Tropical Integration for Gauge Theory Feynman Integrals

project-trillo / **trillo**

cyw4n9 sqzhang-git

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Motivation

- Phenomenology: Precision physics at the LHC requires high-order predictions: higher-loop and multi-scale Feynman integrals are needed, e.g. **Higgs-plus-jet/electroweak corrections in QCD**.
- Formal Theory: Abundance of data from $\mathcal{N} = 4$ **SYM**: four-point correlators **integrand at 12 loops** [Eden, Heslop, Korchemsky, Sokatchev; Bourjaily, He, Shi, Tang]. **All-loop** result from supersymmetric localisation [Dorigoni, Green, Wen] (similar to Γ_{cusp} [Beisert, Eden, Staudacher]).
- Challenges:

  Analytic methods (IBP reduction, differential equations) are **challenging** $\geq$ **4 loops** for four-particle case.

  Current numerical methods (pySecDec, AMFlow) are also difficult around the same order.

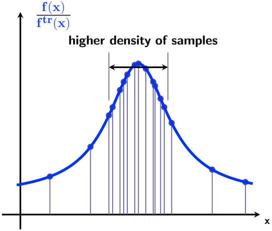
Goal: Develop a new computational tool for obtaining higher-order corrections in gauge theories.

Outline

1. Tropical Monte Carlo Integration
2. Feynman Integrals in QCD
3. Observables in N=4 SYM
4. Conclusions and Outlook

Tropical Monte Carlo Integration

- Tropical Monte Carlo method [Borinsky]: Perform **importance sampling** over tropicalised Feynman integrals.



- Compact (graphical) representation for Symanzik polynomials $(\mathcal{U}, \mathcal{F})$ of Feynman integrals.

$$\mathcal{I}^{(N)} \approx \frac{1}{N} \sum_{n=1}^{N} \frac{f(\boldsymbol{x}^n)}{f^{\mathrm{tr}}(\boldsymbol{x}^n)}, \quad f(\boldsymbol{x}^n) : \mathcal{U}, \mathcal{F}$$

- $f(x_1, ..., x_n) = \sum_k a_k x^k \to f^{\mathrm{tr}}(x_1, ..., x_n) = \mathbf{max}\; x^k$

Current benchmark: `feyntrop`

`feyntrop` **[Borinsky, Munch, Tellander]** **applies to**:

- (Quasi-)**finite** integrals.
- **Euclidean** and **Minkowski** regions.
- Scalar Feynman integrals up to **17 loops**.

Restrictions:

- **numerators** not implemented, limiting its applications to gauge theories (SYM) and gravity.
- linear propagators not implemented.
- non-planar graph in dual-momentum space not implemented.

Goal: Extend tropical Monte Carlo method to include **numerators**.

The program: `Trillo`

- Extended tropical Monte Carlo integration to include **numerator** by "*finding its compact representation*" (derivatives of U, F)

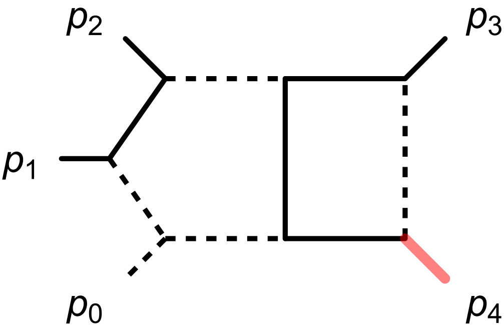
$$U = \det \boldsymbol{C},$$
$$F = U \left[\mathrm{tr} \left((\boldsymbol{B}^T \boldsymbol{C}^{-1} \boldsymbol{B} - \boldsymbol{A}) \boldsymbol{P}_{G_x} \right) + m(x) \right]$$

- Enabled applications to gauge theories (SYM), gravity, and EFTs (linear propagators).
- Verified results up to **eight loops** — excellent agreement with analytic benchmarks.
- Implemented open-source package (GitHub): `Trillo` (**tr**opical **i**ntegration at **l**arge **l**oops **o**ptimised) and a **Web** version

Worked Examples

- Five-point two-loop integrals with off-shell legs and **massive** propagators in the **physical region**.
- **Non-planar** three-loop four-point integrals for subleading colour correction.
- Four-point correlation functions of half-BPS operators in $\mathcal{N} = 4$ SYM at **four loops**—magic identities checked.
- "Integrated" correlation functions from **eight-loop** Feynman integrals—agreement with analytic benchmarks (localisation).
- Observables in SYM, e.g. Wilson loop with Lagrangian insertion: **linear combination** of integrals.

The two-loop pentabox integral



Minkowskian region

$$p_0^2 = 0, \quad p_1^2 = p_2^2 = p_3^2 = m^2 = 1/2, \quad s_{01} = 2.2, \quad s_{02} = 2.3,$$
$$s_{03} = 2.4, \quad s_{12} = 2.5, \quad s_{13} = 2.6, \quad s_{23} = 2.7\,.$$

Result from `trillo`

For the integral with one ISP, $I^{\text{pb}}_{1,1,1,1,1,1,1,1,-1,0}$:

```
threads: 32
epsilon terms: 3, samples: 10000, seed: 0
prefactor: gamma(1+2*eps), kinematics: degenerate minkowskian
eps^0: [ 0.5557435366008827 + -0.39435444975042694 i ]
   +/- [ 0.008487901892240442 + 0.013652005487421282 i ]
eps^1: [ 2.0643121626970475 +    2.702361592308975 i ]
   +/- [ 0.042364943399952025 +  0.10922785723904757 i ]
eps^2: [ -4.788605618185038 +    4.061943410828691 i ]
   +/- [  0.20613490455451738 +  0.45100211018482617 i ]

real 0m17.018s
```

Agrees with `pySecDec`.

For the integral with **two ISPs**, $I^{\text{pb}}_{1,1,1,1,1,1,1,2,-1,-1}$:

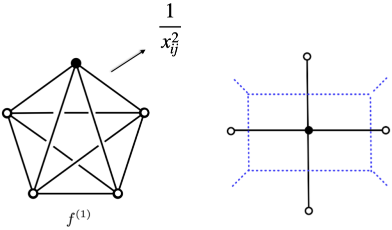
- `trillo` took 24 seconds
- `pySecDec` couldn't go through the pre-processing

Correlation function

- Four half-BPS operators

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle = \text{free} + \frac{I_4(x_i, Y_i)}{x_{12}^4 x_{34}^4}\, T_N\,(U, V; g_{\rm YM})$$

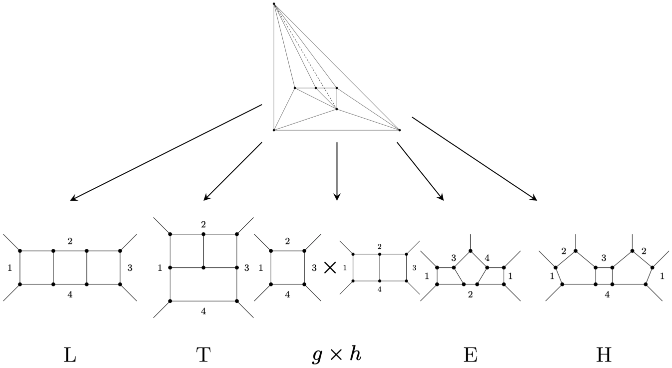
- Loop integrands up to **12 loops** using f-graphs [Eden, Heslop, Korchemsky, Sokatchev; Bourjaily, He, Shi, Tang]
- Integrated result at **3 loops**. [Drummond, Duhr, Eden, Heslop Pennington, Smirnov]
- "Integrated" correlator: **all-loop result** [Dorigoni, Green, Wen]



Picture adapted from [Eden, Heslop, Korchemsky, Sokatchev]

At L=3

- $f^{(3)}$ gives L, T, $g \times h$, E, and H.
- ***L*** (Ladder) is equivalent to ***T*** (Tennis-Court): Magic identities [Drummond, Henn, Smirnov, Sokatchev]

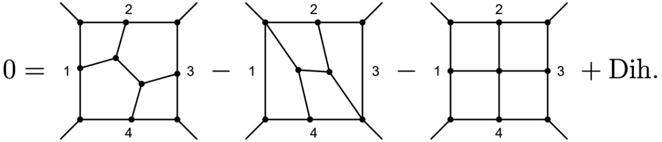


Picture adapted from [He, Jiang, Liu, Zhang]

Numerical results using `Trillo`

Magic identities [Drummond, Henn, Smirnov, Sokatchev], [Caron-Huot, Coronado]

Using `Trillo` at generic kinematic point:



Time	**21.146** s	**7.891** s	**8.681** s

16 threads, 3-digit fit

- Magic identities also imply relation among periods of f-graphs.
- Five-point conformal integrals [Kuo, Yang] were also checked!

“Integrated” correlation function

All-loop result [Dorigoni, Green, Wen]

$$\mathcal{C}_{SU(N)} = \int dM_{U,V}\ T_N\,(U, V; g_{\rm YM})$$

At small $g_{\rm YM}$, expanding in $a = \frac{\lambda}{4\pi}$

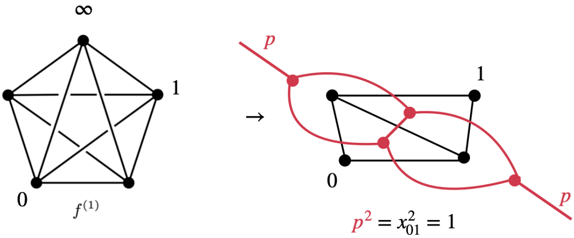
$$\mathcal{C}^{\rm pert}_{SU(N)} \sim \frac{3\zeta(3)a}{2} - \frac{75\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)(1+\frac{2}{7N^2})a^4}{32} + \ldots$$

[Wen, SQZ]: $\mathcal{C}^{\rm pert}_{SU(N)}$ are “periods” of f-graphs

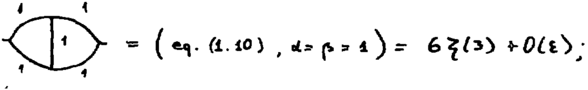
$$\text{Period}\left(f^{(L)}_\alpha\right) := \int_{(0,1,\infty)} d^4x_1 \cdots d^4x_{4+L} f^{(L)}_\alpha(x_1, \ldots, x_{4+L})$$

[Brown; Panzer; Schnetz]

Integrated correlator: L=1



- propagator-type integral [Kazakov]



Results up to L=4

Using `HyperlogProcedure` [Schnetz]

$L = 1:$

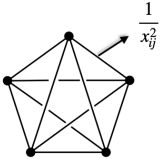
$$\frac{-1}{1!(-4)^1} \times \mathscr{P}_{f^{(1)}} = \boxed{\frac{3\zeta(3)}{2}},$$

$L = 2:$

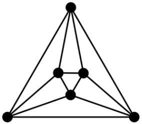
$$\frac{-1}{2!\,(-4)^2} \times \mathscr{P}_{f^{(2)}} = \boxed{\frac{-75\zeta(5)}{8}},$$

$L = 3:$

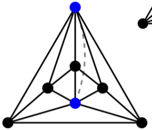
$$\frac{-1}{3!\,(-4)^3} \times \mathscr{P}_{f^{(3)}} = \boxed{\frac{735\zeta(7)}{16}},$$



$6\zeta(3)$



$\frac{6!}{48} \times 20\zeta(5)$

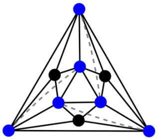


$\frac{7!}{20} \times 70\zeta(7)$

- $L = 4$ (three f-graphs) also verified
- higher loops are given by **MZVs** (even beyond)

Challenge at L=5

`HyperlogProcedure` evaluates **6** out of **7** graphs, except



$$= 9! \times \frac{1}{6} \times \Big[120\zeta(3)^2\zeta(5) + 400\zeta(5)^2 - 10410\zeta(11) - \frac{8\pi^6}{21}\zeta(5) + \frac{24\pi^4}{5}\zeta(7) + 1080\pi^2\zeta(9) + 144\zeta(5,3,3)\Big]$$

$$\sum c_i P_i^{(L)} = \zeta(2L-1)$$

- "Prediction" made by [Wen, SQZ]
- Confirmed numerically by `Trillo`

`Trillo` returns

`eps`^{0}`:  [967.13123 + 0 i] +/- [0.009011969 + 0 i]`

Results up to L=7

$$\mathcal{G}_{N,0}(\tau_2) = (N^2-1)\left[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32} \right.$$
$$+\frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024}$$
$$\left. +\frac{32207175\,\zeta(15)\left(1+\frac{55}{13}N^{-2}+\frac{332}{143}N^{-4}\right)a^7}{2048} + \mathcal{O}(a^8)\right],$$

- a^5 predicted in [Wen, SQZ '22]: checked
- a^6 predicted in [Brown, Heslop, Wen, Xie '23]: checked
- **Agreement at a^7**: summing over 127 **eight-loop** integrals, each with $\mathcal{O}(10^{10})$ sampling points: 8 days with 40 cores (4-digit fit).

Summary: the 1st correlator

Expansion order	$a^1_{G_N}$	$a^2_{G_N}$	$a^3_{G_N}$	$a^4_{G_N}$	$a^5_{G_N}$	$a^6_{G_N}$	$a^7_{G_N}$
# all integrals	1	1	1	3	7	26	127
# integrals (analytic)	1	1	1	3	**6**	**16**	N/A
# integrals (numeric)	1	1	1	3	7	26	127

- $a^7_{SU(N)}$: All planar sector checked (numerically).

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# integrals (numeric)	1	1	1	3	7	26	127

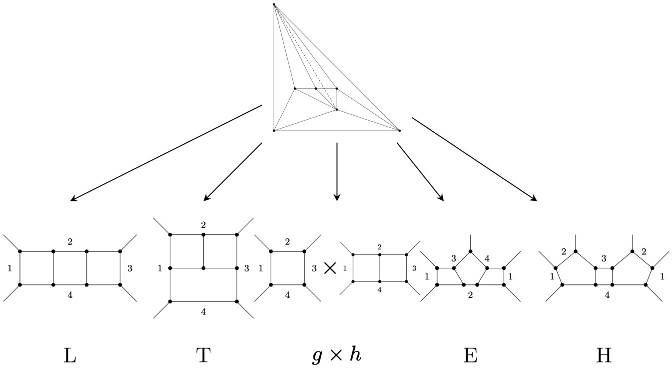
- $a^7_{SU(N)}$: All planar sector checked (numerically).

- **Non-planar** sector:
- $a^4_{SU(N)}$: $1/N^2$ sector agrees with [SQZ '24] (data from [Fleury, Pereira])
- $a^5_{SU(N)}$: New $1/N^2$ data from [Bargheer, Bekov '25]: to be checked.

The second correlator

Attach a one-loop box to the first correlator

$$\mathcal{H}_{G_N} = \int dM_{U,V}(1+U+V)\bar{D}_{1,1,1,1}\ T_N(U,V;g_{\rm YM})$$



Picture adapted from [He, Jiang, Liu, Zhang]

Results up to L=4

$$\mathcal{H}^{pert}_{G_N} \sim$$

$$-60\zeta(5)a + \frac{3\left(36\zeta(3)^2 + 175\zeta(7)\right)a^2}{2} - \frac{45\left(20\zeta(3)\zeta(5) + 49\zeta(9)\right)a^3}{2}$$

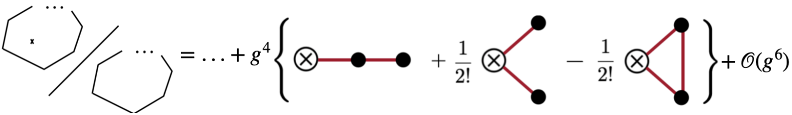
$$+ \frac{45\left(340\zeta(5)^2 + 588\zeta(3)\zeta(7) + 1617\zeta(11) + P_{G_N,1}\left(840\zeta(5)^2 + 1617\zeta(11)\right)\right)a^4}{16}$$

$$+ \mathcal{O}(a^5)$$

- a^4: Summing over **6-loop non-planar** integrals agrees with [Dorigoni, Green, Wen]
- Only *products* of single zetas, MZVs cancel

Wilson loop with Lagrangian insertion

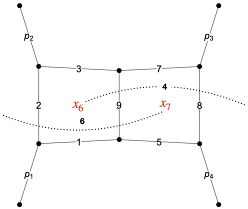
- Finite quantities, similar to QCD hard functions ($3n{-}11$ vars).
- Integrand: logarithm of scattering amplitudes via Wilson-loop/Amplitude duality [Drummond, Korchemsky, Henn, Sokatchev], [Brandhuber, Heslop, Travaglini]
- All-loop leading singularities [Brown, Henn, Mazzucchelli, Trnka]
- Novel "negative geometry" (IR-finite) decomposition
  [Arkani-Hamed, Henn, Trnka]
  [Dixon, Oktem, Paranjape, Trnka, Xu, SQZ, **to appear**]



Four-point two-loop correction

$$\mathcal{I}_3 = I_3(x_5, x_6, x_7) + 2 I_1(x_5) I_1(x_6) I_1(x_7) \\ - I_1(x_5) I_2(x_6, x_7) - I_1(x_6) I_2(x_7, x_5) - I_1(x_7) I_2(x_5, x_6)$$

I_L: L-loop f-graph. Send $x_5 \to \infty$, and $x_{i,i+1} \to 0$.



$\mathcal{I}_3|_{x_5 \to \infty}$: Linear combination of 49 terms (individually divergent) in the "extended" double-box topology

$$s^2 \, J[1,0,1,1,1,1,1,0,0] + \ldots - s \, J[0,0,1,1,1,1,1,-1,1]$$

The red indices are ISPs for the "usual" double-box topology.

Integrated result

At the kinematic point $x = t/s = 1/3$

`Trillo` returns

`eps`^{0}`:  [306.92442 + 0 i] +/- [1.946962 + 0 i]`

`real 0m7.631s`

Analytic result: [Alday, Henn, Sikorowski]

$$F^{(2)}(x) =$$

$$- \frac{\pi^2}{16} H_{0,0} - \frac{3}{16} H_{0,0,0,0} + \frac{\pi^2}{32} H_{0,\text{-}1} + \frac{1}{16} H_{0,\text{-}1,0,0} + \frac{\pi^2}{32} H_{\text{-}1,0}$$
$$+ \frac{1}{16} H_{\text{-}1,0,0,0} - \frac{\pi^2}{16} H_{\text{-}1,\text{-}1} - \frac{1}{8} H_{\text{-}1,\text{-}1,0,0} + \frac{\zeta_3}{16} H_0 - \frac{\zeta_3}{8} H_{-1} - \frac{107\pi^4}{5760} .$$

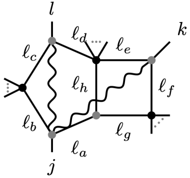
$F^{(2)}(1/3) = 307.87044044$

Conclusions

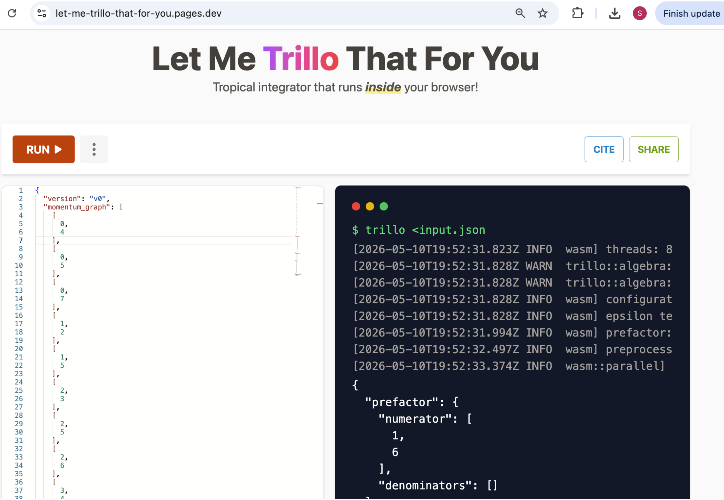
- A new tropical integration method for Feynman integrals with **numerators**
- Applicable to integrals in QCD, e.g. Higgs + jet production.
- Benchmark examples for integrated correlators in SYM: **eight-loop** Feynman integrals + non-planar sector
- Applicable to finite observable in SYM, e.g. Wilson loops with Lagrangian insertion
- Applicable to integrals with linear propagators in gravitational wave scattering processes.

Outlook

- Finite Feynman Integrals [Gambuti, Kosower, Novichkov, Tancredi]
- Electroweak Corrections (multi-scale): hard to compute, but less IR divergence [FCC Collaboration: 2505.00272].
- Wilson loop with Lagrangian insertion: *Symbol* results at 6-point 2-loop [Carrolo, Chicherin, Henn, Yang, Zhang] 5-point 3-loop [Chicherin, Henn, Xu, Zhang, SQZ] Positivity property [Chicherin, Henn]
- Prescriptive unitarity integrals [Bourjaily, Herrmann, Trnka] applications in QCD bootstrap [Carrolo, Chicherin, Henn, Yang, Zhang] [See talk by Sergio]

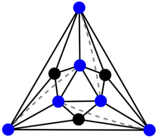
$I^{\rm pb}_{n,jkl} :=$ 

Web Version



https://let-me-trillo-that-for-you.pages.dev/

Challenge at L=5



$$= 9! \times \frac{1}{6} \times \Big[120\zeta(3)^2\zeta(5) + 400\zeta(5)^2 - 10410\zeta(11) - \frac{8\pi^6}{21}\zeta(5) + \frac{24\pi^4}{5}\zeta(7) + 1080\pi^2\zeta(9) + 144\zeta(5,3,3) \Big]$$

$$\sum c_i P_i^{(L)} = \zeta(2L-1)$$

Analytic values: 967.1326788...

Thank you!