

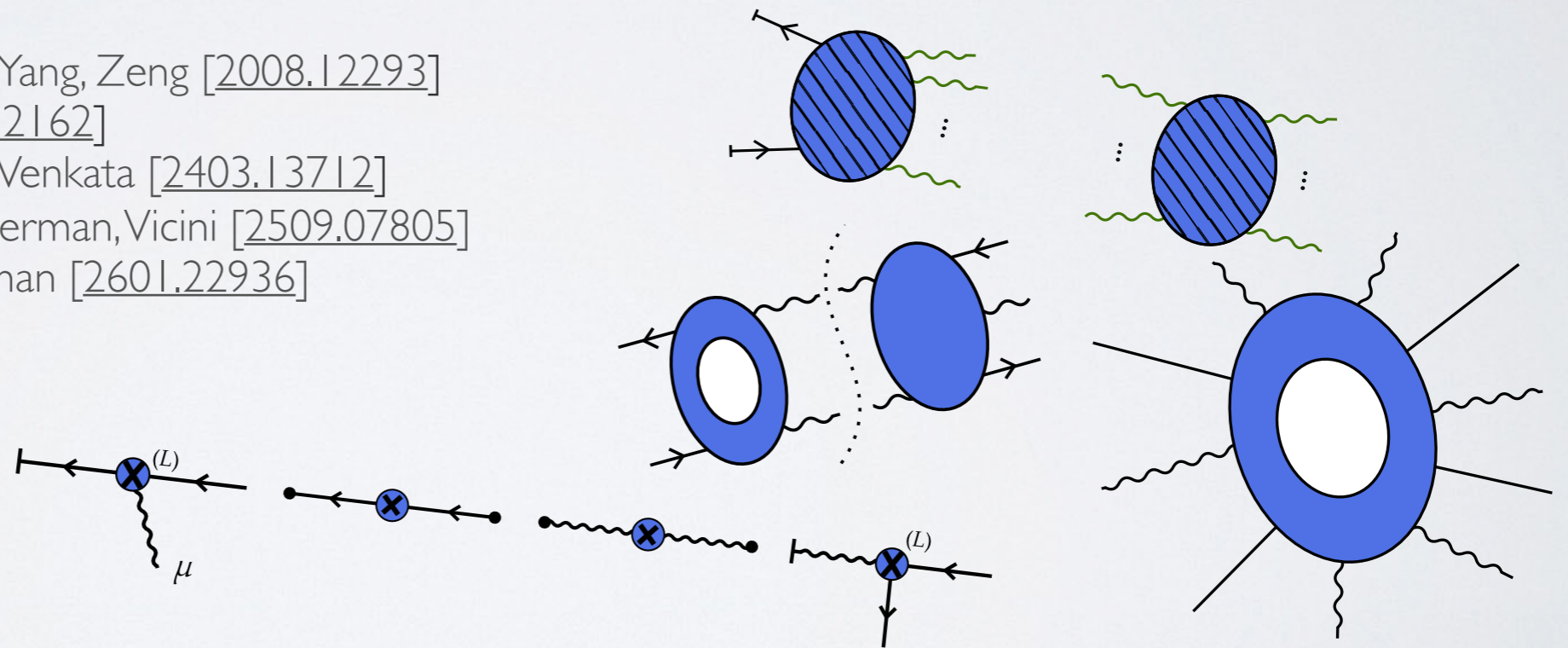
ON THE LOCAL IR STRUCTURE OF SCATTERING AMPLITUDES

To appear in: Anastasiou, Biello, Favorito, **GG**, Sahoo, Sterman [2650.ONISH]

.....

Building on:

- Anastasiou, Haindl, Sterman, Yang, Zeng [2008.12293]
- Anastasiou, Sterman [2212.12162]
- Anastasiou, Karlen, Sterman, Venkata [2403.13712]
- Anastasiou, Karlen, Sahoo, Sterman, Vicini [2509.07805]
- Anastasiou, Karlen, Ma, Sterman [2601.22936]



Amplitude representations whose integrand factorises in all soft & collinear limits for general two-loop QED processes.

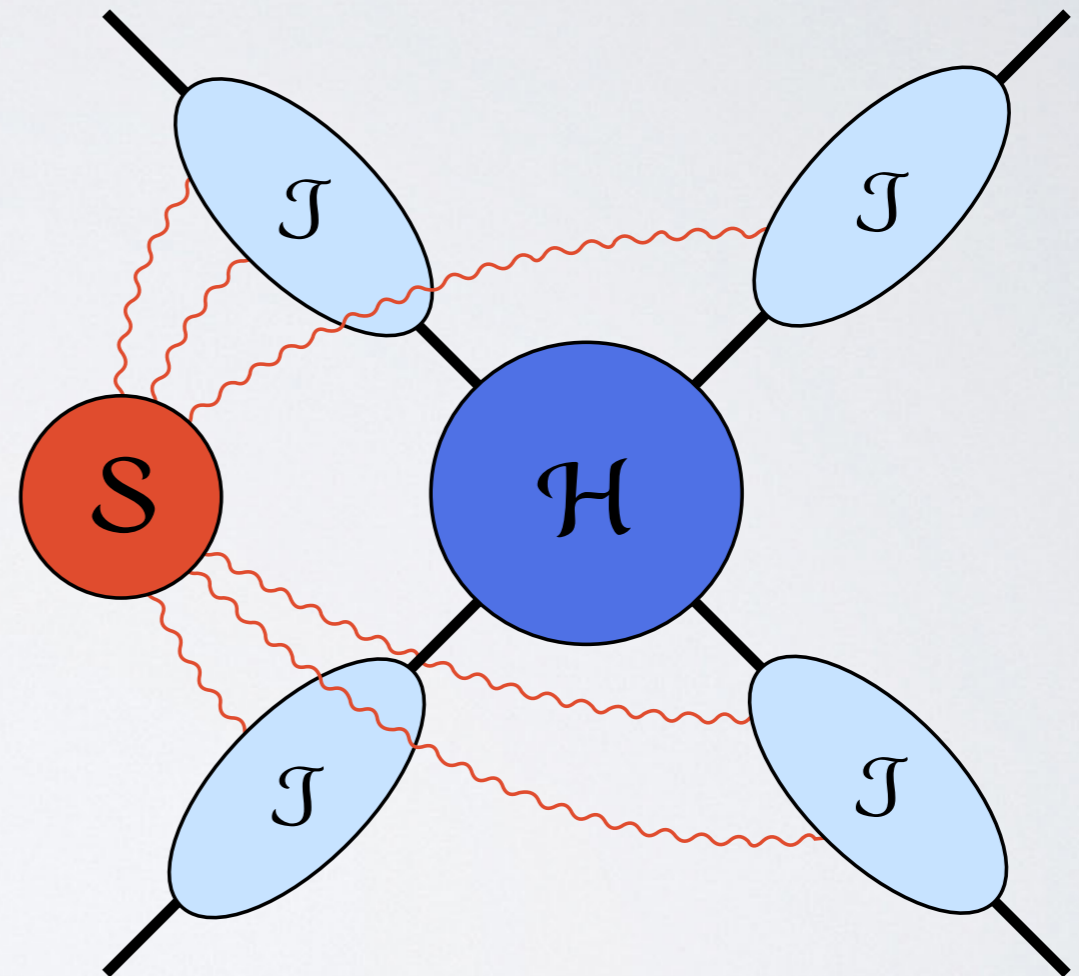
A local Catani-like subtraction formula.

Both can be generalised.

AMPLITUDE *IR* FACTORISATION

$$A = J_1 J_2 \dots J_n \mathcal{S} \cdot \mathcal{H}$$

Collins, Soper, Sterman ['88];
Catani: [9802439];
Kidonakis, Oderda, Sterman [9803241];
Sterman, Tejeda-Yeomans [0210130];
Gardi, Magnea: [0901.1091],[0908.3273];
Becher, Neubert: [0903.1126];
Feige, Schwartz: [1403.6472];
+ many more!!



IN SHORT

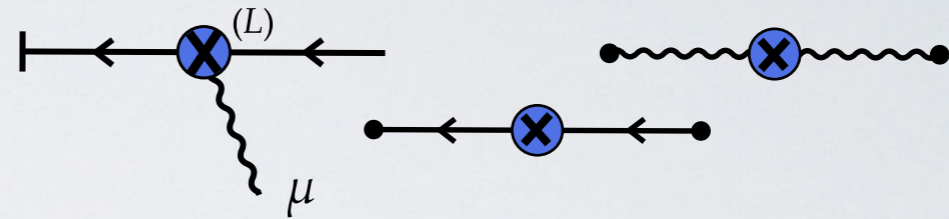
IN SHORT

I. local IR factorisation

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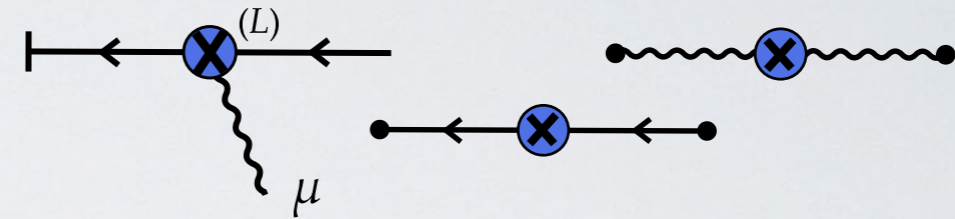
→ boundary terms



IN SHORT

1. local IR factorisation

→ boundary terms

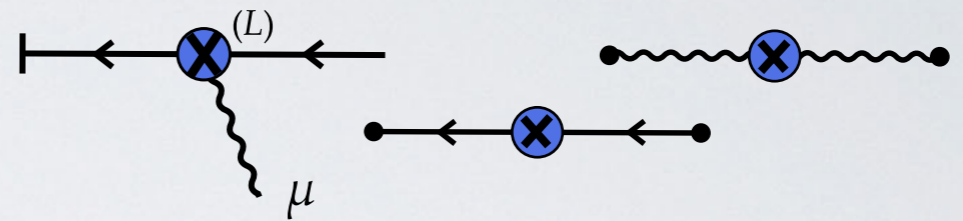


2. finite remainders

IN SHORT

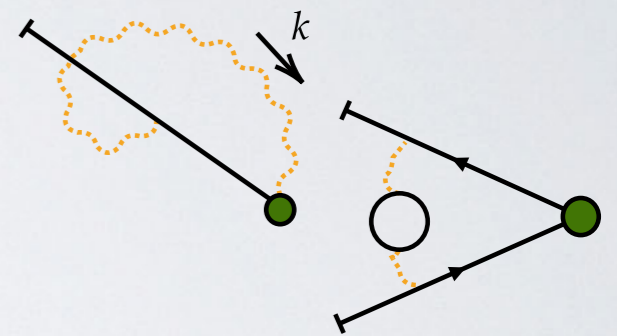
1. local IR factorisation

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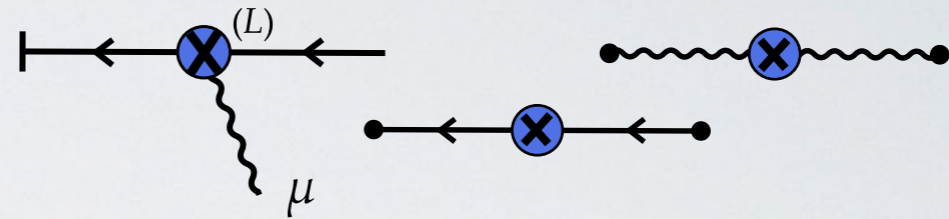
→ via simple+universal subtractions



IN SHORT

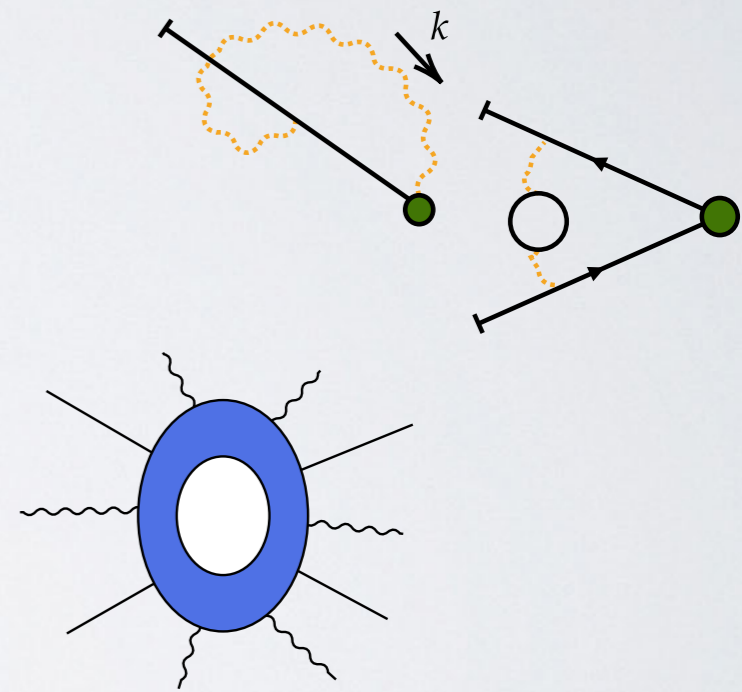
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→ boundary terms



2. finite remainders

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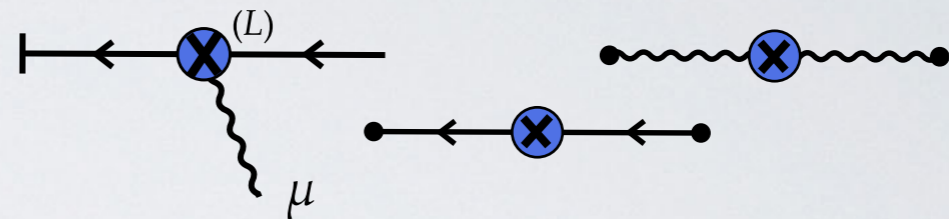


3. numerical integration

IN SHORT

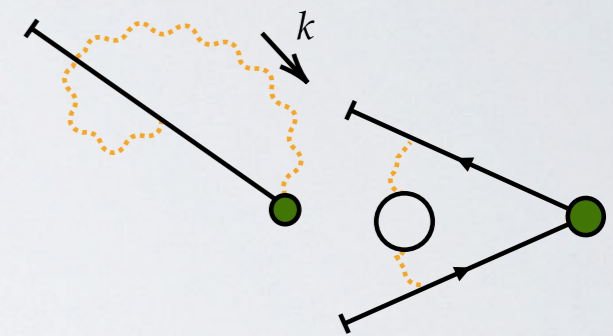
1. local IR factorisation

→ boundary terms



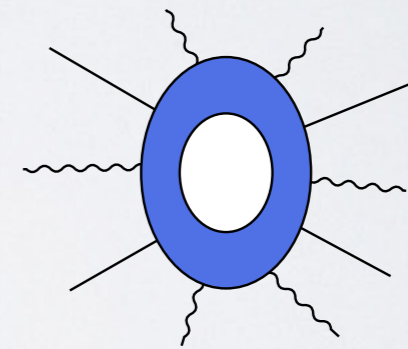
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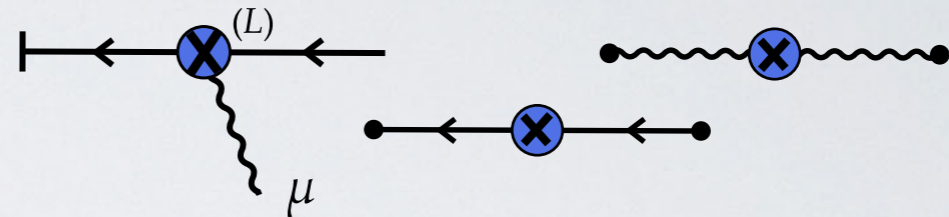
→ replace the many-scales problem
with a purely computational one



IN SHORT

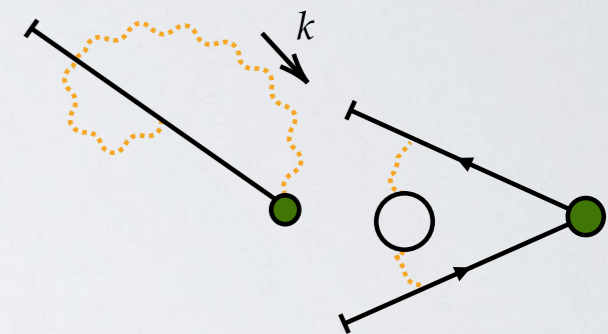
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→ boundary terms



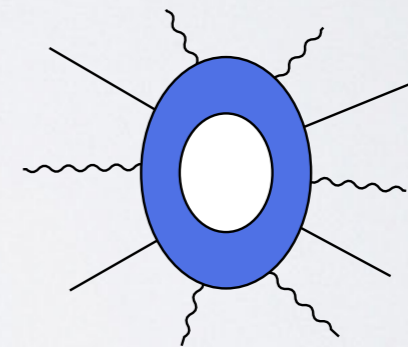
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3. numerical integration

→ replace the many-scales problem
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(2loop) QED \implies

- Relevant for future e^+e^- colliders
- All kinematical complexity of QCD

THE ENGINE_(S) OF LOCAL FACTORISATION

1. tree-level Ward identity

2. longitudinal polarisations

THE ENGINE_(S) OF LOCAL FACTORISATION

1. tree-level Ward identity

$$(r + q)^{-1} q r^{-1} = r^{-1} - (r + q)^{-1}$$

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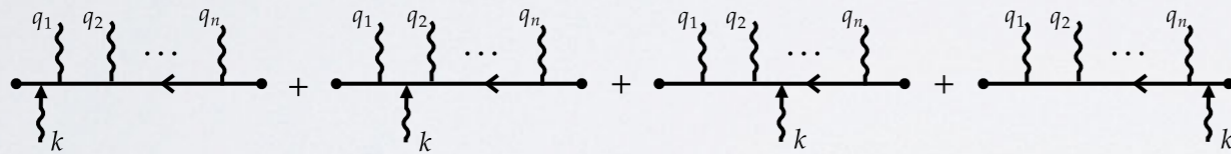
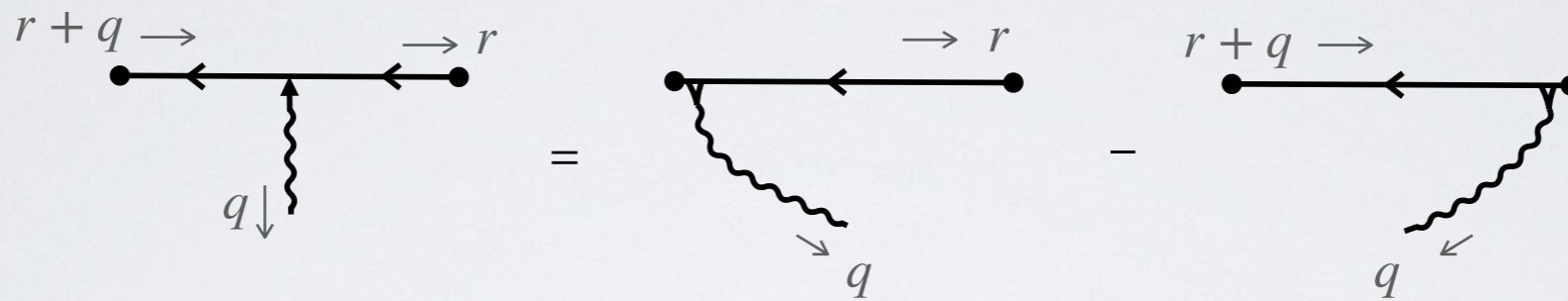
The diagram illustrates the tree-level Ward identity. On the left, a propagator with momentum $r+q$ on the left and r on the right is shown. A wavy line labeled q is attached to the internal line. This is equal to the difference between two propagators: one with momentum r and one with momentum $r+q$, both with a wavy line labeled q attached to the internal line.

2. longitudinal polarisations

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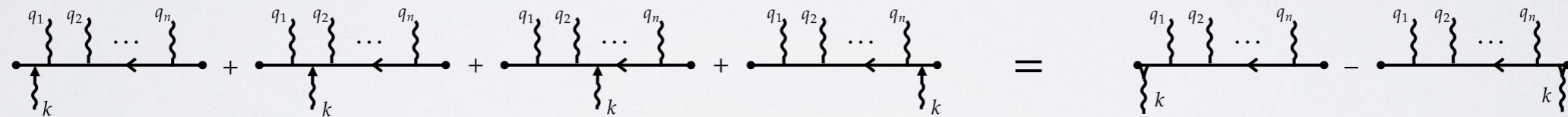
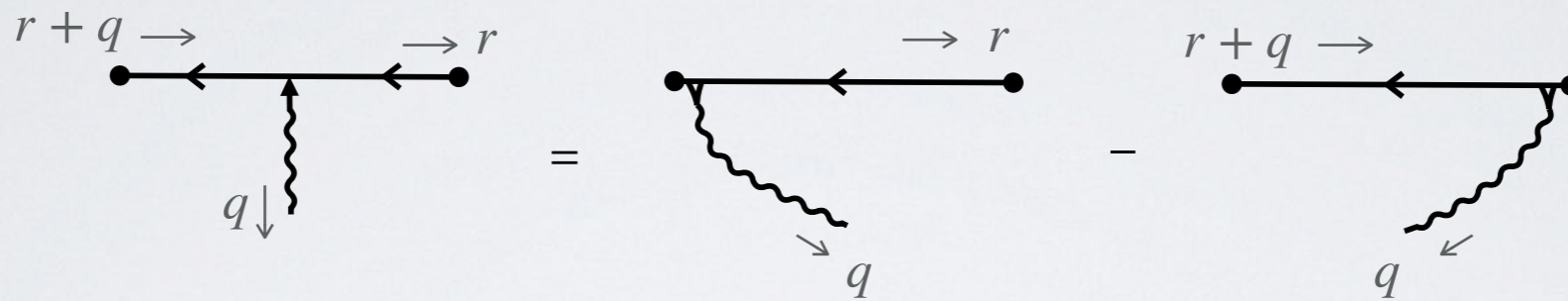


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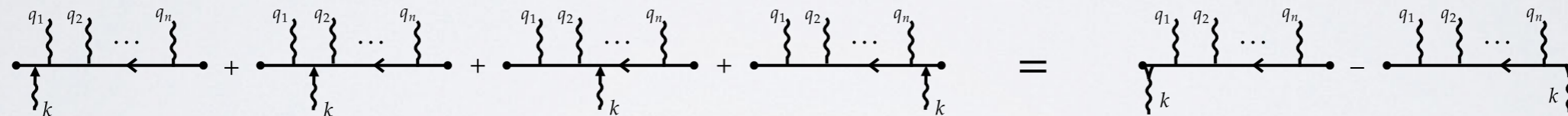
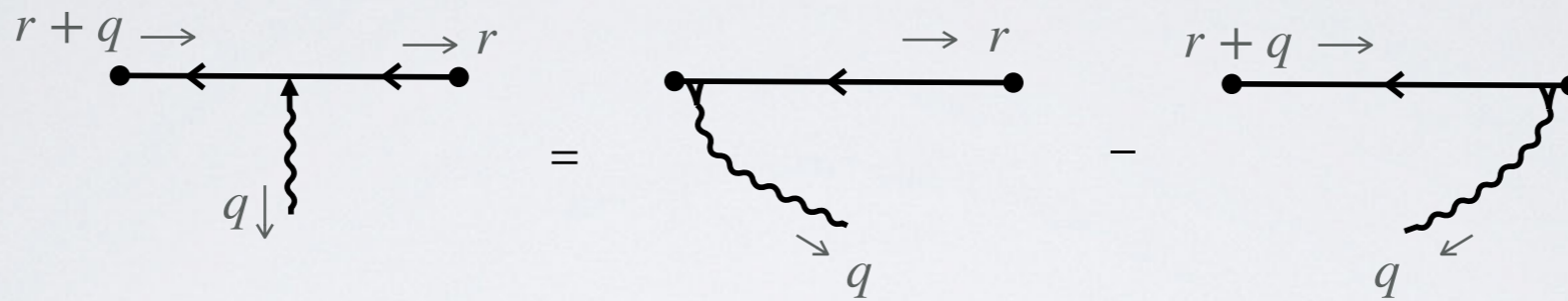


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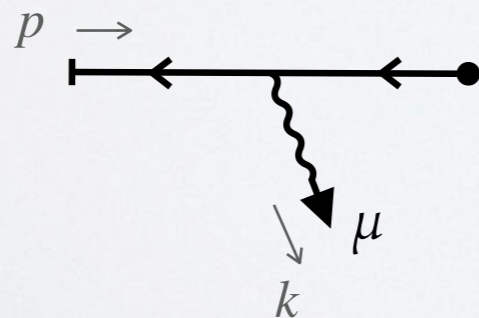
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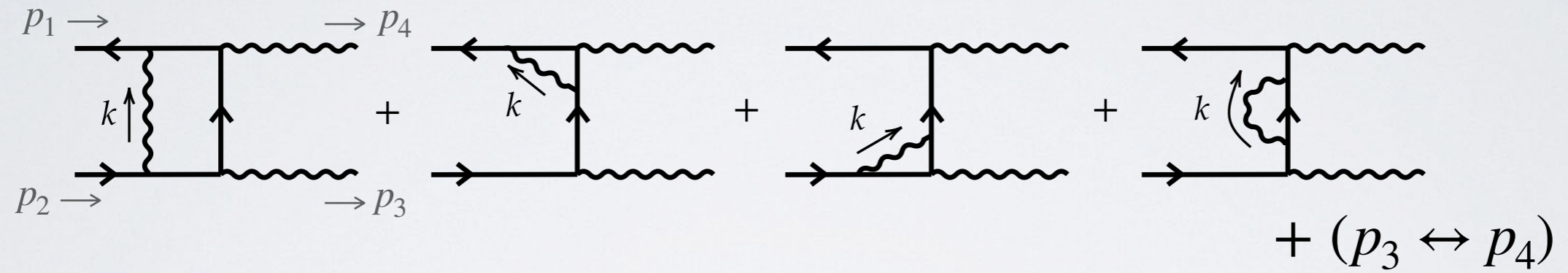
$$k^\mu \rightarrow \alpha p^\mu$$



$$\bar{u}(p)(-ie\gamma^\mu) \frac{-i(\not{p} - \not{k})}{(p - k)^2} \propto k^\mu \bar{u}(p)$$

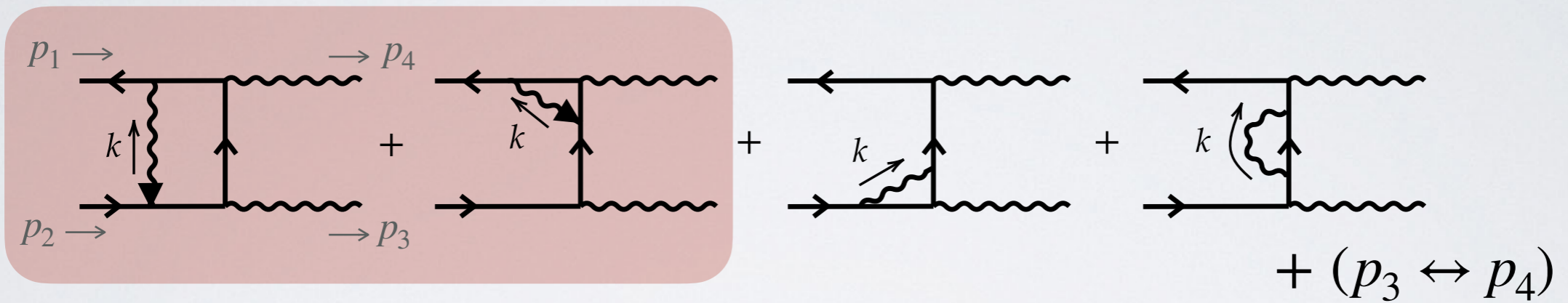
A COLLINEAR FACTORISATION EXAMPLE

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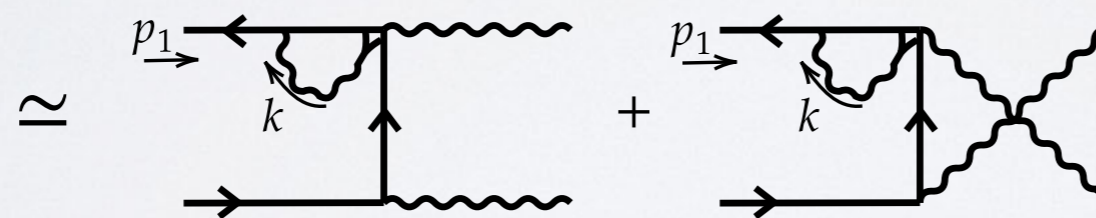
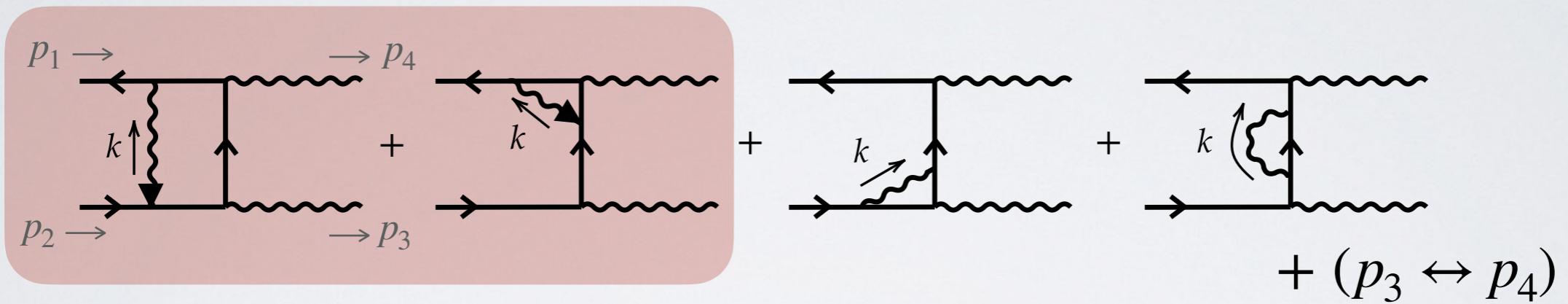
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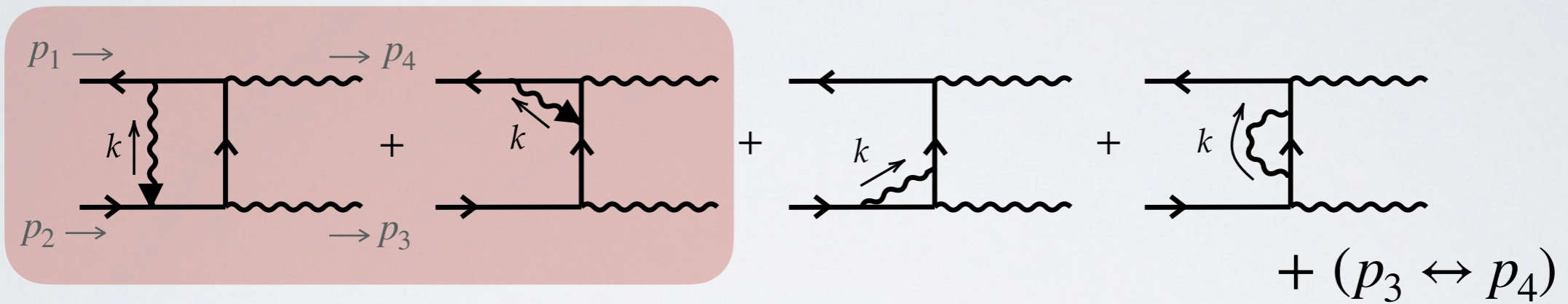
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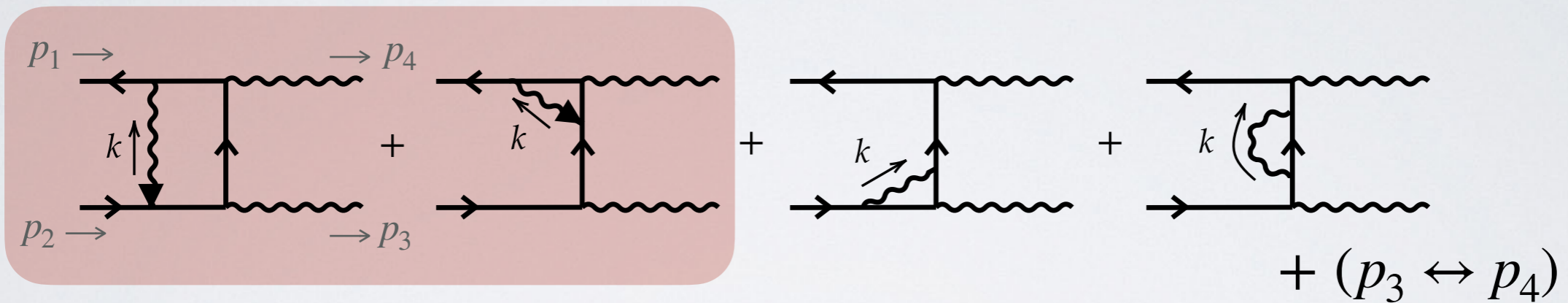
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$$\simeq \text{[Diagram 1]} + \text{[Diagram 2]} = B(p_1, k) \times A_{ee \rightarrow \gamma\gamma}^{(0)}$$

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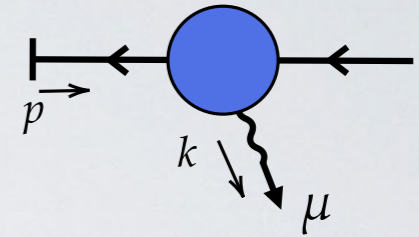
works* for all one-loop amplitudes

*provided a proper loop-momentum routing is imposed

OUR WISH LIST

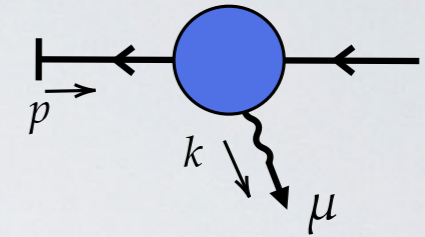
OUR WISH LIST

- collinear photons are longitudinally polarised

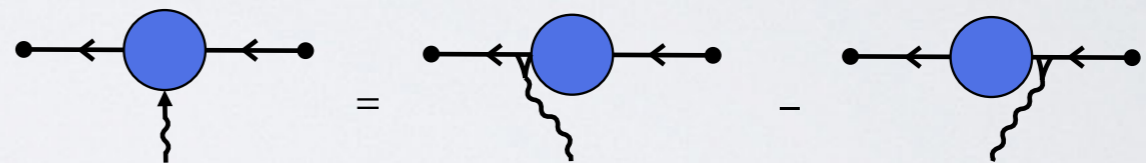


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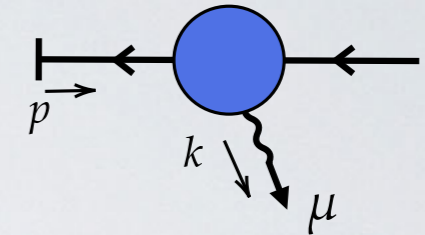


- Ward identities are locally preserved

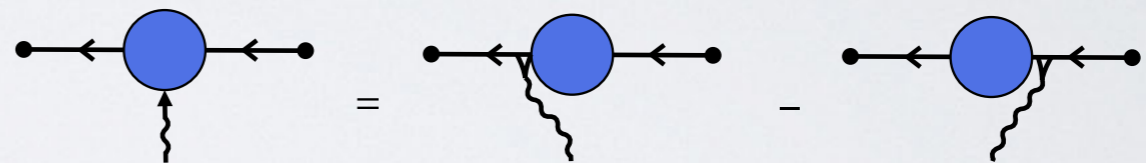


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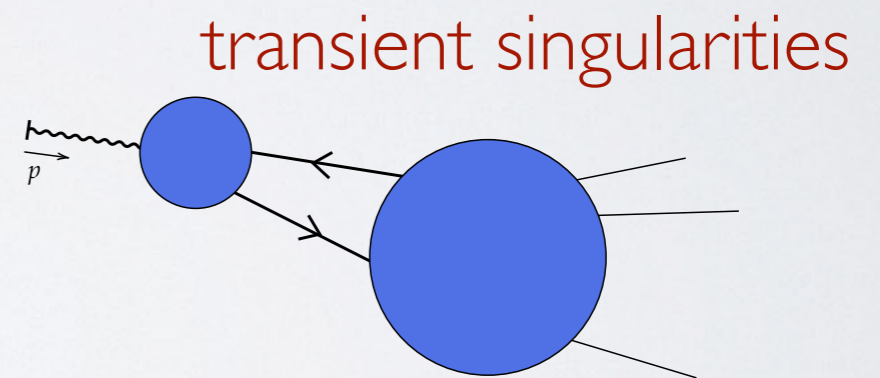
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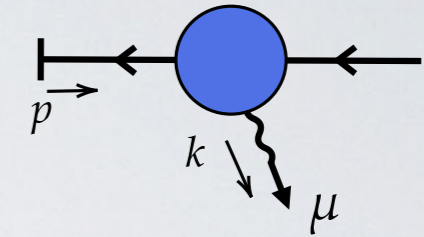
- spurious collinear divergences are absent



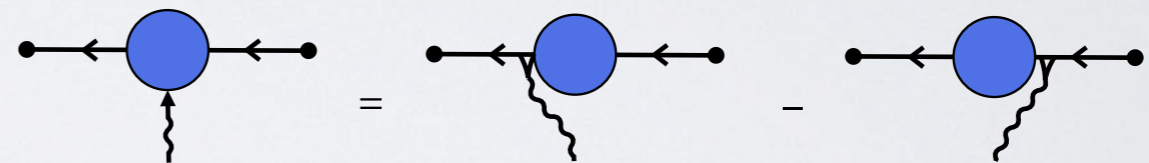
Anastasiou, Karlen, Sahoo, Sterman, Vicini: [2509.07805]

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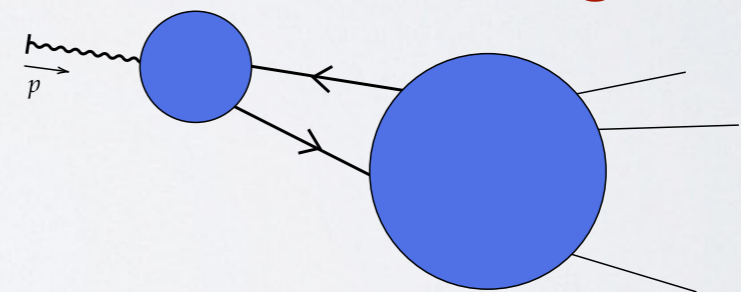


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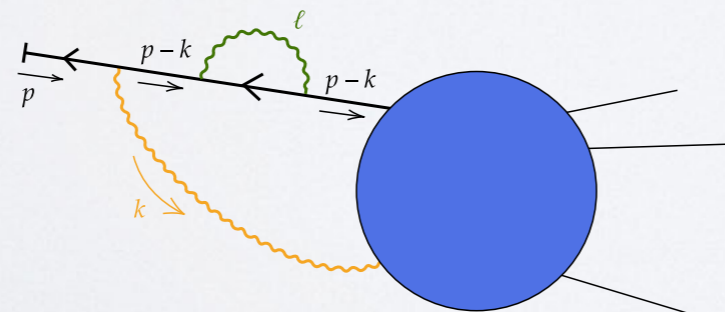


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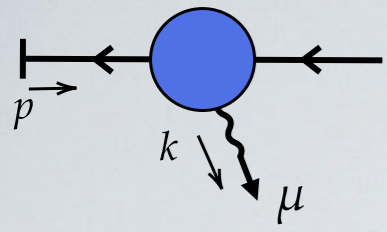
transient singularities



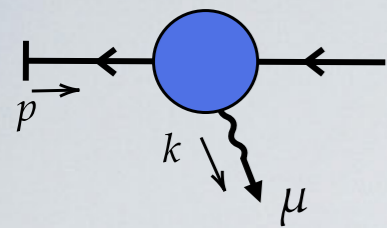
- all singularities are logarithmic



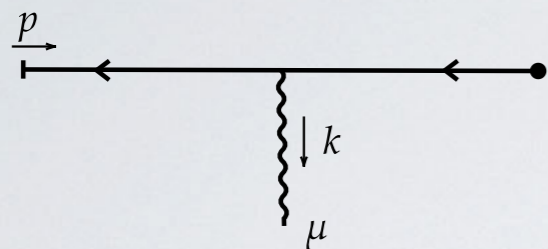
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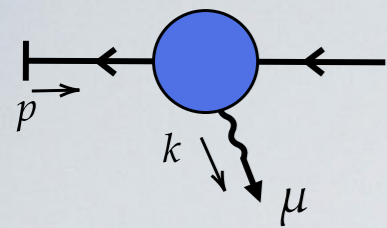


LOOP POLARISATIONS

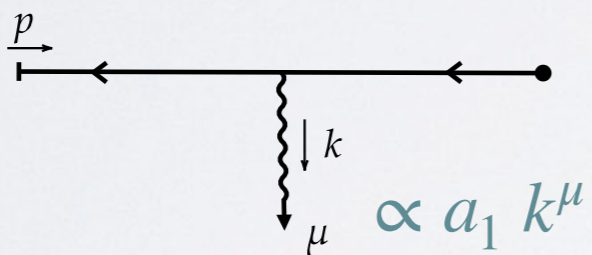
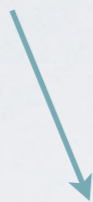
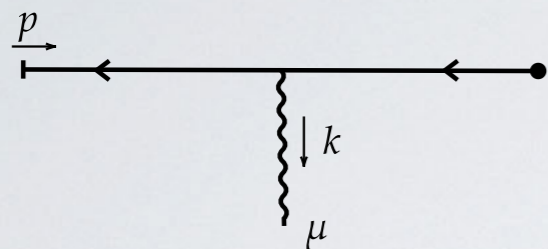


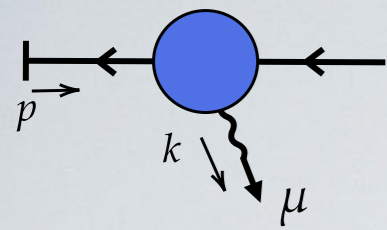
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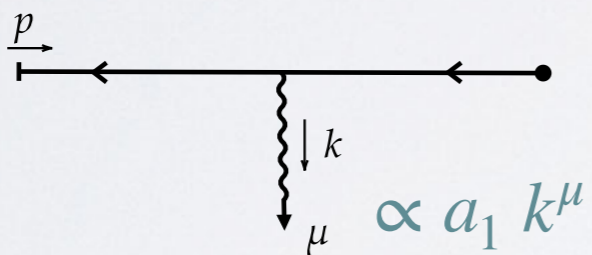
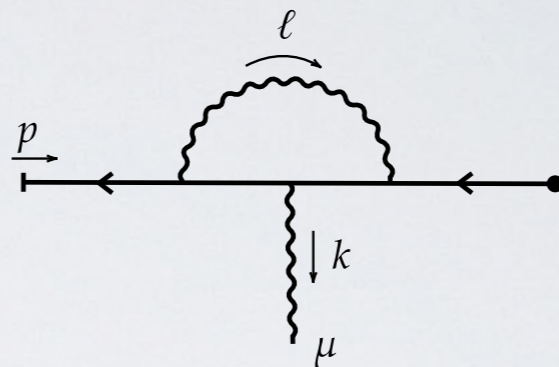
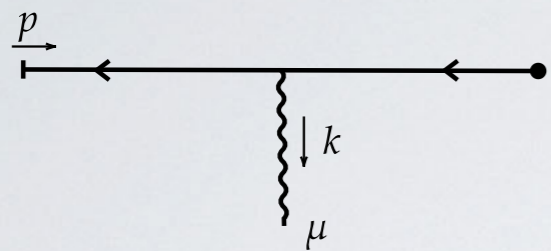


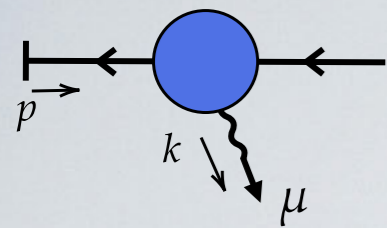
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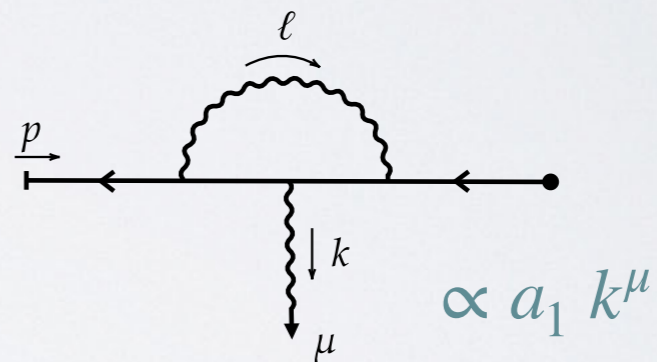
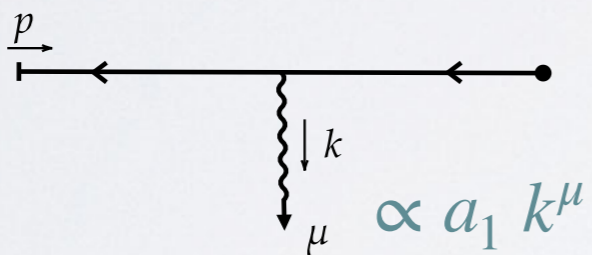
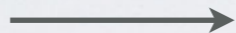
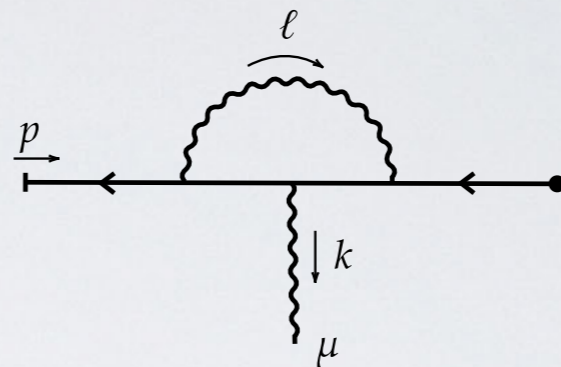
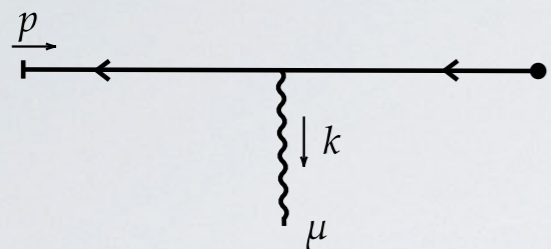


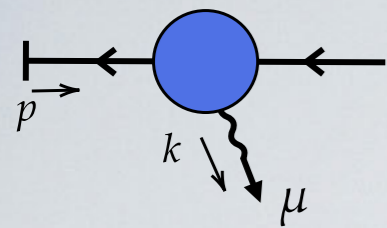
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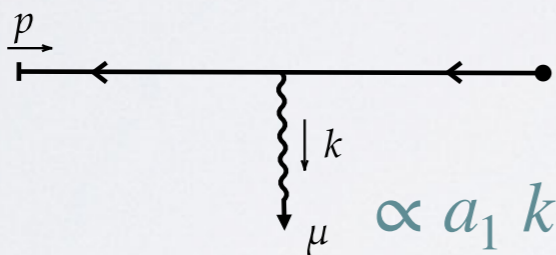
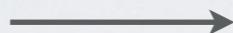
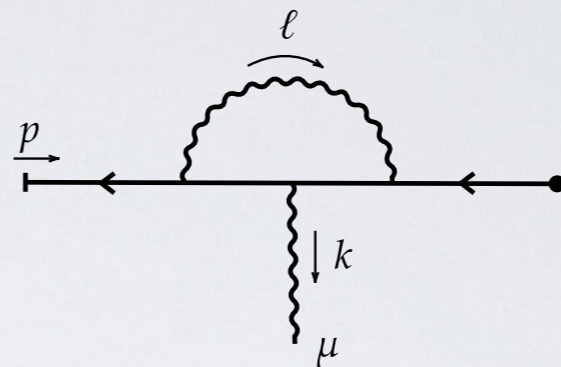
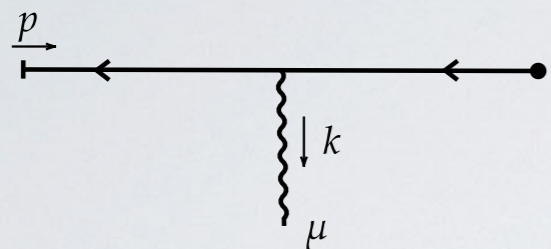


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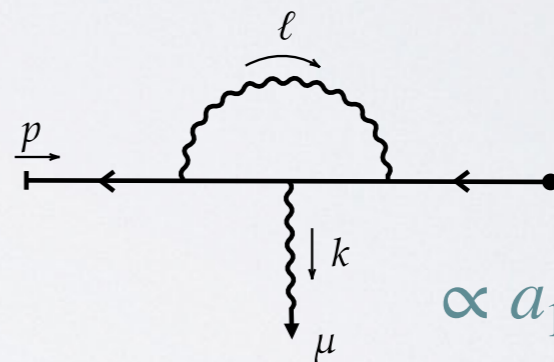




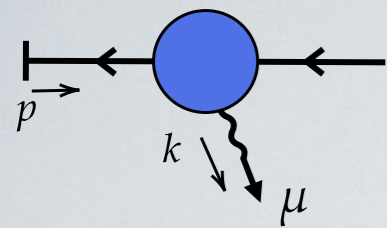
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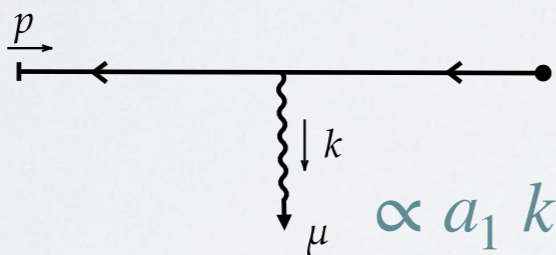
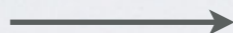
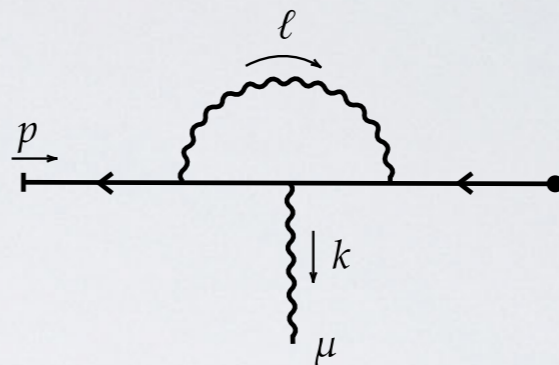
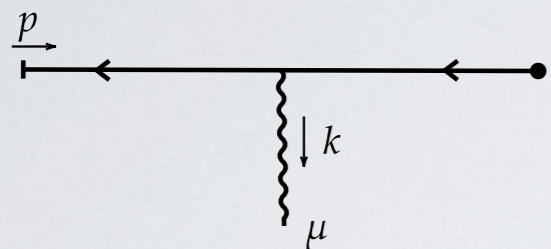
$$\propto a_1 k^\mu$$



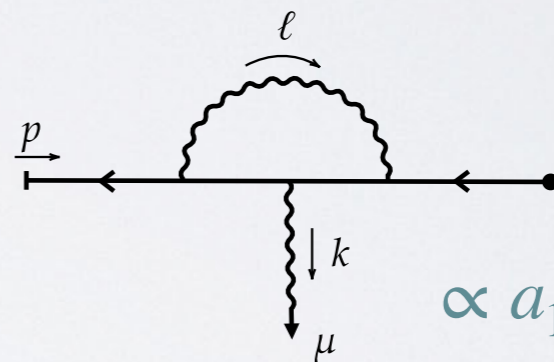
$$\propto a_1 k^\mu + a_2 \ell^\mu$$



LOOP POLARISATIONS



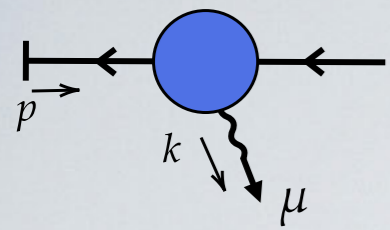
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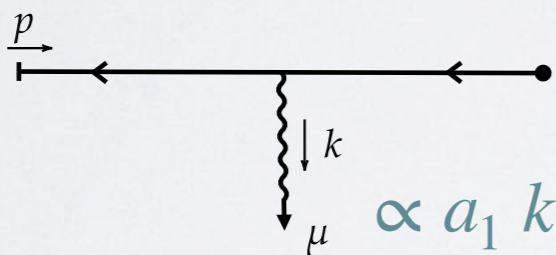
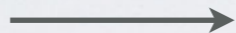
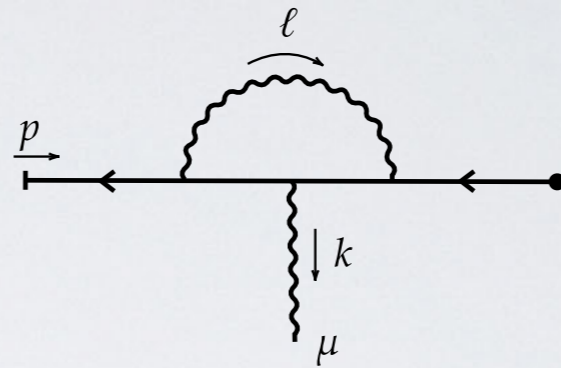
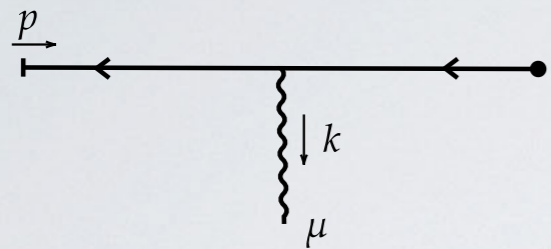
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loop polarisation

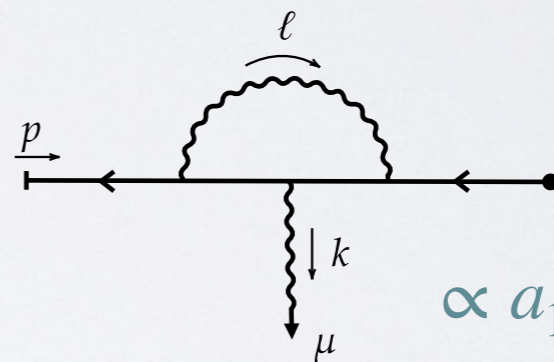




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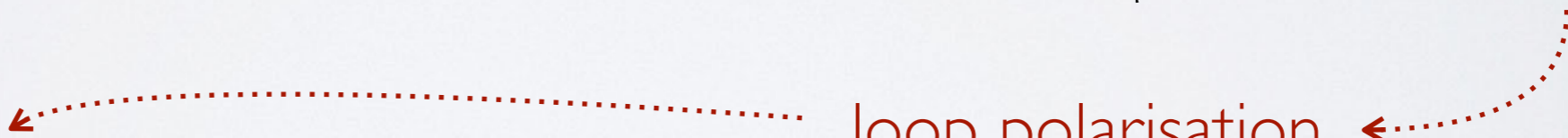
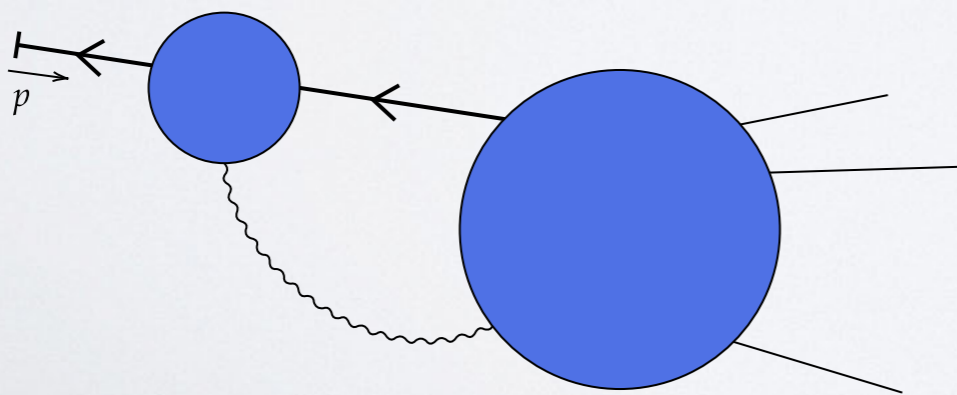


$$\propto a_1 k^\mu$$

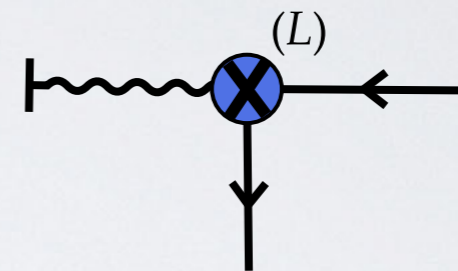
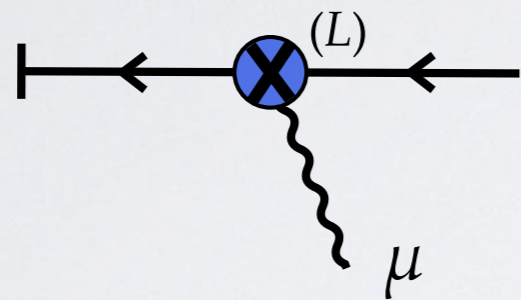


$$\propto a_1 k^\mu + a_2 \ell^\mu$$

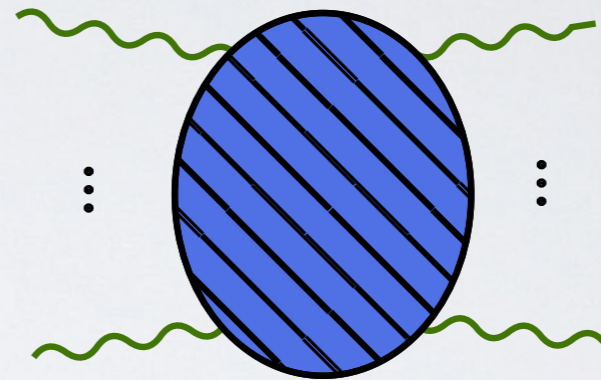
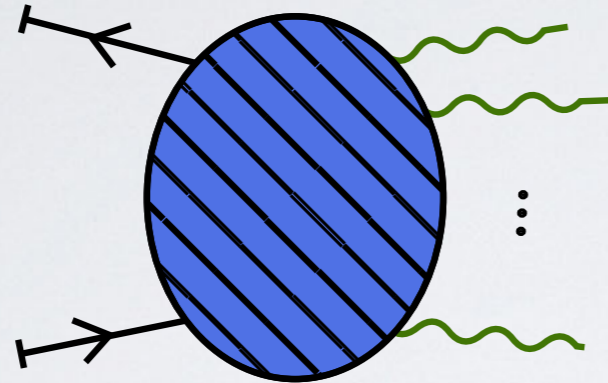
loop polarisation



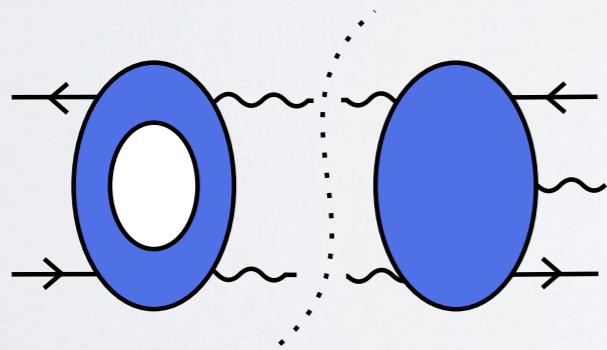
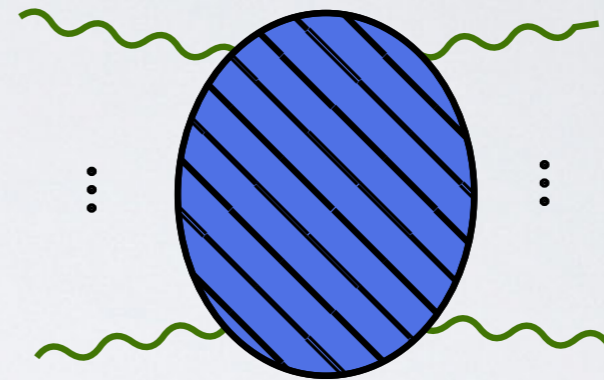
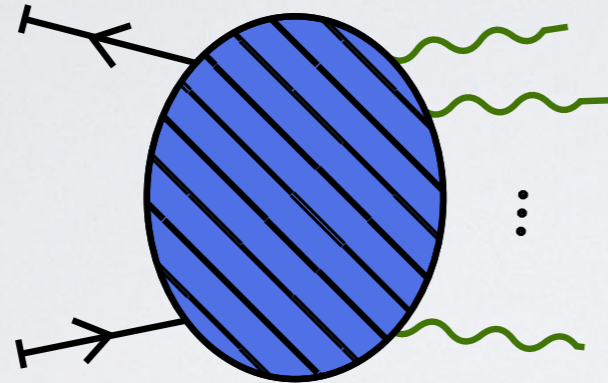
FEYNMAN-RULE COUNTERTERMS

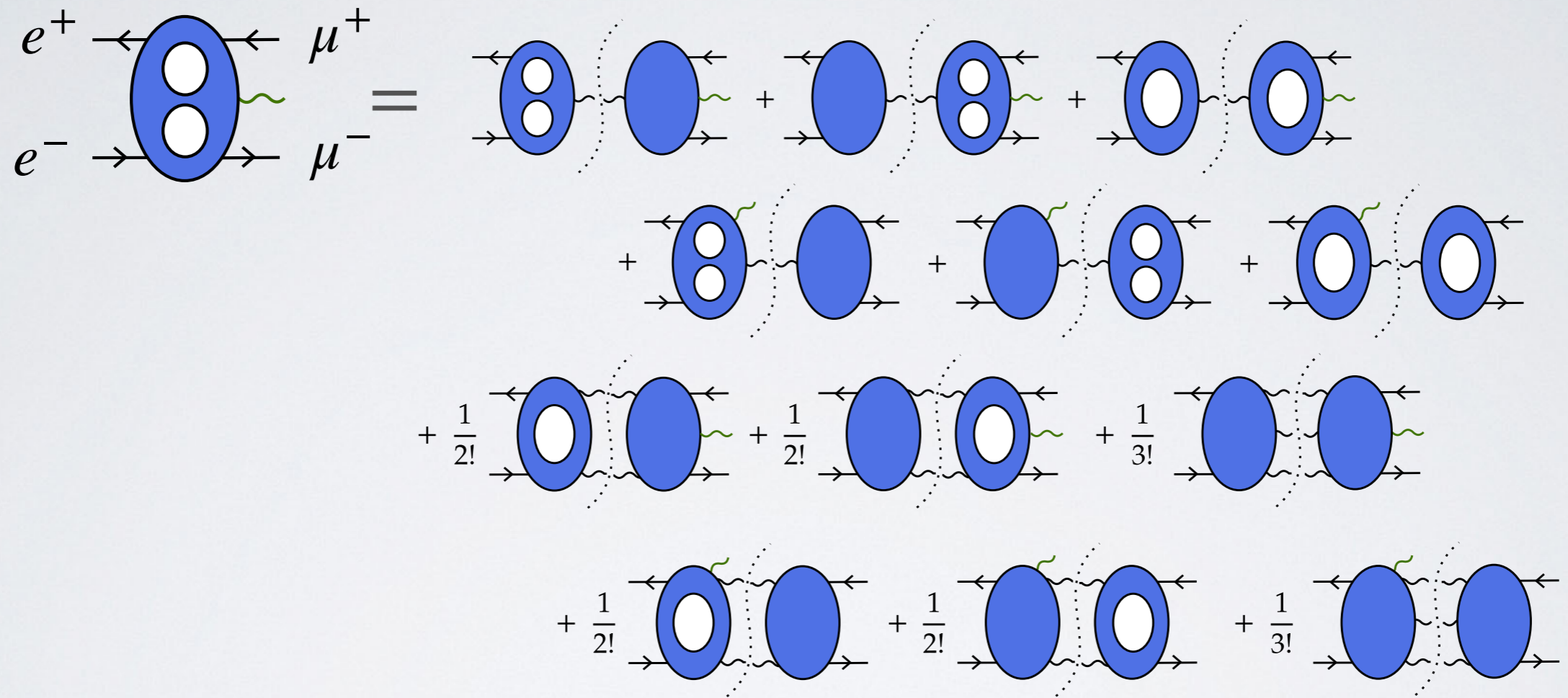


BUILDING-BLOCK AMPLITUDES

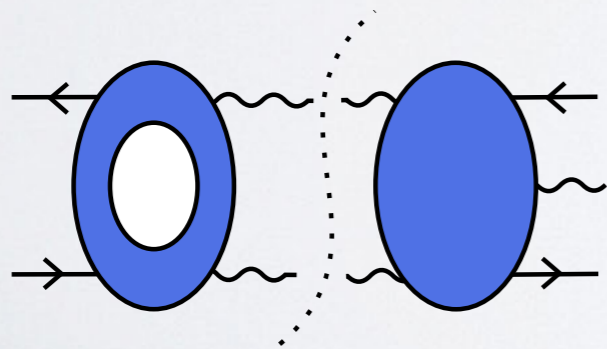
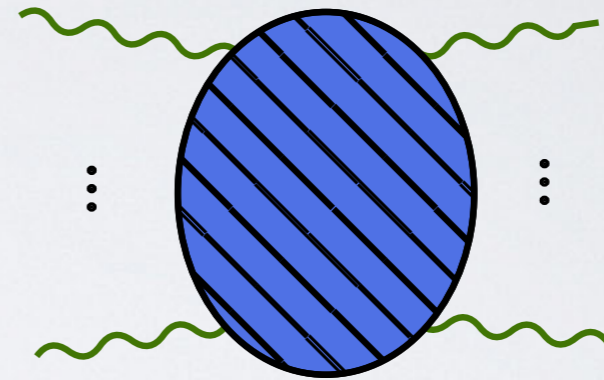
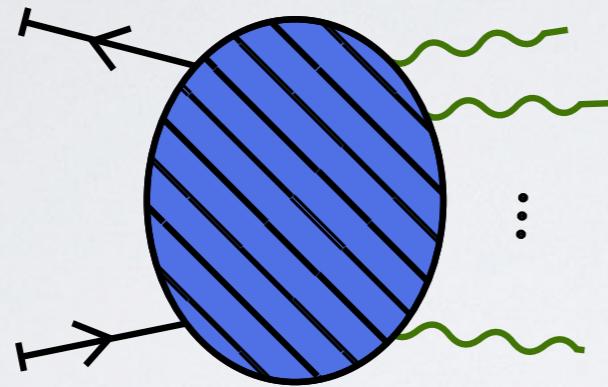


BUILDING-BLOCK AMPLITUDES

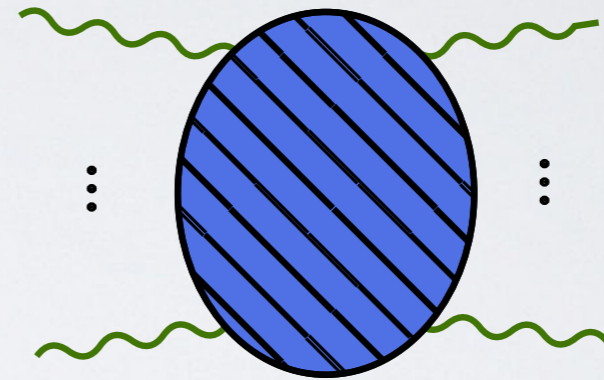
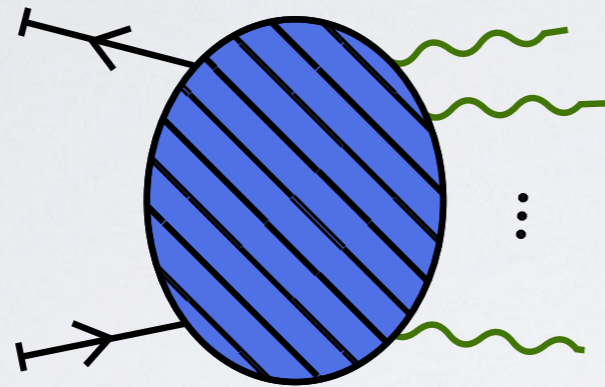




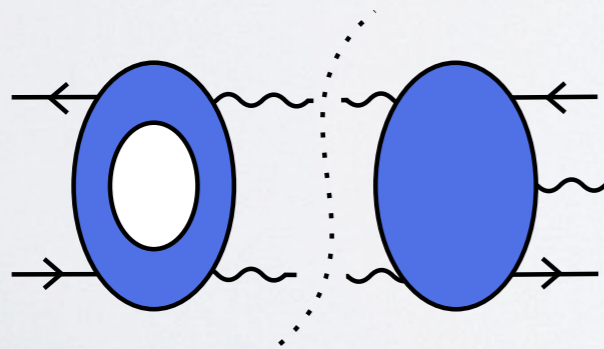
BUILDING-BLOCK AMPLITUDES



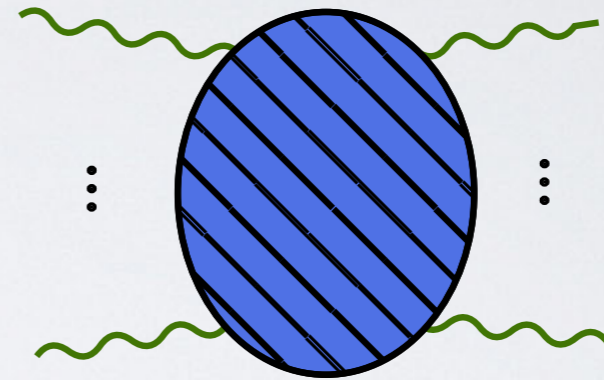
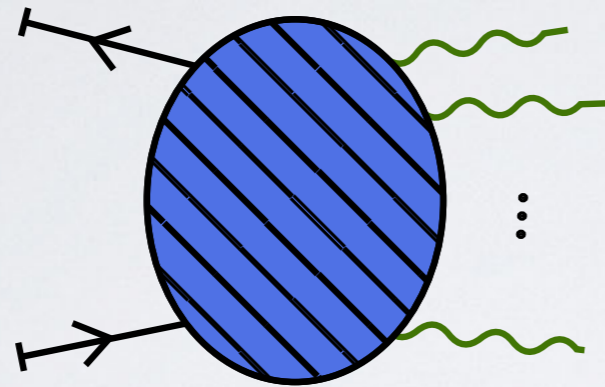
BUILDING-BLOCK AMPLITUDES



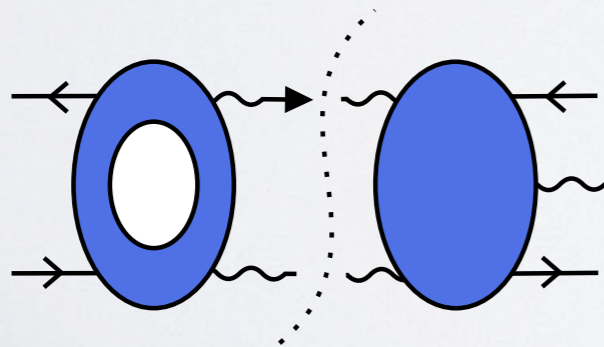
No new collinear
divergences in sewing:



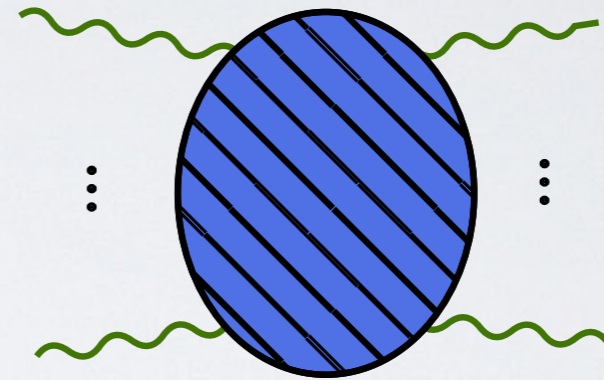
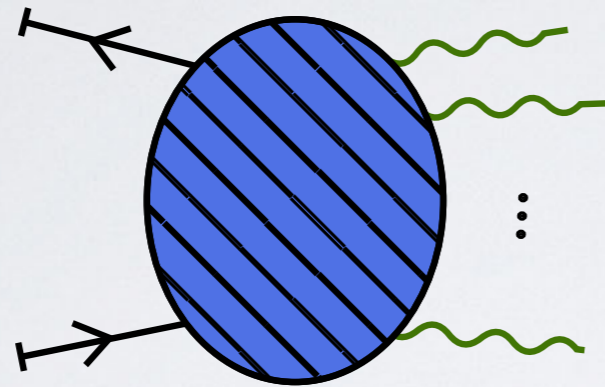
BUILDING-BLOCK AMPLITUDES



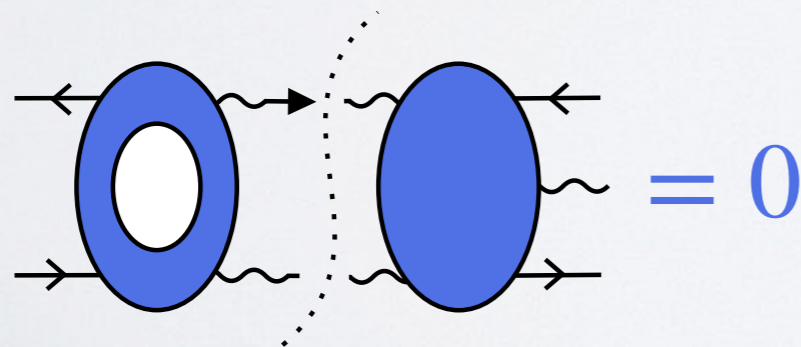
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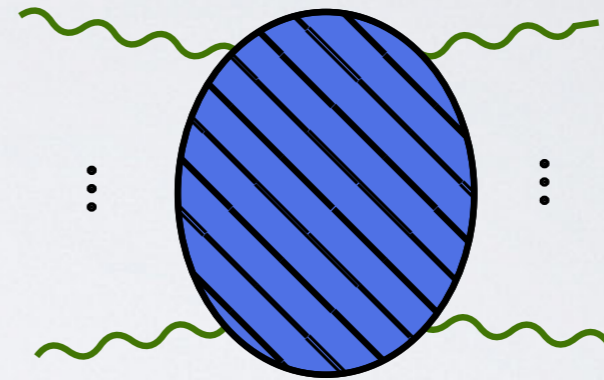
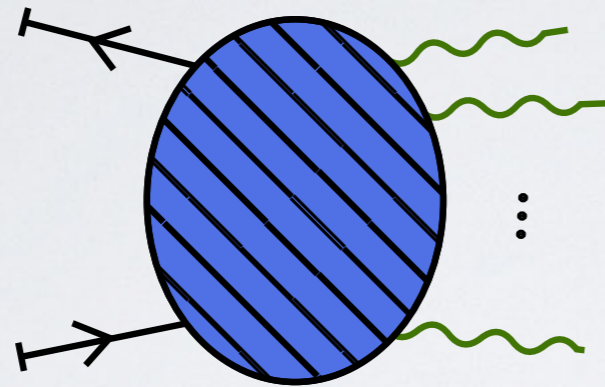
BUILDING-BLOCK AMPLITUDES



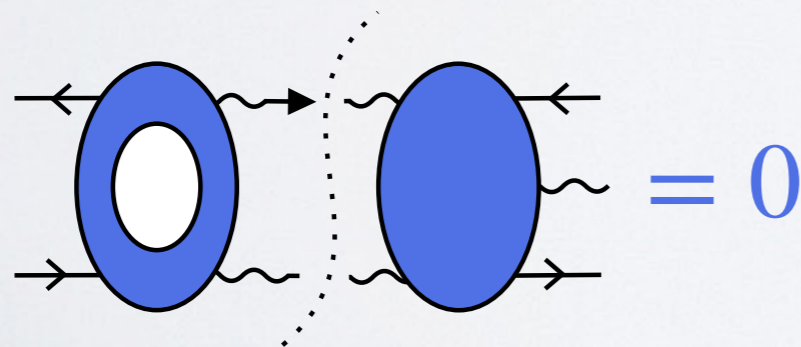
No new collinear
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BUILDING-BLOCK AMPLITUDES



No new collinear divergences in sewing:



A sewing propagator:

A diagram showing a sewing propagator. A wavy line with momentum k pointing to the right is crossed by a dashed purple line. An arrow points from this diagram to the following mathematical expression:

$$\frac{-i}{k^2} \left(g_{\mu\nu} - 2 \frac{k_\mu \rho_\nu + k_\nu \rho_\mu}{(k + \rho)^2 - \rho^2} \right)$$

LOCALLY FINITE REMAINDERS

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$$F = \exp(-\hat{\Gamma}) \circ A$$

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Soft and collinear operators:

$$\hat{\Gamma}^{(i)} = \hat{S}^{(i)} + \hat{C}^{(i)}$$

LOCALLY FINITE REMAINDERS

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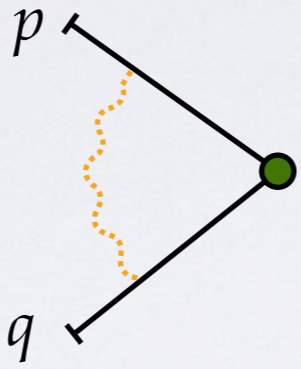
Soft and collinear operators:

$$\hat{\Gamma}^{(i)} = \hat{S}^{(i)} + \hat{C}^{(i)}$$

$$F^{(1)} = A^{(1)} - \hat{\Gamma}^{(1)} \circ A^{(0)}$$

$$F^{(2)} = A^{(2)} - \hat{\Gamma}^{(1)} \circ F^{(1)} - \left(\frac{1}{2} \hat{\Gamma}^{(1)} \circ \hat{\Gamma}^{(1)} + \hat{\Gamma}^{(2)} \right) \circ A^{(0)}$$

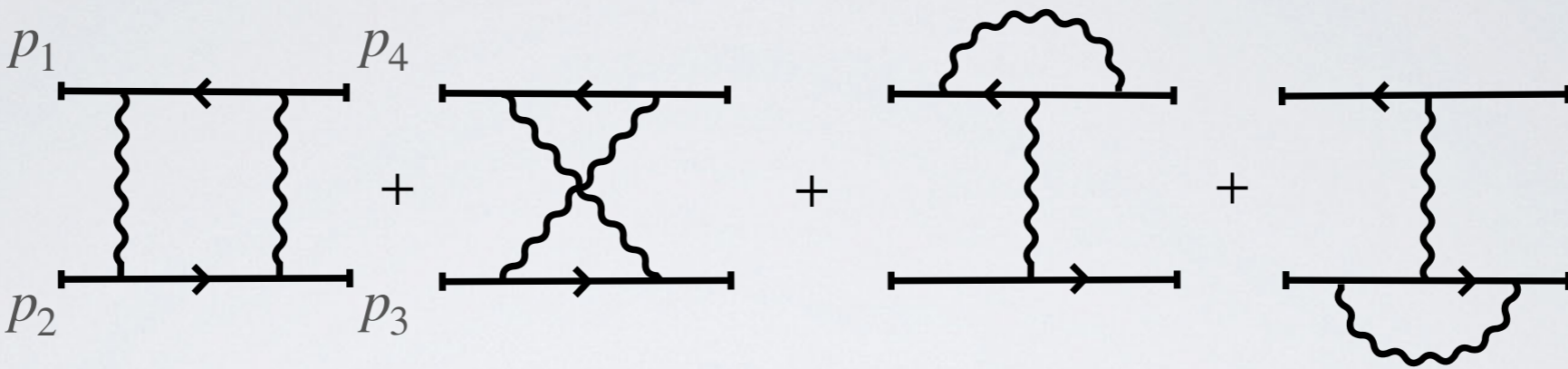
I-LOOP SUBTRACTION OPERATORS


$$\hat{S}^{(1)} \rightarrow \text{Diagram} \equiv \frac{-ie_q e_p (2p \cdot 2q)}{k^2 (p - k)^2 (q + k)^2}$$

ONE-LOOP IR-FINITE REMAINDER

$$A^{(1)} =$$

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$$A^{(1)} =$$


The diagram shows the one-loop IR-finite remainder $A^{(1)}$ as a sum of five Feynman diagrams. The first diagram is a box diagram with external momenta p_1 (top-left), p_4 (top-right), p_2 (bottom-left), and p_3 (bottom-right). It consists of two horizontal fermion lines and two vertical gluon lines. The second diagram is a crossed box diagram with two horizontal fermion lines and two crossed gluon lines. The third diagram is a self-energy correction on the top fermion line, featuring a gluon loop. The fourth diagram is a self-energy correction on the bottom fermion line, also featuring a gluon loop. The fifth diagram is a vertex correction on the right fermion line, featuring a gluon loop.

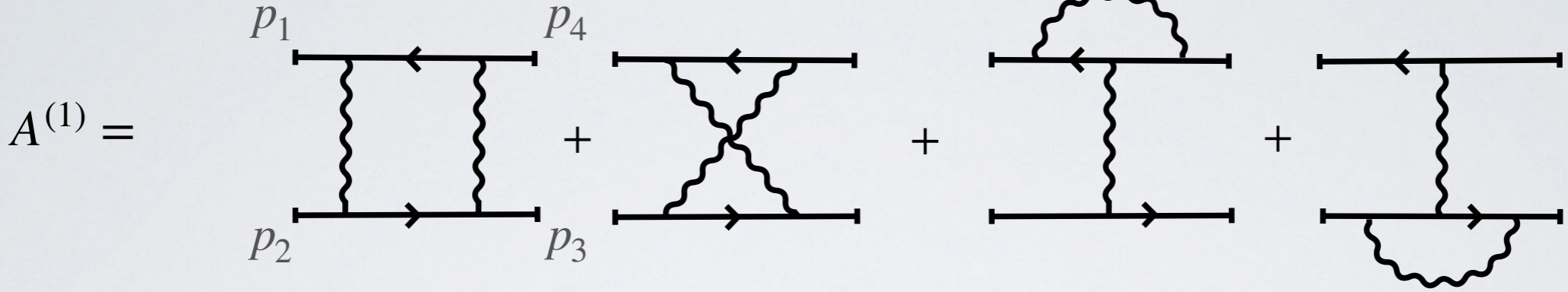
ONE-LOOP IR-FINITE REMAINDER

$$A^{(1)} =$$

The diagram shows the one-loop IR-finite remainder $A^{(1)}$ as a sum of four Feynman diagrams. The first diagram is a box diagram with external momenta p_1, p_2, p_3, p_4 and two wavy internal lines. The second diagram is a crossed box diagram with two wavy internal lines. The third diagram is a self-energy diagram with a wavy loop on the top line. The fourth diagram is a self-energy diagram with a wavy loop on the bottom line.

$$F^{(1)} = A^{(1)} - \left(\hat{S}^{(1)} + \hat{C}^{(1)} \right) \circ A^{(0)}$$

ONE-LOOP IR-FINITE REMAINDER



$$F^{(1)} = A^{(1)} - \left(\hat{S}^{(1)} + \hat{C}^{(1)} \right) \circ A^{(0)}$$

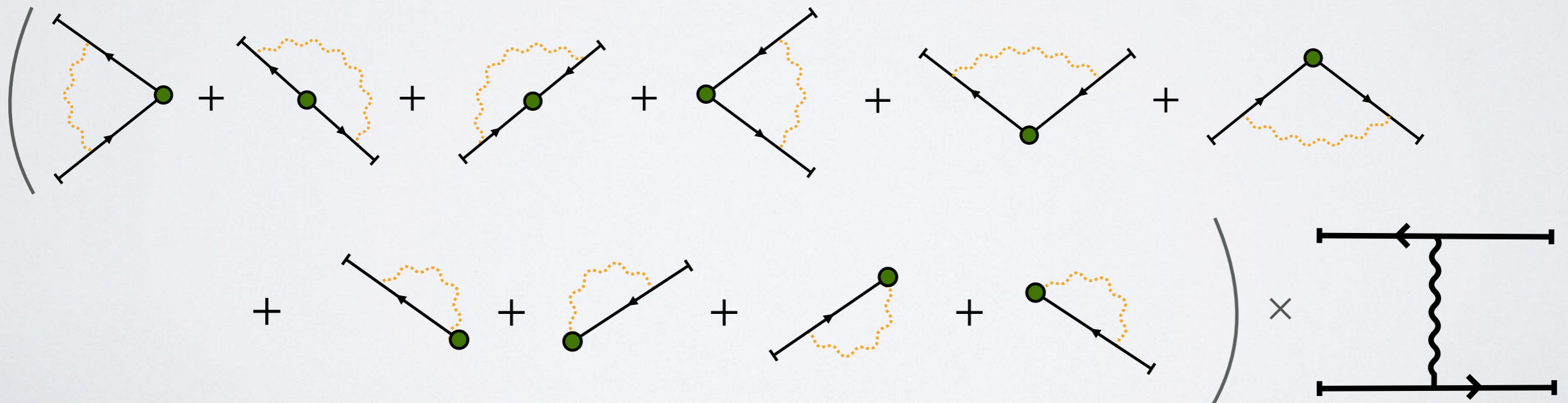
soft and collinear finite!

ONE-LOOP IR-FINITE REMAINDER

$$A^{(1)} =$$

$$F^{(1)} = A^{(1)} - \left(\hat{S}^{(1)} + \hat{C}^{(1)} \right) \circ A^{(0)}$$

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ONE-LOOP IR-FINITE REMAINDER

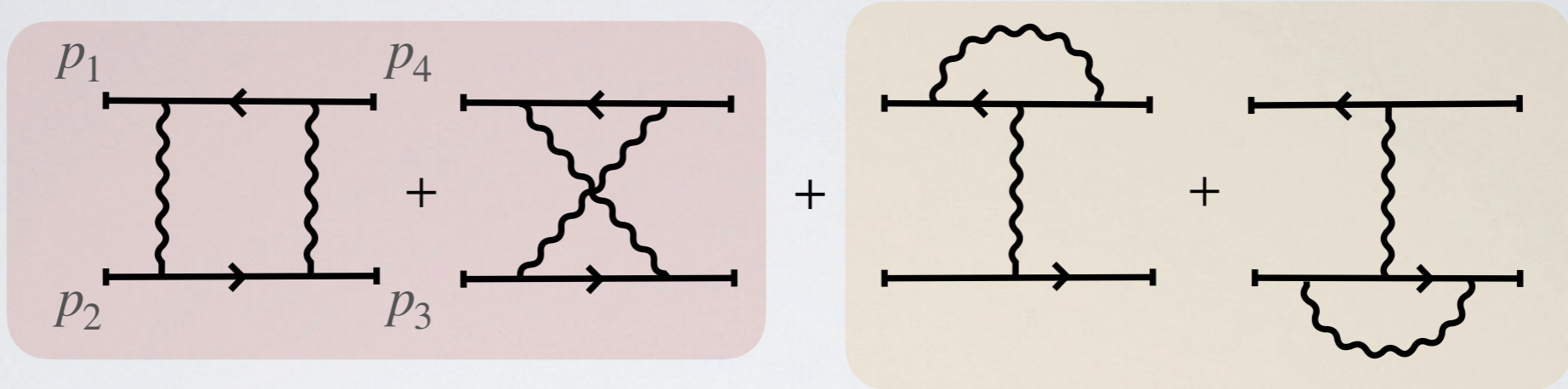
$$A^{(1)} =$$

$$F^{(1)} = A^{(1)} - \left(\hat{S}^{(1)} + \hat{C}^{(1)} \right) \circ A^{(0)}$$

soft and collinear finite!

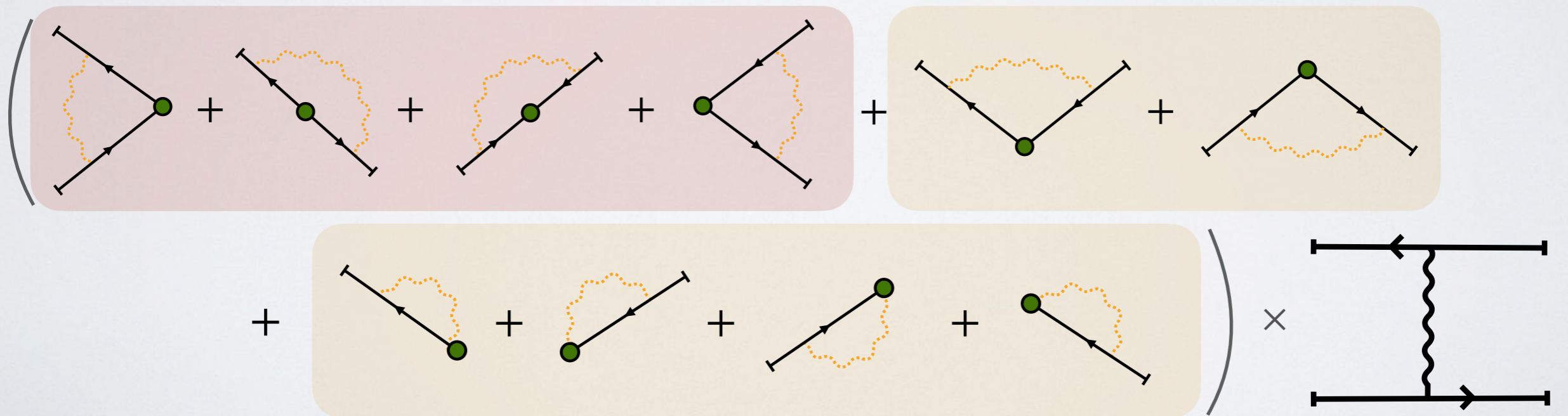
ONE-LOOP IR-FINITE REMAINDER

$$A^{(1)} =$$



$$F^{(1)} = A^{(1)} - \left(\hat{S}^{(1)} + \hat{C}^{(1)} \right) \circ A^{(0)}$$

soft and collinear finite!



2-LOOP SUBTRACTION OPERATORS

$$\hat{S}^{(2)} \rightarrow \frac{4(1-\epsilon)}{3-2\epsilon} \text{ [Diagram: A vertex with two outgoing lines and a ghost loop] }$$

$$\hat{C}^{(2)} \rightarrow \frac{4(1-\epsilon)}{3-2\epsilon} \text{ [Diagram: A vertex with one outgoing line and a ghost loop] } + 2(1-\epsilon) \frac{(p-k) \cdot \eta}{p \cdot \eta} \left[\text{[Diagram: Ghost loop with momentum k] } - \text{[Diagram: Ghost loop with momentum k] } \right]$$

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$$\hat{S}^{(2)} \rightarrow \frac{4(1-\epsilon)}{3-2\epsilon} \text{ [Diagram: A vertex with two outgoing lines and a ghost loop] }$$

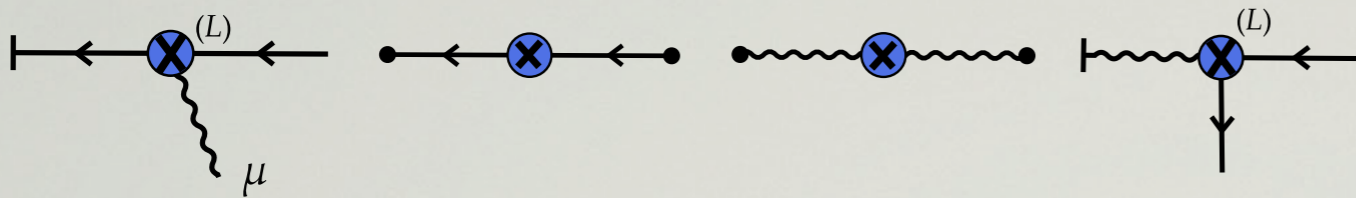
$$\hat{C}^{(2)} \rightarrow \frac{4(1-\epsilon)}{3-2\epsilon} \text{ [Diagram: A vertex with one outgoing line and a ghost loop] } + 2(1-\epsilon) \frac{(p-k) \cdot \eta}{p \cdot \eta} \left[\text{[Diagram: Triangle with ghost loop and momentum k]} - \text{[Diagram: Triangle with ghost loop and momentum k]} \right] + \delta \hat{C}_{\text{NLP double collinear}}^{(2)}$$

WORK IN PROGRESS

NEXT STEPS

- QED practical applications
- Generalisation to QCD (colour = routings)
- Generalisation to 3+ loops (all orders?)

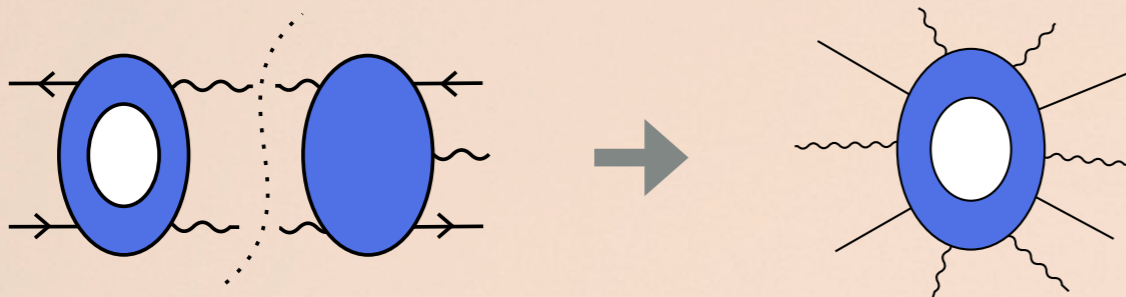
local counterterms



locally factorising building blocks



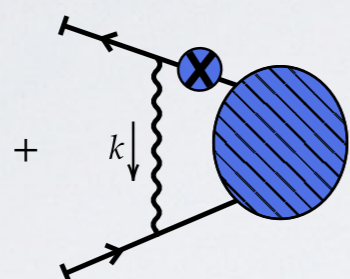
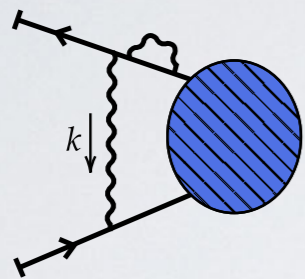
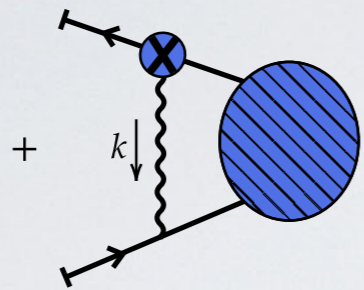
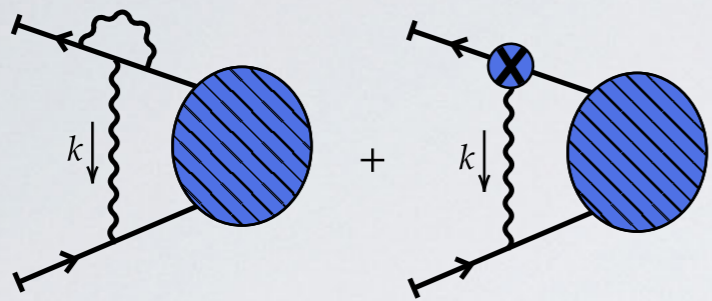
sewing into general amplitudes

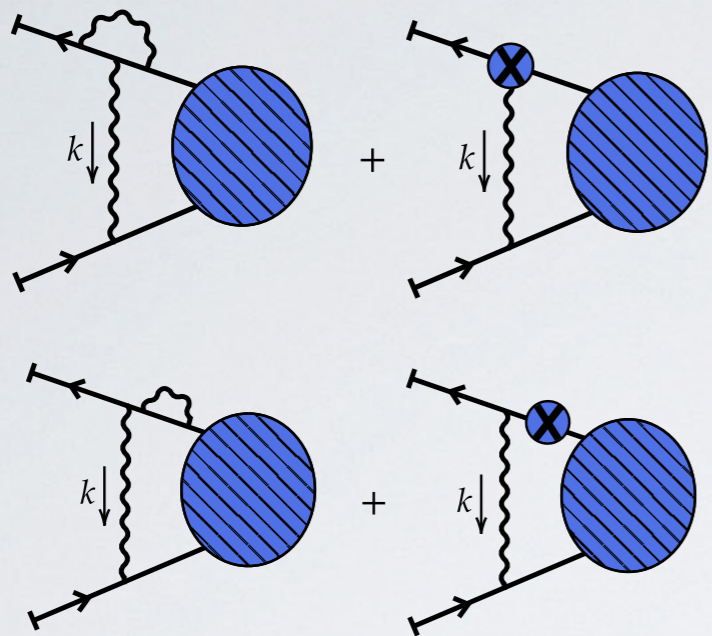


Integrand soft and collinear factorisation!

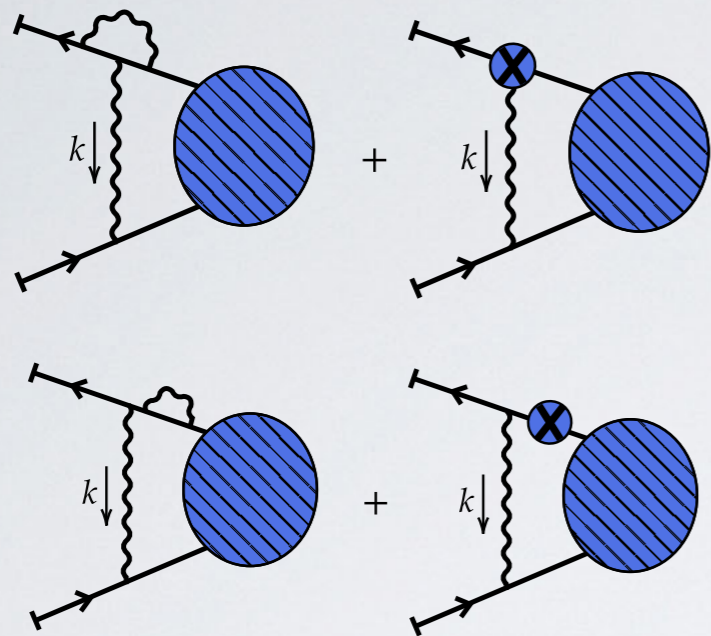
$$F^{(1)} = A^{(1)} - \hat{\Gamma}^{(1)} \circ A^{(0)}$$

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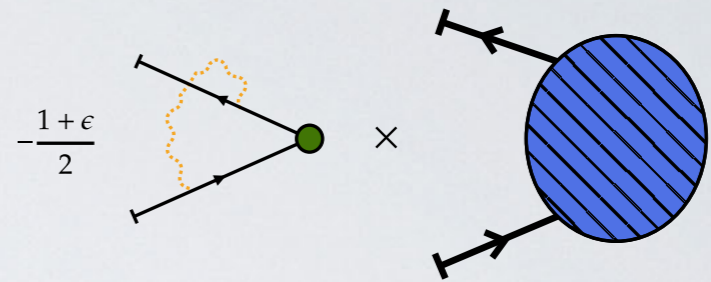


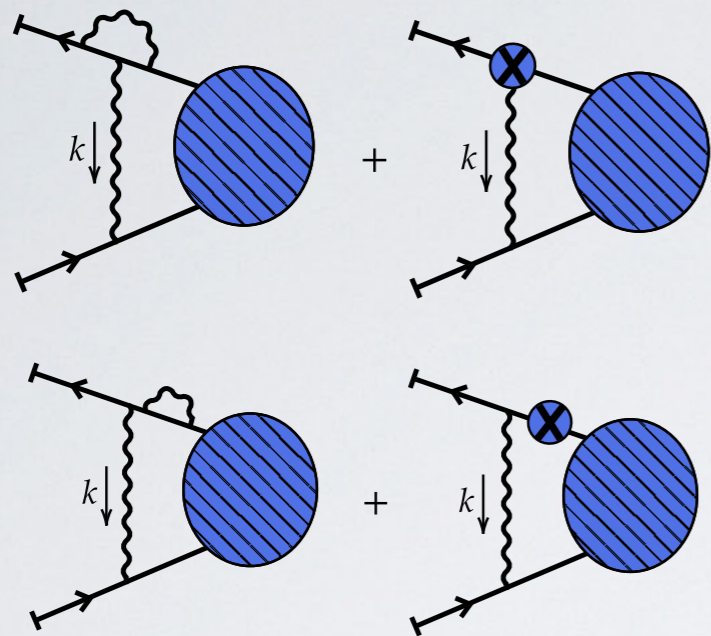


k soft
→

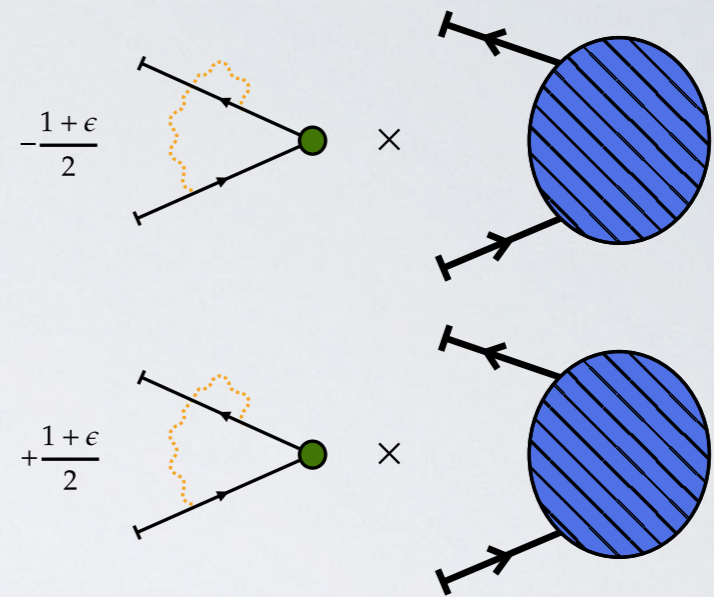


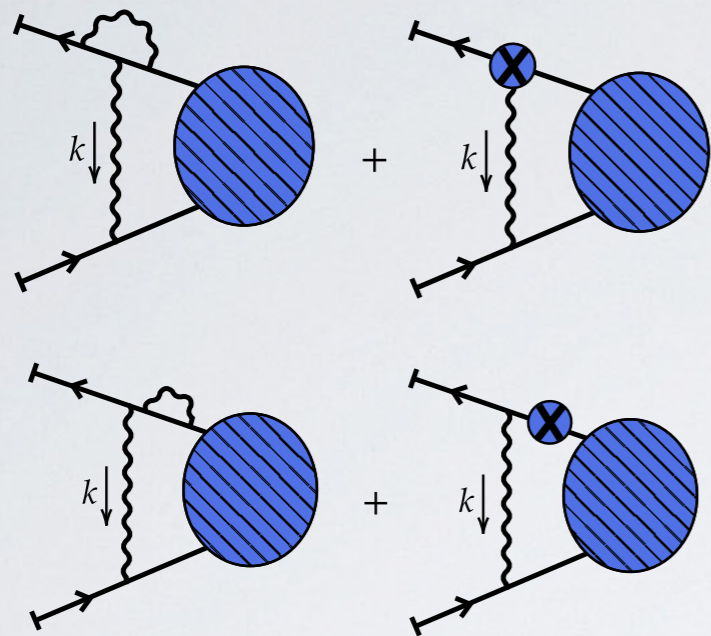
k soft

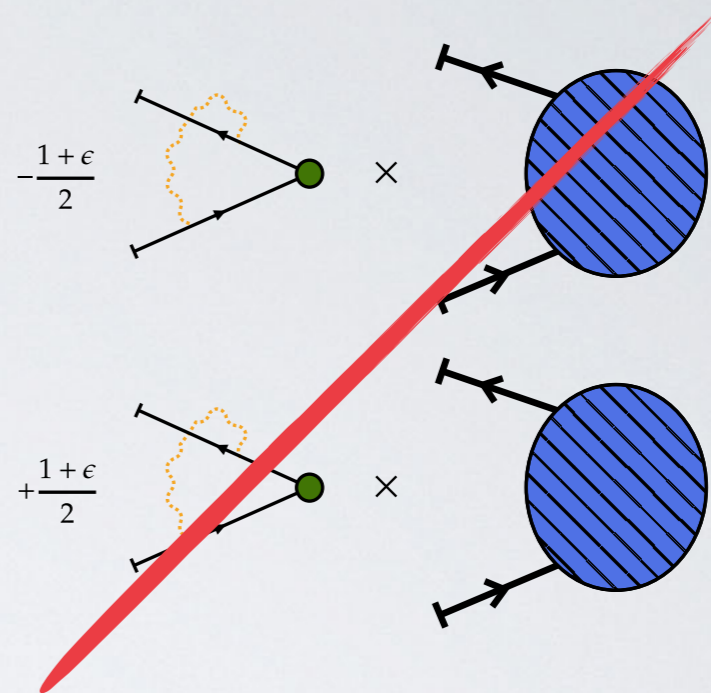


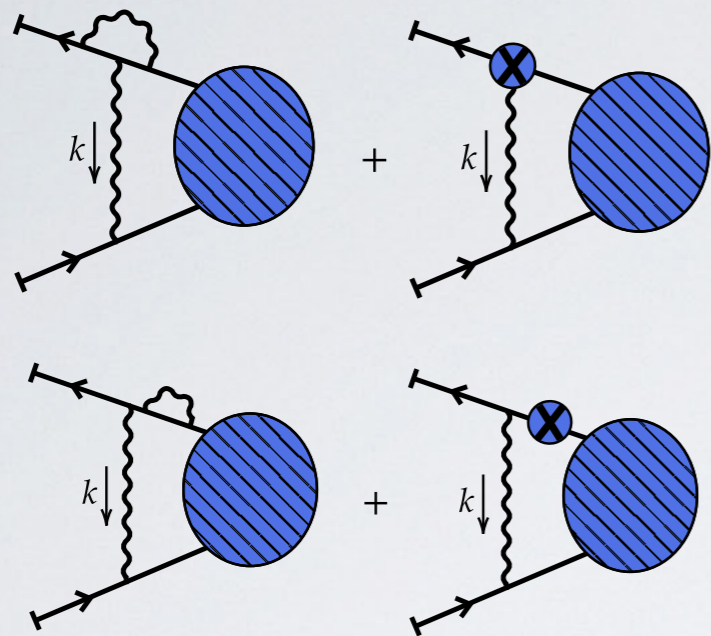
k soft

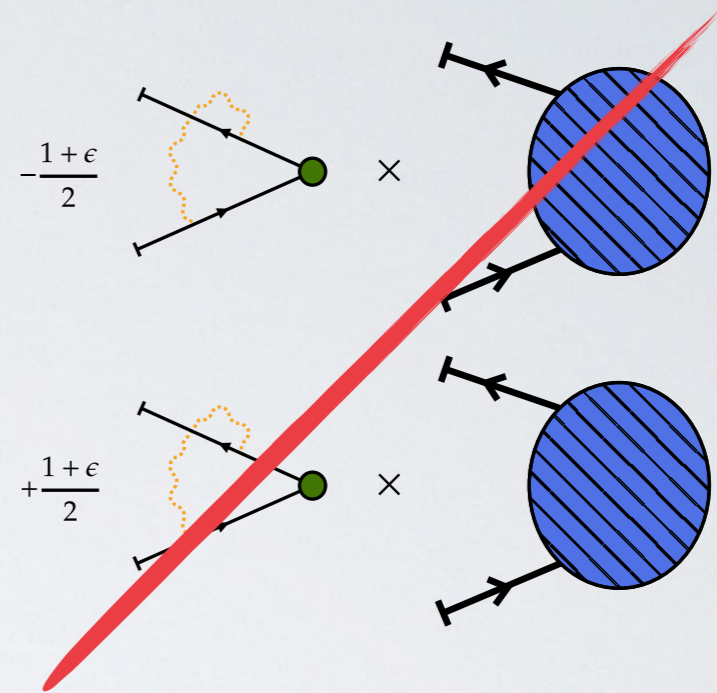


k soft

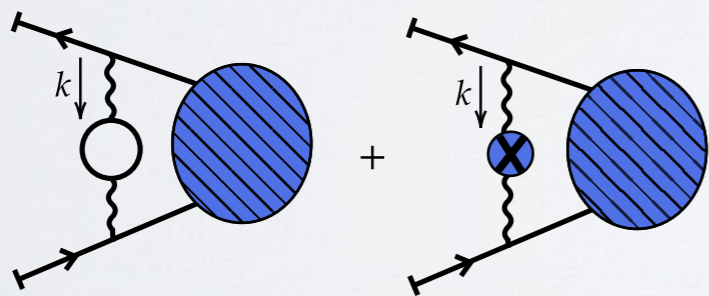




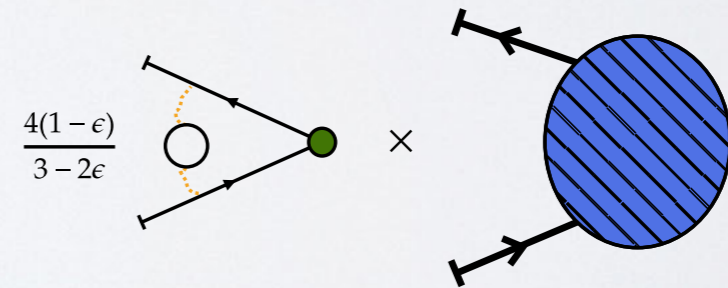
k soft
→



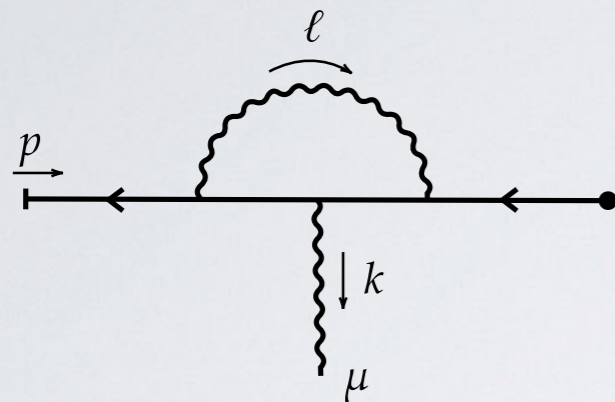
.....



k soft
→



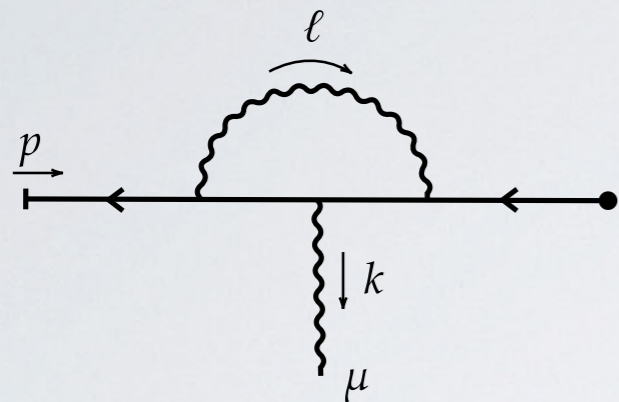
THERE ARE NO LOOP POLARISATIONS!



The diagram shows a fermion loop (represented by a wavy line) with an incoming fermion line on the left and an outgoing fermion line on the right. The incoming fermion line is labeled with momentum p and an arrow pointing right. The outgoing fermion line is labeled with momentum k and an arrow pointing left. A photon line (represented by a wavy line) is attached to the loop, with momentum ℓ and index μ .

$$= \bar{u}(p) \Gamma^\mu(p, k) \frac{i(\not{k} - \not{p})}{(k - p)^2}$$

THERE ARE NO LOOP POLARISATIONS!

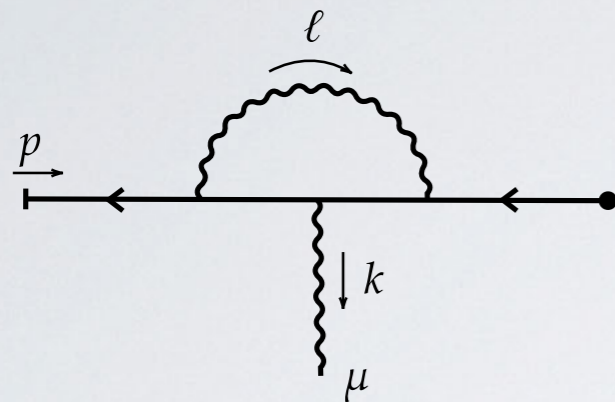


$$= \bar{u}(p) \Gamma^\mu(p, k) \frac{i(\not{k} - \not{p})}{(k - p)^2}$$

tensor reduction

$$\Gamma^\mu(p, k) \equiv \not{p} p^\mu + \not{k} k^\mu$$

THERE ARE NO LOOP POLARISATIONS!



The diagram shows a fermion loop (represented by a wavy line) with an external photon line (represented by a straight line with arrows). The loop momentum is labeled ℓ . The external photon line has momentum k and index μ . The fermion line has momentum p .

$$= \bar{u}(p) \Gamma^\mu(p, k) \frac{i(\not{k} - \not{p})}{(k - p)^2}$$

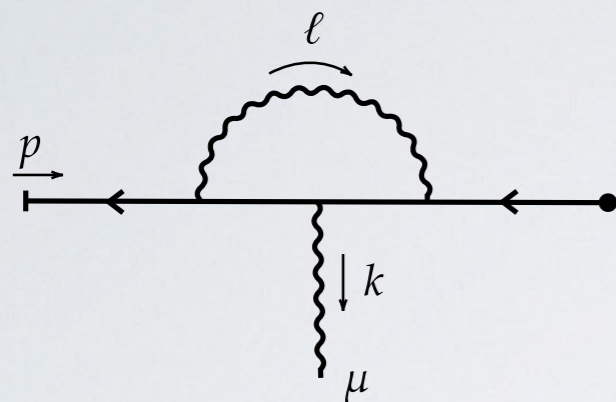
tensor reduction

$$\Gamma^\mu(p, k) \equiv \Gamma_p p^\mu + \Gamma_k k^\mu$$

all loop polarisation can be removed via boundary terms!

$$\Gamma^\mu(p, k) \rightarrow \Gamma^\mu(p, k) + \delta\Gamma^\mu(p, k)$$

THERE ARE NO LOOP POLARISATIONS!



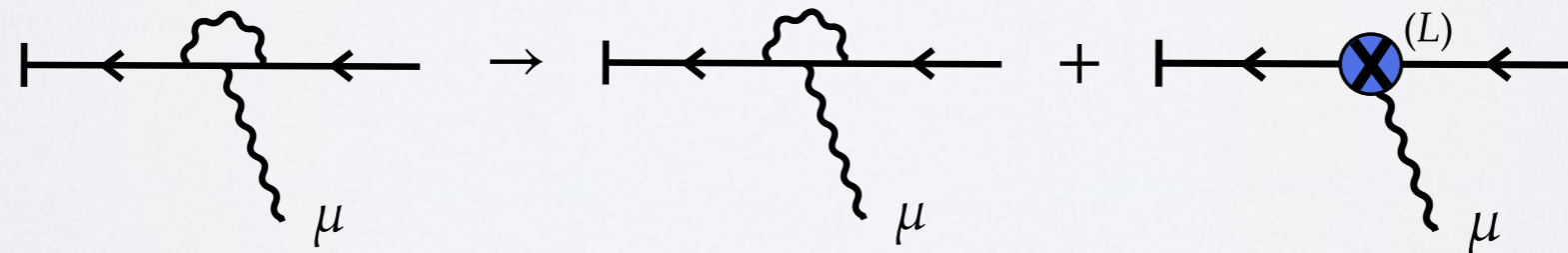
A Feynman diagram showing a fermion line with momentum p entering from the left and exiting to the right. A loop is formed by a fermion line (top) and a wavy line (bottom). The loop momentum is labeled ℓ . A wavy line with momentum k and index μ is attached to the bottom vertex of the loop.

$$= \bar{u}(p) \Gamma^\mu(p, k) \frac{i(\not{k} - \not{p})}{(k - p)^2}$$

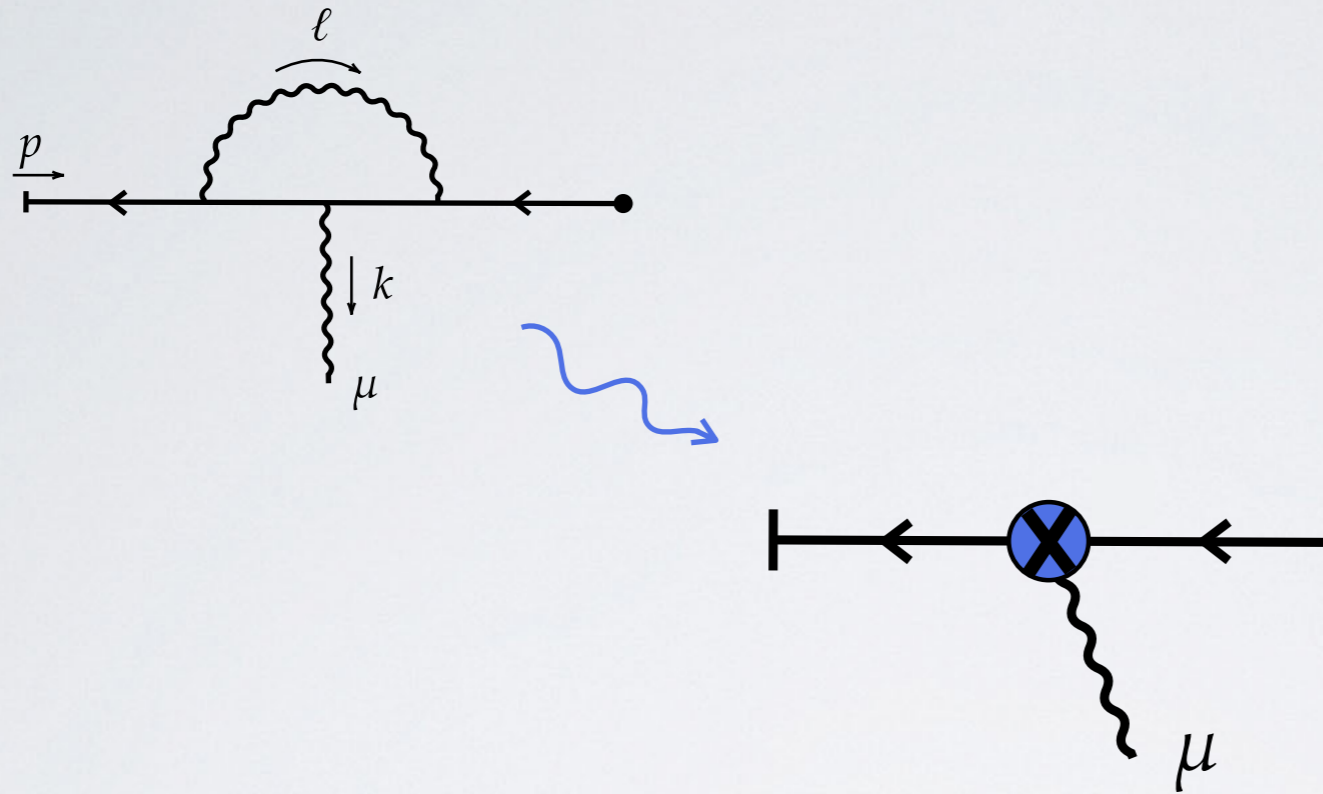
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The diagrammatic representation shows the decomposition of the loop integral. On the left is the original diagram with a loop and a wavy line. This is equal to the sum of two diagrams: the original diagram (representing $\Gamma^\mu(p, k)$) and a diagram where the loop is replaced by a blue circle with a cross, labeled (L) , representing the boundary term $\delta\Gamma^\mu(p, k)$.

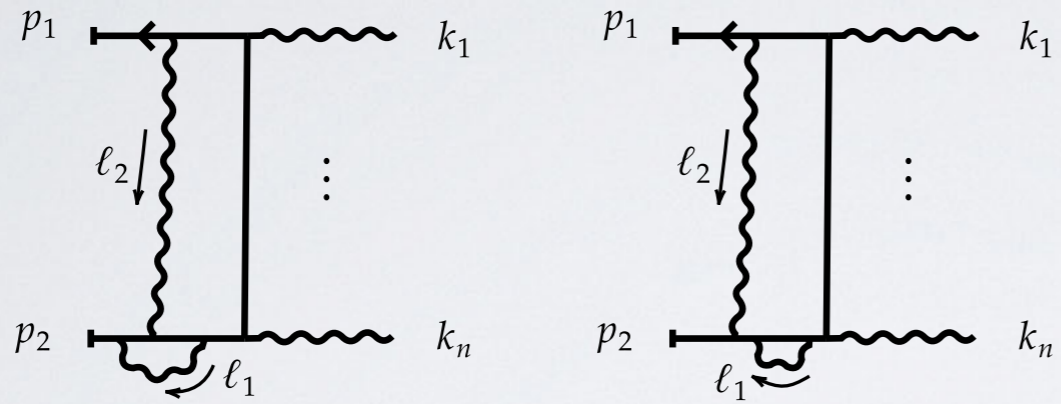


$$\delta\Gamma^\mu(p, k; \ell) \equiv ie^2(2-d) \frac{\not{\eta}}{2p \cdot \eta} \left[\frac{(p - 2\ell - k)^\mu}{\ell^2(p - \ell - k)^2} - \frac{(2p - 2\ell - k)^\mu}{(p - \ell)^2(p - \ell - k)^2} \right]$$

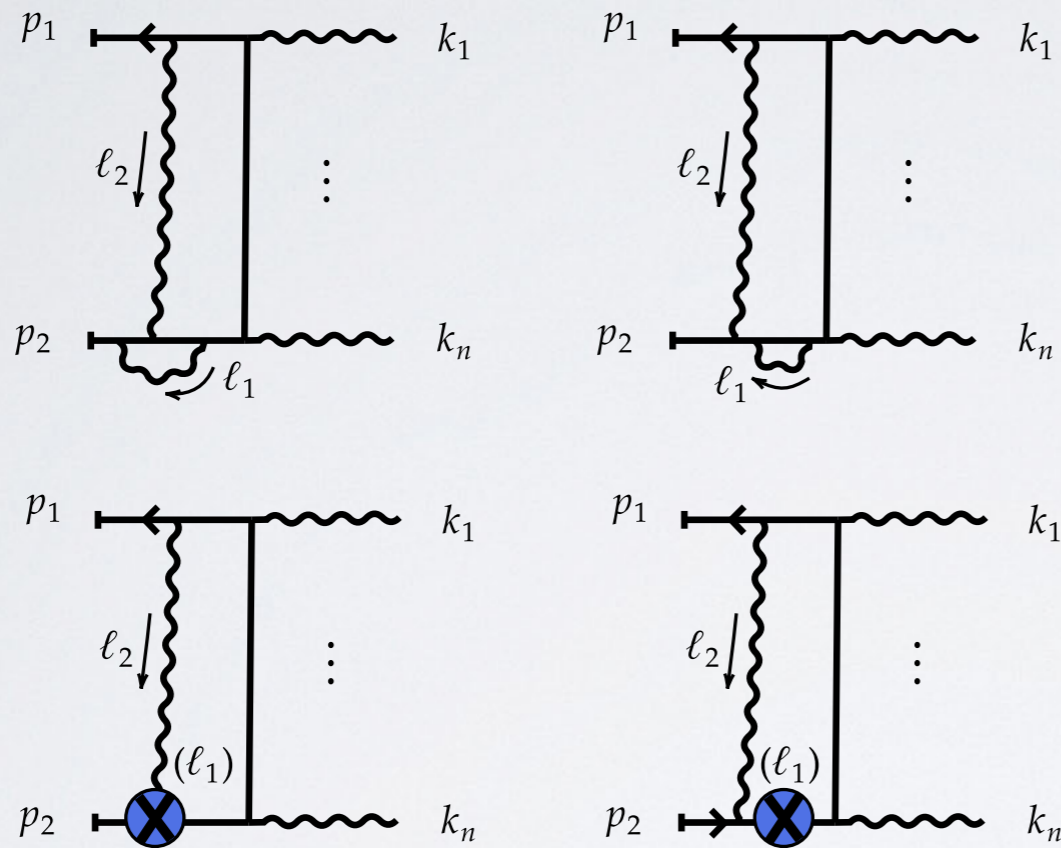
$$\eta^2 = 0$$

AMPLITUDE COUNTERTERMS

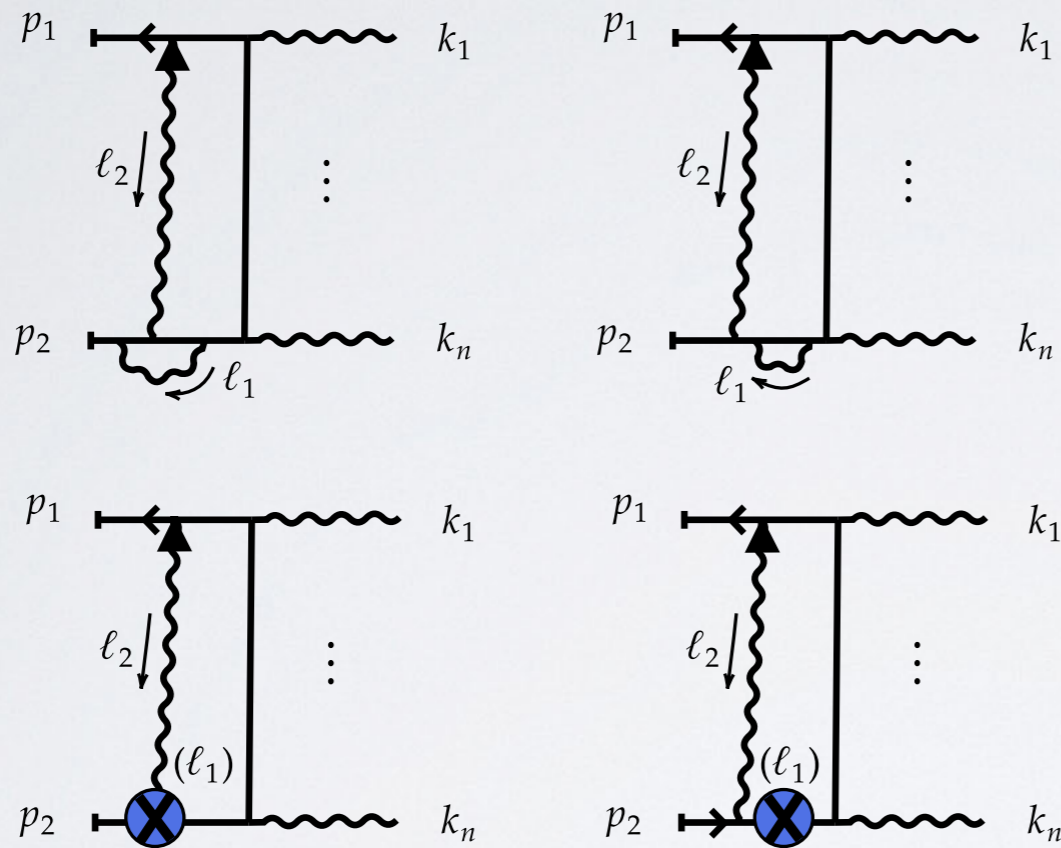
AMPLITUDE COUNTERTERMS



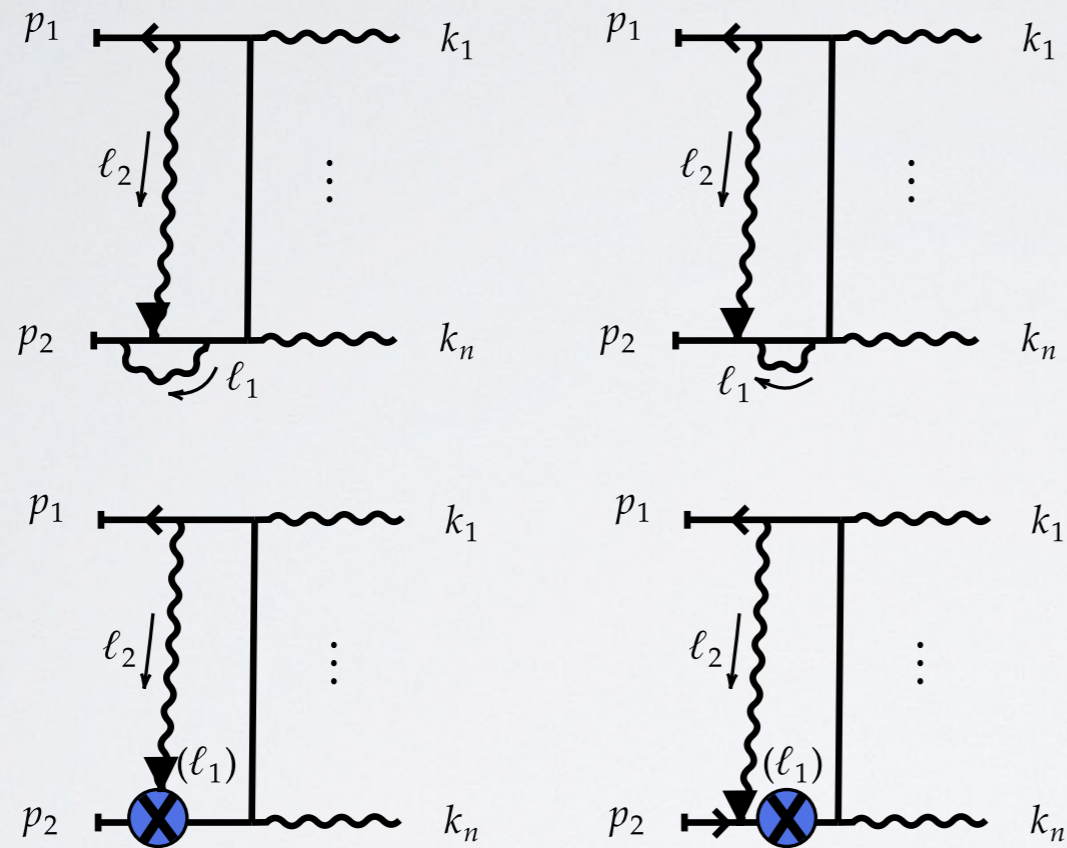
AMPLITUDE COUNTERTERMS



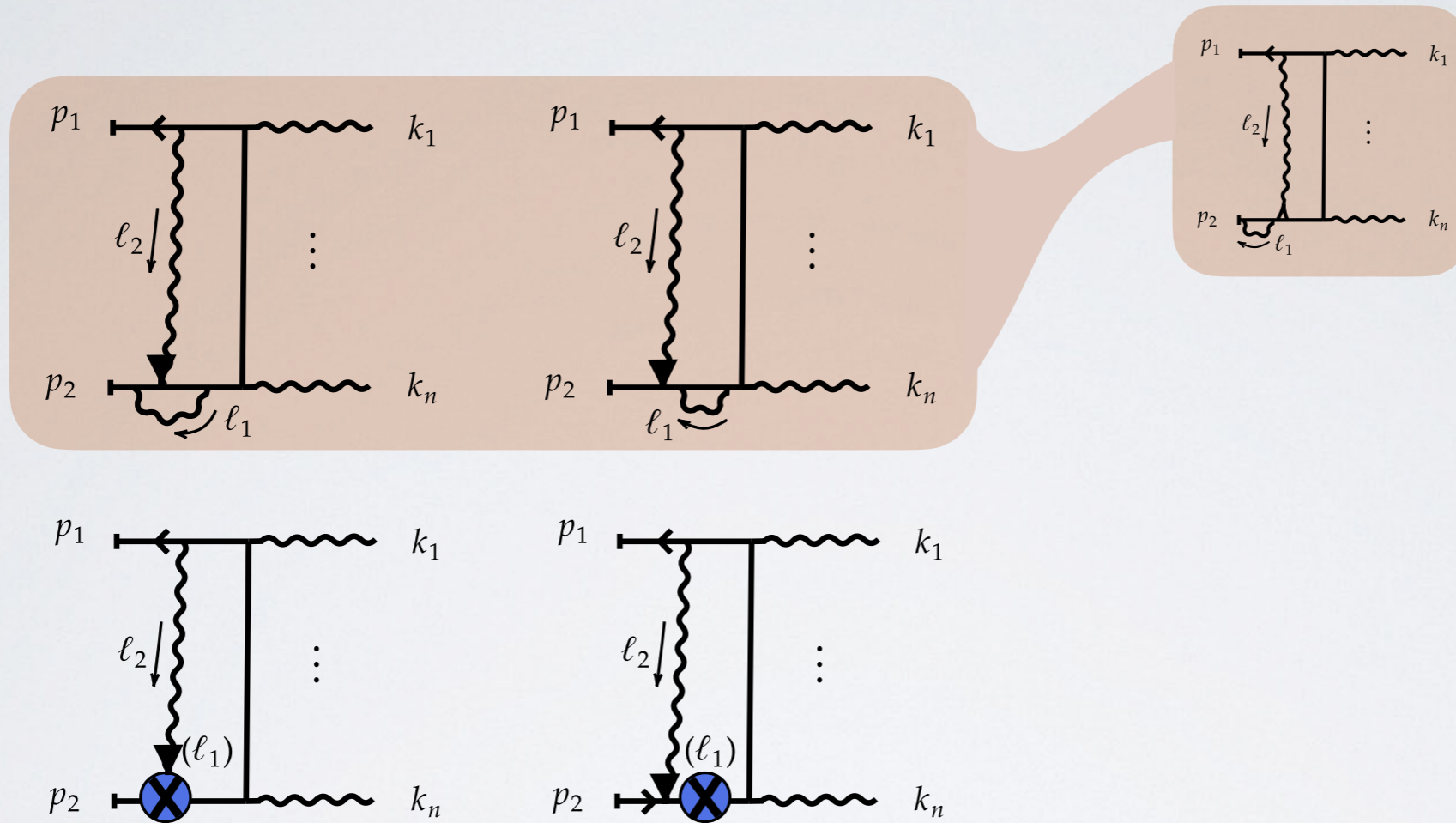
AMPLITUDE COUNTERTERMS



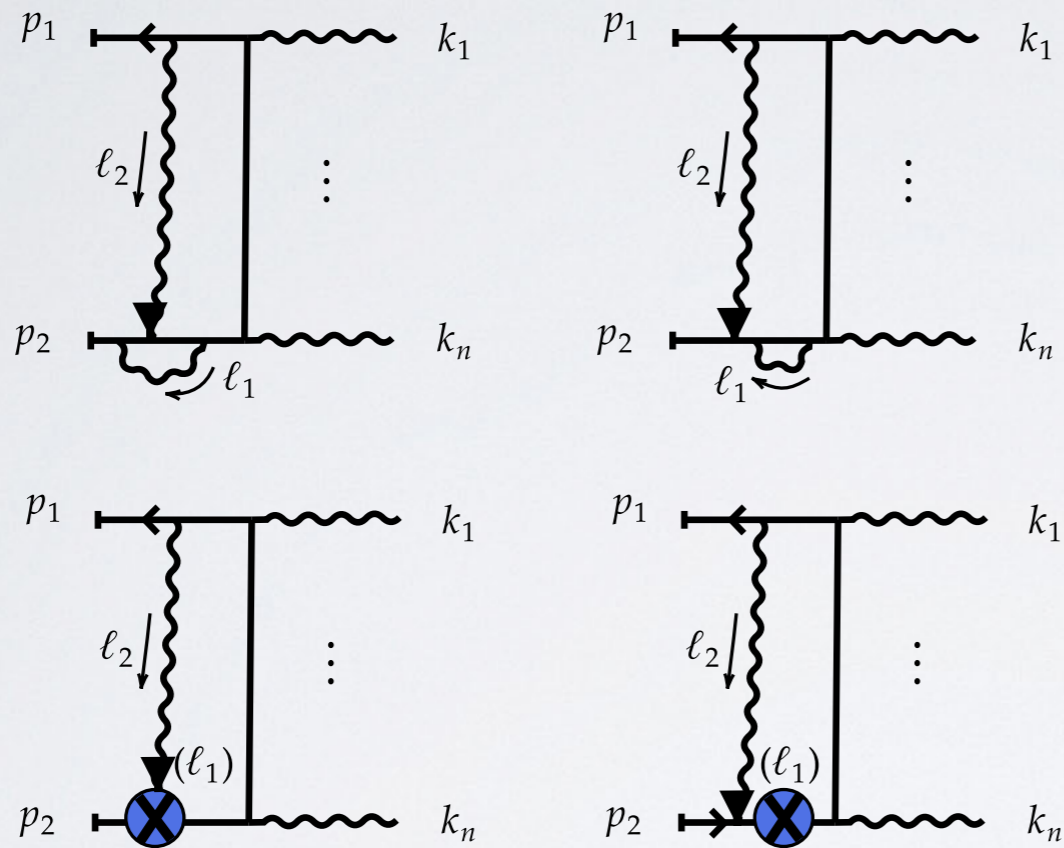
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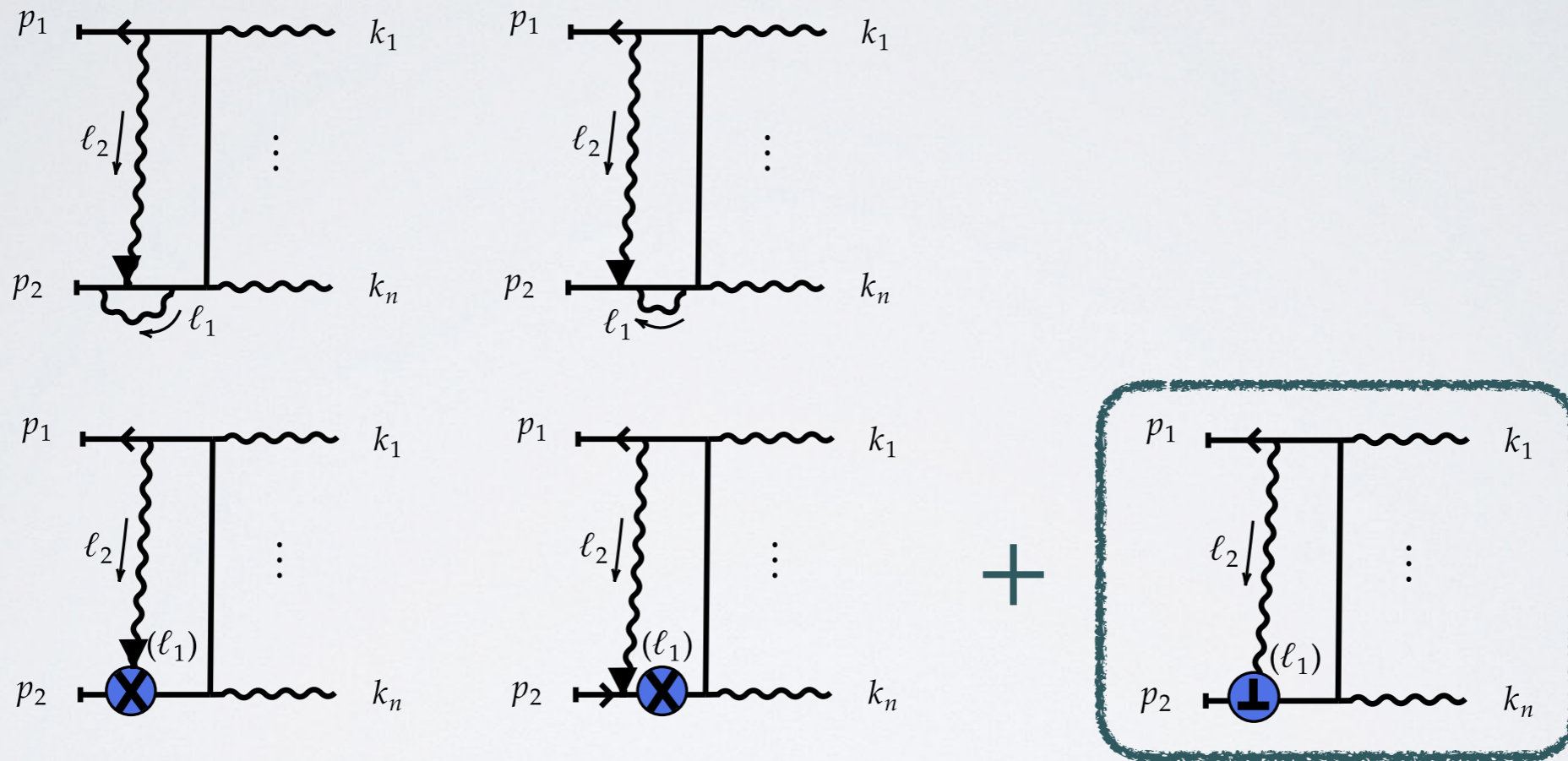
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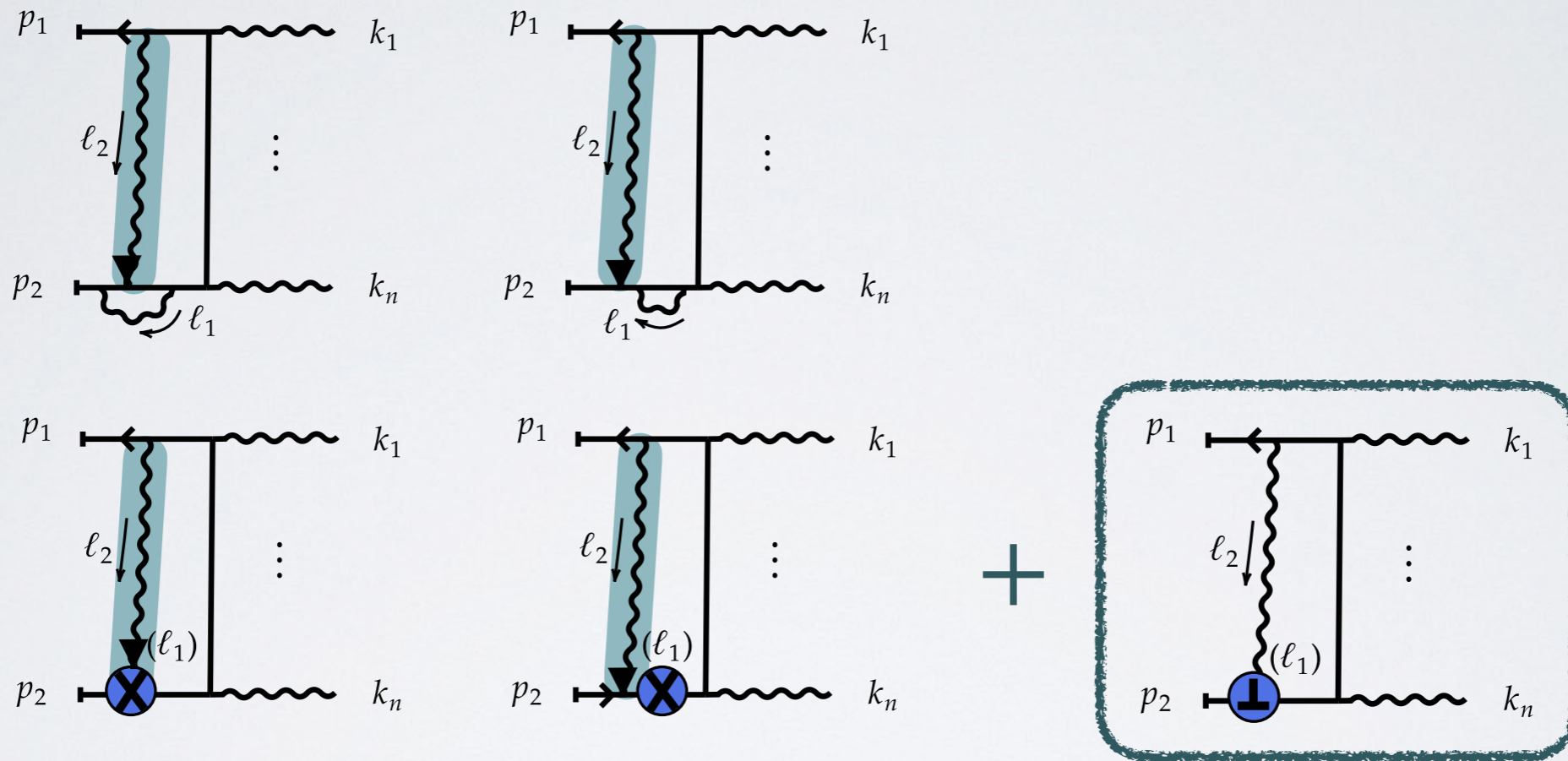
AMPLITUDE COUNTERTERMS



AMPLITUDE COUNTERTERMS

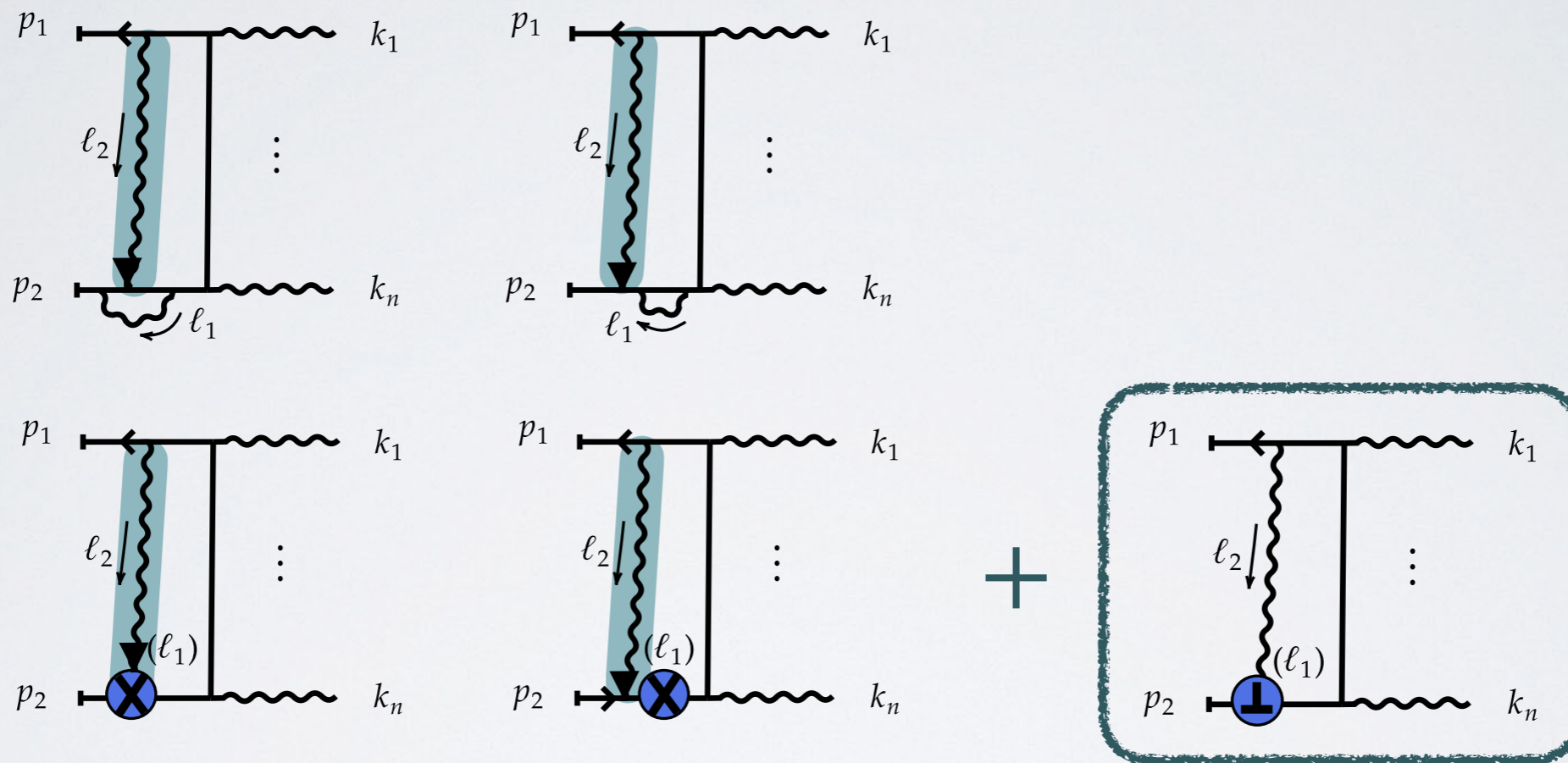


AMPLITUDE COUNTERTERMS



$$\begin{array}{c} \ell_2 \\ \downarrow \\ \mu \end{array} \begin{array}{c} \nu \\ \uparrow \\ \mu \end{array} \frac{-ig^{\mu\nu}}{\ell_2^2} \rightarrow \frac{-i}{\ell_2^2} \left(g^{\mu\nu} - 2 \frac{\ell_2^\mu \rho^\nu}{(\ell_2 + \rho)^2 - \rho^2} \right)$$

AMPLITUDE COUNTERTERMS



$$\begin{array}{c} \ell_2 \\ \downarrow \\ \mu \end{array} \begin{array}{c} \nu \\ \uparrow \\ \mu \end{array} \frac{-ig^{\mu\nu}}{\ell_2^2} \rightarrow \frac{-i}{\ell_2^2} \left(g^{\mu\nu} - 2 \frac{\ell_2^\mu \rho^\nu}{(\ell_2 + \rho)^2 - \rho^2} \right)$$

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