Streamlining canonical differential equations through factorized Picard-Fuchs operators

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

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DESY.

Previous talks on canonical basis:

Yesterday:

- Martin Link
- Gabriele Fiore

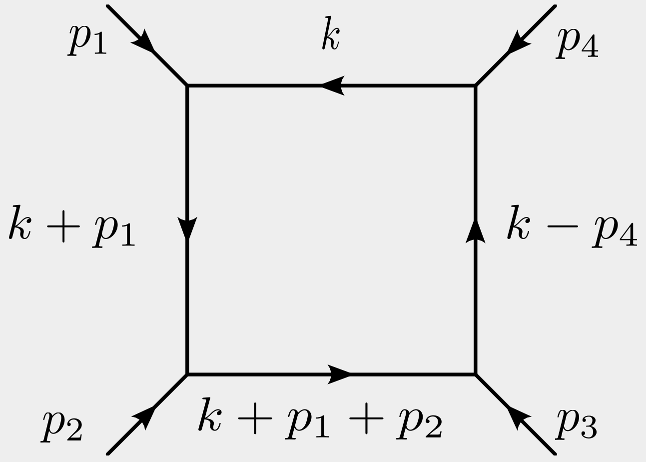
Today:

- Tobias Neumann
- Piotr Bargieła
- Cesare Mella
- Christoph Nega

Example: One-loop massless box integral

- Definition: $$I_{a_1 a_2 a_3 a_4} = \int \mathrm{d}^D k \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}$$

$D = 4 - 2\epsilon$



$$s = (p_1 + p_2)^2,\ t = (p_2 + p_3)^2,\ p_i^2 = 0$$

$$x = s/t$$

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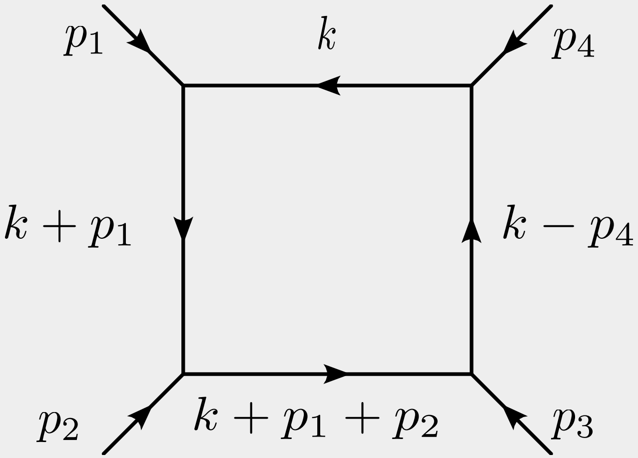

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- IBP reduction

$$I_{a_1 a_2 a_3 a_4} = \sum_i c_i f_i,$$

$$\vec{f} = \begin{pmatrix} I_{0101} \\ I_{1010} \\ I_{1111} \end{pmatrix}$$

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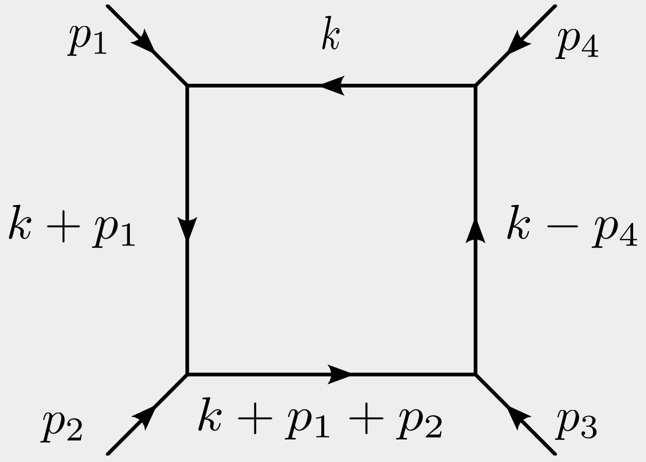

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- Canonical differential equations

IBP ↘

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f}$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{x} & 0 \\ \frac{-2(1-2\epsilon)}{x(x+1)} & \frac{2(1-2\epsilon)}{x^2(x+1)} & -\frac{x+1+\epsilon}{x(x+1)} \end{pmatrix}$$

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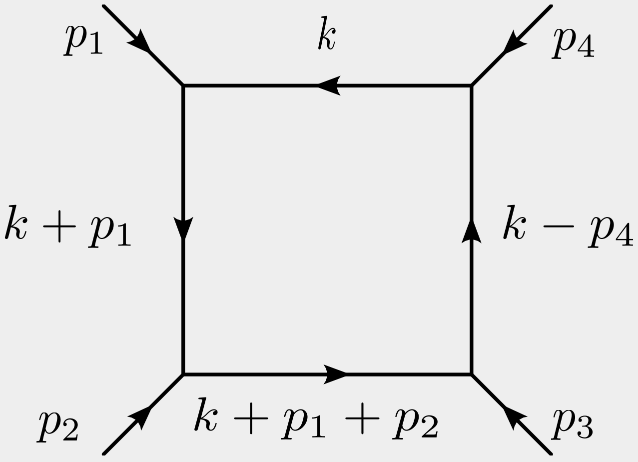

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- Canonical differential equations [Henn, '13]

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$$\partial_x \vec{f} = A(x,\epsilon)\vec{f} \xrightarrow{\vec{g} = T\vec{f}} \partial_x \vec{g} = \epsilon \tilde{A}(x)\vec{g}$$

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Advantages of the canonical form

- Canonical differential equations [Henn, '13]

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f} \quad \xrightarrow{\vec{g} = T\vec{f}} \quad \partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g}$$

1) ϵ factorized

$$\vec{g} = \sum_{k=0}^{\infty} \epsilon^k \vec{g}^{(k)} \quad \longrightarrow \quad \vec{g}^{(k)} = \int \tilde{A}\, \vec{g}^{(k-1)} \mathrm{d}x$$

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2) only simple poles $\Rightarrow$ leads to polylogarithms

$$\partial_x \vec{g} = \epsilon\, \tilde{A}_{\text{simple}} \vec{g},$$

$$\tilde{A}_{\text{simple}} = \frac{1}{1+x}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t),$$

multiplies $\epsilon^n \Rightarrow$ uniform transcendental weight

Minimal solvable form

- When can we still solve the DEs in a straightforward way?
  - double poles $\longrightarrow$ integration by parts

canonical form: $\partial_x \vec{g} = \epsilon\, \tilde{A}_{\text{simple}} \vec{g}$

Minimal solvable form

- When can we still solve the DEs in a straightforward way?
  - double poles $\longrightarrow$ integration by parts
  - ϵ-dependence decoupled:

$$\partial_x \vec{f} = A(\vec{x}, \epsilon)\vec{f}, \qquad A(x,\epsilon) = A^{(0)}(x) + \epsilon A^{(1)}(x) + \epsilon^2 A^{(2)}(x) + \cdots ,$$

canonical form: $\partial_x \vec{g} = \epsilon\, \tilde{A}_{\text{simple}} \vec{g}$

strictly lower triangular

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{pmatrix}$$

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$$\begin{aligned} \partial_x f_1 &= \mathcal{O}(\epsilon), \\ \partial_x f_2 &= A^{(0)}_{2,1} f_1 + \mathcal{O}(\epsilon), \\ \partial_x f_3 &= A^{(0)}_{3,1} f_1 + A^{(0)}_{3,2} f_2 + \mathcal{O}(\epsilon), \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

Decoupling the ϵ-dependence

[Adams, Chaubey, Weinzierl, '17]

- Factorize Picard-Fuchs operator

$$\partial_x \vec{f}^{(0)} = A^{(0)}(\vec{x},\epsilon)\vec{f}^{(0)} \longrightarrow 0 = \left[1 + \sum_{m=1}^{n} b_m \partial_x^m\right] f_1^{(0)}$$

maple

polylogarithmic: $= \left[(\partial_x + R_1)(\partial_x + R_2)\ldots(\partial_x + R_n)\right] f_1^{(0)}$

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lower triangular, $-R_i$ on diagonal

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- Why not solve from here?
  - IBP to remove double poles costly
  - many terms in ϵ-recursion required
  - canonical form makes function space manifest

$$A^{(0)} = \begin{pmatrix} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \end{pmatrix}$$

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Removing $A^{(0)}$

[Görges et al., '23, Duhr et al., '25]

- Example: $\partial_x \vec{f} = A(\vec{x},\epsilon)\vec{f}, \qquad A(x,\epsilon) = A^{(0)}(x) + \epsilon A^{(1)}(x) + \epsilon^2 A^{(2)}(x) \xrightarrow{?} \partial_x \vec{g} = \epsilon\, \tilde{A}_{\text{simple}}\vec{g}$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{x}{(1+x^2)^2} & 0 & 0 \\ \frac{1-x^2}{4x(1+x^2)} & \frac{1}{x} & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \frac{\epsilon\left(1+6x^2+x^4\right)}{(1-x)x(1+x)(1+x^2)} & -\frac{4\epsilon(1+2\epsilon)}{x} & -\frac{12\epsilon^2(1-x^2)}{x(1+x^2)} \\ \frac{x}{(1+x^2)^2} & -\frac{2\epsilon(1-x^2)}{x(1+x^2)} & \frac{12\epsilon x}{(1+x^2)^2} \\ \frac{1-x^2}{4x(1+x^2)} & \frac{1+2\epsilon}{x} & -\frac{3\epsilon\left(x^4+6x^2+1\right)}{(1-x)x(1+x)(1+x^2)} \end{pmatrix}$$

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  - Step 1: Remove double poles

$$T = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{x}{2(1+x^2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow A^{(0)}_{\text{simple}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{4x} & \frac{1}{x} & 0 \end{pmatrix}$$

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  - Step 4: Apply transformations on full matrix

$$\tilde{\vec{f}} = T\vec{f} \longrightarrow \tilde{A}(x,\epsilon) = \tilde{A}^{(0)}(x) + \mathcal{O}(\epsilon), \qquad \tilde{A}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{48x}{(1+x^2)^2} & -\frac{16(1+x^4)}{x(1+x^2)^2} & 0 \end{pmatrix}$$

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Removing $A^{(0)}$

[Görges et al., '23, Duhr et al., '25]

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Removing double poles in $A^{(1)}$

CANONICA: [Meyer, '17]

$$\partial_x \vec{f} = A(\vec{x}, \epsilon)\vec{f}, \qquad A(x,\epsilon) = \epsilon A^{(1)}(x) + \epsilon^2 A^{(2)}(x) + \ldots \quad \xrightarrow{?} \quad \partial_x \vec{g} = \epsilon\, \tilde{A}^{(1)}_{\text{simple}} \vec{g}$$

- Split off simple poles: $A^{(1)}(x) = A^{(1)}_{\text{simple}} + A^{(1)}_{\text{double}}$

$$A^{(1)}_{\text{simple}} = \begin{pmatrix} -\frac{1}{x} & -\frac{16}{x} & \frac{12\left(13-x^2\right)}{(1-x)x(1+x)} \\ \frac{1}{(1+x)x(1+x)} & \frac{2\left(3-x^2\right)}{(1-x)x(1+x)} & -\frac{36}{(1-x)x(1+x)} \\ -\frac{1}{4x} & 0 & \frac{3\left(3+x^2\right)}{(1-x)x(1+x)} \end{pmatrix}$$

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$$\begin{aligned}\tilde{A} &= T_1^{-1}(AT_1 - \partial_x T_1),\\ &= \epsilon\left(A^{(1)}_{\text{simple}} + A^{(1)}_{\text{double}} - \partial_x T_1^{(1)}\right) + \mathcal{O}(\epsilon^2)\end{aligned}$$

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- Additionally require $A^{(2)}_{\text{simple}}(C_1) = 0 \longrightarrow$ constraints on C_1

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$$= \epsilon A^{(1)}_{\text{simple}} + \epsilon^2 \left(A^{(2)}_{\text{simple}} + A^{(2)}_{\text{double}} - \partial_x T_2^{(2)} \right) + \mathcal{O}(\epsilon^3)$$

$$\longrightarrow T_2^{(2)} = \int A^{(2)}_{\text{double}} \mathrm{d}x + C_2$$

- Additionally require $A^{(2)}_{\text{simple}}(C_1) = 0 \longrightarrow$ constraints on C_1
- At each step check if $A^{(n+1)}(C_1, \dots, C_{n-1}) = 0$ has a solution for the C_i

Beyond Polylogarithms (elliptic, Calabi-Yau)

- Most important differences:
  - rational factorization not possible

$$0 = \left[(\partial_x + R_1) \dots (\partial_x^2 + R_{i,1}\partial_x + R_{i,0}) \dots (\partial_x + R_n)\right] f_1^{(0)}$$

[Adams, Chaubey, Weinzierl, '17]

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  - elliptic / Calabi-Yau:

$$0 = [\beta(x)\theta_x\alpha_n\theta_x\alpha_{n-1}\theta_x\dots\theta_x\alpha_0]\, f_1^{(0)}, \qquad \theta_x = x\partial_x$$

equivalent to semi-simple part

[Duhr et al., '25, ...]

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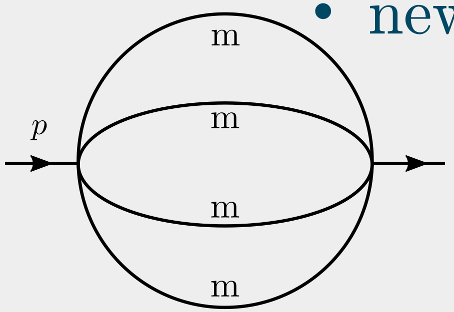
  - example: three-loop banana integrals
    - complete elliptic integral: $$0 = \left[\frac{1+8x}{4x^2(1+4x)(1+16x)} + \frac{2(5+32x)}{(1+4x)(1+16x)}\partial_x + \partial_x^2\right]\psi$$
    - new functions:

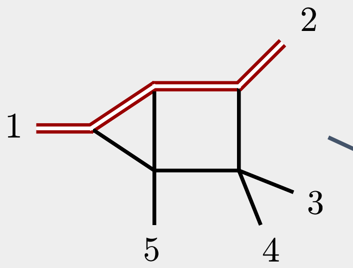


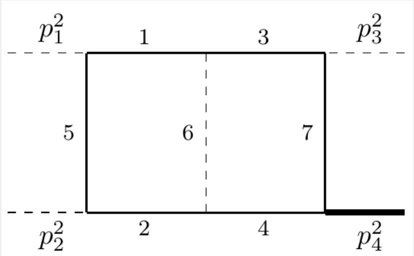
$$G_1 = \int \frac{2(1-8x)^3(1+8x)}{x^2(1+4x)^2(1+16x)^2}\psi^4 \mathrm{d}x$$

$$G_2 = -\int \frac{G_1}{\sqrt{(1+4x)(1+16x)}}\frac{1}{\psi^2}\mathrm{d}x$$

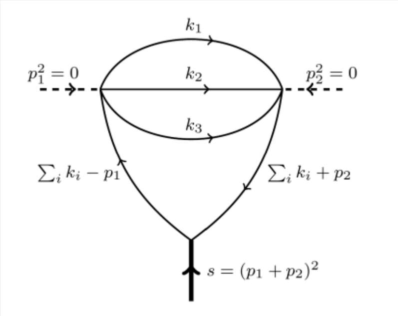
[Pögel, Wang, Weinzierl, ‘22]

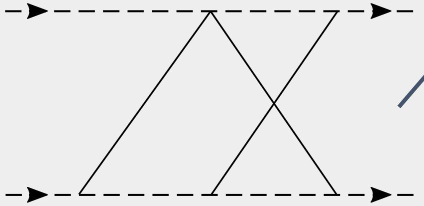
Benchmarks (on the maximal cut)





[Görges et al., '23]





	Geometry	Masters	Runtime [s]
4PM	MPL	6	112
$t\bar{t} + j$ *	elliptic	3	7
2L box $(2m)$ *	elliptic	4	9
3L ice cone	$2 \times$ elliptic	5	5
2L banana $(1m)$	elliptic	2	1
3L banana $(1m)$	K3	3	11
4L banana $(1m)$	CY3	4	94
4PM	K3	3	4
5PM 2SF	K3	9	52
5PM 2SF	CY3	5	2450

*univariate slice

Summary

- Algorithm:

minimal solvable form

$$\mathrm{d}\vec{f} = \mathrm{d}A(\epsilon)\,\vec{f}$$

→

canonical form

$$\mathrm{d}\vec{g} = \epsilon\,\mathrm{d}\tilde{A}^{(1)}_{\text{simple}}\,\vec{g}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{pmatrix}$$

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- Remove $A^{(0)}$:
  - $\Rightarrow$ remove $A^{(0)}_{\text{double}}$
  - $\Rightarrow$ push $A^{(0)}_{\text{simple}}$ to $A^{(1)}_{\text{simple}}$

[Adams, Chaubey, Weinzierl, ’17]

- Remove $A^{(1)}_{\text{double}}$

[Görges et al., ’23, Duhr et al., ’25]

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CANONICA: [Meyer, ’17]

- Coming soon: paper, Mathematica package, more benchmarks