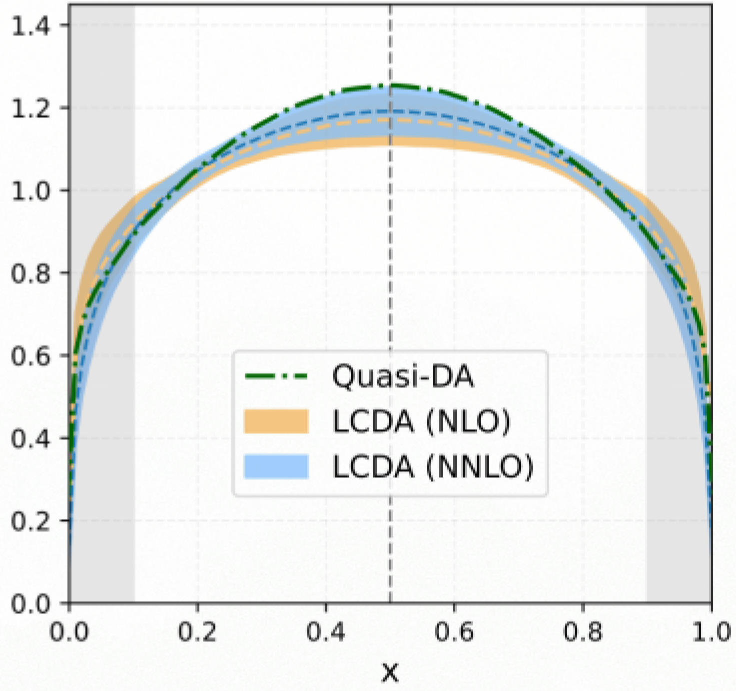
Y. Ji, F. Yao, and J. H. Zhang, arXiv:2504.09367

- Scale variation: $\mu = 2 - 4$ GeV
- NNLO reduces scale dependence.
- NNLO correction: ~30% of the NLO correction.

Numerical test in momentum space

- Momentum space kernel: $\mathbb{C}_{\mathbf{r}}(x, y, \mu, P_z)$.
- Input: lattice quasi-DA data at $a \to 0, P_z = 2.15$ GeV from [J. Hua, F. Yao et.al. (LPC), PRL 129 (2022)].
- Light-cone DAs: $\tilde{\phi}_\pi(x, P_z) = \int_0^1 dy\ \mathbb{C}_{\mathbf{r}}(x, y, \mu, P_z)\, \phi_\pi(x, \mu)$

- NNLO correction: ~10% of the NLO correction.
- NNLO band is slightly narrower
- Endpoint regions are excluded



Y. Ji, F. Yao, and J. H. Zhang, arXiv:2504.09367

Summary and outlook

Euclidean correlators — **Perturbative matching (NNLO)** — **Light-cone DAs**

essential bridge

Result
NNLO for meson DAs | reduced scale dependence / improved precision

Importance
Systematic higher-order matching framework

Outlook
Beyond PDFs and DAs | toward more observables (unpol. and polarized GPDs)

Thank you!