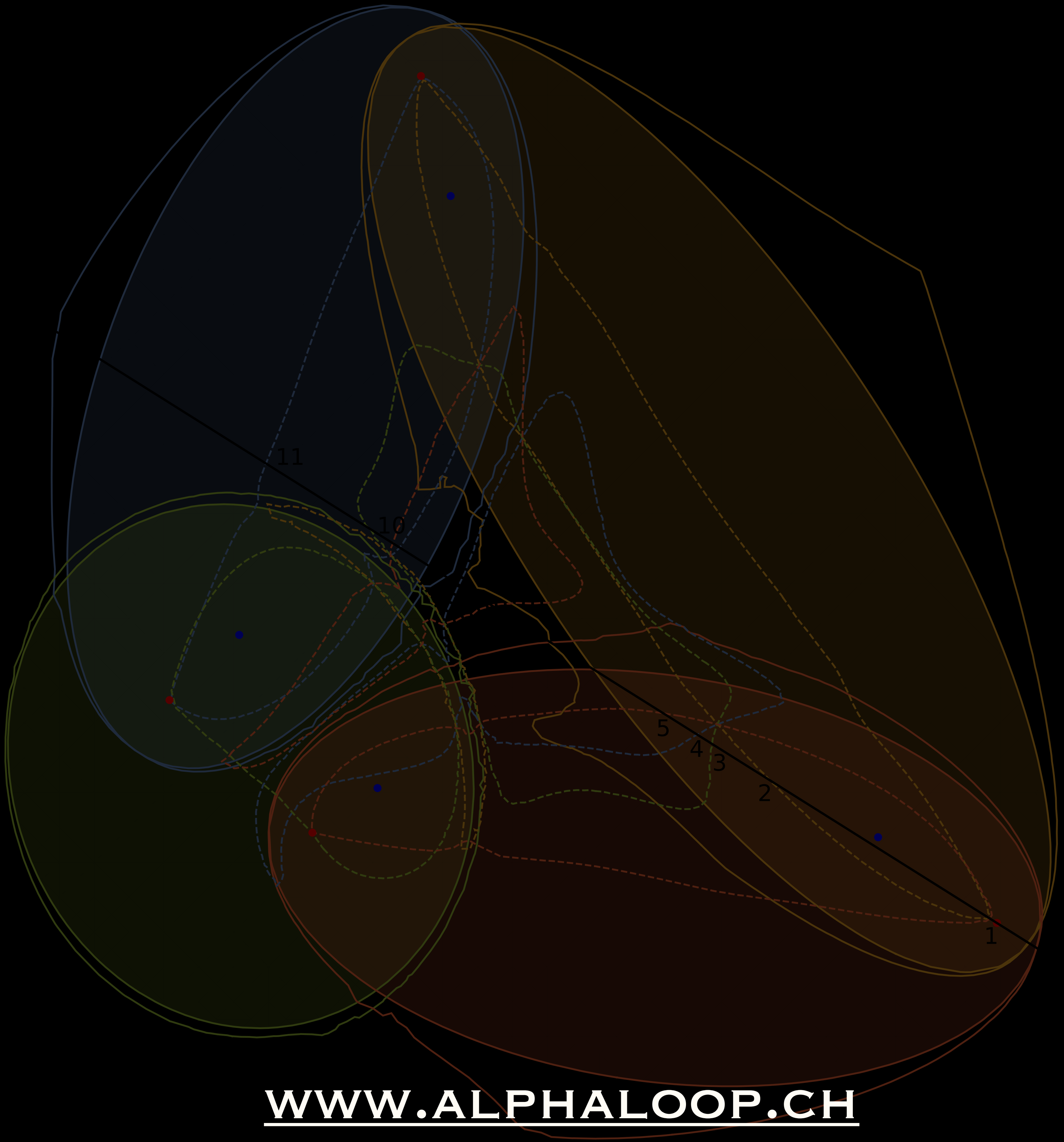

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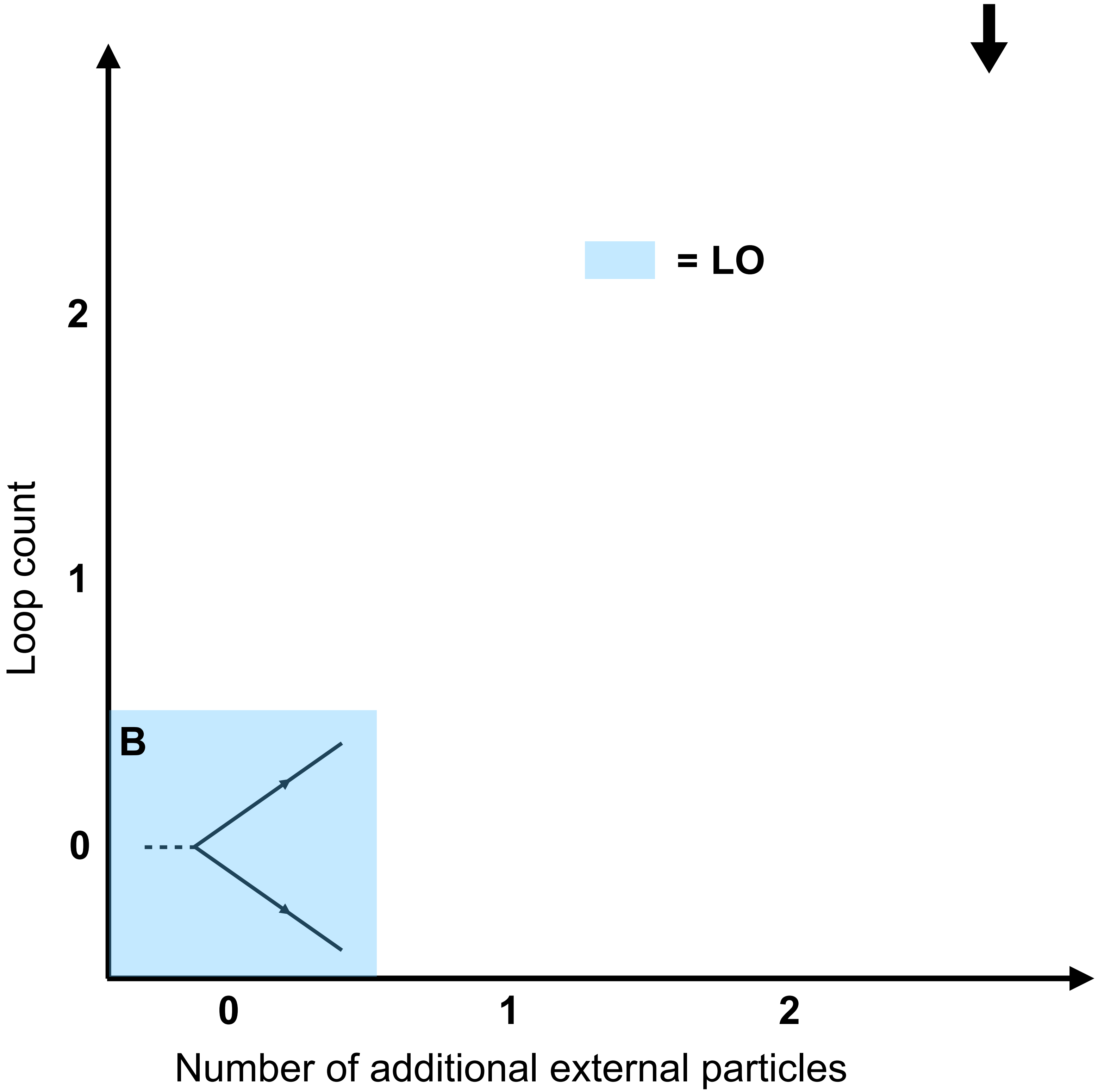
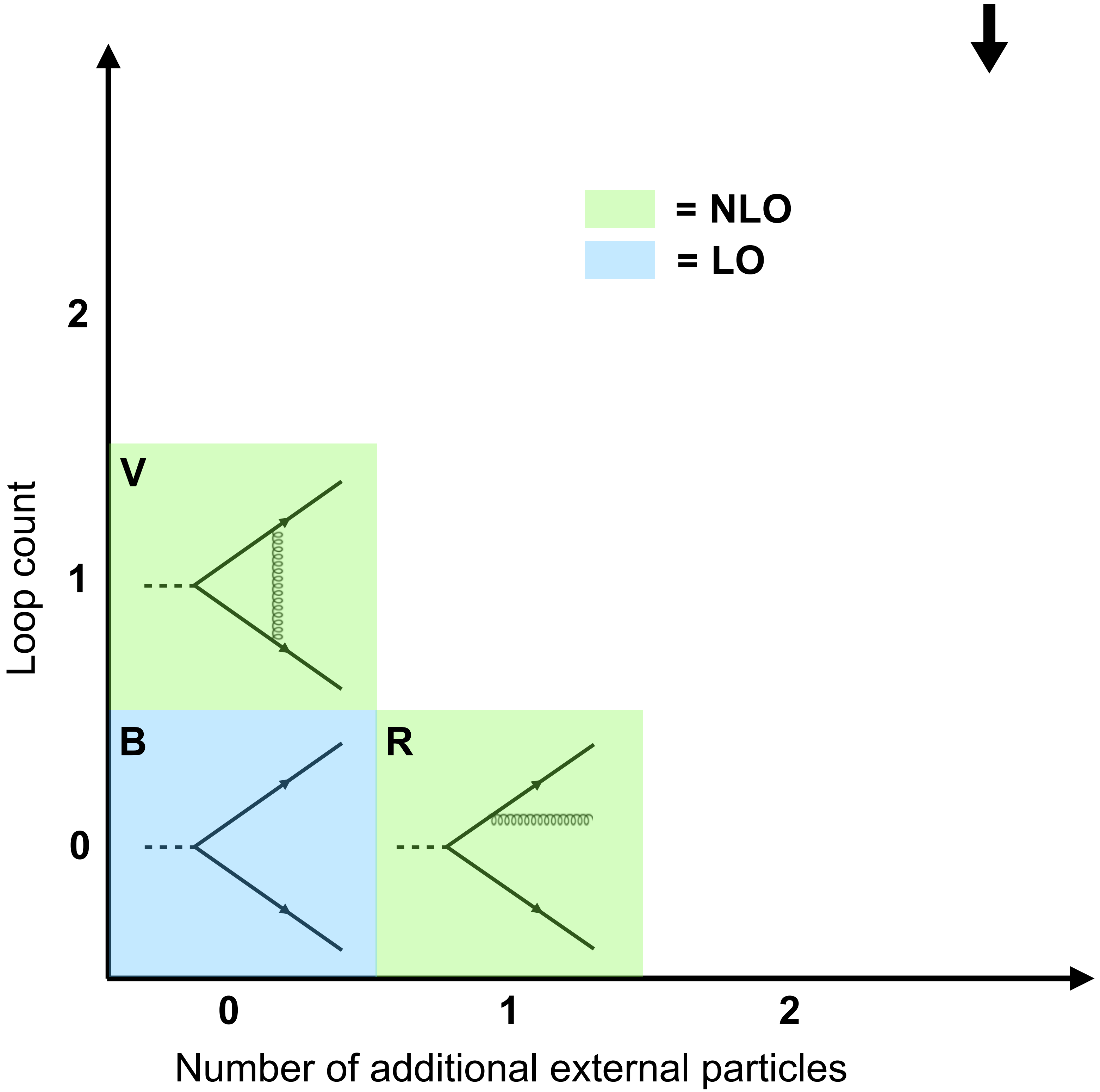
PERTURBATIVE EXPANSIONS

$$\langle p_1, \ldots, p_n | S | q_1, \ldots, q_m \rangle$$



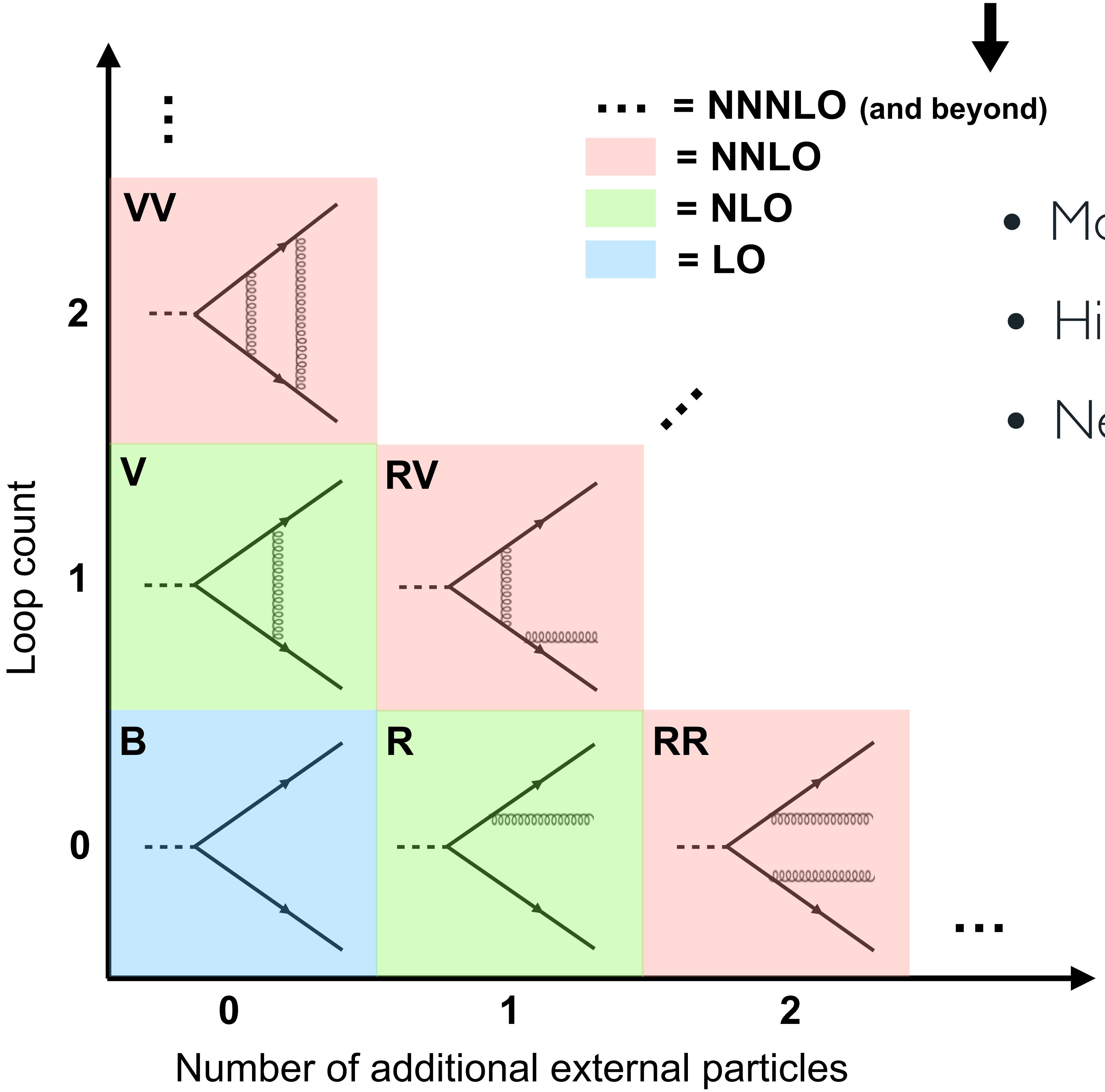
PERTURBATIVE EXPANSIONS

$$\langle p_1, \ldots, p_n | S | q_1, \ldots, q_m \rangle$$



PERTURBATIVE EXPANSIONS

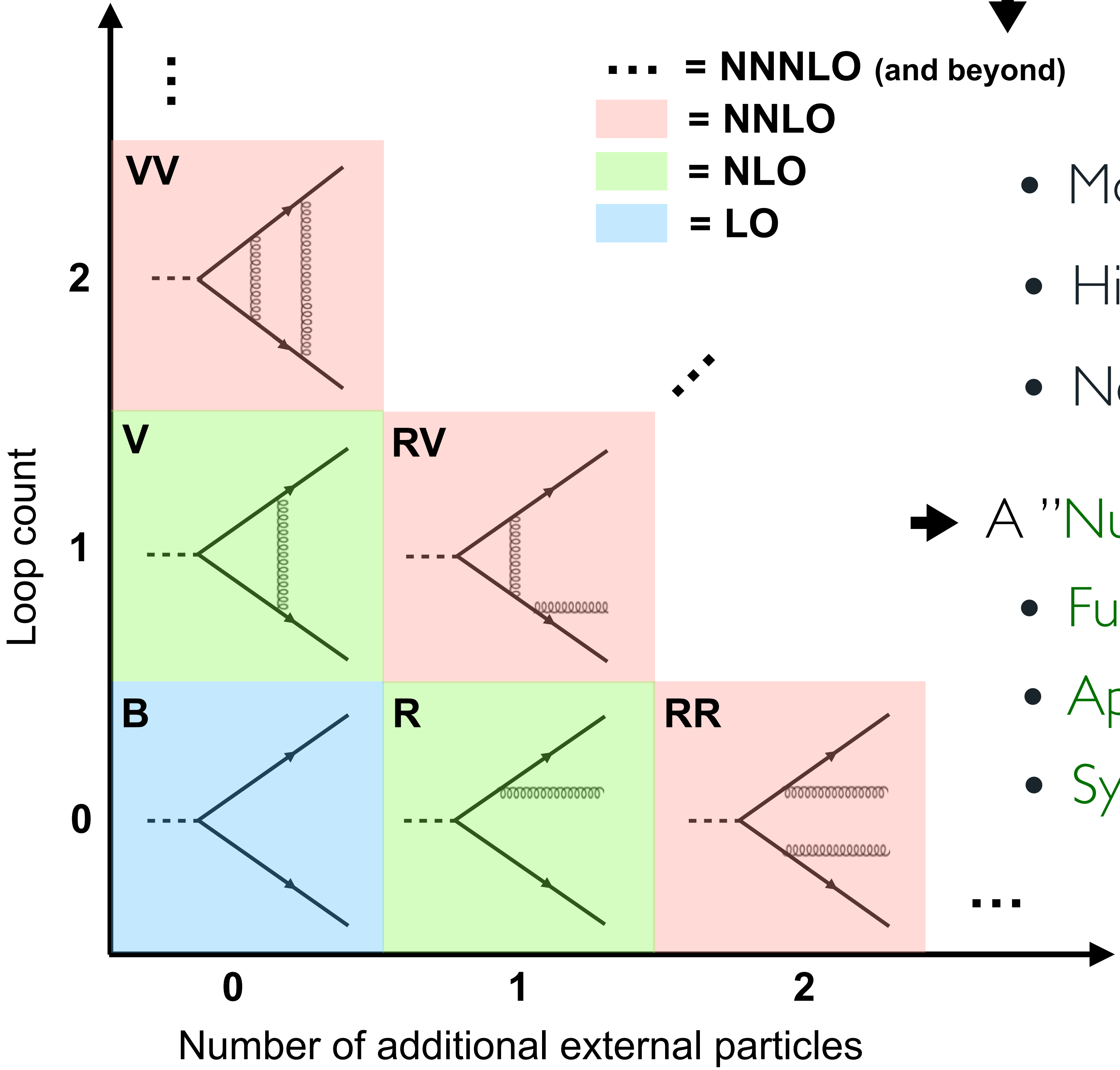
$$\langle p_1, \ldots, p_n | S | q_1, \ldots, q_m \rangle$$



- More complicated singularities
- Higher dimensionality
- New (unknown) analytic integrals

PERTURBATIVE EXPANSIONS

$$\langle p_1, \ldots, p_n | S | q_1, \ldots, q_m \rangle$$



- More complicated singularities
- Higher dimensionality
- New (unknown) analytic integrals

➜ A "Numerical Collider" ideally features:

- Fully numerical approach
- Applicable to all scatterings
- Systematic at all orders

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 (p_1 + \ldots p_N)$$

Numerical

LO

$d\bar{d} \to Z$

{}

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\underset{\text{Numerical}}{\int d^4p_1 \dots d^4p_N}} \left|\mathcal{M}^{(L)}(\{p_i\})\right|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4 \left(p_1 + \dots p_N\right)$$

LO	NLO
$d\bar{d} \to Z$	$d\bar{d} \to Zg$

{}

```
| L0 : ((S(4),),)
| L1 : ((C(1,4),),)
| L2 : ((C(2,4),),)
| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 (p_1 + \ldots p_N)$$

Numerical

LO

$$d\bar{d} \to Z$$

{}

NLO

$$d\bar{d} \to Zg$$

```
| L0 : ((S(4),),)
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| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

NNLO

$$d\bar{d} \to Zgg$$

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
| L3 : ((C(1,4),),)
| L4 : ((C(2,4),),)
[ … 71 limits … ]
| L69 : ((C(C(S(4),2),S(5),),),)
| L70 : ((C(S(C(S(4),5),),2),),)
| L71 : ((C(S(C(S(5),4),),2),),)
```

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\begin{array}{l} \int d^4p_1 \dots d^4p_N \\ \text{Numerical} \end{array}} |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \dots \delta p_N^2 \delta^4\left(p_1 + \dots p_N\right)$$

LO

$$d\bar{d} \to Z$$

{}

NLO

$$d\bar{d} \to Zg$$

```
| L0 : ((S(4),),)
| L1 : ((C(1,4),),)
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| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

NNLO

$$d\bar{d} \to Zgg$$

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
| L3 : ((C(1,4),),)
| L4 : ((C(2,4),),)
[ … 71 limits … ]
| L69 : ((C(C(S(4),2),S(5),),),)
| L70 : ((C(S(C(S(4),5),),2),),)
| L71 : ((C(S(C(S(5),4),),2),),)
```

N3LO

$$d\bar{d} \to Zggg$$

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((S(6),),)
| L3 : ((C(4,5),),)
| L4 : ((C(4,6),),)
[ … 1396 limits … ]
| L1394 : ((C(S(C(S(C(S(6),4),),5),),2),),)
| L1395 : ((C(S(C(S(C(S(5),6),),4),),2),),)
| L1396 : ((C(S(C(S(C(S(6),5),),4),),2),),)
```

…

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

LO

$d\bar{d} \to Z$

{}

NLO

$d\bar{d} \to Zg$

```
| L0 : ((S(4),),)
| L1 : ((C(1,4),),)
| L2 : ((C(2,4),),)
| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

NNLO

$d\bar{d} \to Zgg$

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
| L3 : ((C(1,4),),)
| L4 : ((C(2,4),),)
[ … 71 limits … ]
| L69 : ((C(C(S(4),2),S(5),),),)
| L70 : ((C(S(C(S(4),5),),2),),)
| L71 : ((C(S(C(S(5),4),),2),),)
```

N3LO

$d\bar{d} \to Zggg$

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((S(6),),)
| L3 : ((C(4,5),),)
| L4 : ((C(4,6),),)
[ … 1396 limits … ]
| L1394 : ((C(S(C(S(C(S(6),4),),5),),2),),)
| L1395 : ((C(S(C(S(C(S(5),6),),4),),2),),)
| L1396 : ((C(S(C(S(C(S(6),5),),4),),2),),)
```

…

On top of :

- R^kV^{k-n} proliferation of contributions
- Convolution with PDF counterterms
- Growth of number of partonic channels

} Very challenging at and beyond NNLO…

OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \boxed{\int d^dk_1 \cdots \int d^dk_L} I(\{p_i\}, \{k_i\})|$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]
$d\bar{d} \rightarrow ZZ$	7 μs
$d\bar{d} \rightarrow ZZg$	35 μs
$d\bar{d} \rightarrow ZZgg$	220 μs

OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\int d^4p_1 \dots d^4p_N} |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots p_N)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \boxed{\int d^d k_1 \cdots \int d^d k_L} I(\{p_i\}, \{k_i\})|$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]		One-loop MadLoop [arXiv:1103.0621]
$d\bar{d} \to ZZ$	7 μs	x 10^2 →	0.6 ms
$d\bar{d} \to ZZg$	35 μs	x 10^3 →	38 ms
$d\bar{d} \to ZZgg$	220 μs	x 10^4 →	1200 ms

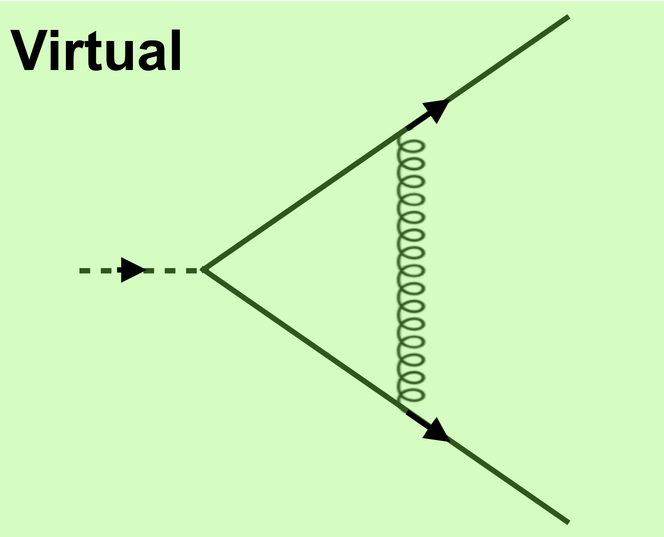
OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\begin{array}{l}\int d^4p_1 \ldots d^4p_N \\ \text{Numerical}\end{array}} \left|\mathcal{M}^{(L)}(\{p_i\})\right|^2 \delta p_1^2 \ldots \delta p_N^2 \delta^4\left(p_1 + \ldots p_N\right)$$

$$\left|\mathcal{M}^{(L)}(\{p_i\})\right|^2 \sim \boxed{\begin{array}{l}\int d^dk_1 \cdots \int d^dk_L \\ \text{Analytical ?}\end{array}} I(\{p_i\},\{k_i\})\big|$$

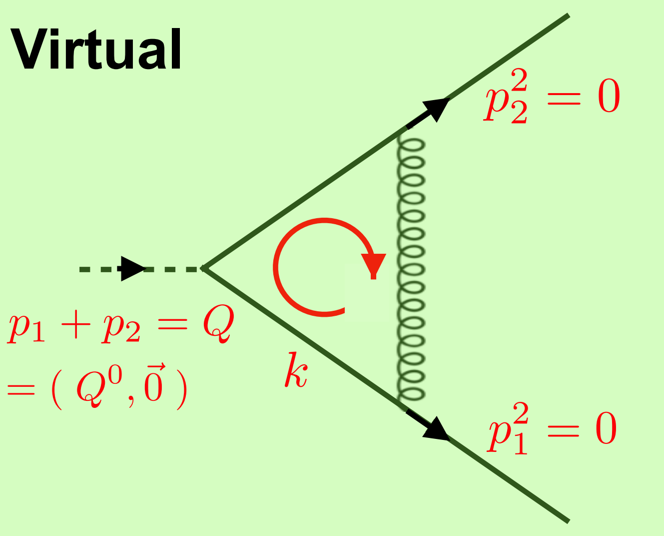
Timings	Tree-level MG5aMC [arXiv:1405.0301]		One-loop MadLoop [arXiv:1103.0621]		Two-loop VVamp [arXiv:1503.08835]
$d\bar{d} \to ZZ$	7 µs	$\xrightarrow{\text{x } 10^2}$	0.6 ms	$\xrightarrow{\text{x } 10^3}$	$\mathcal{O}$(~1s)
$d\bar{d} \to ZZg$	35 µs	$\xrightarrow{\text{x } 10^3}$	38 ms	$\xrightarrow{?}$	N/A
$d\bar{d} \to ZZgg$	220 µs	$\xrightarrow{\text{x } 10^4}$	1200 ms		N/A

LOOP INTEGRAND SINGULARITIES

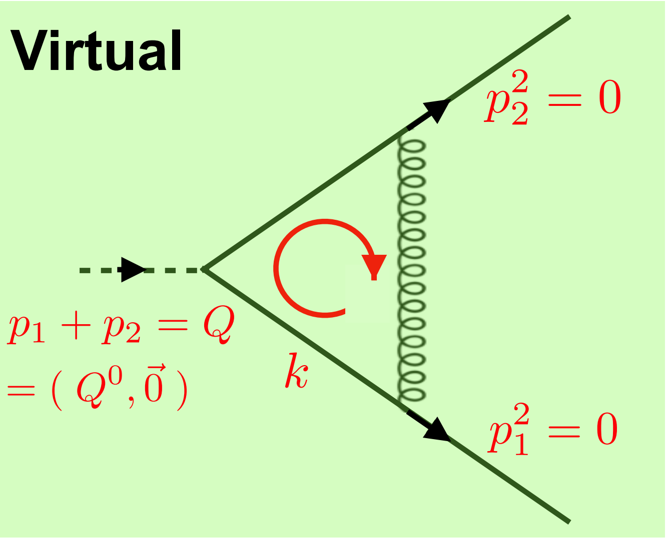


LOOP INTEGRAND SINGULARITIES

Virtual

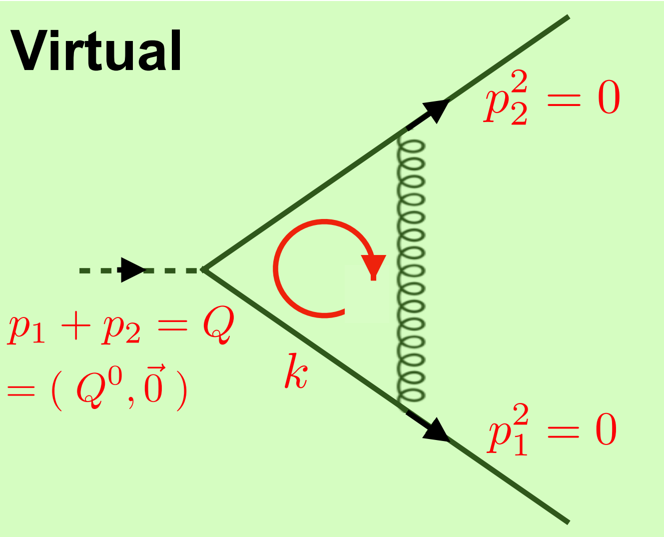


LOOP INTEGRAND SINGULARITIES



$$\longrightarrow \int dk^0 d^3\vec{k} \;\; \frac{\bar{v}_{p_2}\gamma^\mu \not{k} \, (\not{k} + \not{Q}) \, \gamma_\mu u_{p_1}}{k^2 (k+Q)^2 (k+p_1)^2}$$

LOOP INTEGRAND SINGULARITIES



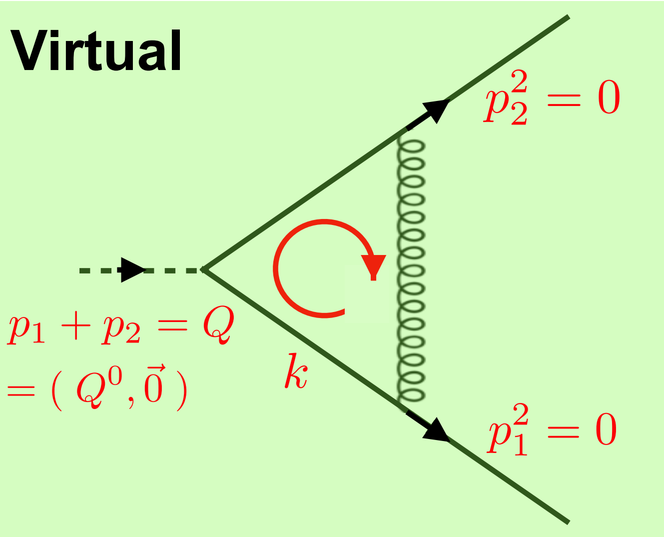
$$\int dk^0 d^3\vec{k} \; \frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k}+\not{Q})\,\gamma_\mu u_{p_1}}{k^2(k+Q)^2(k+p_1)^2}$$

Loop-Tree Duality

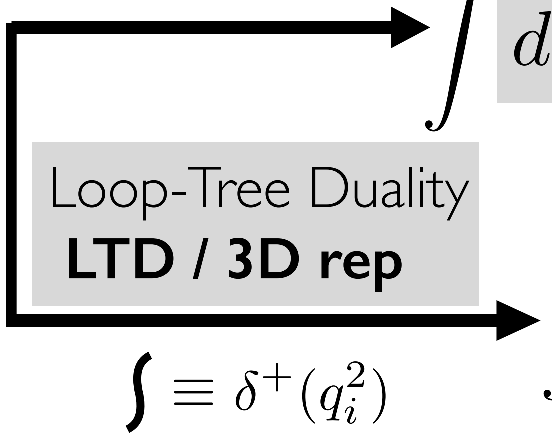
LTD / 3D rep

$$\int d^3\vec{k} \left[\qquad\qquad\qquad\qquad \right]$$

LOOP INTEGRAND SINGULARITIES



$$\int dk^0 d^3\vec{k} \; \frac{\bar{v}_{p_2} \gamma^\mu \not{k} \, (\not{k} + \not{Q}) \, \gamma_\mu u_{p_1}}{k^2 (k+Q)^2 (k+p_1)^2}$$



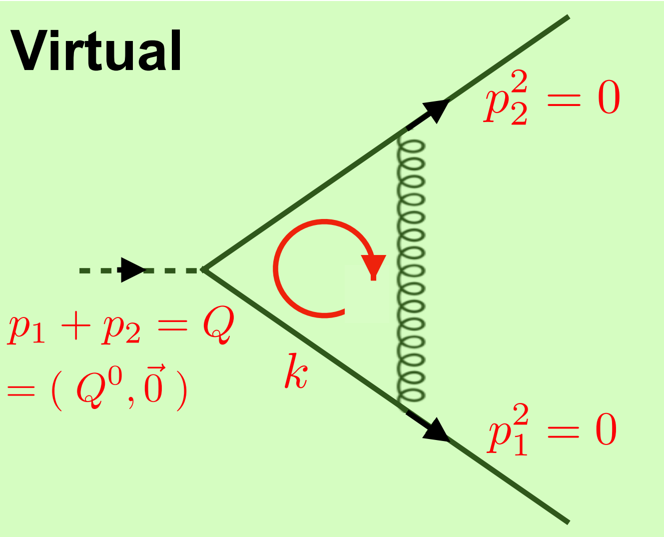
Loop-Tree Duality

LTD / 3D rep

$$\int \equiv \delta^+(q_i^2)$$

$$\int d^3\vec{k}$$

LOOP INTEGRAND SINGULARITIES



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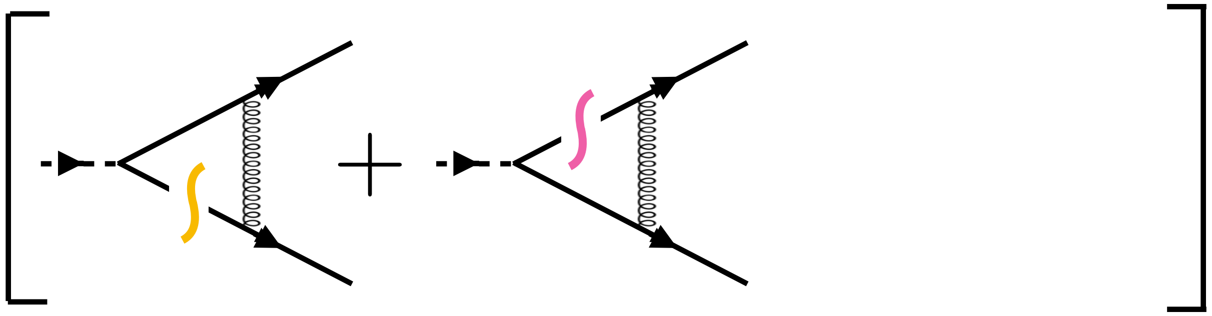
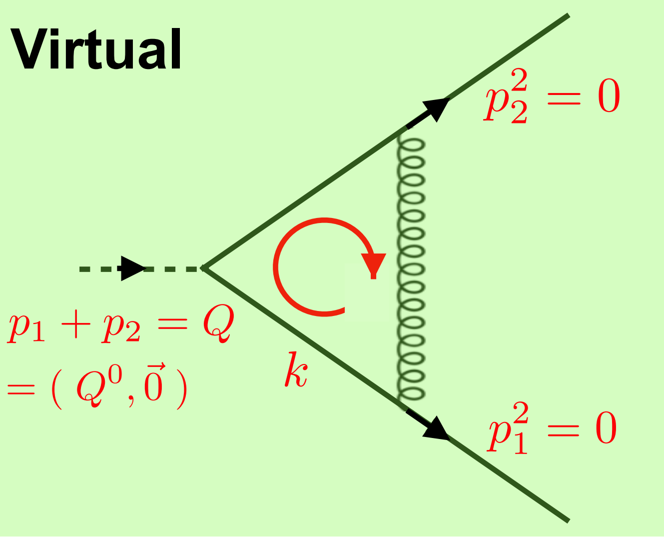


$\int \equiv \delta^+(q_i^2)$

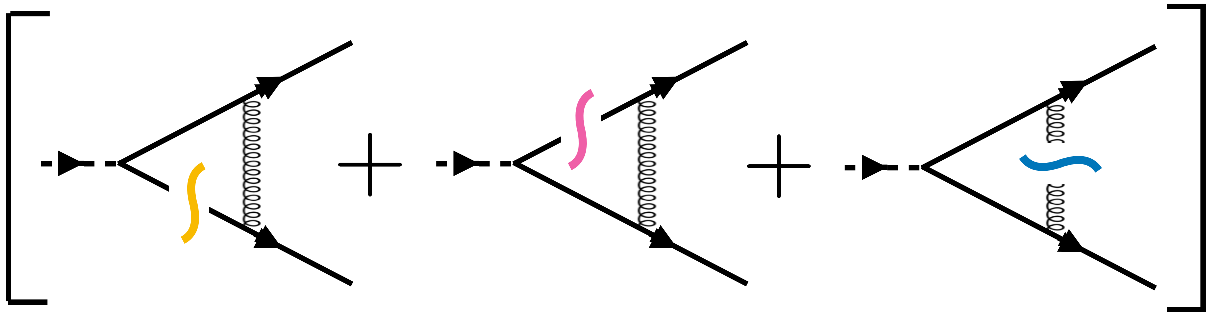
LOOP INTEGRAND SINGULARITIES



$$\int dk^0 d^3\vec{k} \; \frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k} + \not{Q})\,\gamma_\mu u_{p_1}}{k^2 (k+Q)^2 (k+p_1)^2}$$

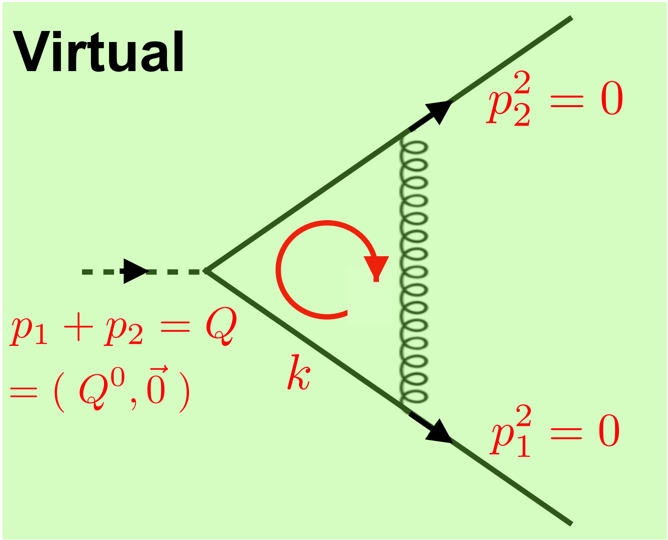
Loop-Tree Duality
LTD / 3D rep

$$\int d^3\vec{k}$$



$\int \equiv \delta^+(q_i^2)$

LOOP INTEGRAND SINGULARITIES



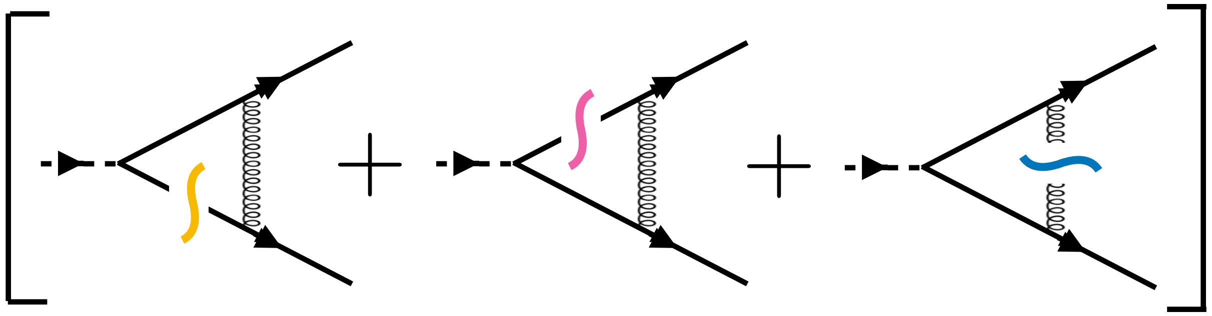
$$\int dk^0 d^3\vec{k} \; \frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k} + \not{Q})\,\gamma_\mu u_{p_1}}{k^2 (k+Q)^2 (k+p_1)^2}$$

Loop-Tree Duality

LTD / 3D rep

$\int \equiv \delta^+(q_i^2)$

$$\int d^3\vec{k}$$



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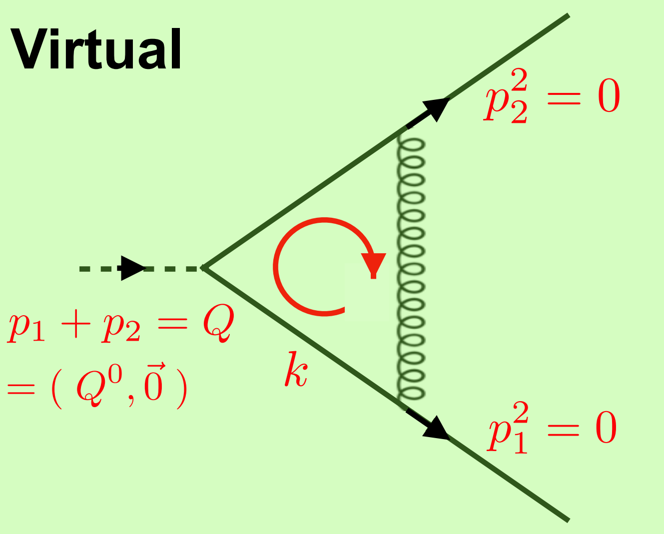
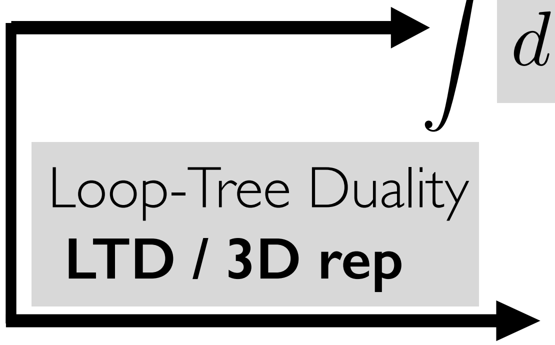
cLTD
$\supset$
CFF

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \quad \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2|\vec{k} + \vec{p_1}|} \quad \frac{1}{|\vec{k}| + |\vec{k} + \vec{p_1}| - |\vec{p_1}|} \quad \frac{1}{|\vec{k}| + |\vec{k} + \vec{Q}| - Q^0 + \mathrm{i}\delta}$$

LOOP INTEGRAND SINGULARITIES





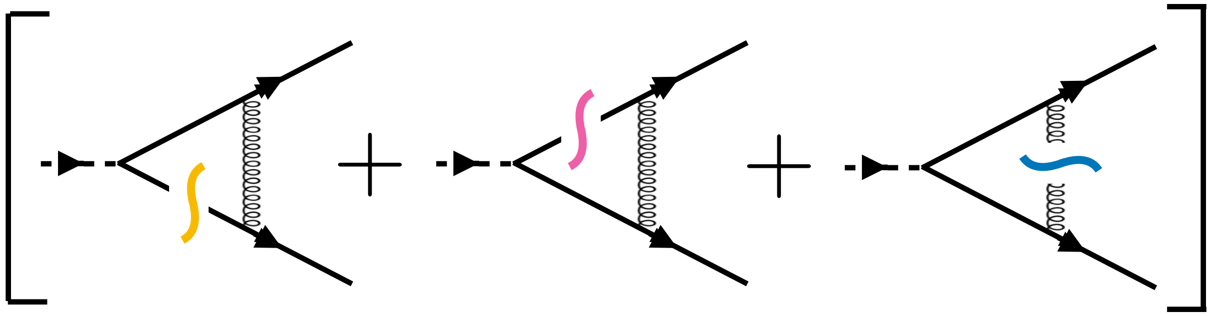
$$\int dk^0 d^3\vec{k}\ \frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k}+\not{Q})\,\gamma_\mu u_{p_1}}{k^2(k+Q)^2(k+p_1)^2}$$

Loop-Tree Duality

LTD / 3D rep

$\int \equiv \delta^+(q_i^2)$

$$\int d^3\vec{k}$$



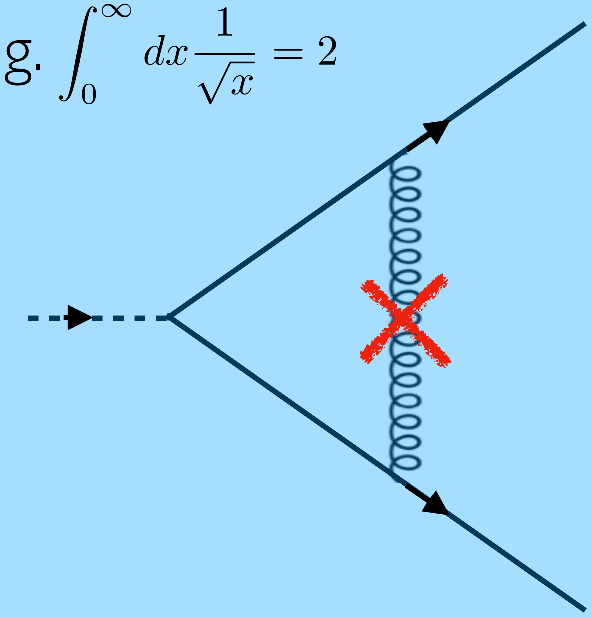
cLTD

$\supset$

CFF

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}|\ \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2|\vec{k}+\vec{p_1}|}\ \frac{1}{|\vec{k}| + |\vec{k}+\vec{p_1}| - |\vec{p_1}|}\ \frac{1}{|\vec{k}| + |\vec{k}+\vec{Q}| - Q^0 + \mathrm{i}\delta}$$

e.g. $\int_0^\infty dx \frac{1}{\sqrt{x}} = 2$

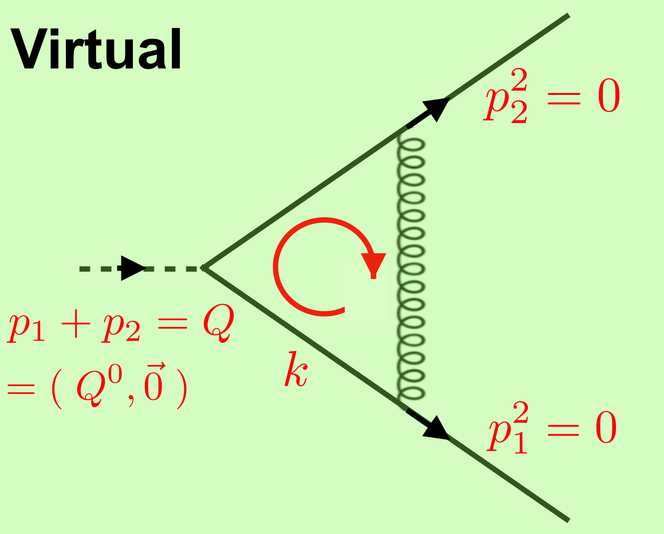


$$\vec{k} \to -\vec{p_1}$$

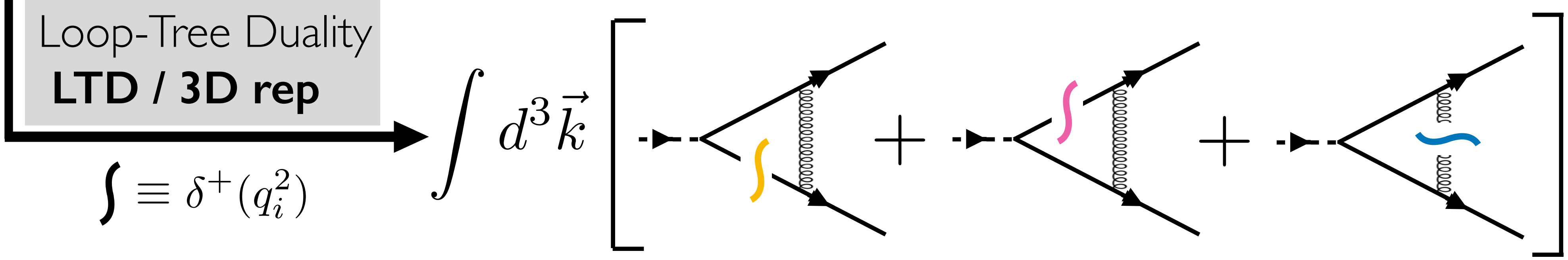
Integrable singularity

- **LMB multi-channeling**
- **Tropical sampling**

LOOP INTEGRAND SINGULARITIES



$$\int \boxed{dk^0} d^3\vec{k}\;\frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k}+\not{Q})\,\gamma_\mu u_{p_1}}{k^2(k+Q)^2(k+p_1)^2}$$



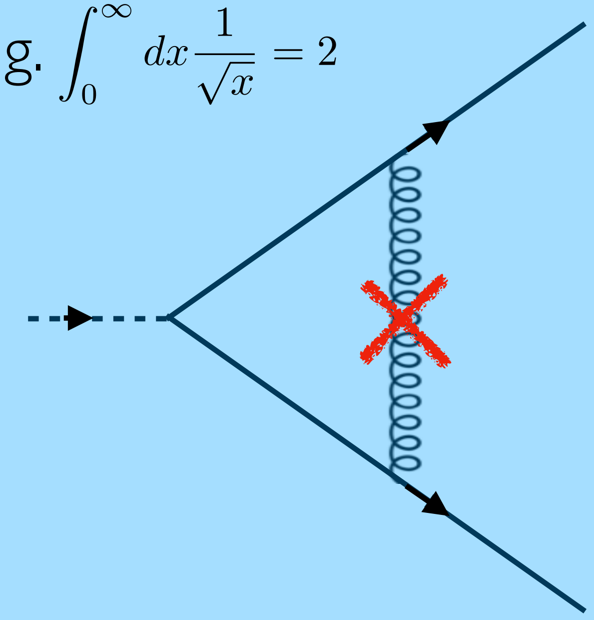
$$\int d^3\vec{k}\left[\;\cdots+\cdots+\cdots\;\right]$$

$$\int \equiv \delta^+(q_i^2)$$

cLTD $\supset$ **CFF**

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}|\;\frac{a|\vec{k}|^2+b|\vec{k}|+c}{8|\vec{k}|^2|\vec{k}+\vec{p_1}|}\;\frac{1}{|\vec{k}|+|\vec{k}+\vec{p_1}|-|\vec{p_1}|}\;\frac{1}{|\vec{k}|+|\vec{k}+\vec{Q}|-Q^0+\mathrm{i}\delta}$$

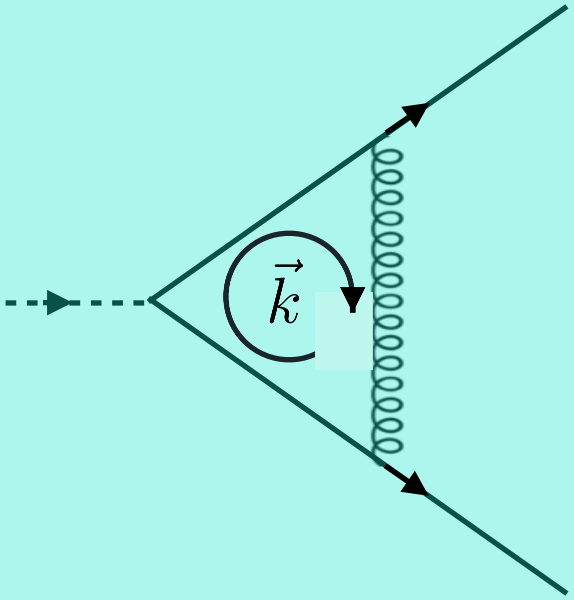
e.g. $\int_0^\infty dx\frac{1}{\sqrt{x}}=2$



$$\vec{k}\to-\vec{p_1}$$

Integrable singularity

- **LMB multi-channeling**
- **Tropical sampling**

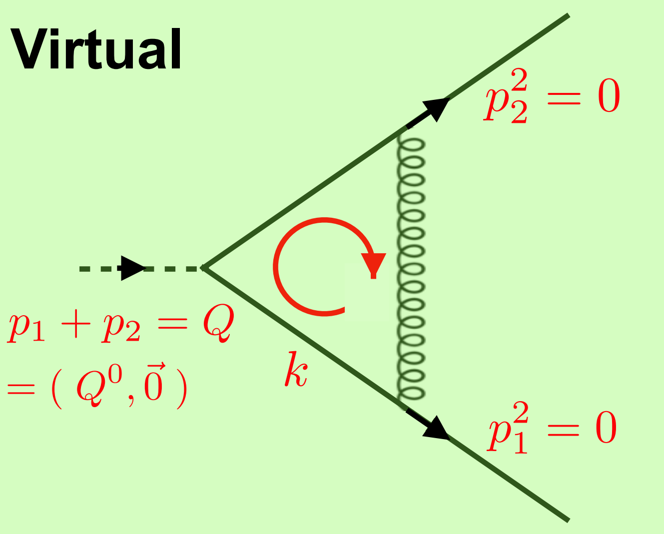


$$|\vec{k}|\to\infty$$

UV singularity

- **Local BPHZ**
- **Scheme matching**

LOOP INTEGRAND SINGULARITIES



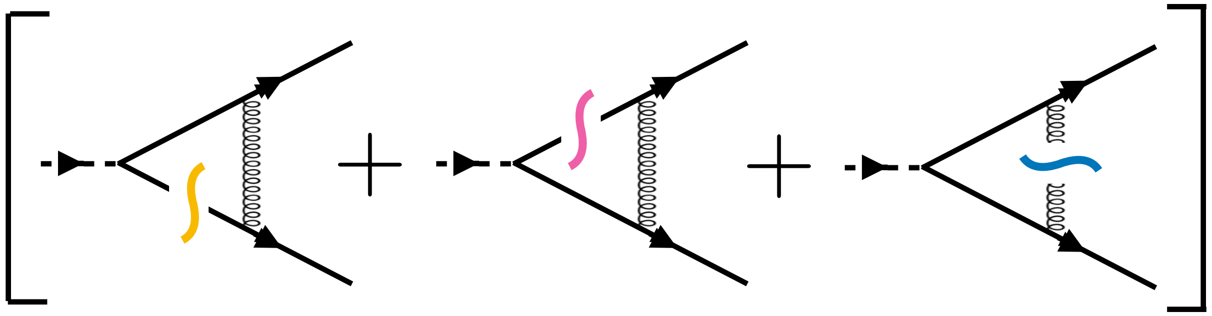
$$\int dk^0 d^3\vec{k} \; \frac{\bar{v}_{p_2}\gamma^\mu \not{k} \left(\not{k} + \not{Q}\right) \gamma_\mu u_{p_1}}{k^2 (k+Q)^2 (k+p_1)^2}$$

Loop-Tree Duality

LTD / 3D rep

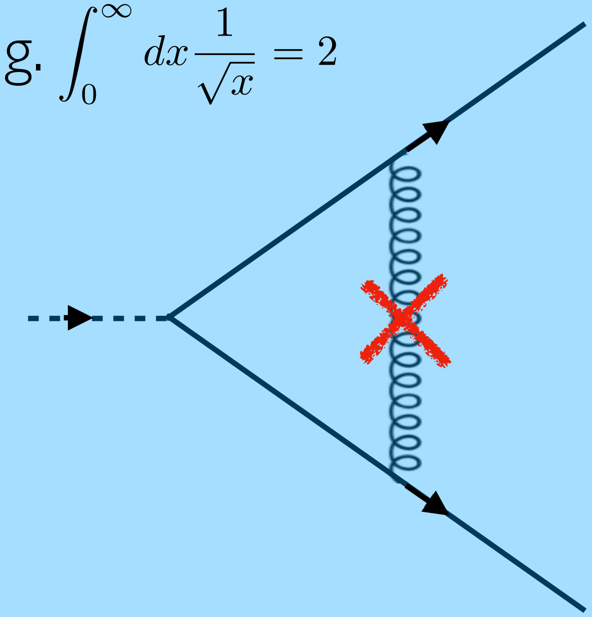
$\int \equiv \delta^+(q_i^2)$

$$\int d^3\vec{k}$$



$$\text{cLTD} \supset \text{CFF} \int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \; \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2|\vec{k}+\vec{p_1}|} \; \frac{1}{|\vec{k}| + |\vec{k}+\vec{p_1}| - |\vec{p_1}|} \; \frac{1}{|\vec{k}| + |\vec{k}+\vec{Q}| - Q^0 + \mathrm{i}\delta}$$

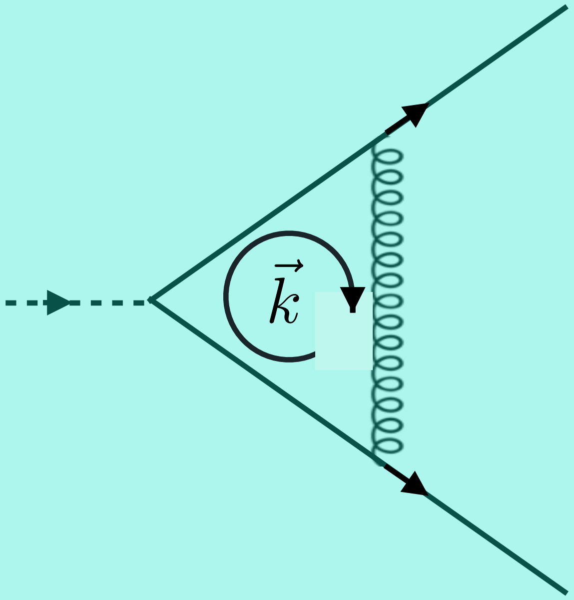
e.g. $\int_0^\infty dx \frac{1}{\sqrt{x}} = 2$



$\vec{k} \to -\vec{p_1}$

Integrable singularity

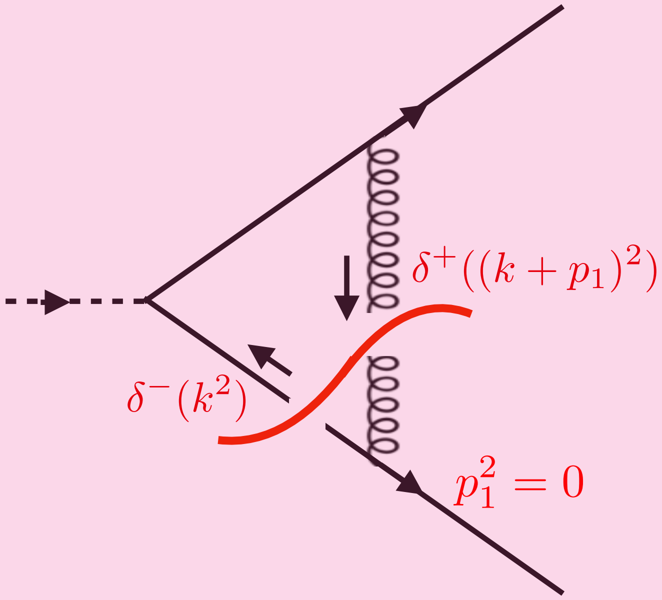
- **LMB multi-channeling**
- **Tropical sampling**



$|\vec{k}| \to \infty$

UV singularity

- **Local BPHZ**
- **Scheme matching**

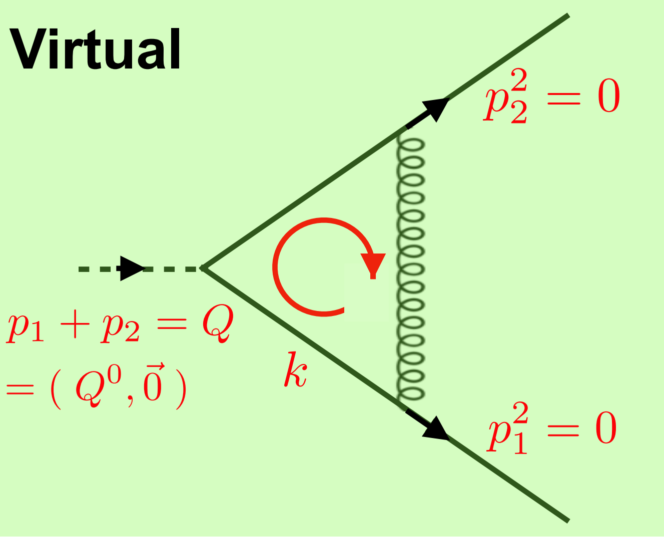


$\vec{k} \to -x\vec{p_1} \;\; x \in [0,1]$

IR singularity

- **Loop IR counterterms**
- **Local Unitarity / KLN**

LOOP INTEGRAND SINGULARITIES



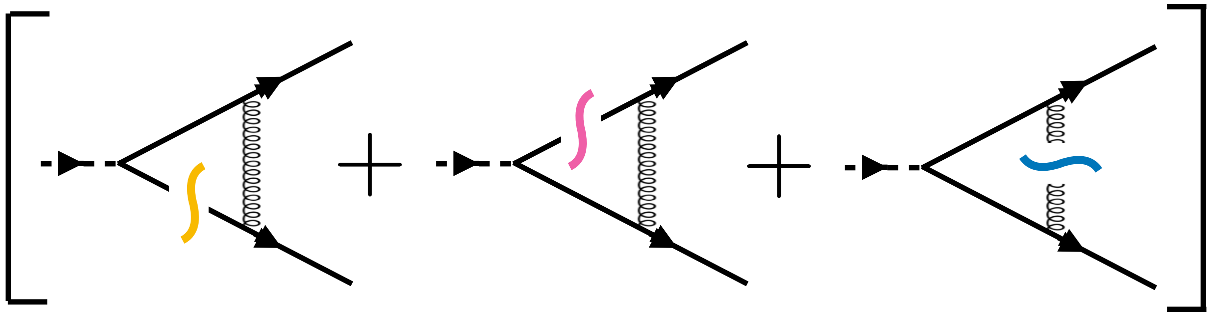
$$\int dk^0 d^3\vec{k}\,\frac{\bar{v}_{p_2}\gamma^\mu \not{k}\,(\not{k}+\not{Q})\,\gamma_\mu u_{p_1}}{k^2(k+Q)^2(k+p_1)^2}$$

Loop-Tree Duality

LTD / 3D rep

$\int \equiv \delta^+(q_i^2)$

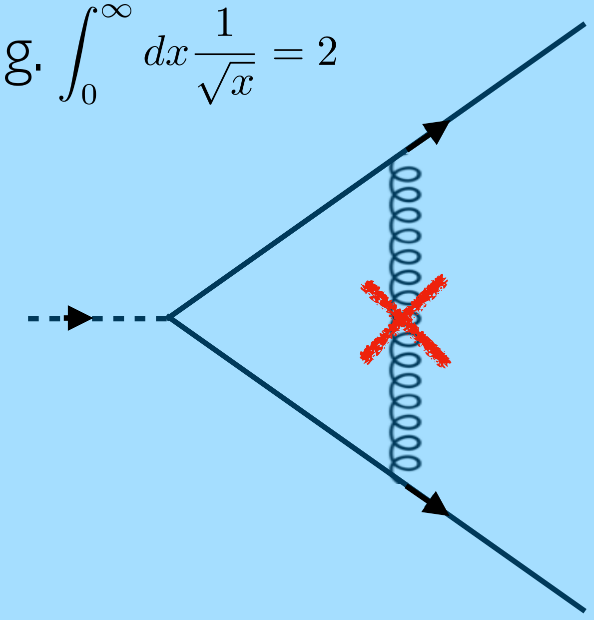
$$\int d^3\vec{k}$$



cLTD
$\supset$
CFF

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}|\,\frac{a|\vec{k}|^2+b|\vec{k}|+c}{8|\vec{k}|^2|\vec{k}+\vec{p_1}|}\,\frac{1}{|\vec{k}|+|\vec{k}+\vec{p_1}|-|\vec{p_1}|}\,\frac{1}{|\vec{k}|+|\vec{k}+\vec{Q}|-Q^0+\mathrm{i}\delta}$$

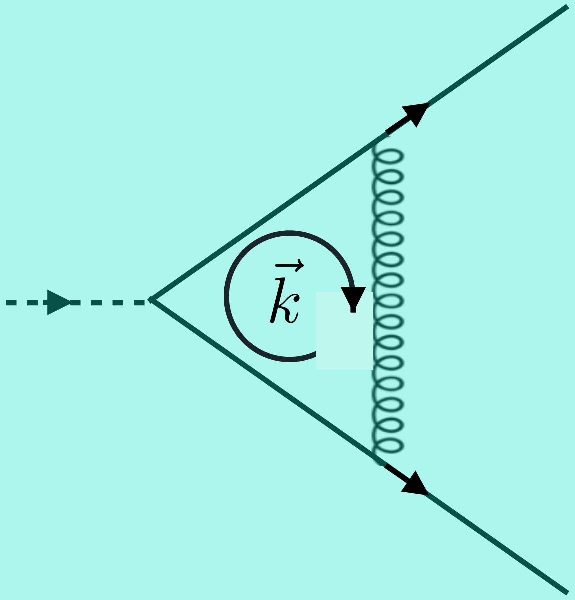
e.g. $\int_0^\infty dx\frac{1}{\sqrt{x}}=2$



$\vec{k}\to-\vec{p_1}$

Integrable singularity

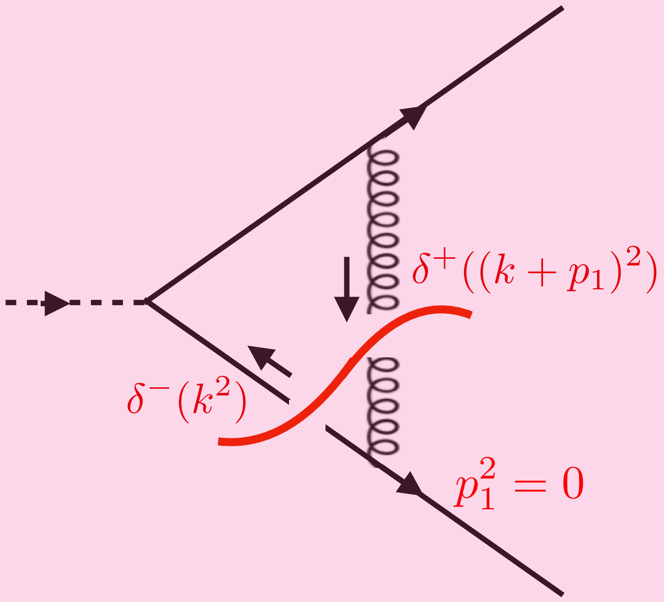
- LMB multi-channeling
- Tropical sampling



$|\vec{k}|\to\infty$

UV singularity

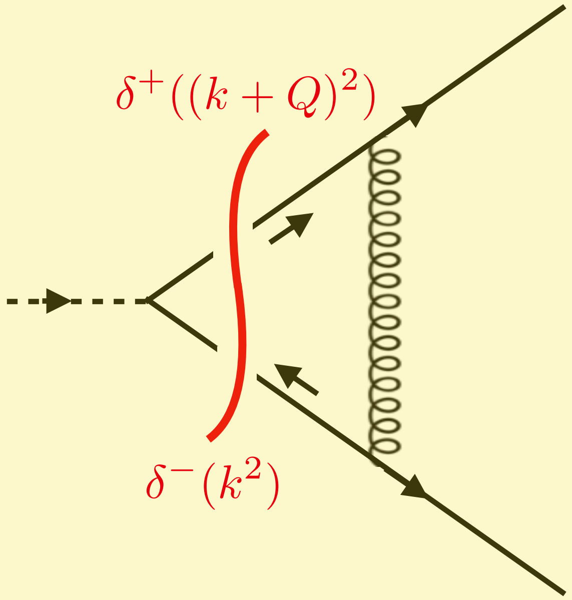
- Local BPHZ
- Scheme matching



$\vec{k}\to-x\vec{p_1}\;\;x\in[0,1]$

IR singularity

- Loop IR counterterms
- Local Unitarity / KLN



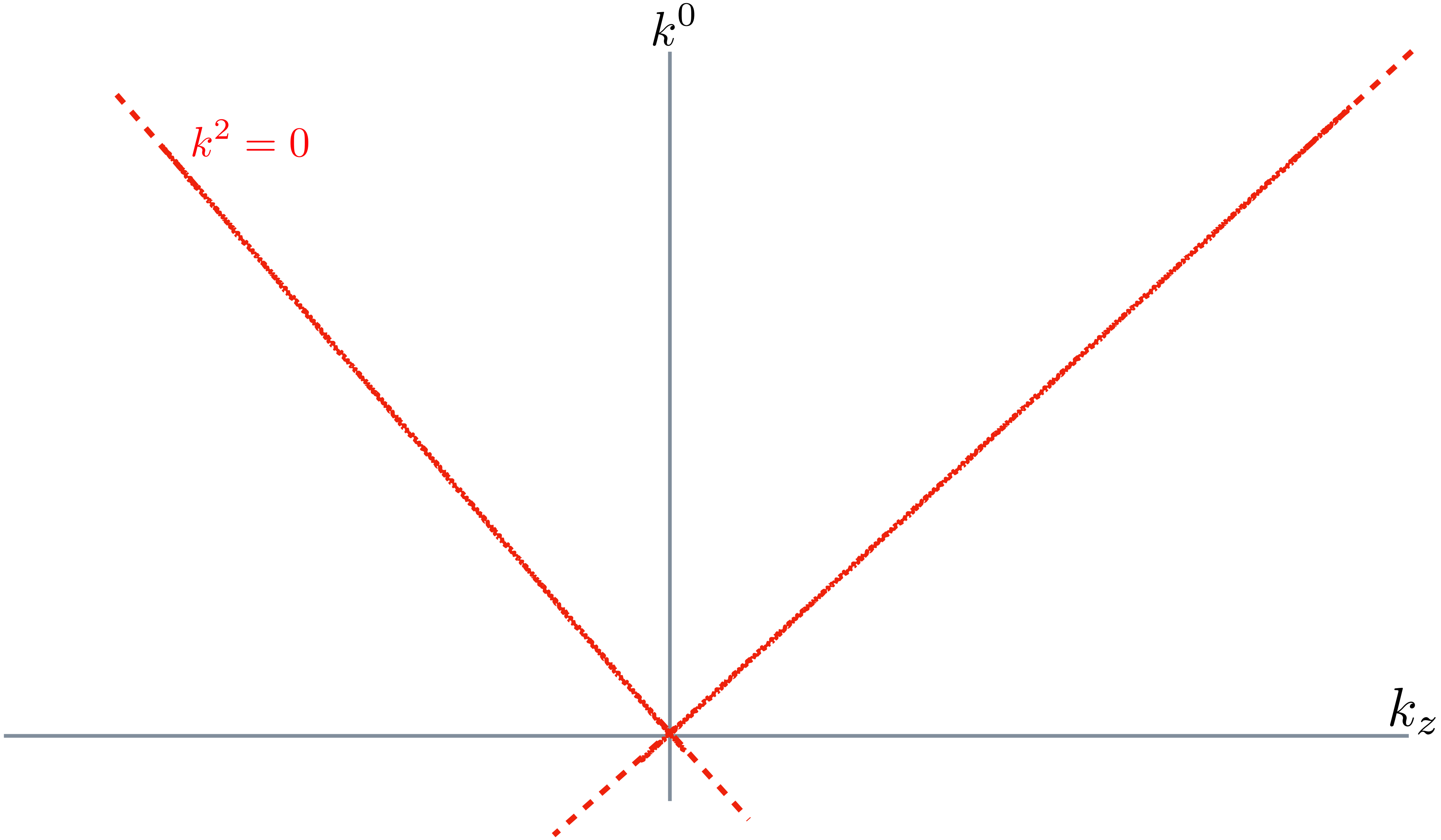
$|\vec{k}|\to Q^0/2$

Threshold singularity

- Contour deformation
- Threshold subtraction

SINGULAR SURFACES IN MINKOWSKI SPACE

$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

Contour deformation in 4D very difficult: [S. Weinzierl & al, arxiv: 1006.4609]
[W. Gong. Z. Nagy, D.Soper, arxiv: 0812.3686]

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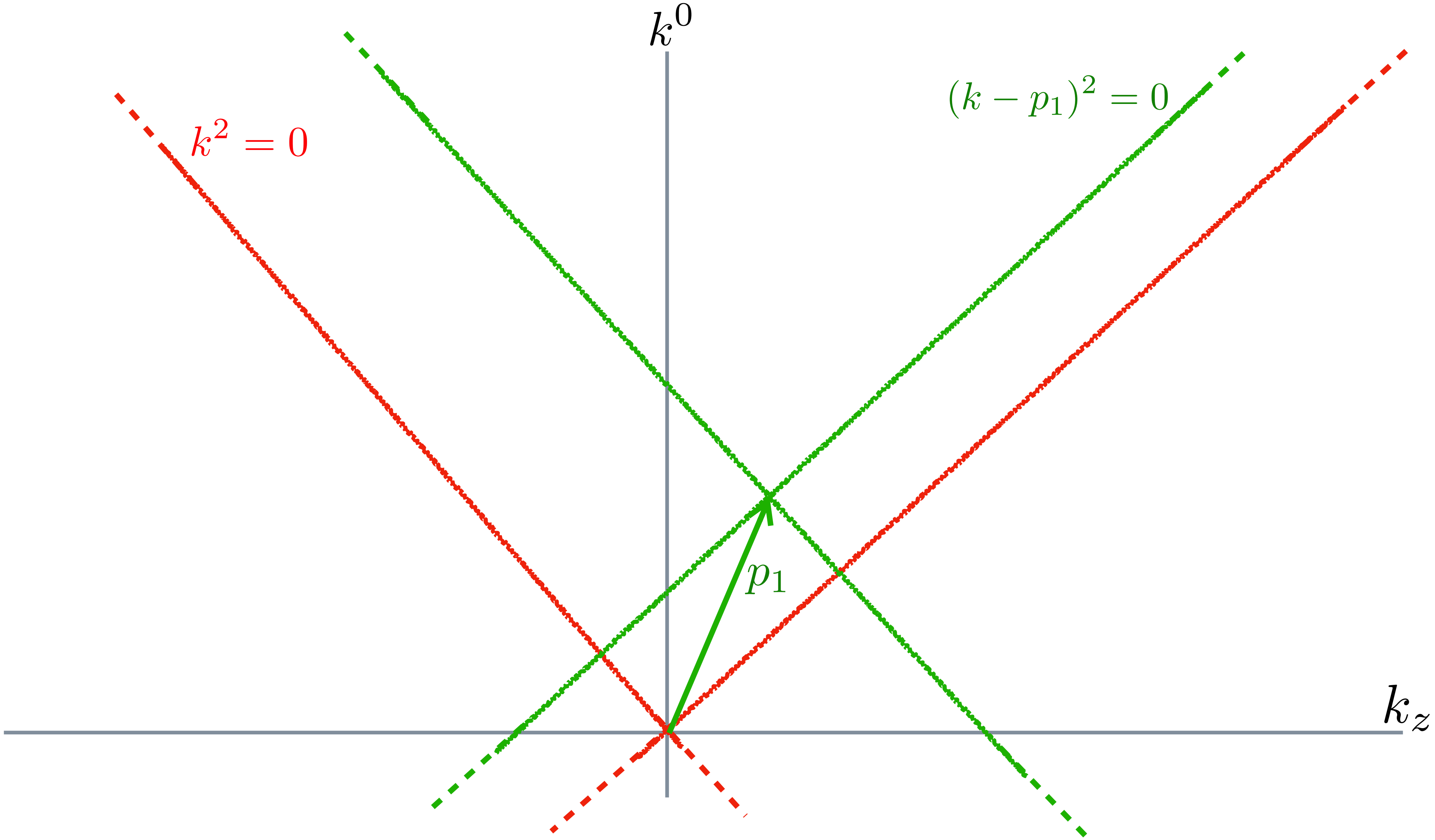

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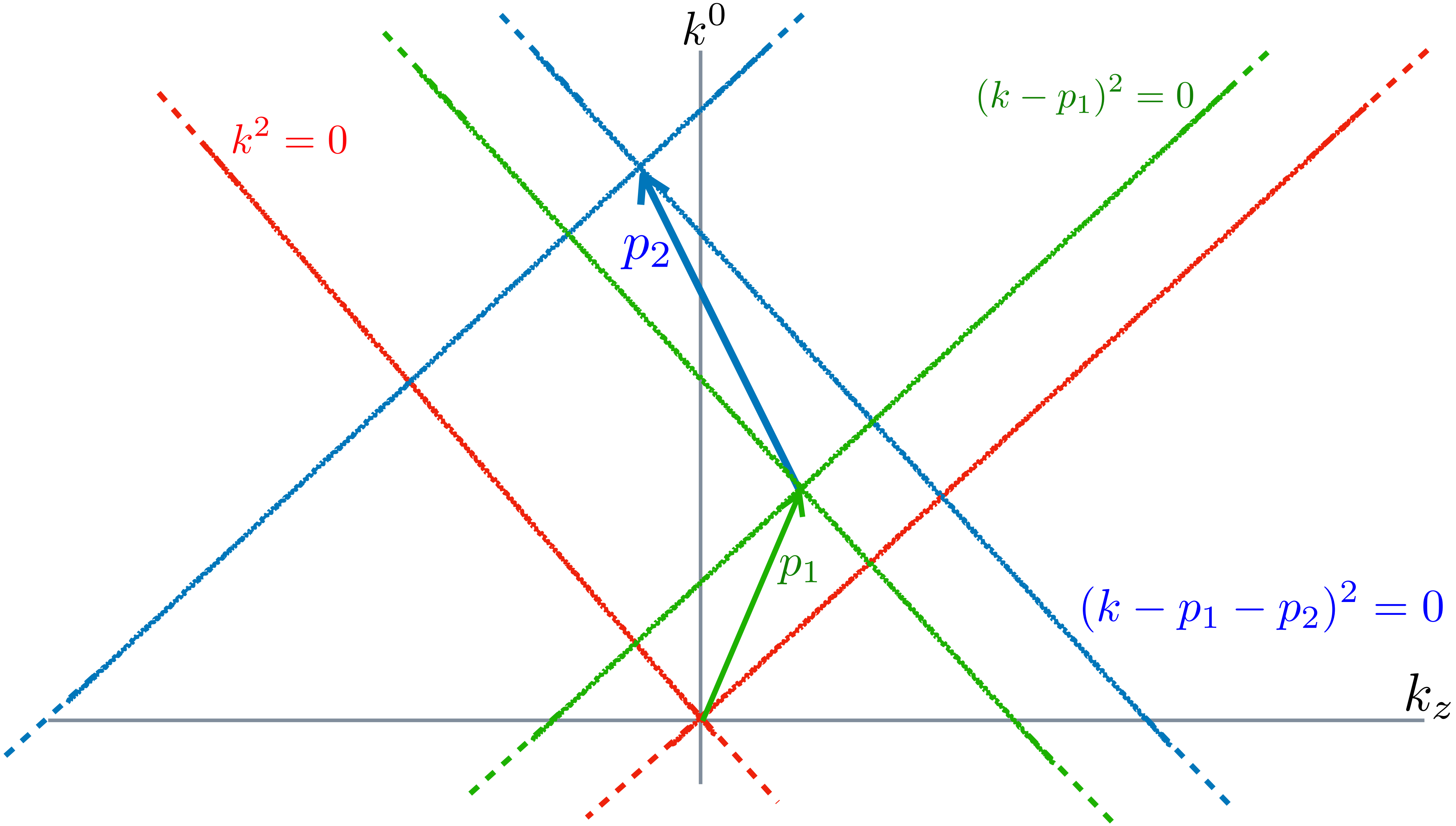

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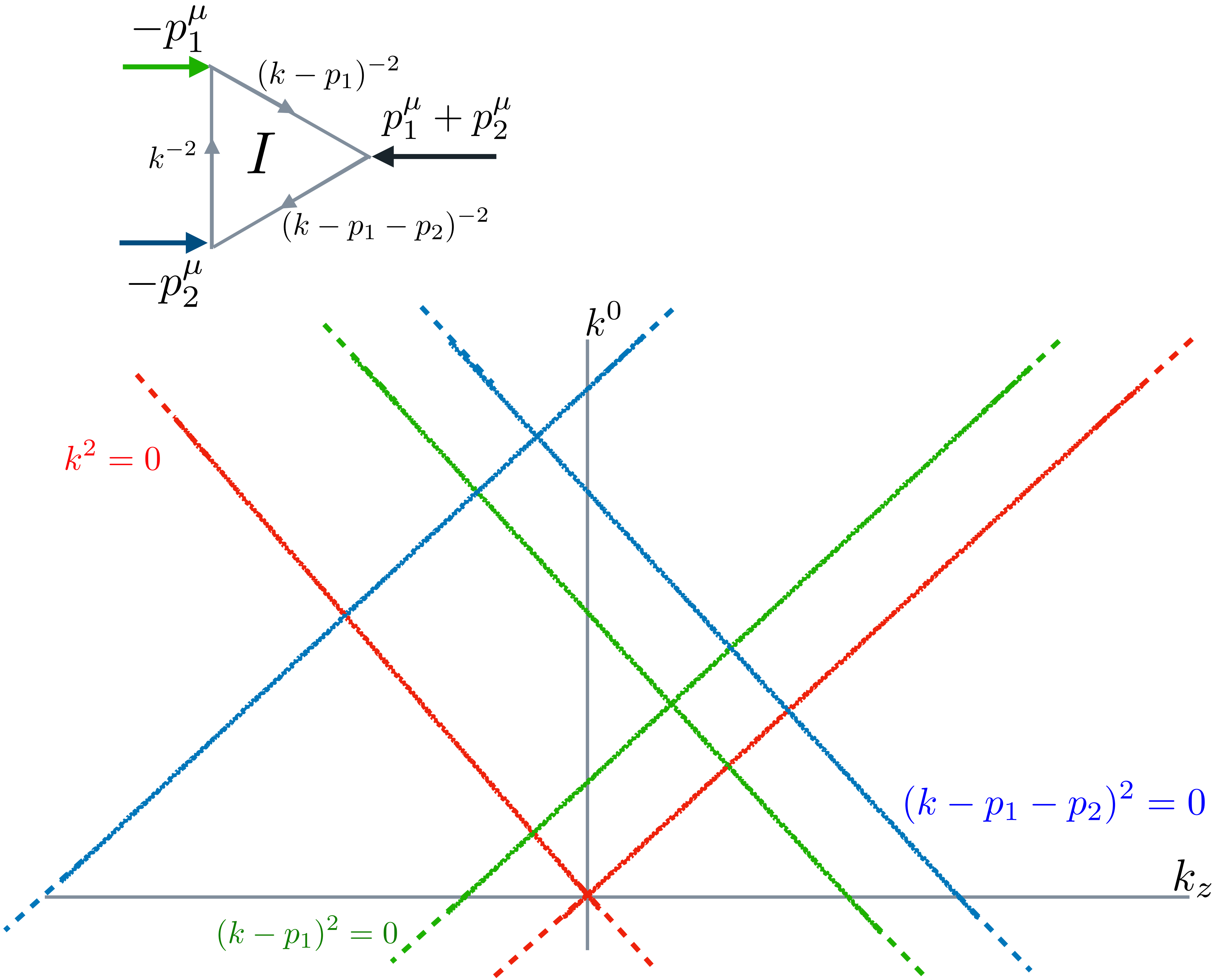
SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$p_i{}^2 > 0 \quad \forall i$

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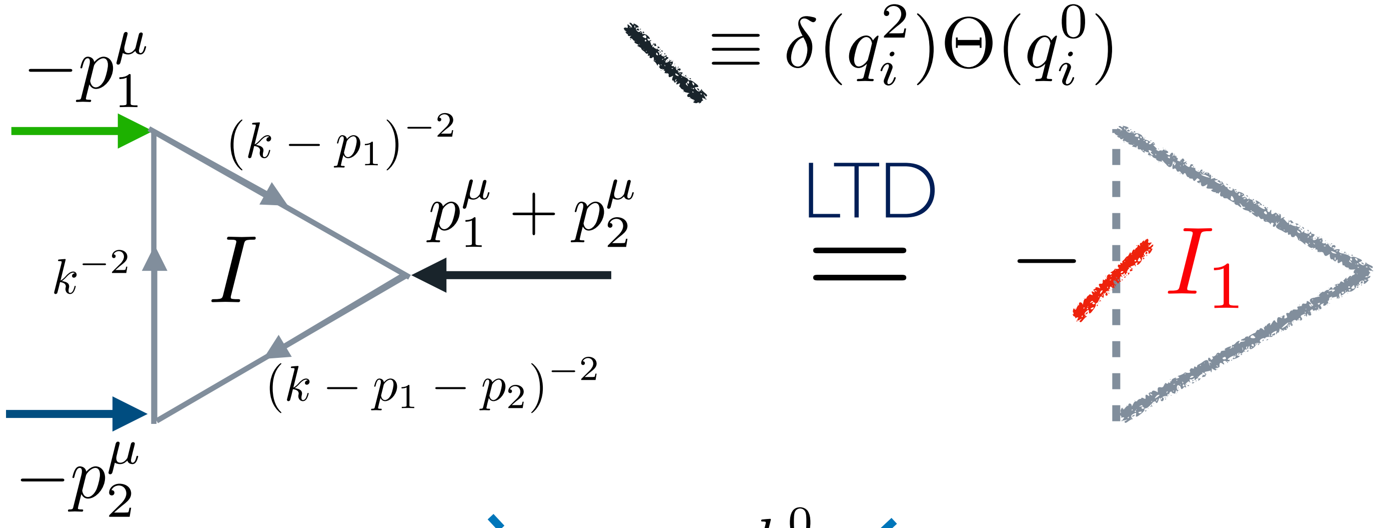

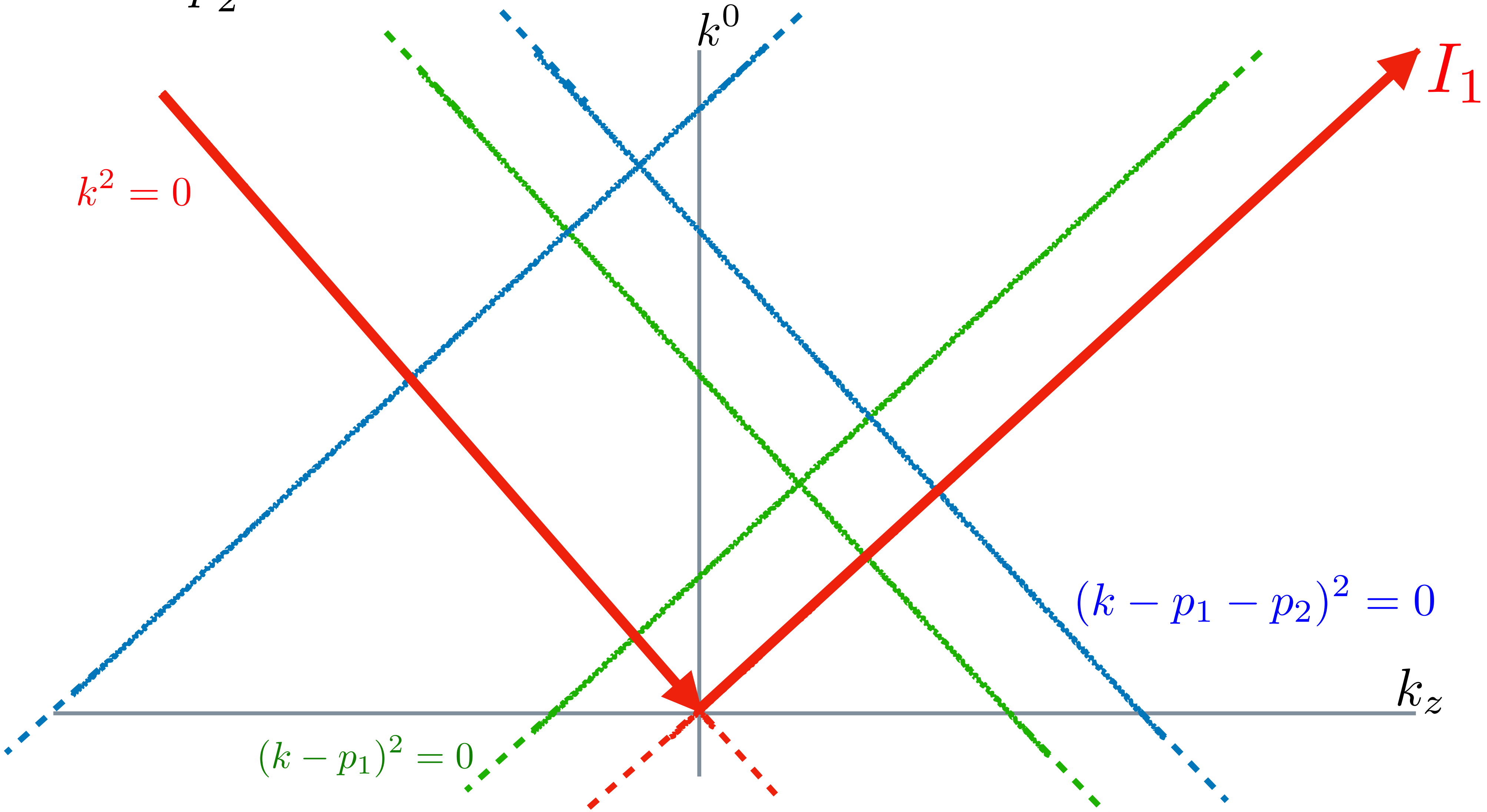

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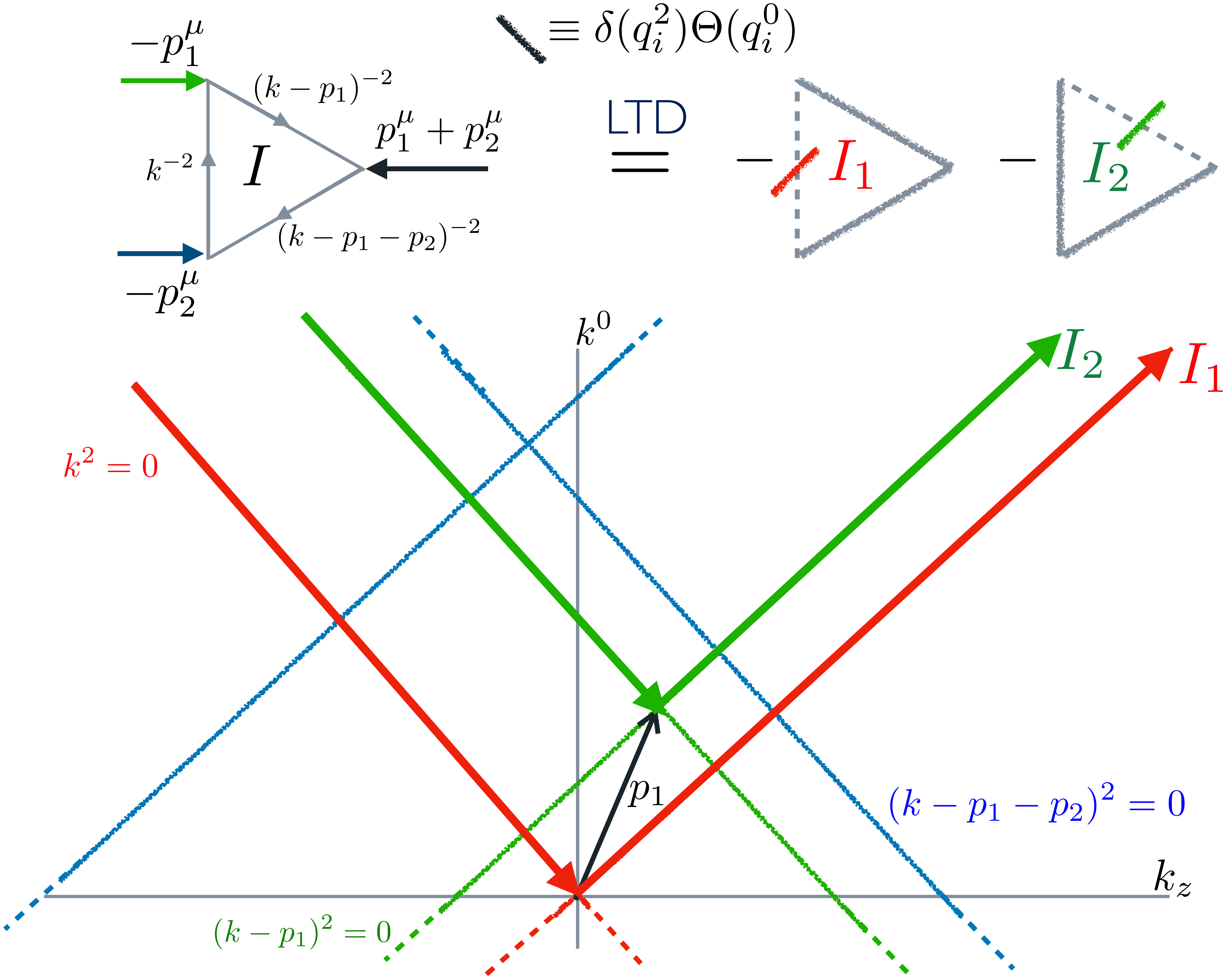

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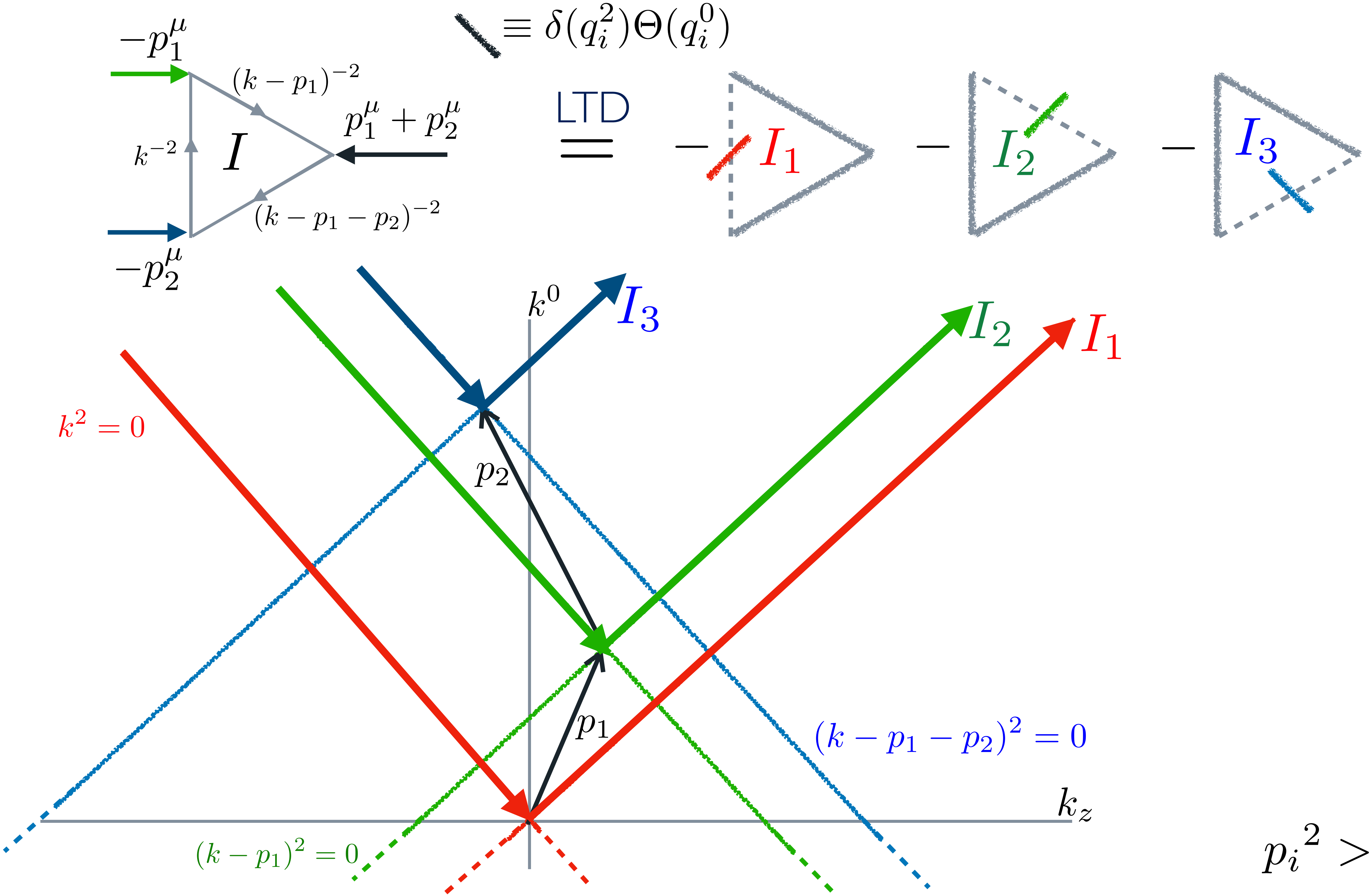
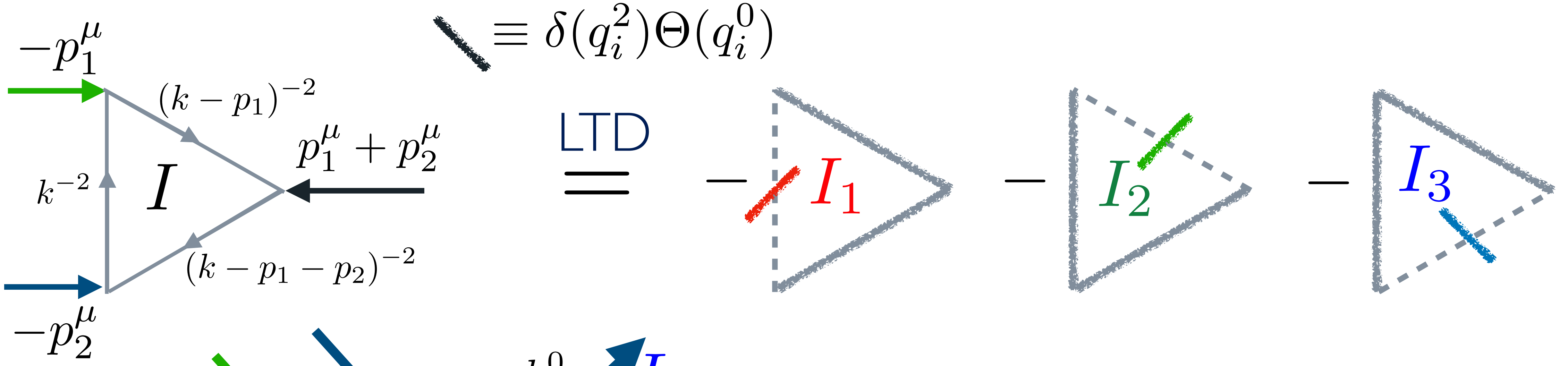
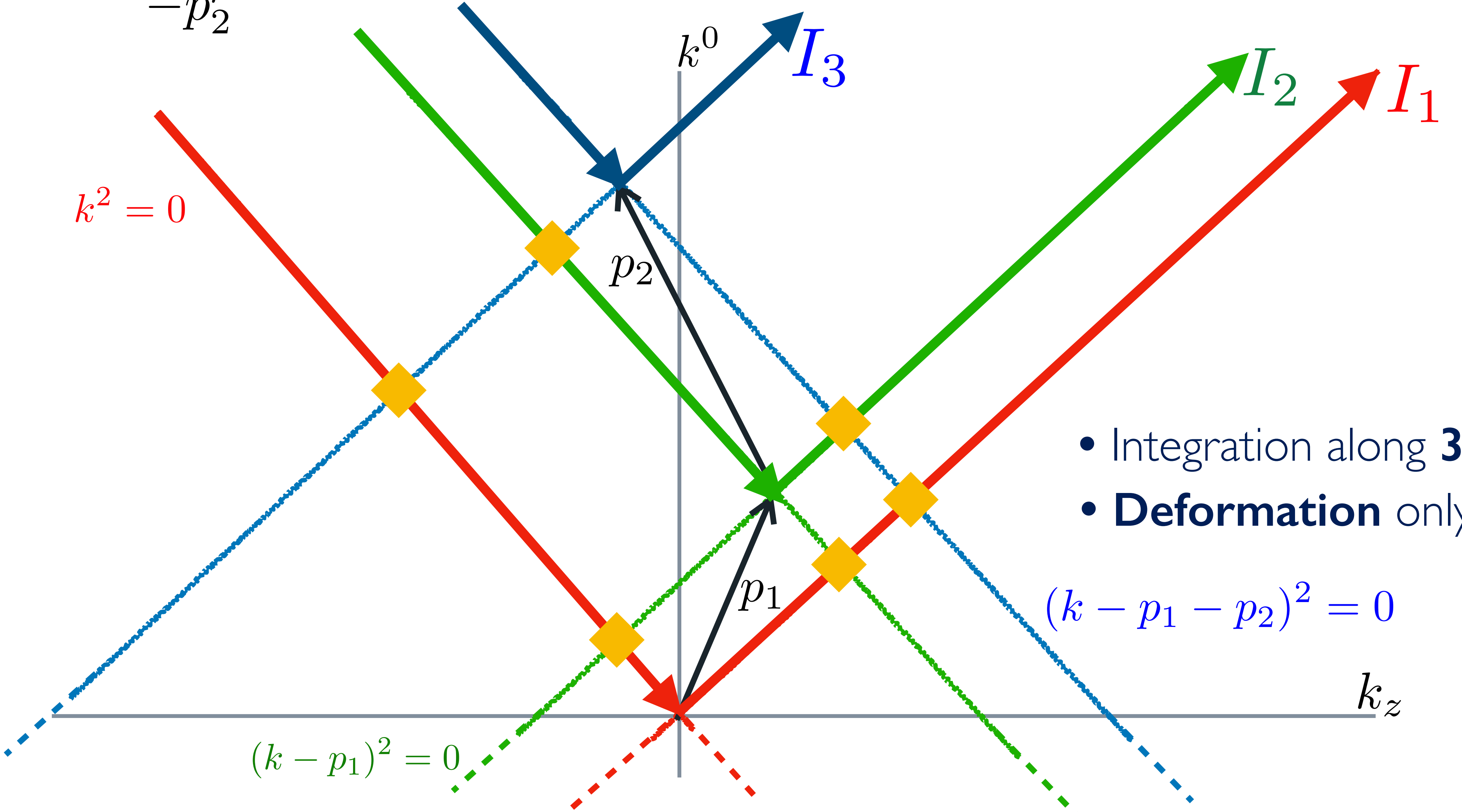


$${p_i}^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



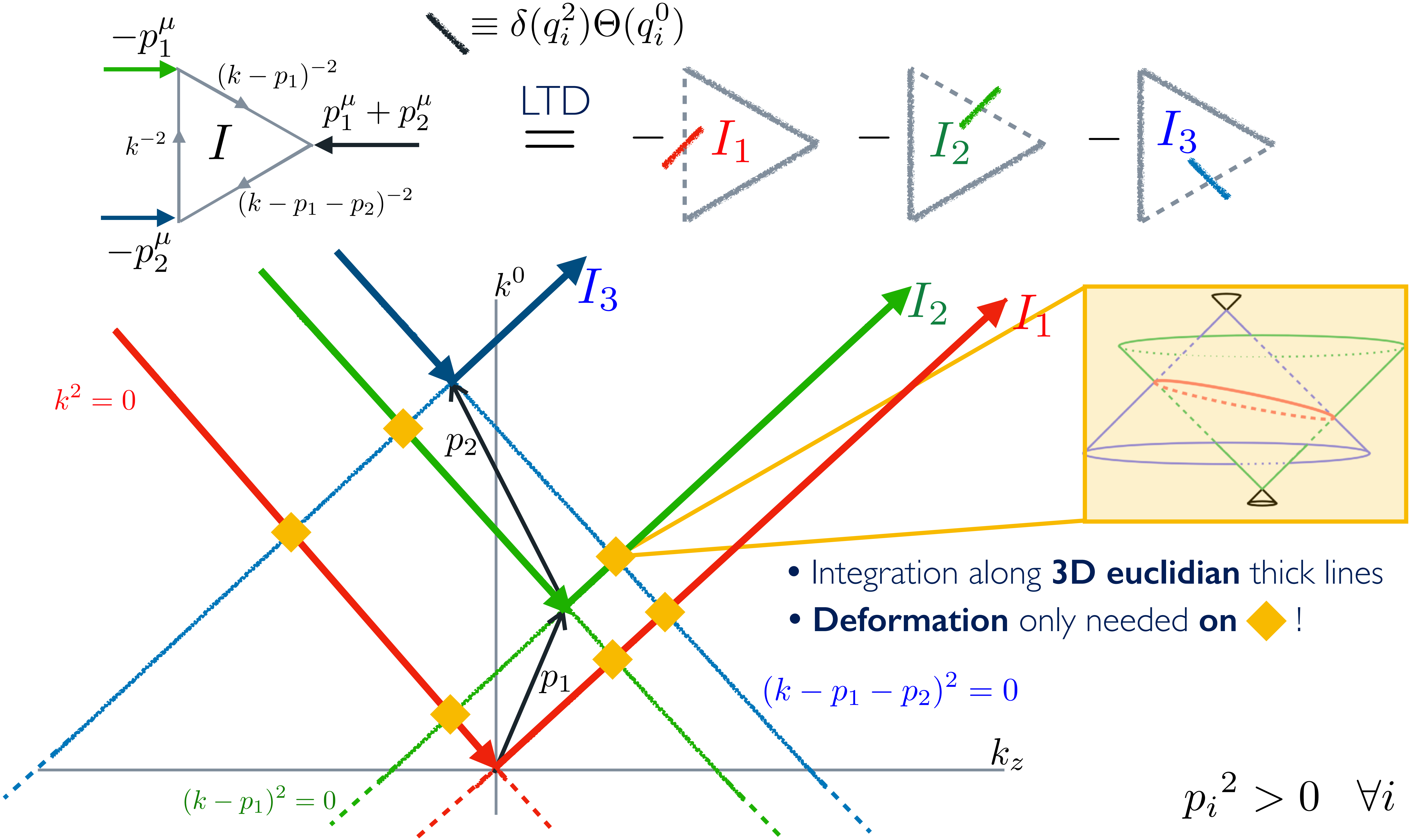


- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

$$p_i{}^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):

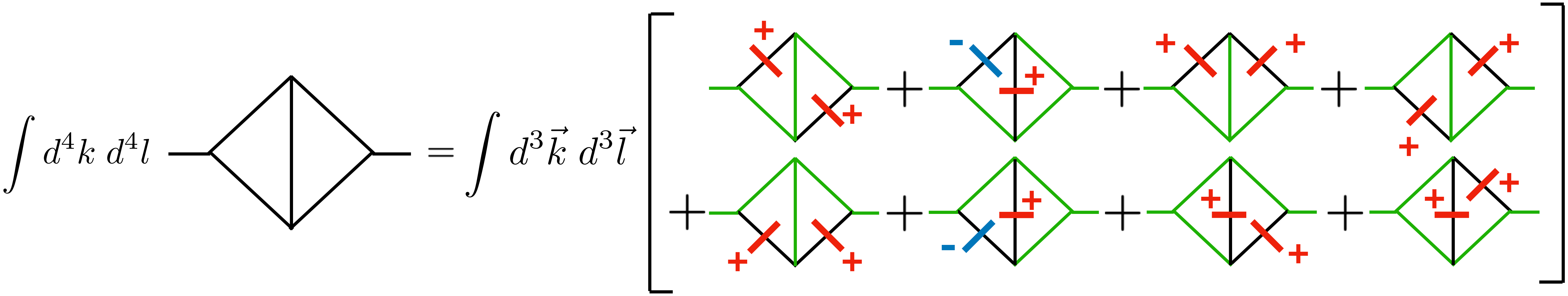


- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

$$p_i{}^2 > 0 \quad \forall i$$

THE THREE-DIMENSIONAL REPS. ZOO

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509]
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]

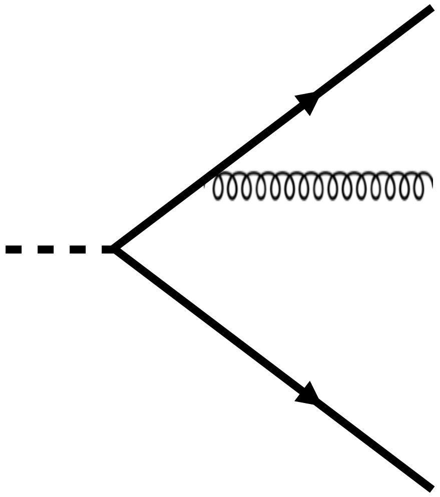
Codes : [`https://github.com/apelloni/cLTD`]

[`https://bitbucket.org/wjtorresb/lotty`] [`gammaloop`]

LOCAL UNITARITY

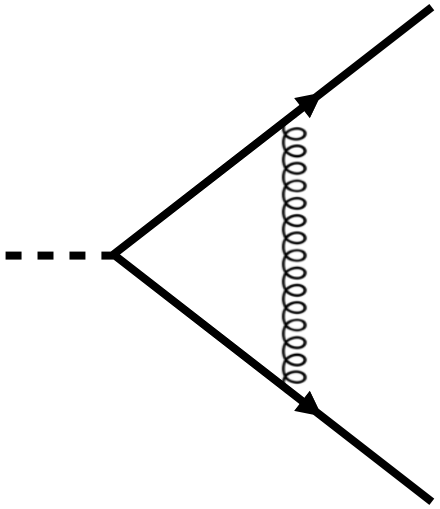
Real-emission (R)

$$\int_0^1 dx \left[\frac{\cos(x)}{x}\right]$$



Virtual (V)

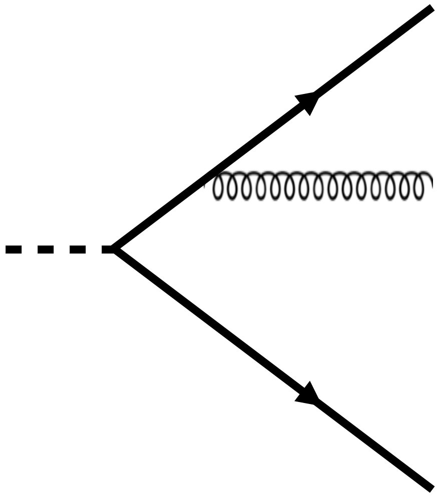
$$\int_0^{10} dy \left[-\frac{e^{-y}}{y}\right]$$



LOCAL UNITARITY

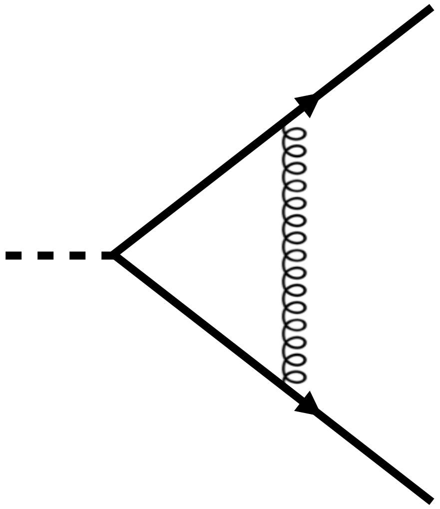
Real-emission (R)

$$\int_0^1 dx \left[\frac{\cos(x)}{x}\right] \quad \rightarrow \quad \int_0^1 dx \left[\frac{\cos(x)}{x} - \frac{1}{x}\right]$$



Virtual (V)

$$\int_0^{10} dy \left[-\frac{e^{-y}}{y}\right]$$



LOCAL UNITARITY

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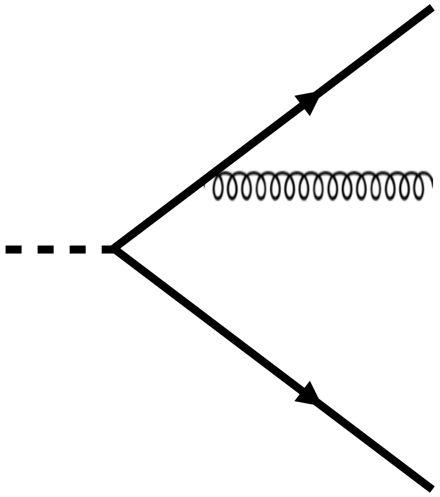
$$\int_0^1 dx \left[\frac{\cos(x)}{x}\right] \quad \rightarrow \quad \int_0^1 dx \left[\frac{\cos(x)}{x} - \frac{1}{x}\right]$$

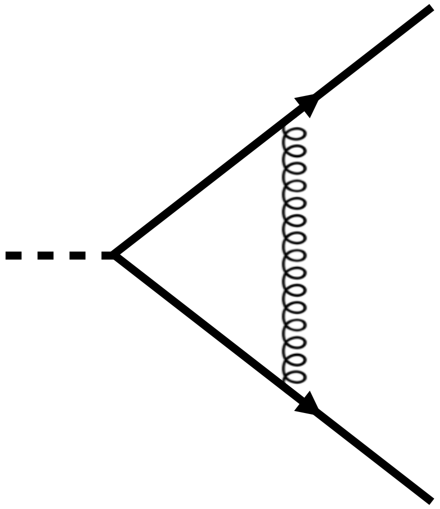


→ Finite at $x = 0$!

- Integral performed numerically
- Need to design complicated counterterms

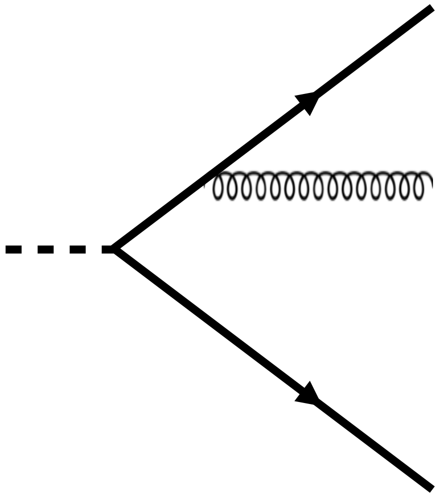
Virtual (V)

$$\int_0^{10} dy \left[-\frac{e^{-y}}{y}\right]$$



LOCAL UNITARITY

Real-emission (R)



$$\int_0^1 dx \left[\frac{\cos(x)}{x}\right] \rightarrow \int_0^1 dx \left[\boxed{\frac{\cos(x)}{x} - \frac{1}{x}}\right]$$

$\rightarrow$ Finite at $x = 0$!

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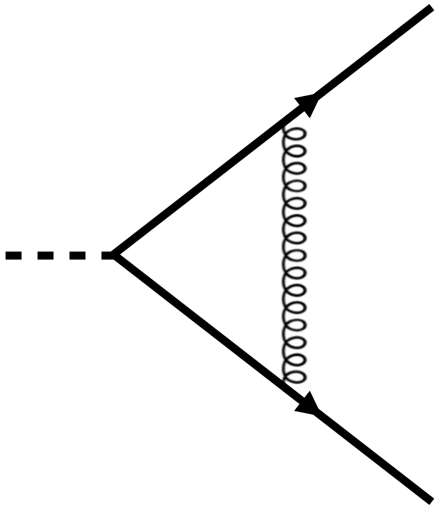
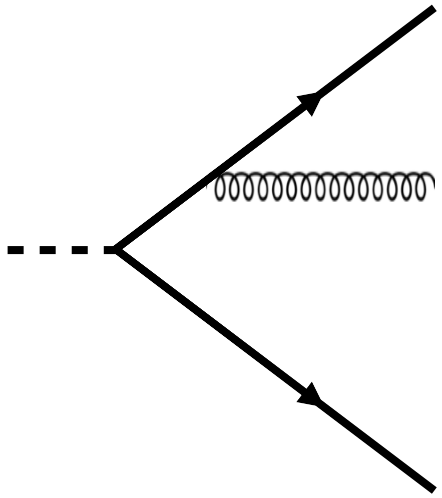
Virtual (V)



$$\int_0^{10} dy \left[-\frac{e^{-y}}{y}\right] \rightarrow \int_0^{10} dy \left[-\frac{e^{-y}}{y}\right] + \int_0^1 dx \left[\frac{1}{x}\right]$$

LOCAL UNITARITY

Real-emission (R)

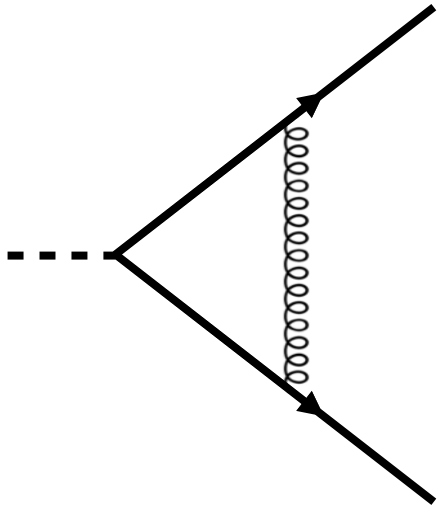


$$\int_0^1 dx\left[\frac{\cos(x)}{x}\right] \rightarrow \int_0^1 dx\left[\frac{\cos(x)}{x} - \frac{1}{x}\right]$$

$\rightarrow$ Finite at $\mathrm{x} = 0$!

- Integral performed numerically
- Need to design complicated counterterms

Virtual (V)



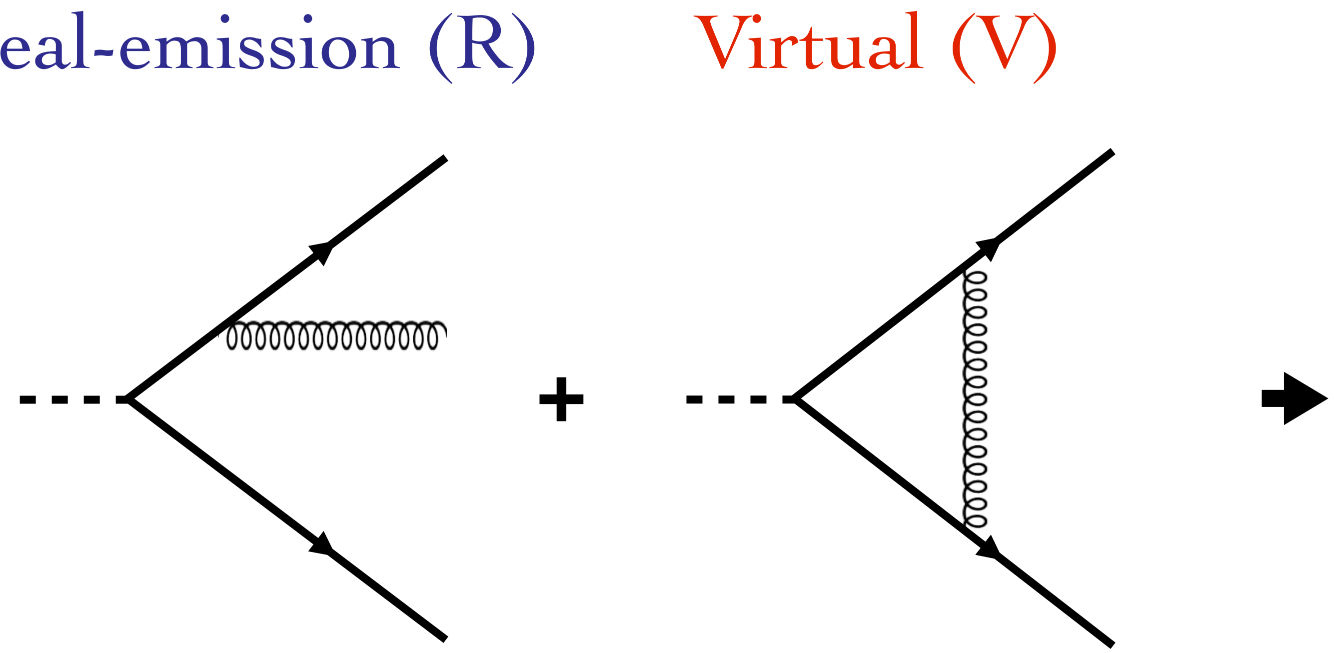
$$\int_0^{10} dy\left[-\frac{e^{-y}}{y}\right] \rightarrow \int_{\cancel{0}\epsilon}^{10} dy\left[-\frac{e^{-y}}{y}\right] + \int_{\cancel{0}\epsilon}^{1} dx\left[\frac{1}{x}\right]$$

$$\rightarrow \quad A + \cancel{\log(\epsilon)} \quad + B - \cancel{\log(\epsilon)}$$

$$\cancel{+\infty} \qquad \cancel{-\infty}$$

- Integral performed analytically
- No universal nor systematic solution

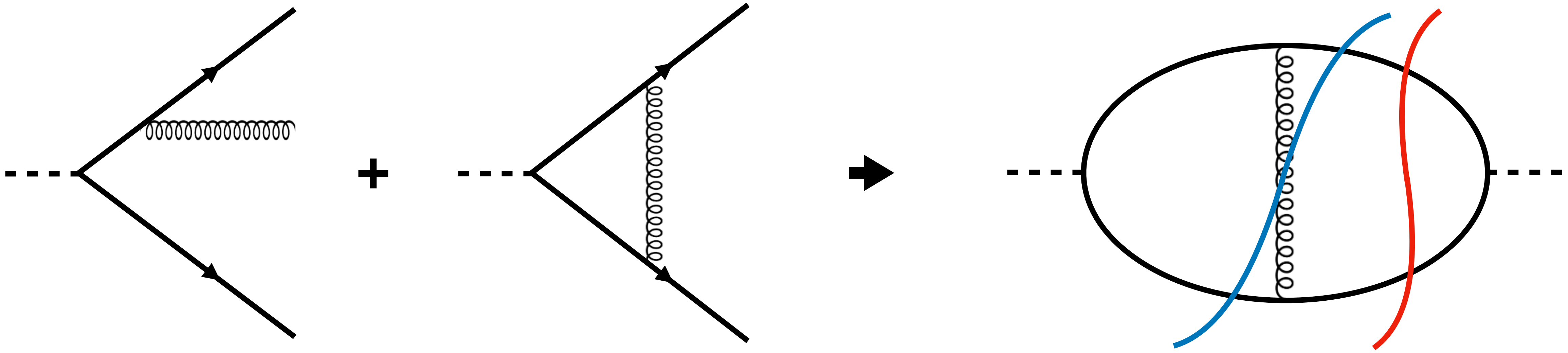
LOCAL UNITARITY



$$\int_0^1 dx \left[\frac{\cos(x)}{x}\right] + \int_0^{10} dy \left[-\frac{e^{-y}}{y}\right] \rightarrow$$

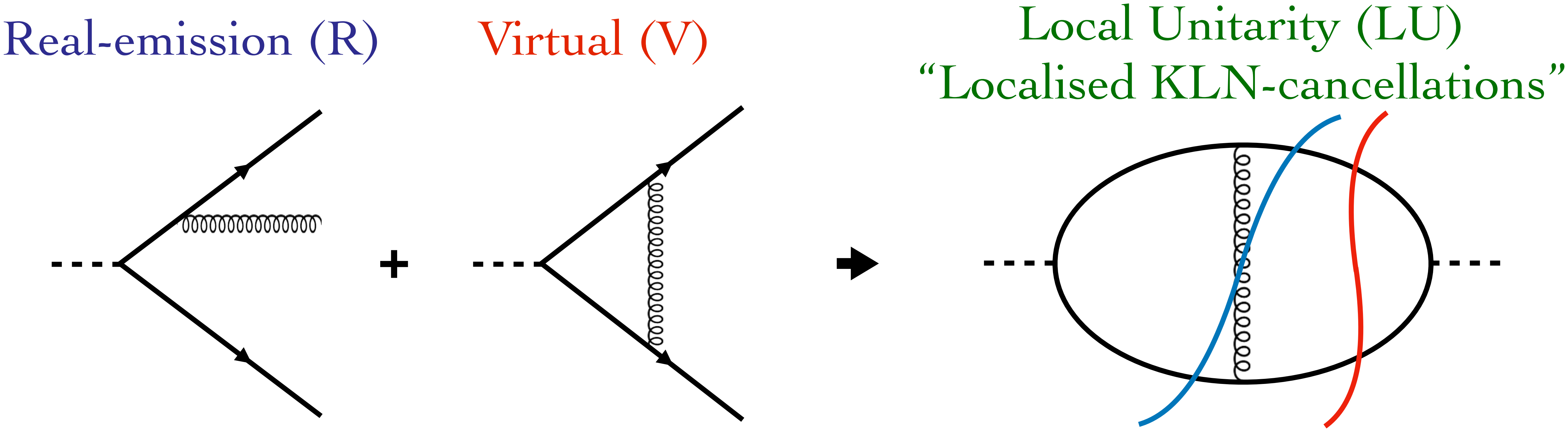
LOCAL UNITARITY

Real-emission (R) Virtual (V) Local Unitarity (LU)
"Localised KLN-cancellations"



$$\int_0^1 dx\left[\frac{\cos(x)}{x}\right] + \int_0^{10} dy\left[-\frac{e^{-y}}{y}\right] \rightarrow \int_0^{10} dx\left[\frac{\cos(x)}{x}\Theta(1-x) - \frac{e^{-x}}{x}\right]$$

LOCAL UNITARITY



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- Direct cancellation of divergences: no regulator

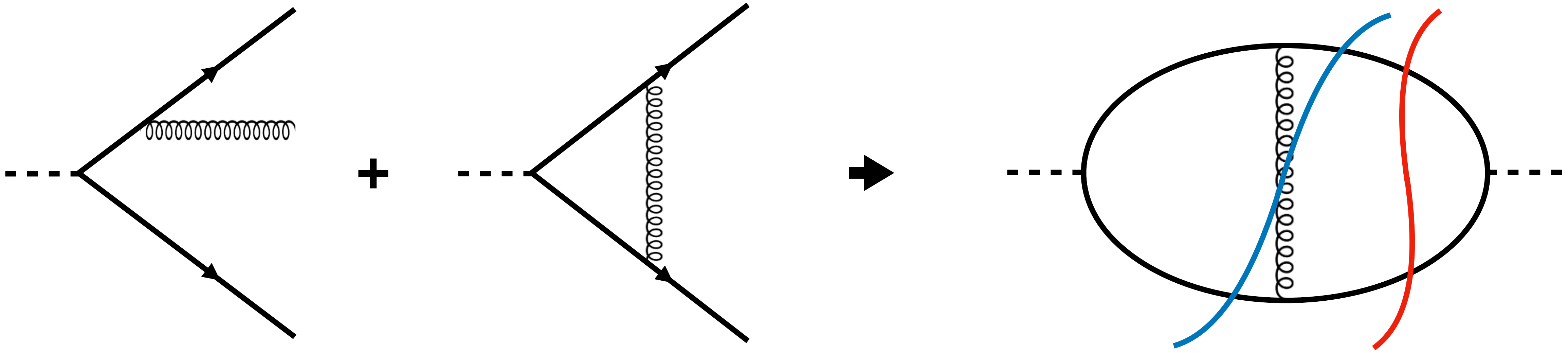
Finite at $x = 0$!

LOCAL UNITARITY

Real-emission (R)

Virtual (V)

Local Unitarity (LU)
“Localised KLN-cancellations”



$$\int_0^1 dx\left[\frac{\cos(x)}{x}\right] + \int_0^{10} dy\left[-\frac{e^{-y}}{y}\right] \rightarrow \int_0^{10} dx\left[\frac{\cos(x)}{x}\Theta(1-x) - \frac{e^{-x}}{x}\right]$$

- Direct cancellation of divergences: no regulator

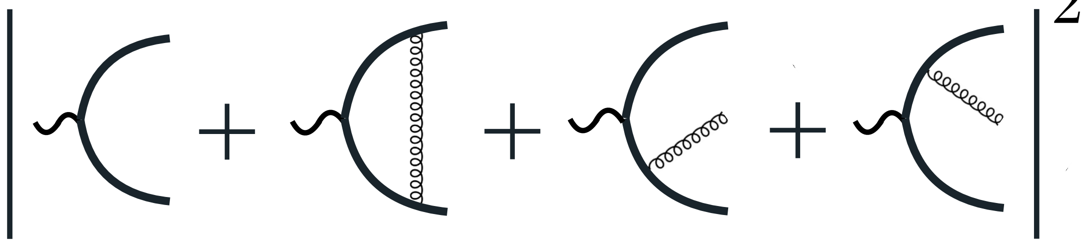
Finite at $\mathrm{x} = 0$!

- Very simplified example: applying the idea to QFT is a very challenging

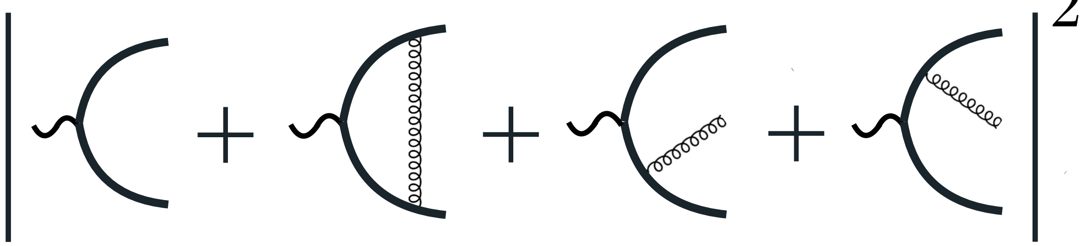
[Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138] [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 1912.09291]
[Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509] [Capatti, VH, Pelloni, Ruijl, arxiv: 2010.01068]
[Capatti, VH, Ruijl, arxiv: 2203.11038]

LOCAL UNITARITY: A CONCEPTUAL SHIFT

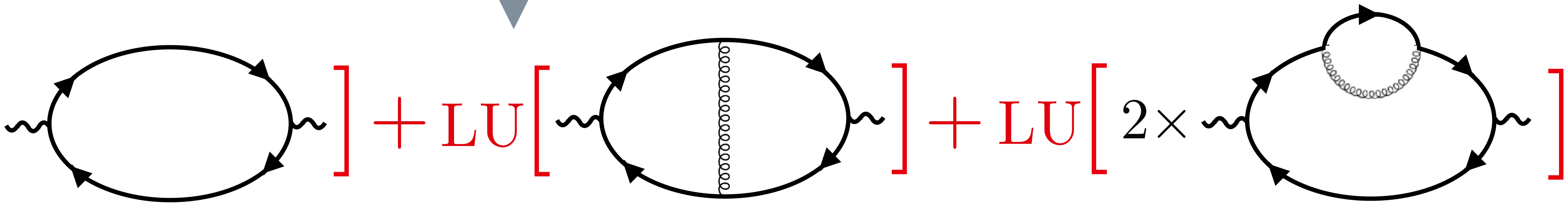
$$\sigma^{(\text{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$



LOCAL UNITARITY: A CONCEPTUAL SHIFT



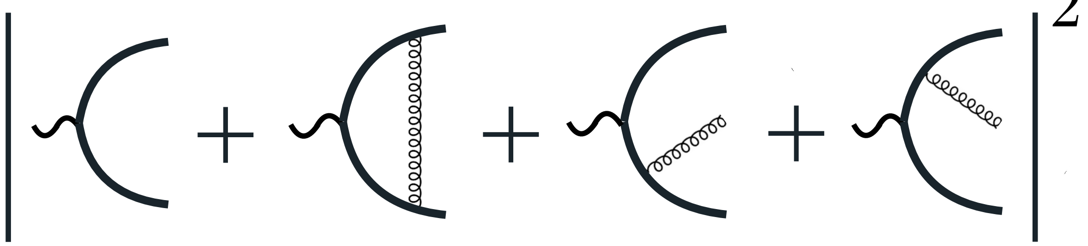
$$\sigma^{(\text{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})} \left| \quad + \quad + \quad + \quad \right|^2$$

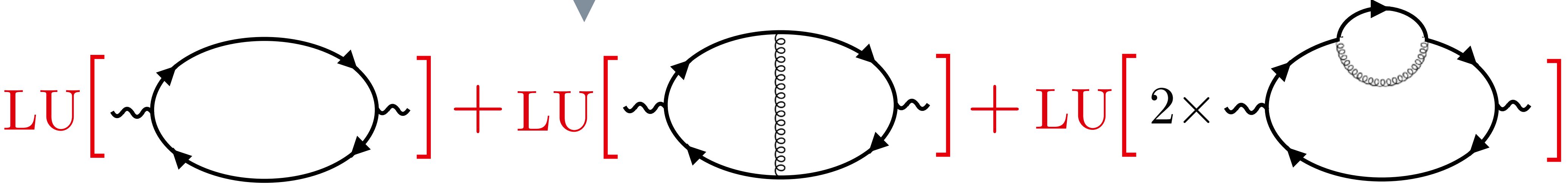


$$\sigma^{(\text{LU})}_{\gamma^\star\to d\bar{d}} = \text{LU}\left[\quad\right] + \text{LU}\left[\quad\right] + \text{LU}\left[2\times \quad\right]$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})} \left| \; \cdots \; \right|^2$$





$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}[\;\cdots\;] + \text{LU}[\;\cdots\;] + \text{LU}[\,2\times \cdots\;]$$

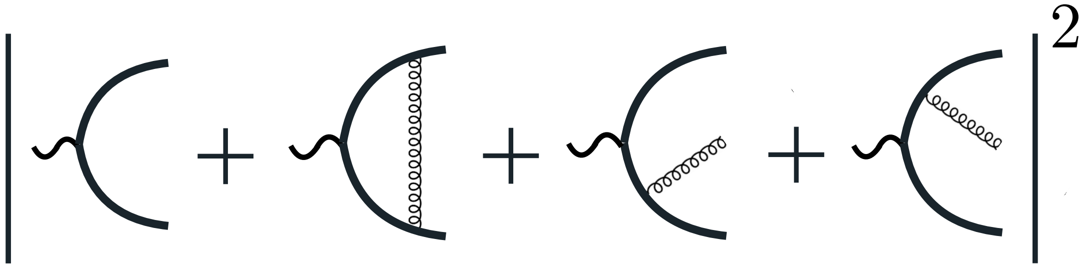
$$\sum_{c\in\{RRR,RRV,RVV,\dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

IR-subtraction numerical $d = 4$

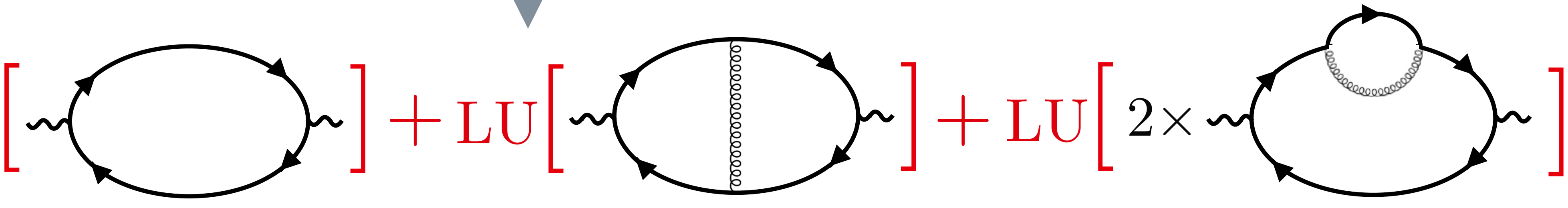
analytic $d = 4 - 2\epsilon$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma^{(\text{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$



$$\sigma^{(\text{LU})}_{\gamma^\star\to d\bar{d}} = \text{LU}[\quad] + \text{LU}[\quad] + \text{LU}[2\times\quad]$$



$$\sum_{c\in\{RRR,RRV,RVV,\ldots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

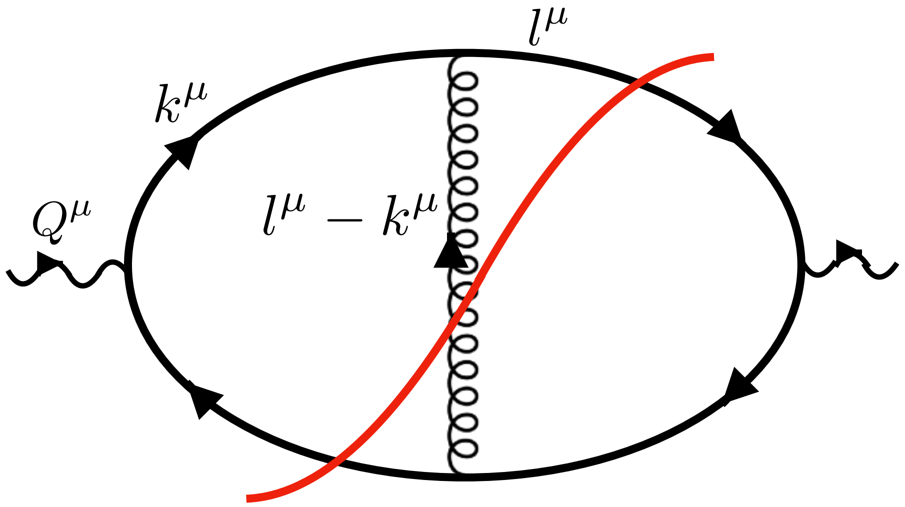
IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi\, g_j^{(\text{LU})}$$

numerical $d = 4$

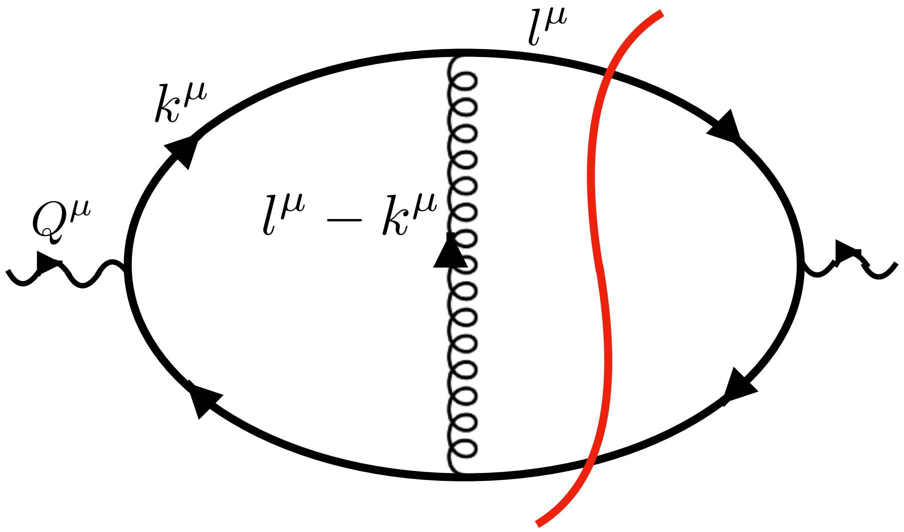
NO IR-subtraction

LOCAL UNITARITY: THE ROLE OF LTD



$$\propto \delta^+(l^2)\delta^-((l-k)^2)\delta^-((k-Q)^2)$$

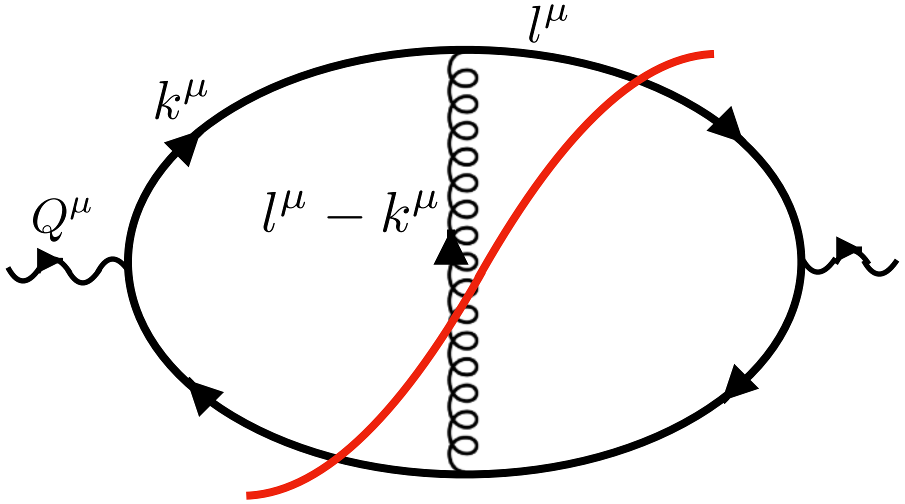
?



$$\propto \delta^+(l^2)\ \delta^-((l-Q)^2)$$

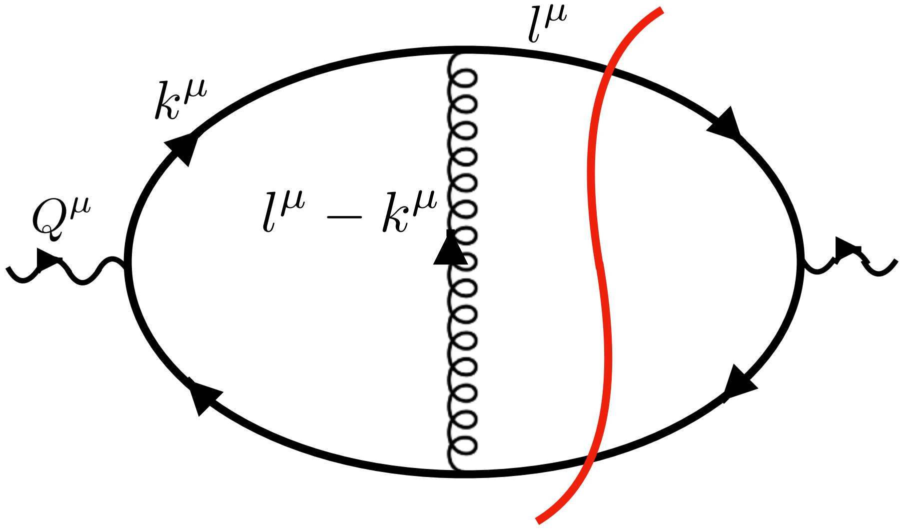
Simple and easy-to-follow proceeding show-casing Local Unitarity: [Capatti, arxiv: 2110.15662]

LOCAL UNITARITY: THE ROLE OF LTD



$$\propto \delta^+(l^2)\delta^-((l-k)^2)\delta^-((k-Q)^2)$$

?



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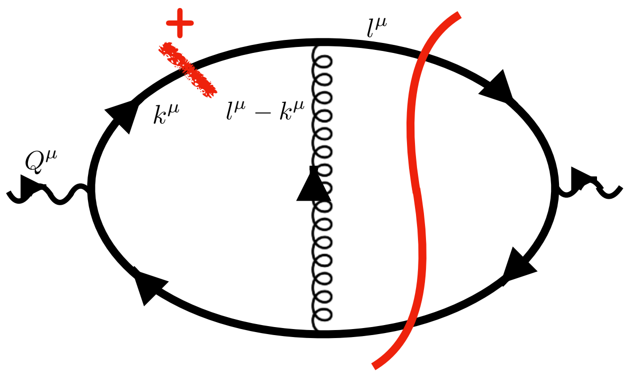
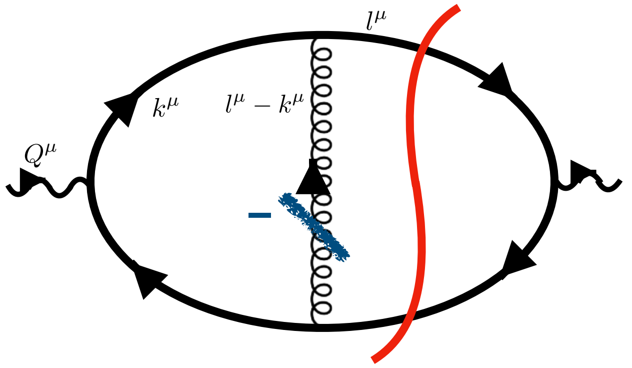
LTD

=



$$\propto \delta^+(l^2)\;\delta^-((l-Q)^2)\;\delta^+(k^2)$$

+



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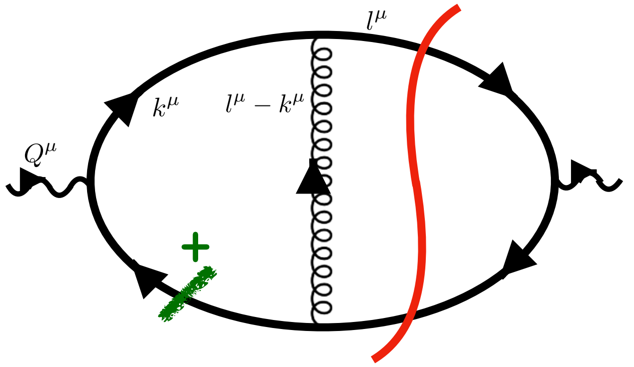


$$\propto \delta^+(l^2)\;\delta^-((l-Q)^2)\;\delta^+((k-Q)^2)$$

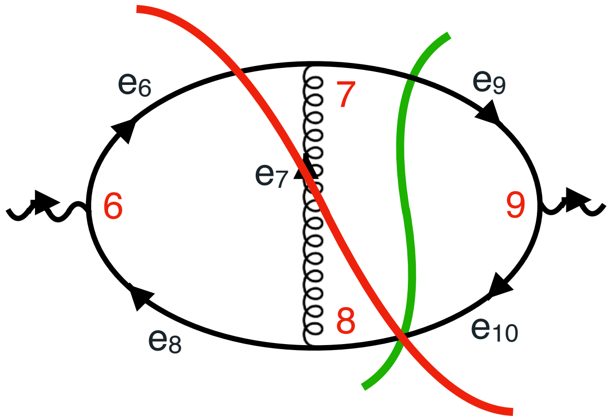
Simple and easy-to-follow proceeding show-casing Local Unitarity: [Capatti, arxiv: 2110.15662]

LOCALITY UNITARITY

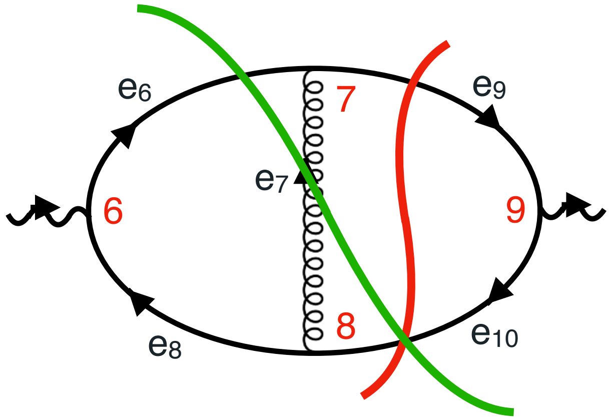
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

🟥 = Cutkosky cut 🟩 = threshold singularity



cancels



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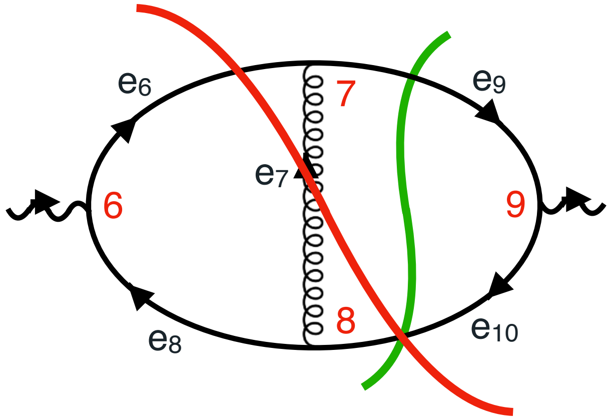

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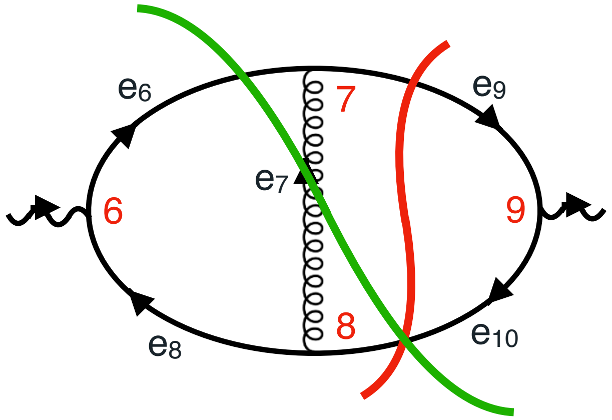

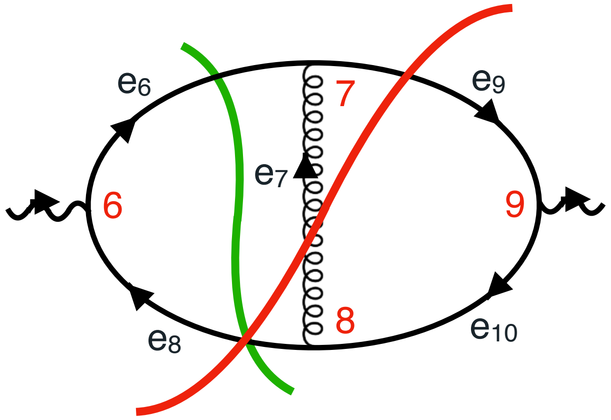

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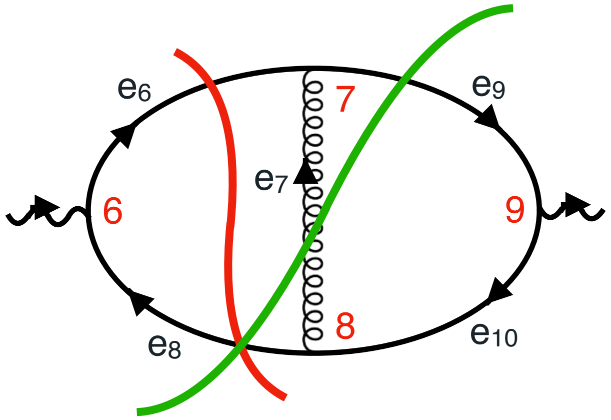

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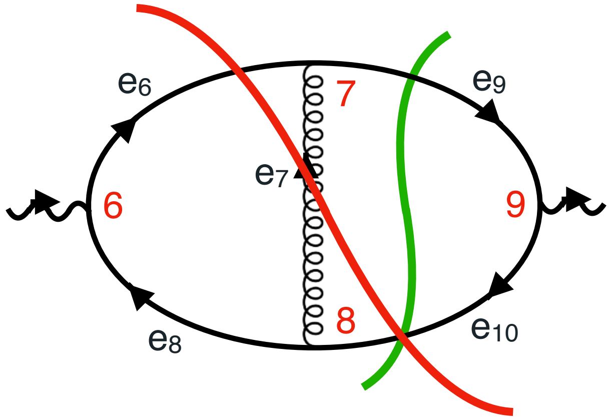

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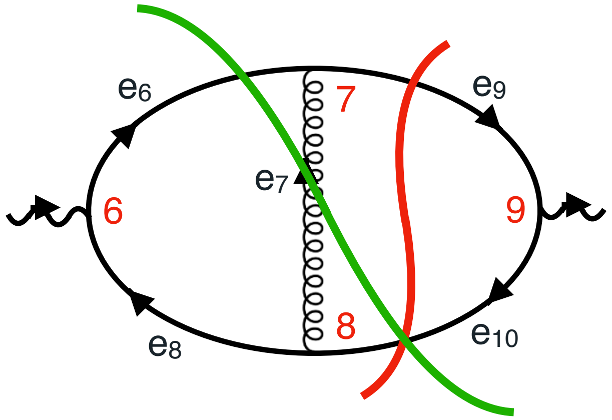

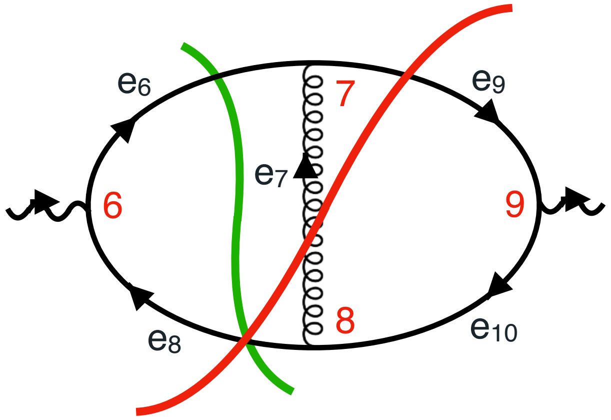

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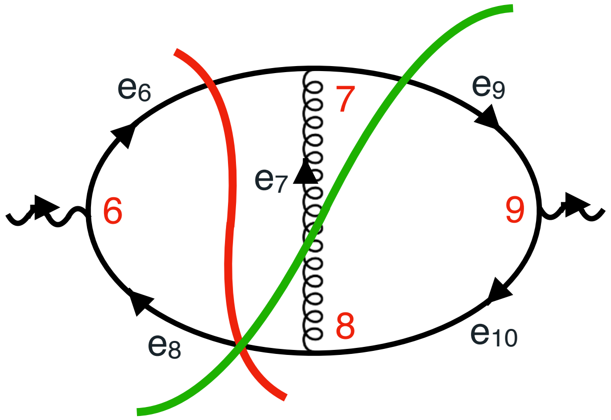
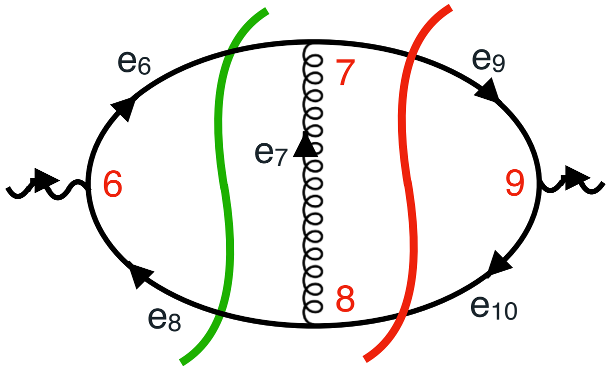


Even for **non-pinched singular threshold** ! (when $\mathcal{O}_{\mathbf{s}} \equiv 1$) :



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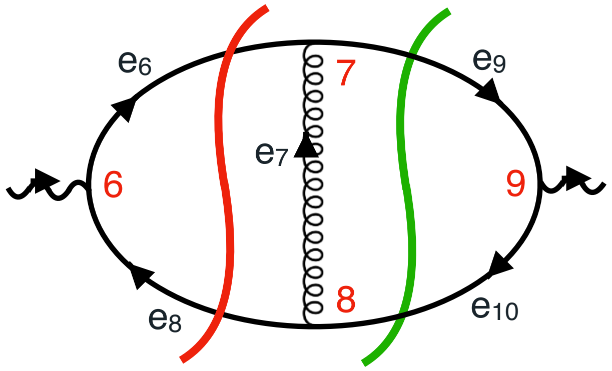

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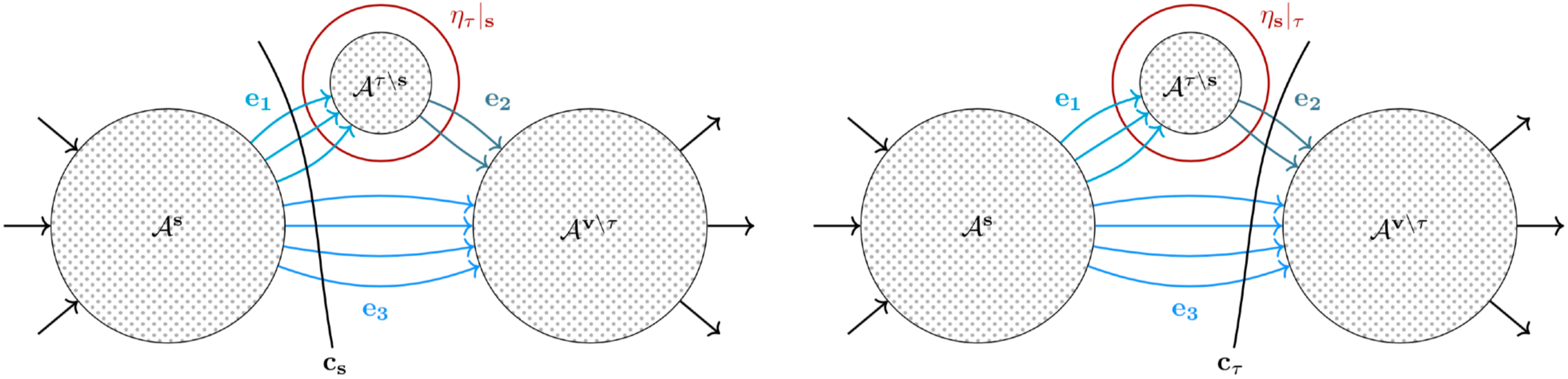
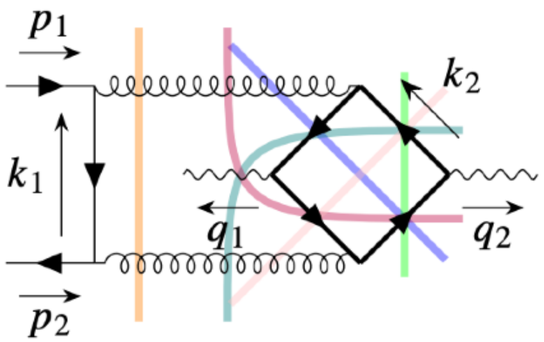


APPLICATIONS

GENERIC COMPUTATIONAL FRAMEWORK

To accommodate the many applications, on top of collider simulations

Collider:



$pp \to VV(V)$ @ NNLO

[D.Kermanschah, M. Vicini,
C. Anastasiou, & al.]

$pp \to t\bar{t}$ @ NLO

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$pp \to$ [high-mult] @ NLO

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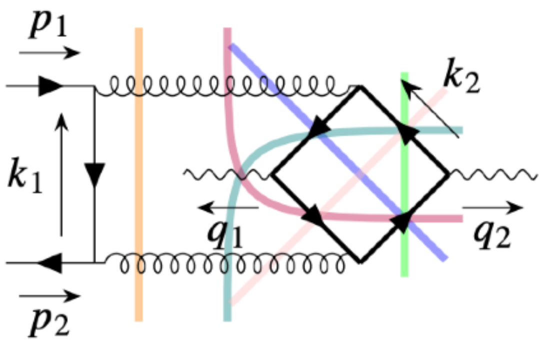

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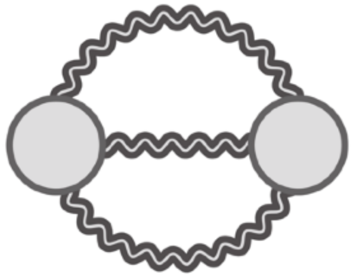

Neutron stars EoS

[Kärkkäinen & al.: 2501.17921]

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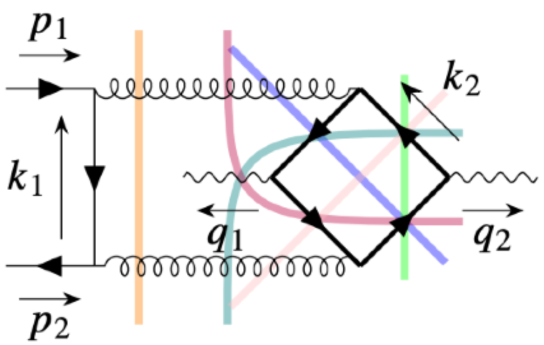

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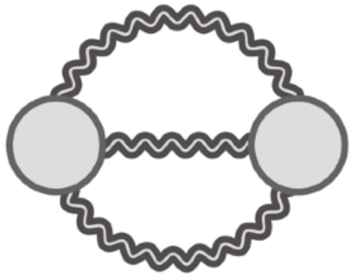

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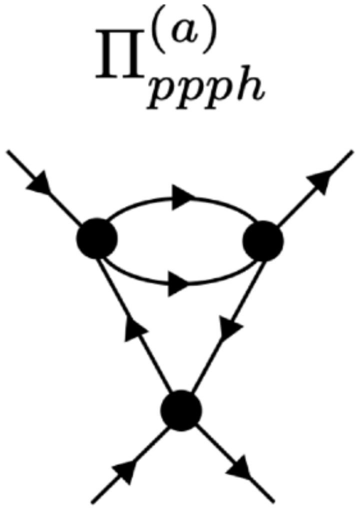

Fermi gas

[R. Beane & al.: 2407.20168]

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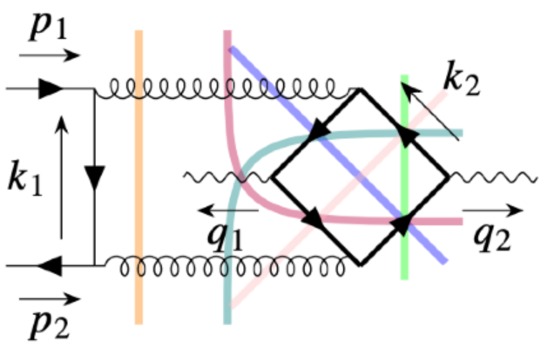

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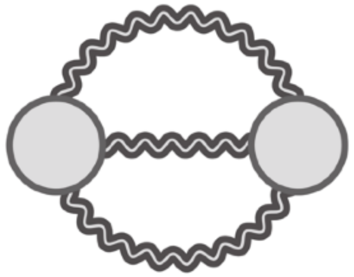

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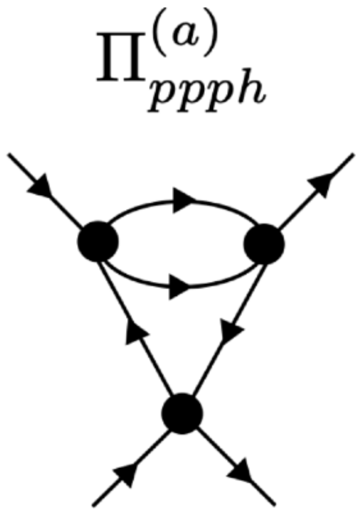

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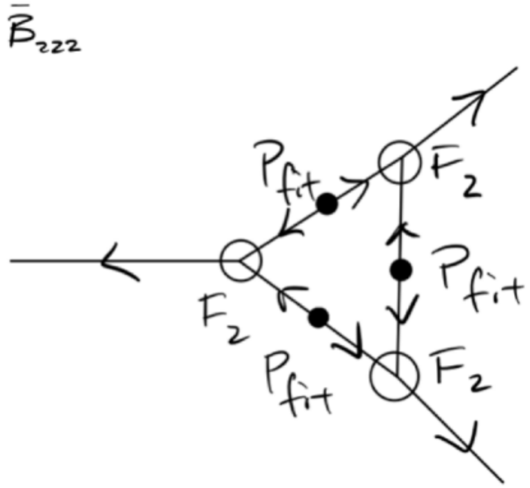

EFT for Large Scale Structure

[Anastasiou & al.: 2212.07421]

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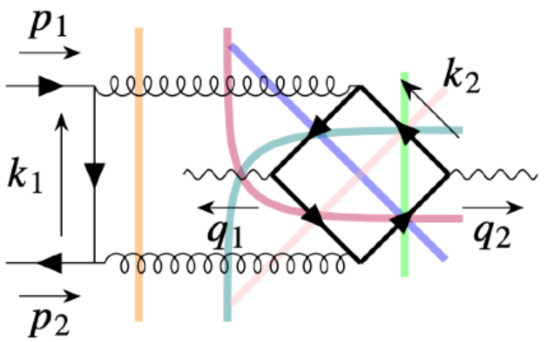

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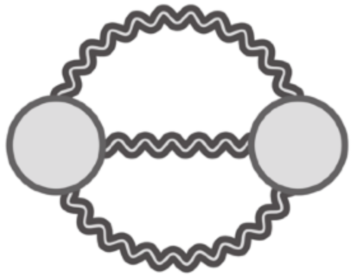

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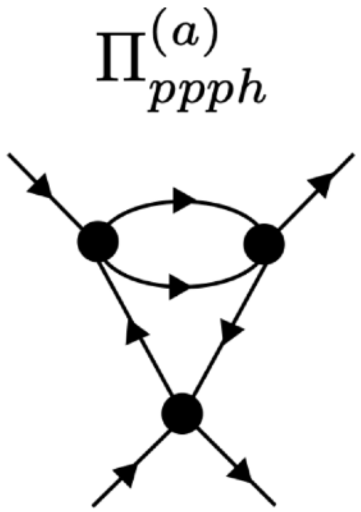

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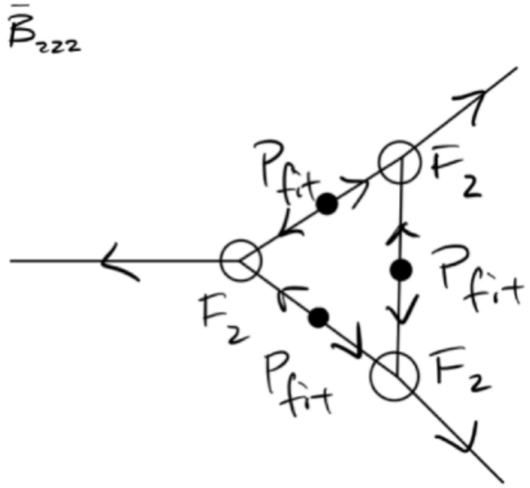

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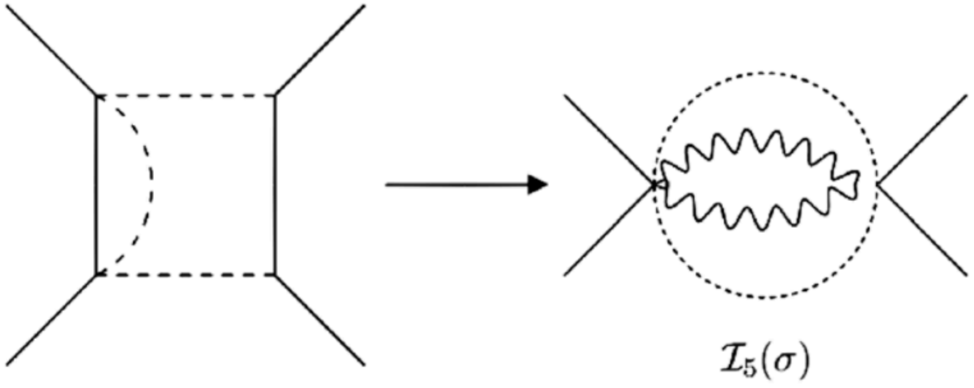

Gravitational waves from classical limit of loops

[Emil & al.: 2104.04510]

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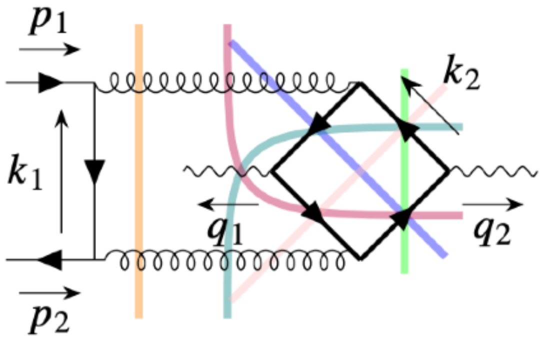

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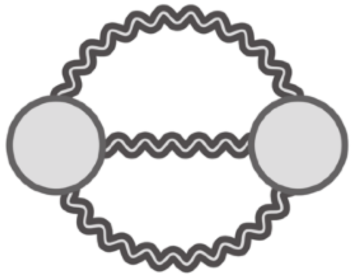

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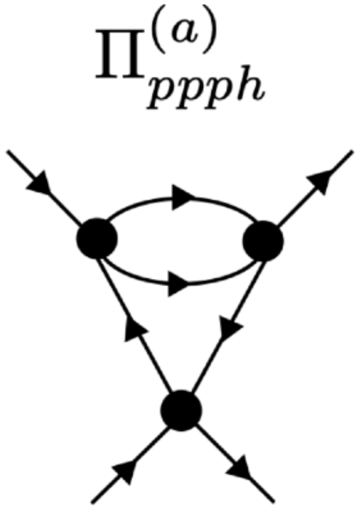

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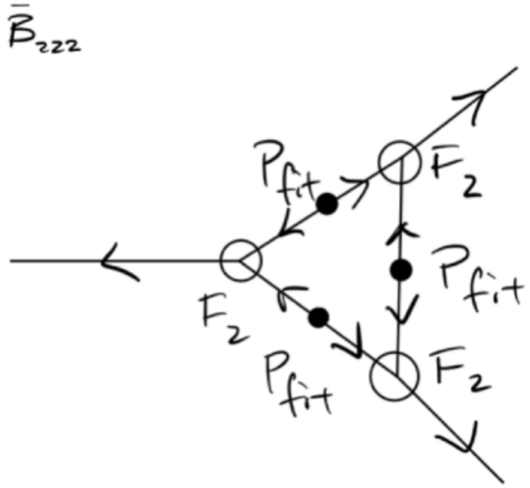

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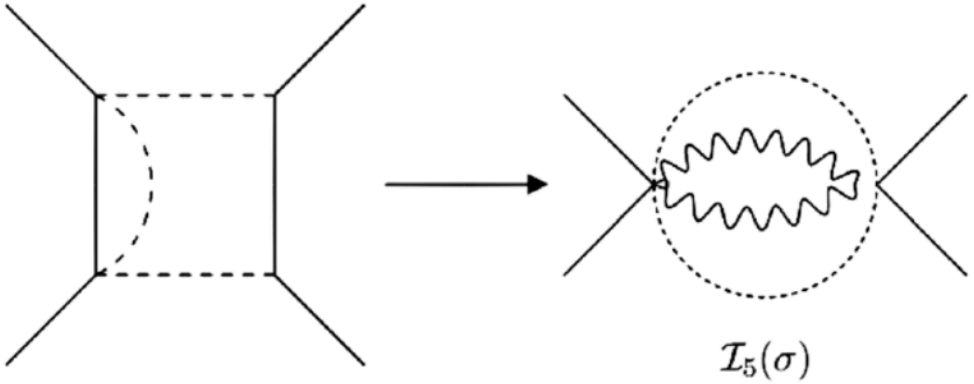

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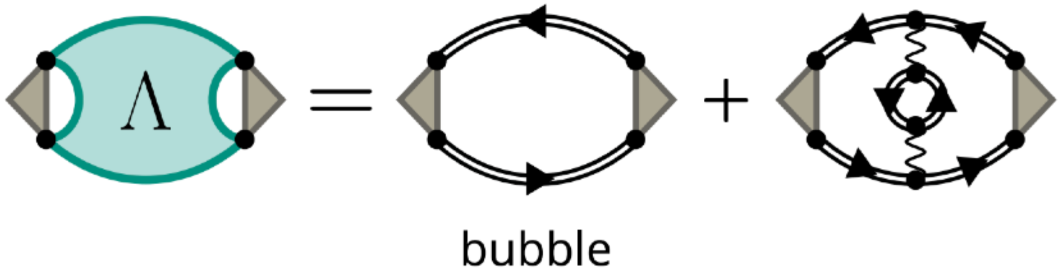


"RFDiagMC" for high-Tc superconductivity

[Ferrero & al.: 2501.19118]

...

NUMERICAL RESULTS

[Z. Capatti, VH, B. Ruijl, arxiv: 2203.11038]

We have computed the following fixed-order processes with **Local Unitarity**:

NLO

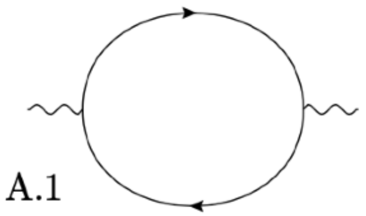
- $e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

NNLO

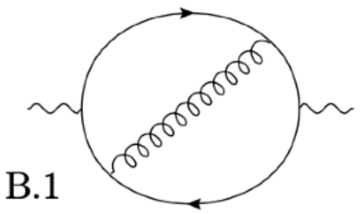
- $\gamma^\star \to jj$ inclusive
- $\gamma^\star \to t\bar{t}$ inclusive

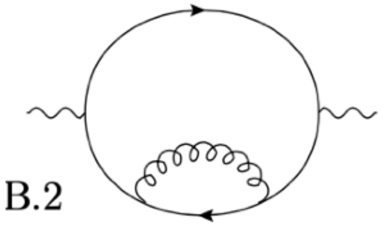
First NNLO cross-sections computed **fully numerically in momentum space**.

LO


A.1

NLO


B.1


B.2

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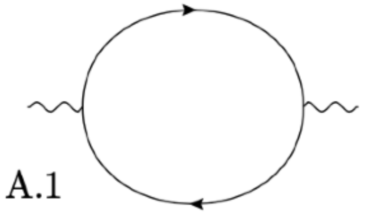

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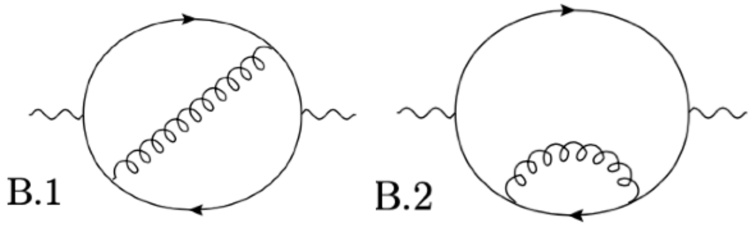
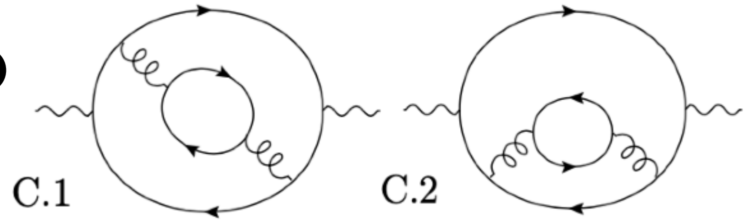


NNLO
$\mathcal{O}(n_f)$



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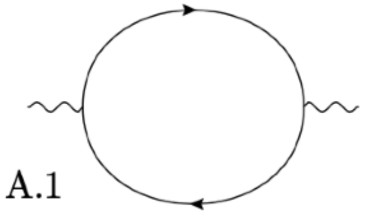

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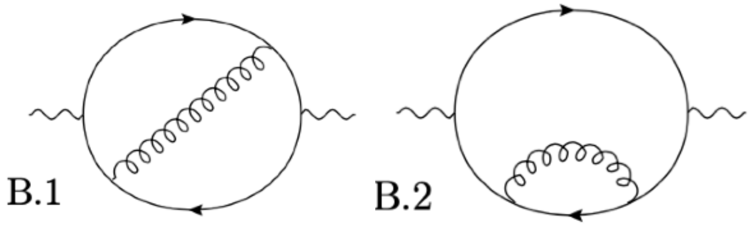

NNLO
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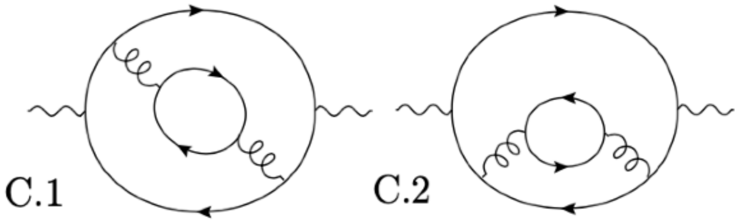
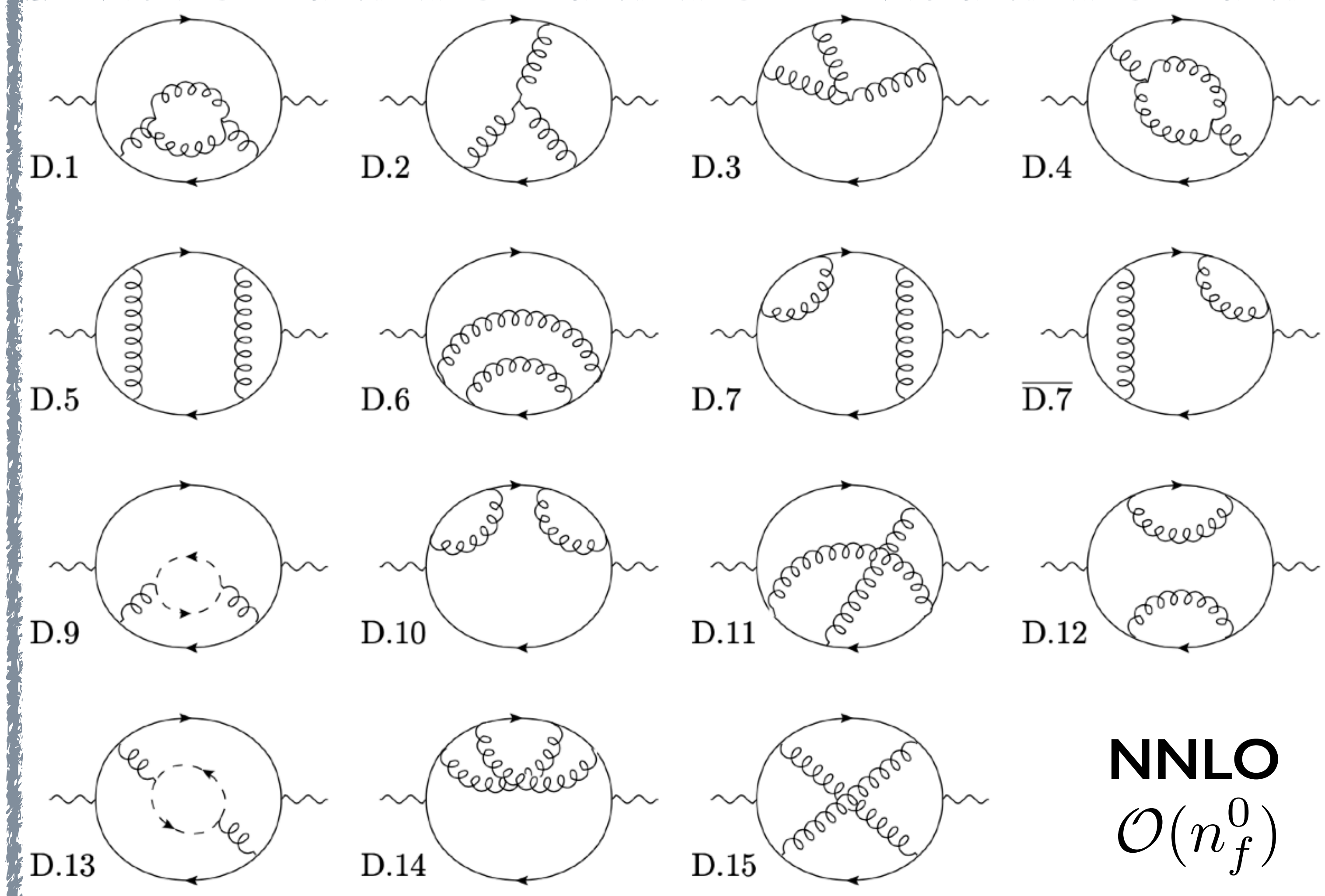




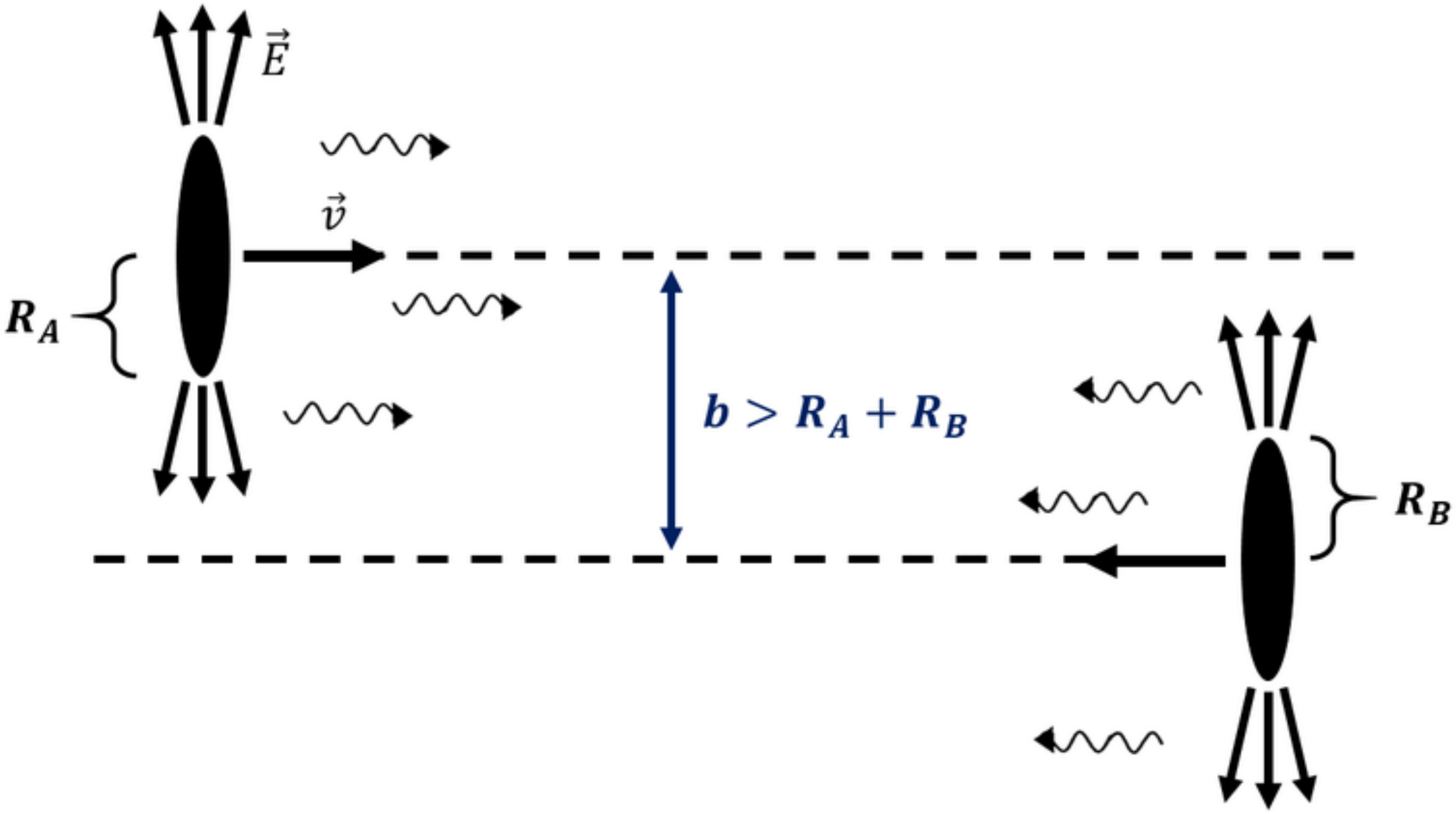
NNLO
$\mathcal{O}(n_f^0)$

IN ULTRA PERIPHERAL COLLISIONS (UPC) AT HADRON COLLIDERS

$\gamma\gamma \to \gamma\gamma$ AND $\boxed{\gamma\gamma \to Q\bar{Q}}$

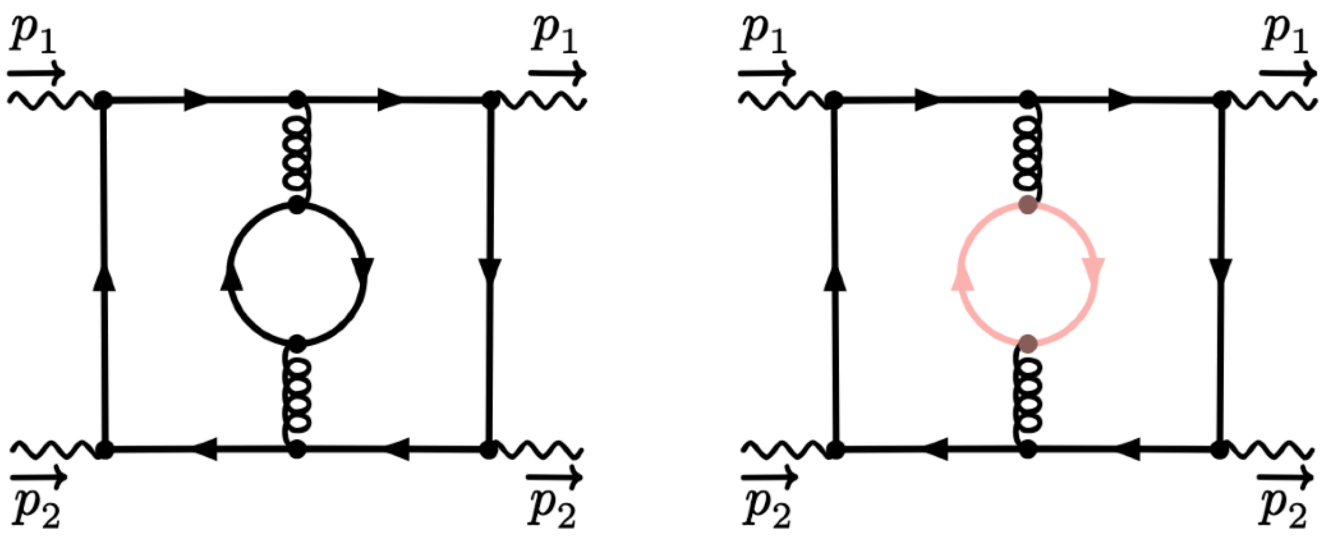
[Ajjath, Ekta, Fraaije, VH, H.S.Shao arxiv: 1007.0194]

[Capatti, Fraaije, VH, Huber, Ruijl, H.S.Shao arxiv: 2312.16956]

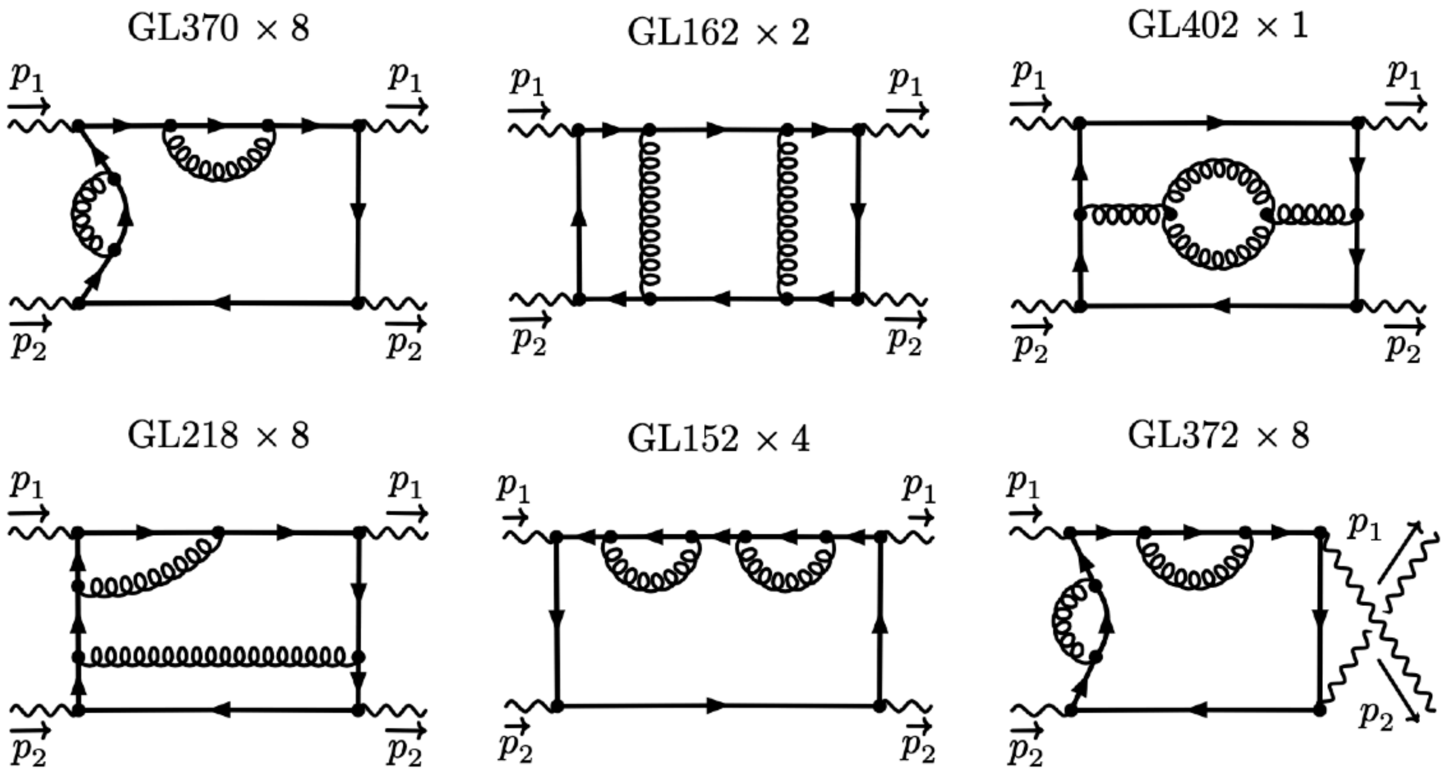


NNLO QCD WITH LOCAL UNITARITY

NNLO : nh, nf
2 x 10 distinct FS graphs

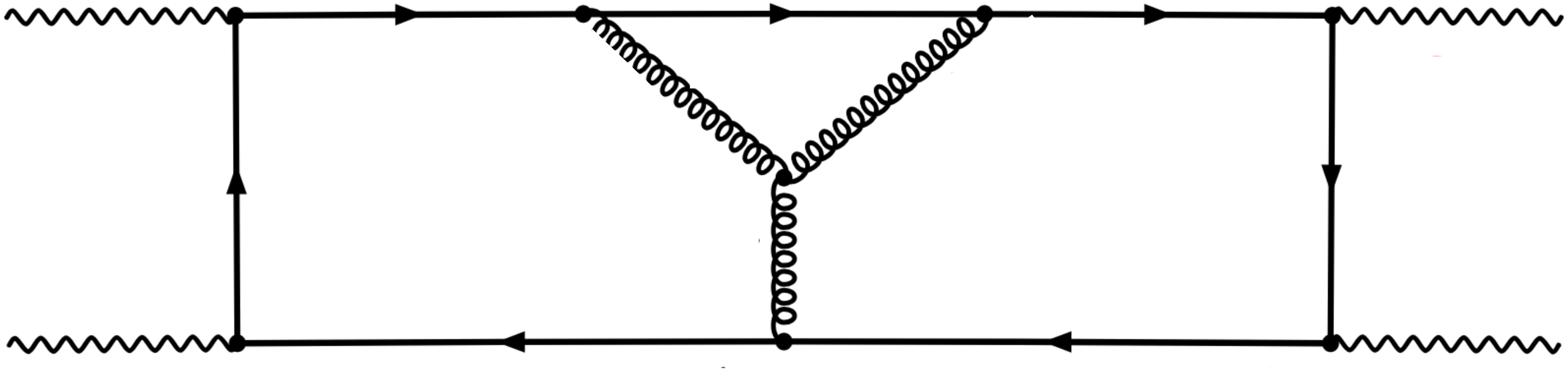


NNLO : non-singlet
114 distinct FS graphs

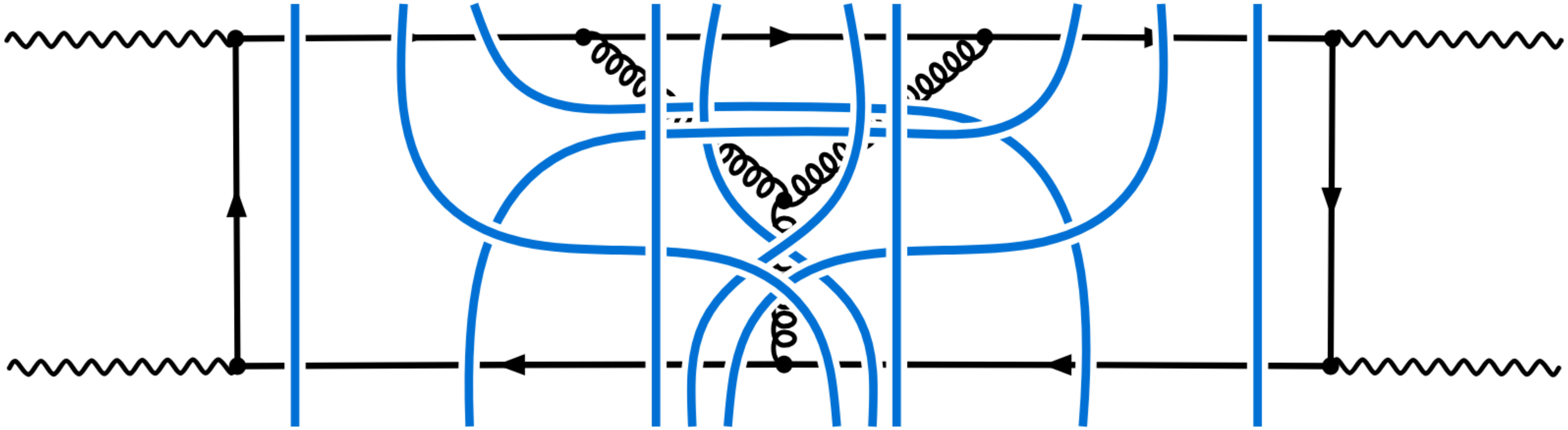


NNLO QCD WITH LOCAL UNITARITY

Reminder: when you see:



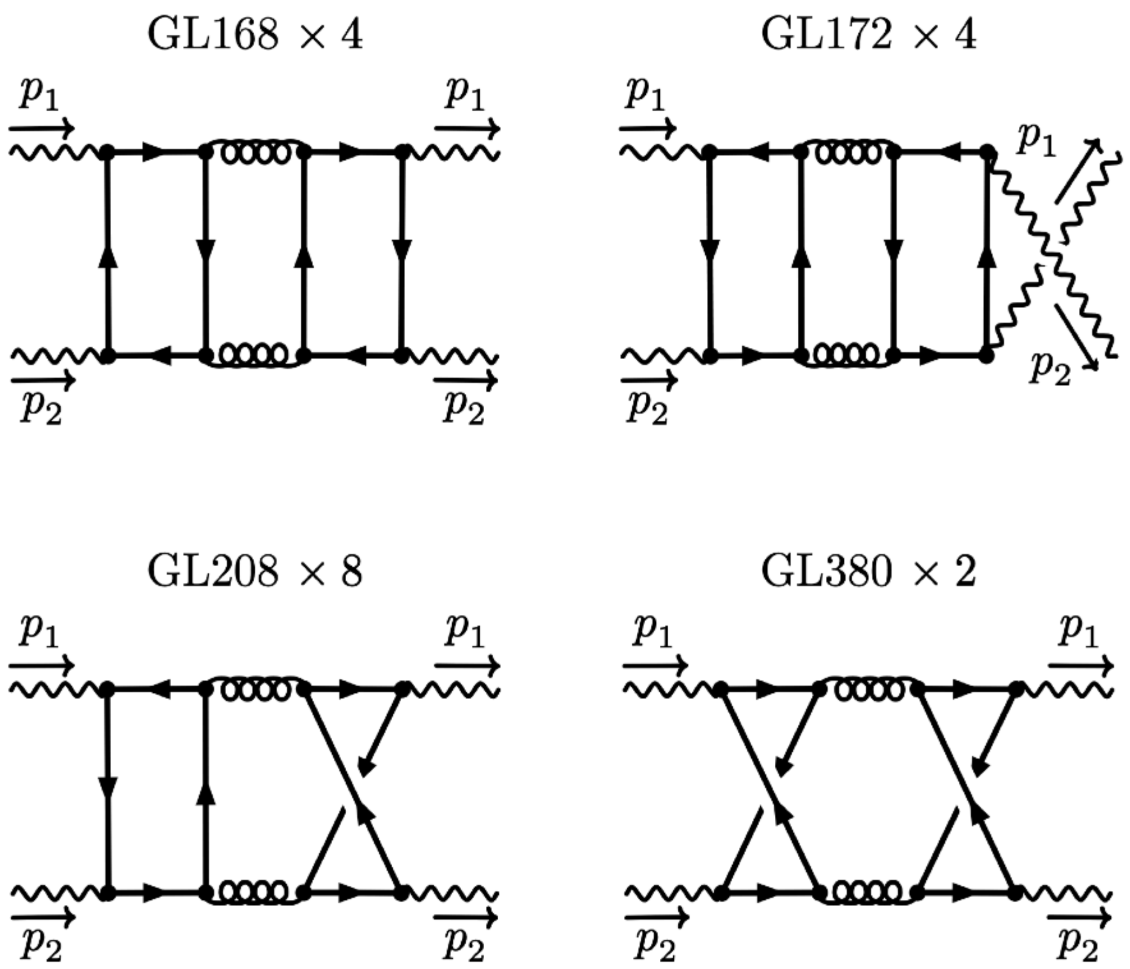
Think:



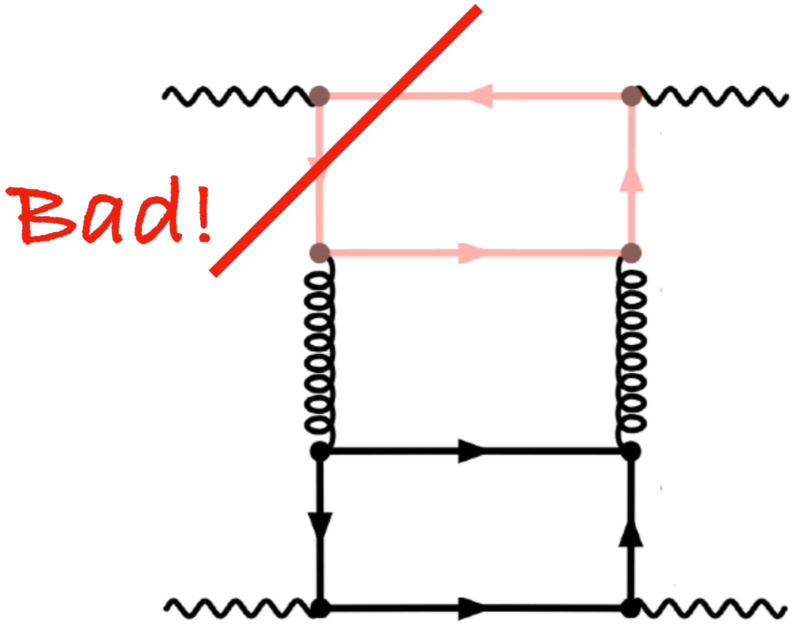
All cuts included !

NNLO QCD WITH LOCAL UNITARITY

NNLO: singlets
4 distinct FS graphs

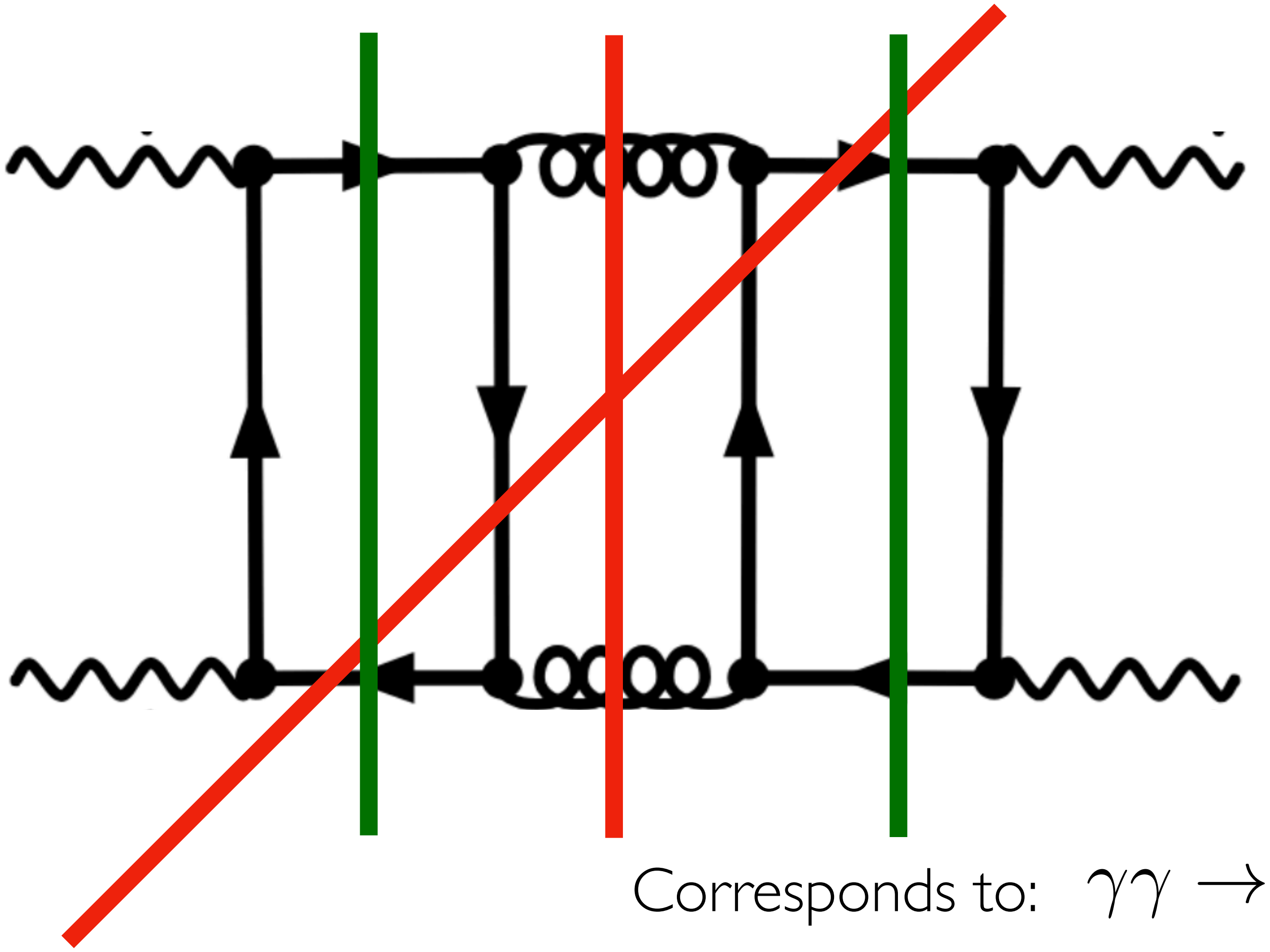


NNLO : singlets with ISR excluded



NNLO QCD WITH LOCAL UNITARITY

What about the singlet contributions ?



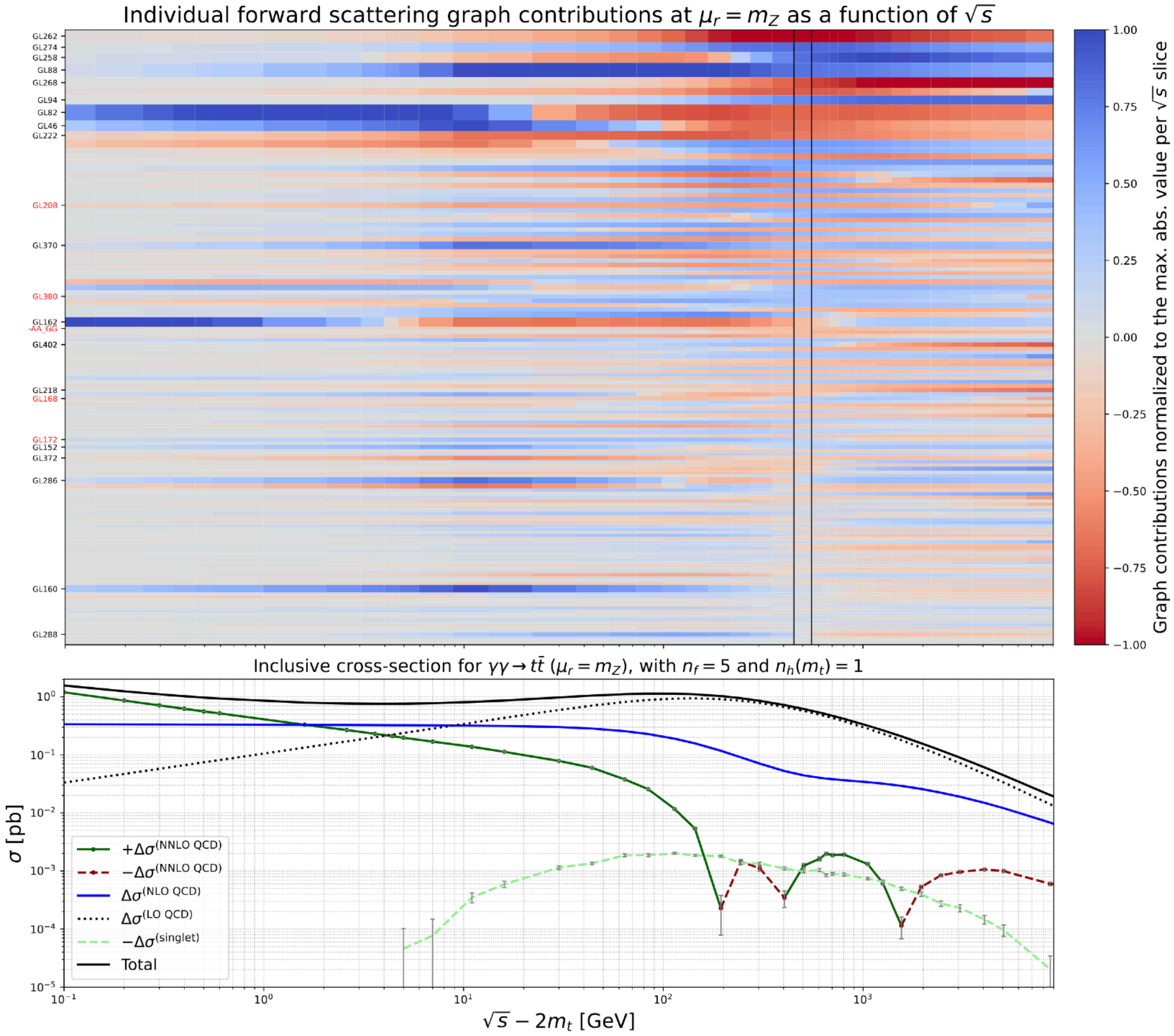
Corresponds to: $\gamma\gamma \to gg$

Keep to avoid threshold regularisation!
And subtract it back, computed as with MadGraph!

Corresponds to: $\gamma\gamma \to t\bar{t}t\bar{t}$
Safe to remove!
It cannot intersect with $\gamma\gamma \to t\bar{t}$ cuts!

$$\sum_{G \in \text{class C}} \sigma_G(\gamma\gamma \to Q\bar{Q}, Q\bar{Q}g) = \sum_{G \in \text{class C}} \underbrace{\sigma_G(\gamma\gamma \to Q\bar{Q}, Q\bar{Q}g, gg)}_{\text{LU}} \underbrace{- \sum_{G \in \text{class C}} \sigma_G(\gamma\gamma \to gg)}_{\text{MG5\_AMC}} .$$

RESULTS:



Individual forward scattering graph contributions at $\mu_r = m_Z$ as a function of $\sqrt{s}$

Inclusive cross-section for $\gamma\gamma \to t\bar{t}$ ($\mu_r = m_Z$), with $n_f = 5$ and $n_h(m_t) = 1$

γLoop : Demo

MODULAR CODES FOR MAXIMUM FLEXIBILITY

γLoop $\supset$ User-friendly

Low-level

Symbolica

~ "Open-source **CAS / Compiler** "

Components often distributed independently as sub-crates on:

https://github.com/alphal00p/gammaloop/tree/main/crates/XXX

- Fast and flexible numerator algebra → `spenso`

spenso / idenso

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  T. Heimel & al tool tailored to gammaBoard

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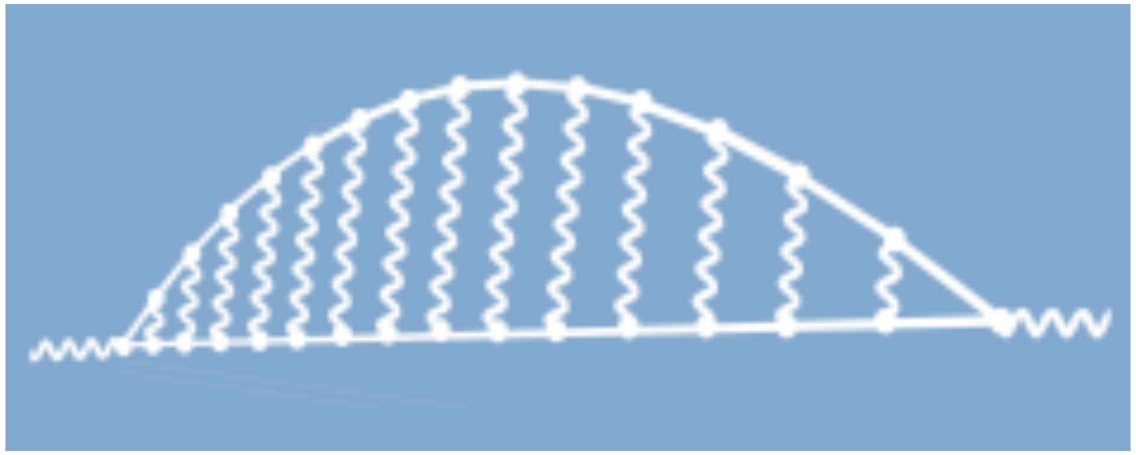
  T. Heimel & al tool tailored to gammaBoard
- Monte-Carlo integration harness / dashboard → `GammaBoard`
  Full generic: use it!

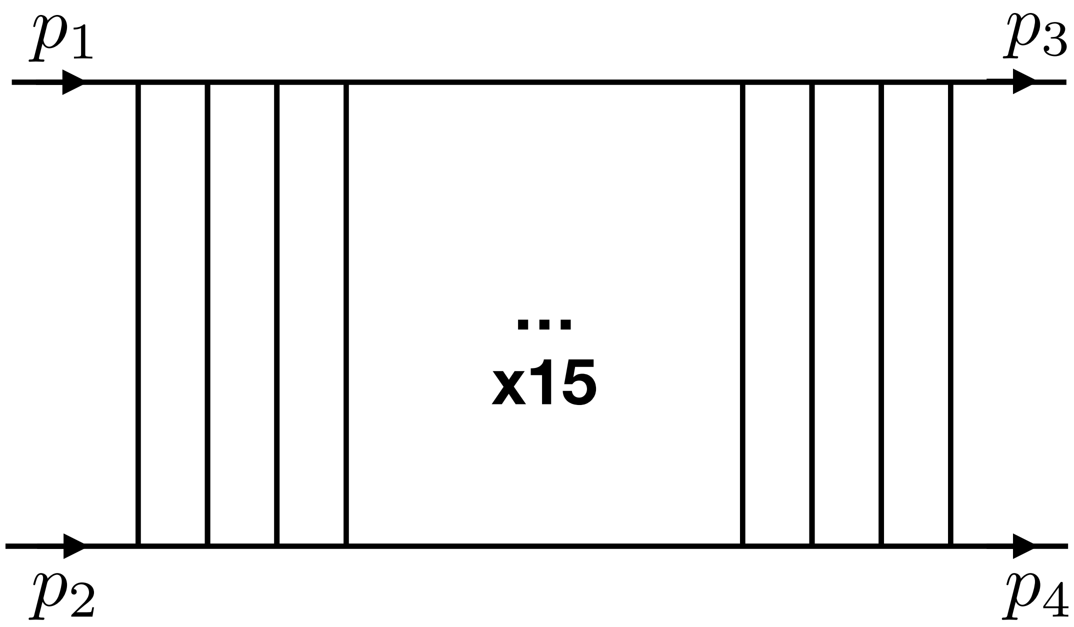
THE LOOPFEST 2020 CHALLENGE

LOOPFEST XX



No the easiest…

My interpretation



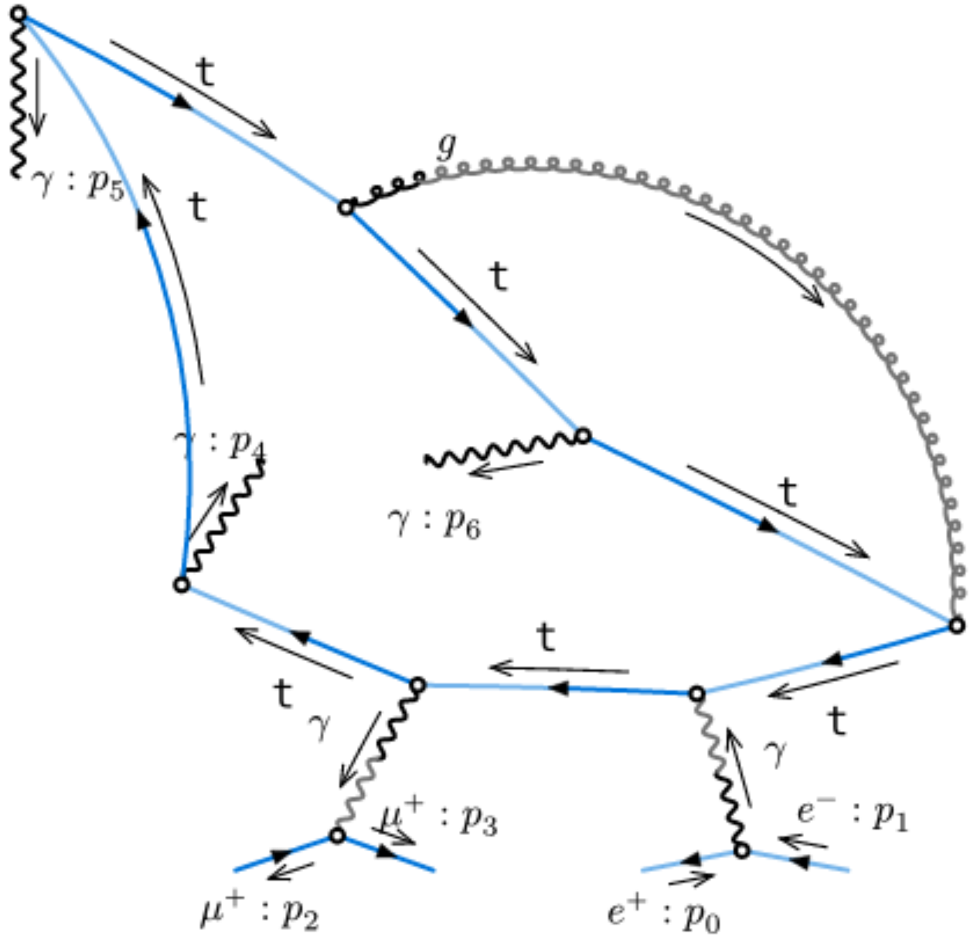
1x15 fishnet was the closest shot then

`[Broadhurst,Davydychev:1705.03545]`

`[Basso,Dixon:1705.03545]`

THE LOOPFEST 2026 CHALLENGE

My interpretation



Drawn using linnet (within typst)

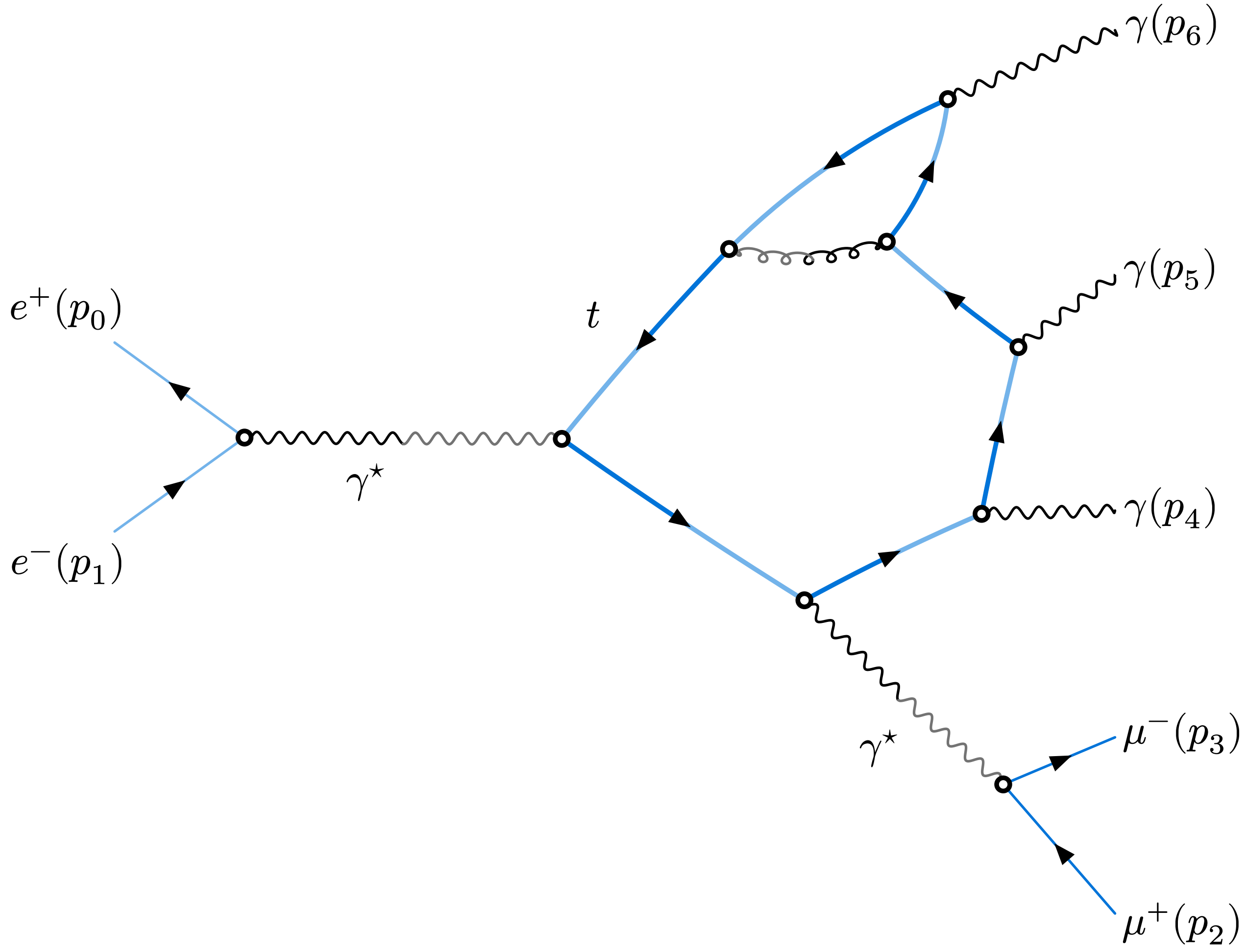
Now more doable!

$$\gamma^\star(p_0 + p_1) \rightarrow \gamma^\star(p_2 + p_3)\gamma(p_4)\gamma(p_5)\gamma(p_6)$$

5-point vector fusion through top loop !

THE LOOPFEST 2026 CHALLENGE

Two-loop 5-point challenge diagram



Two-loop 5-point challenge diagram

```
digraph BNL {
  num = "1";
  overall_factor = "1";
  overall_factor_evaluated = "1";
  projector = "vbar(0,spenso::bis(4,hedge(0)))
    * u(1,spenso::bis(4,hedge(1)))
    * v(2,spenso::bis(4,hedge(2)))
    * ubar(3,spenso::bis(4,hedge(3)))
    * ebar(4,spenso::mink(4,hedge(4)))
    * ebar(5,spenso::mink(4,hedge(5)))
    * ebar(6,spenso::mink(4,hedge(6)))";

  ext [style=invis]
  ext  -> 0:0 [id=0 particle="e+"];
  ext -> 0:1 [id=1 particle="e-"];

  1:2 -> ext [id=2 particle="mu+"];
  1:3 -> ext [id=3 particle="mu-"];
  2:4 -> ext [id=4 particle="a"];
  3:5 -> ext [id=5 particle="a"];
  4:6 -> ext [id=6 particle="a"];

  0 -> 5 [id=7 particle="a"];
  6 -> 1 [id=8 particle="a"];

  5 -> 6 [id=9 particle="t" lmb_id=0];
  6 -> 2 [id=10 particle="t"];
  2 -> 3 [id=11 particle="t"];
  3 -> 7 [id=12 particle="t"];
  7 -> 4 [id=13 particle="t"];
  4 -> 8 [id=14 particle="t"];
  8 -> 5 [id=15 particle="t"];
  7 -> 8 [id=16 particle="g" lmb_id=1];
}
```

"dot" format documentation

The LoopFest 2026 demo

https://gammaloop.hirschi.lu

Welcome to the LoopFest XXIV demo

Use the live GammaLoop demo tools below to monitor distributed runs, generate graph data, or continue the challenge workflow.

Gammaboard

Monitor and control distributed Monte Carlo integration runs.

Open board

GammaLoop source on GitHub

Gammaboard source on GitHub

Feynmangraph (WIP)

Generate and inspect Feynman graph data with the local GammaLoop backend.

Open graph tool

FeynmanGraph source on GitHub (credits: Elijah Cavan)

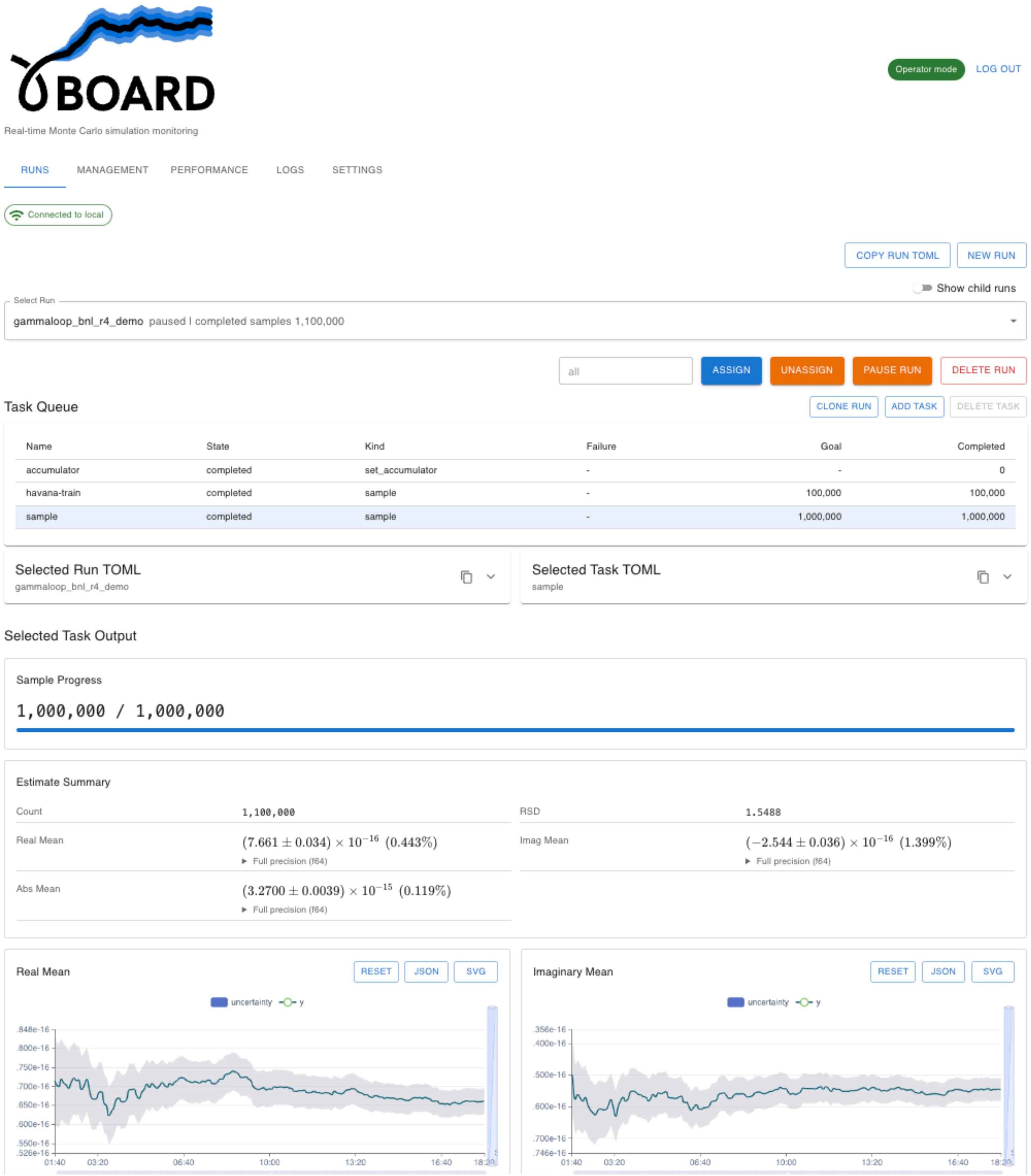
GammaLoop Challenge

Enter the LoopFest challenge ratios and check the generated result.

Open challenge

THE LOOPFEST 2026 DEMO

https://gammaloop.hirschi.lu/board



Select Run: gammaloop_bnl_r4_demo paused | completed samples 1,100,000

Task Queue

Name	State	Kind	Failure	Goal	Completed
accumulator	completed	set_accumulator	-	-	0
havana-train	completed	sample	-	100,000	100,000
sample	completed	sample	-	1,000,000	1,000,000

Selected Run TOML
gammaloop_bnl_r4_demo

Selected Task TOML
sample

Selected Task Output

Sample Progress

`1,000,000 / 1,000,000`

Estimate Summary

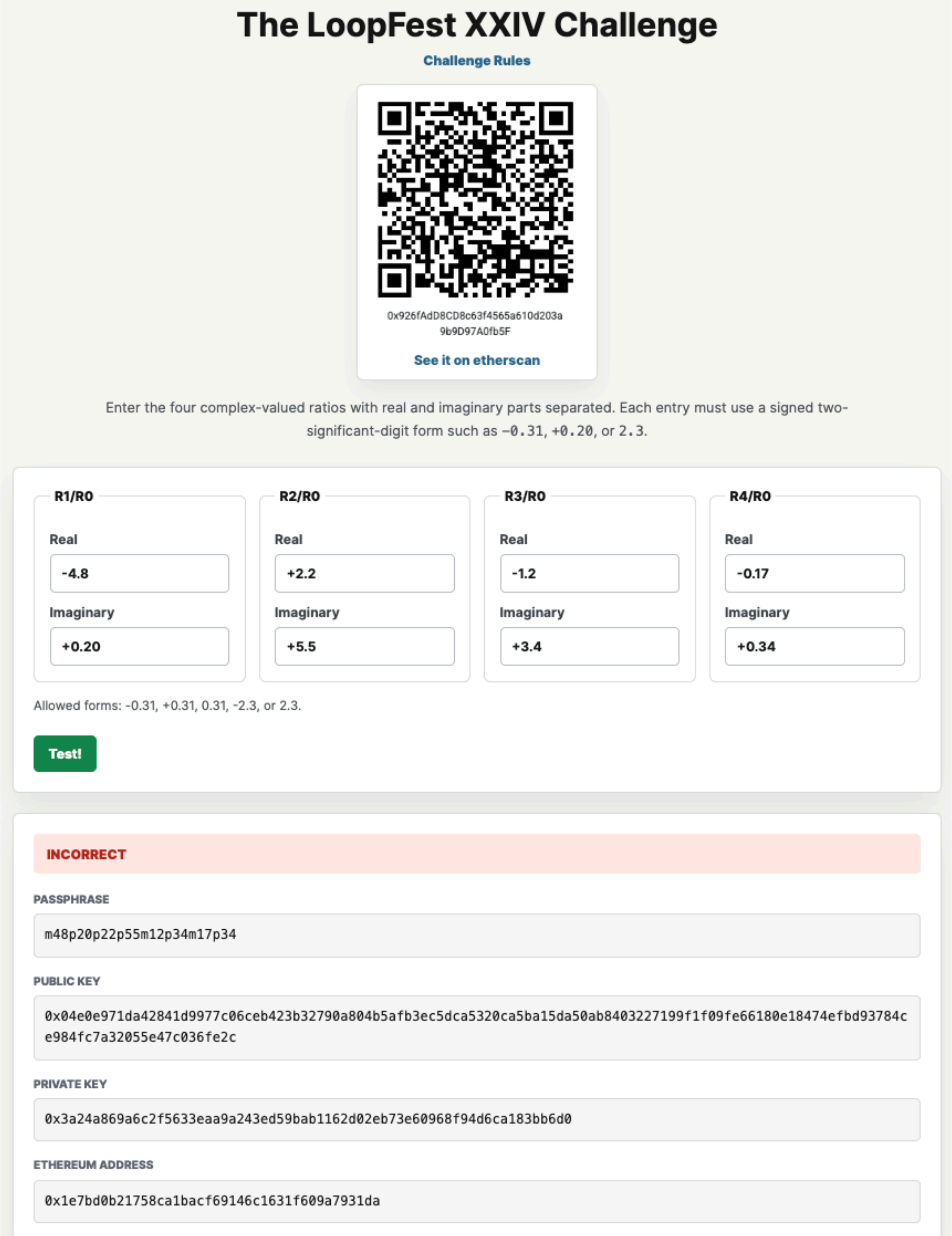
Count	`1,100,000`	RSD	`1.5488`
Real Mean	$(7.661 \pm 0.034) \times 10^{-16}$ (0.443%) ▸ Full precision (f64)	Imag Mean	$(-2.544 \pm 0.036) \times 10^{-16}$ (1.399%) ▸ Full precision (f64)
Abs Mean	$(3.2700 \pm 0.0039) \times 10^{-15}$ (0.119%) ▸ Full precision (f64)		

Real Mean

Imaginary Mean

THE LOOPFEST 2026 DEMO

https://gammaloop.hirschi.lu/challenge



The LoopFest XXIV Challenge

Challenge Rules

0x926fAdD8CD8c63f4565a610d203a9b9D97A0fb5F

See it on etherscan

Enter the four complex-valued ratios with real and imaginary parts separated. Each entry must use a signed two-significant-digit form such as `−0.31`, `+0.20`, or `2.3`.

R1/R0
Real: -4.8
Imaginary: +0.20

R2/R0
Real: +2.2
Imaginary: +5.5

R3/R0
Real: -1.2
Imaginary: +3.4

R4/R0
Real: -0.17
Imaginary: +0.34

Allowed forms: -0.31, +0.31, 0.31, -2.3, or 2.3.

INCORRECT

PASSPHRASE

`m48p20p22p55m12p34m17p34`

PUBLIC KEY

`0x04e0e971da42841d9977c06ceb423b32790a804b5afb3ec5dca5320ca5ba15da50ab8403227199f1f09fe66180e18474efbd93784ce984fc7a32055e47c036fe2c`

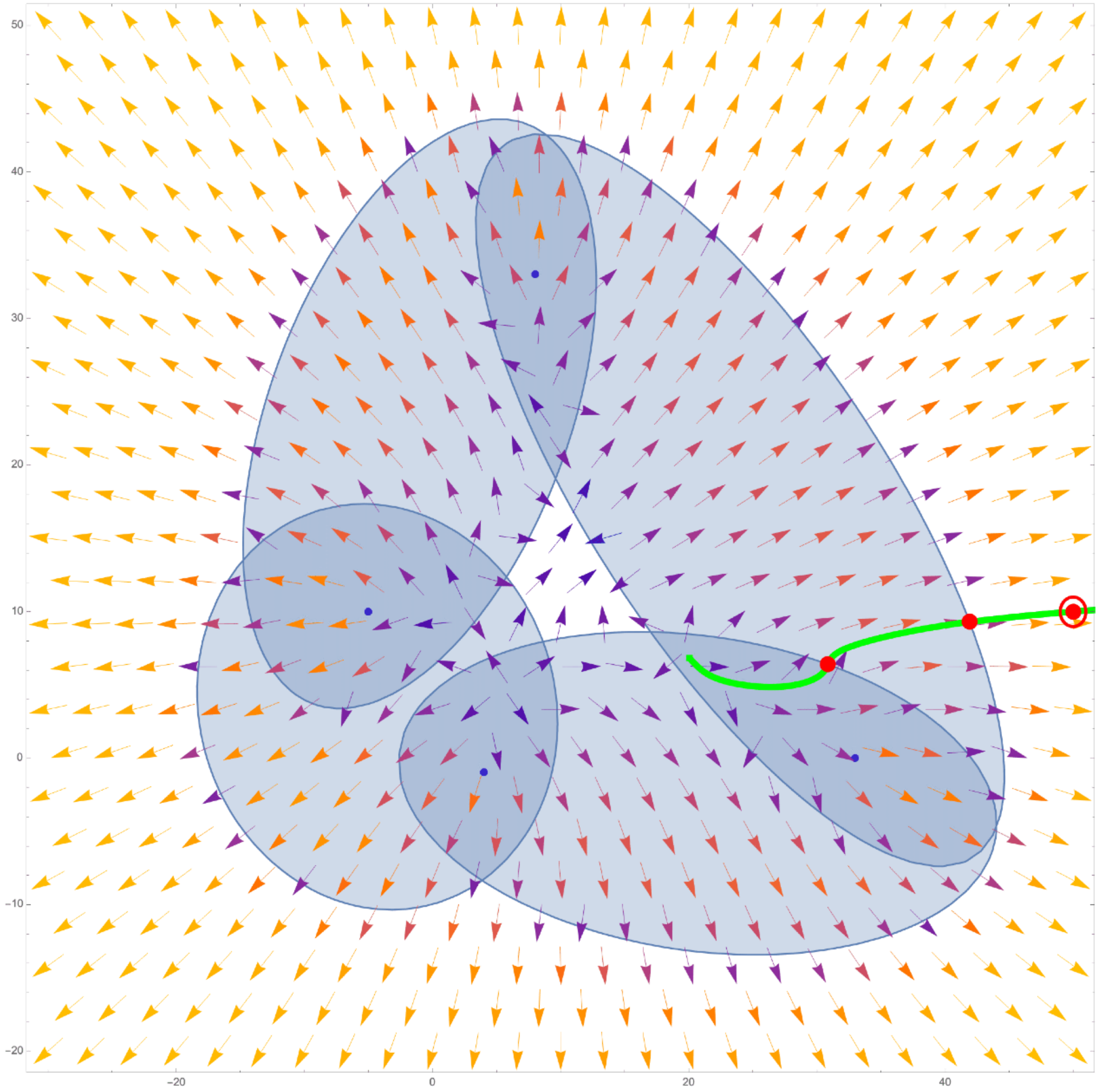
PRIVATE KEY

`0x3a24a869a6c2f5633eaa9a243ed59bab1162d02eb73e60968f94d6ca183bb6d0`

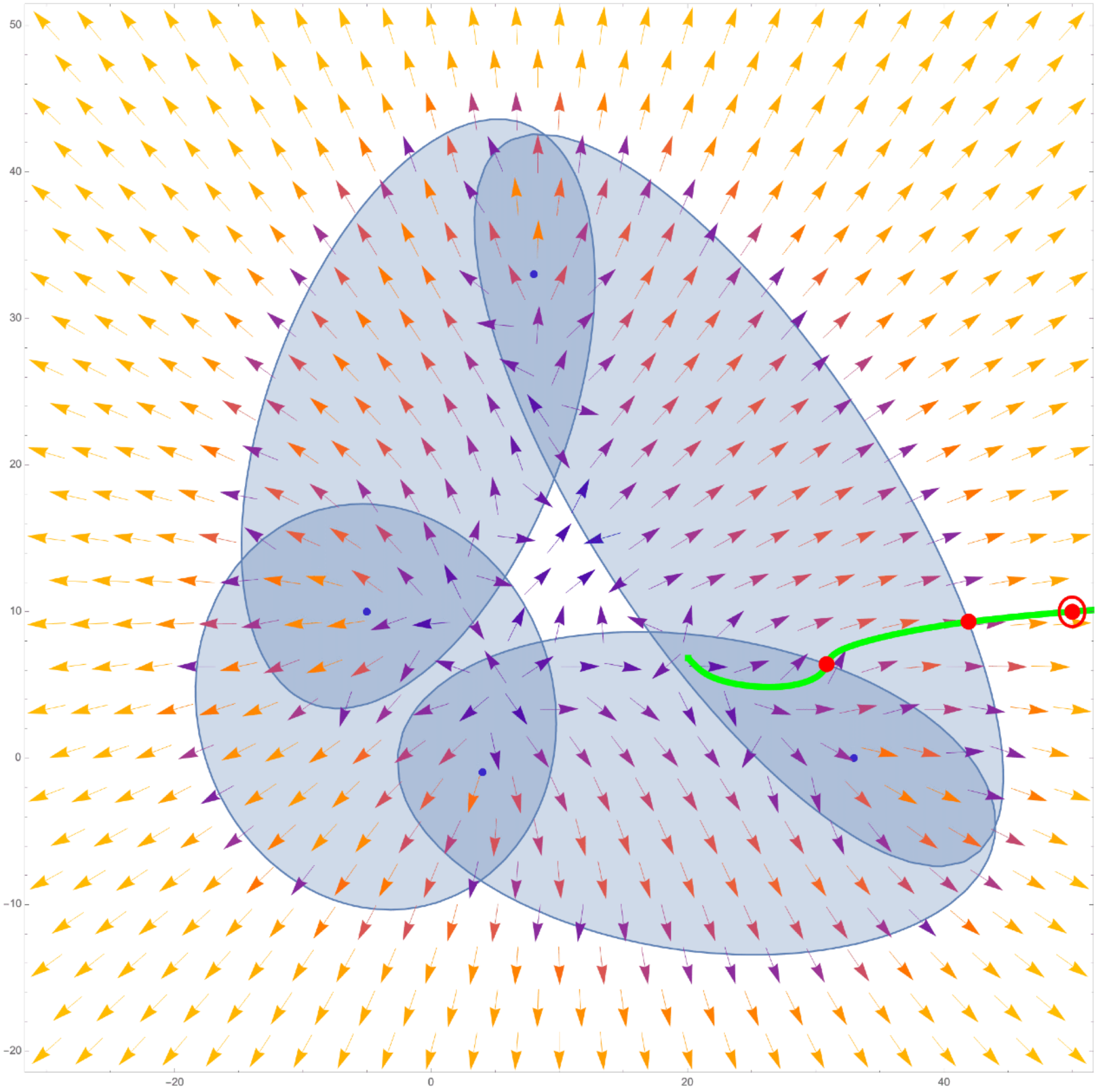
ETHEREUM ADDRESS

`0x1e7bd0b21758ca1bacf69146c1631f609a7931da`

THANK YOU FOR YOUR ATTENTION



THANK YOU FOR YOUR ATTENTION



BACKUP SLIDES

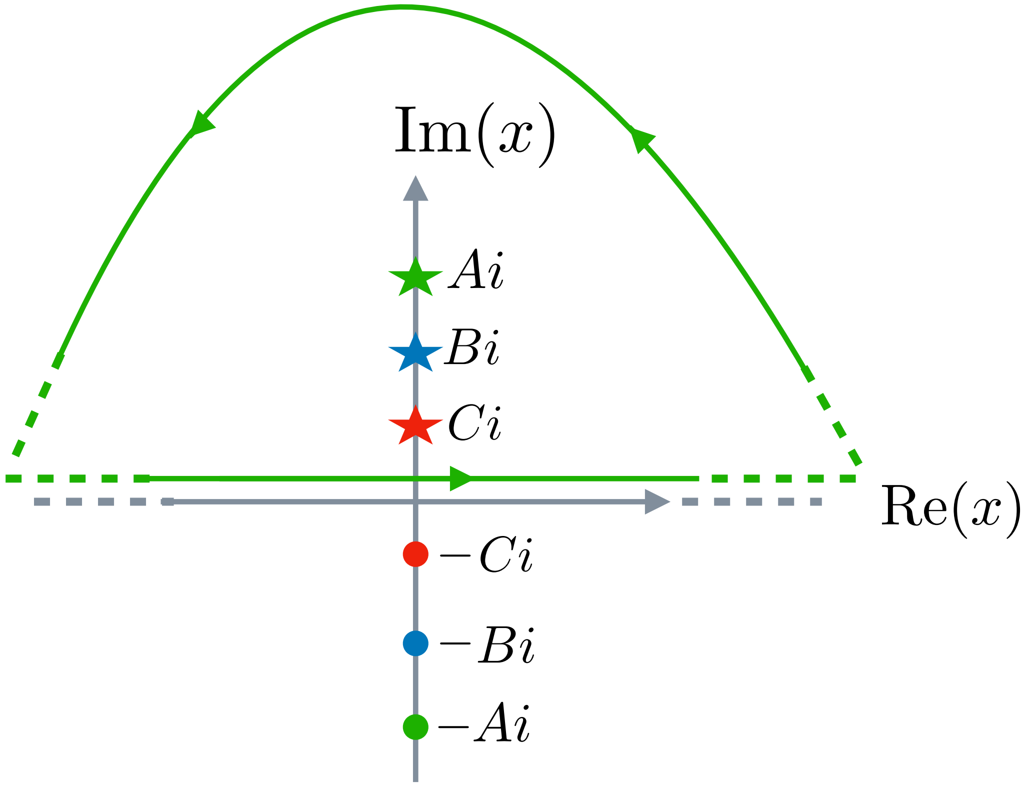
LOOP-TREE DUALITY

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \qquad F(x) = \frac{1}{x^2+A^2}\frac{1}{x^2+B^2}\frac{1}{x^2+C^2}$$

$$F(x) = \frac{1}{(x-Ai)(x+Ai)}\frac{1}{(x-Bi)(x+Bi)}\frac{1}{(x-Ci)(x+Ci)}$$

$(\text{Assumptions} \to \{A>0, B>0, C>0\})$



Cauchy: $\left(R(x^\star) \equiv \mathrm{Res}(F, x = x^\star) \right)$

$$I = (-2\pi i)[R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta}$$

$$= \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(ONE-)LOOP TREE DUALITY

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Pole selected for each propagator

(ONE-)LOOP TREE DUALITY

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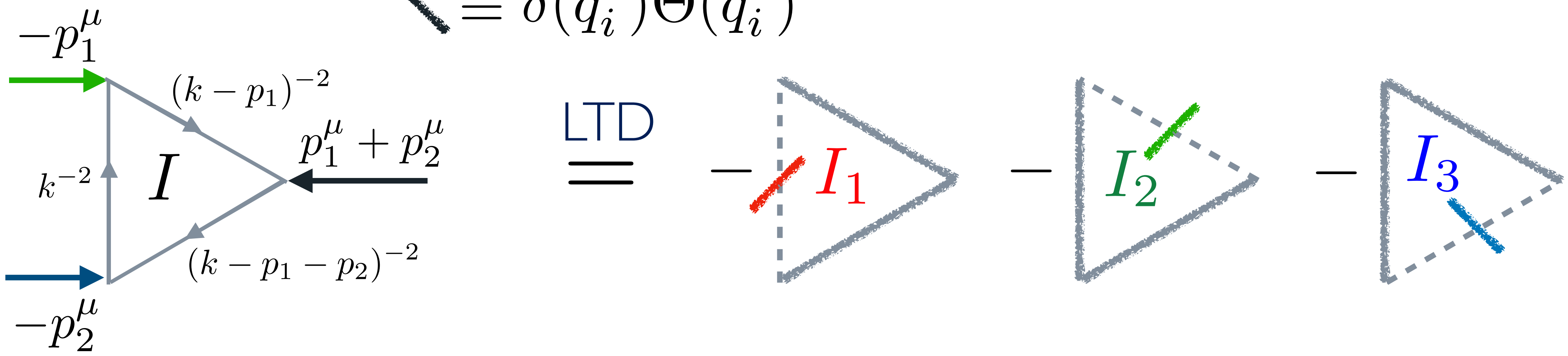
Pole selected for each propagator

Analogous 1-loop triangle case:

$$\backslash \equiv \delta(q_i^2)\Theta(q_i^0)$$



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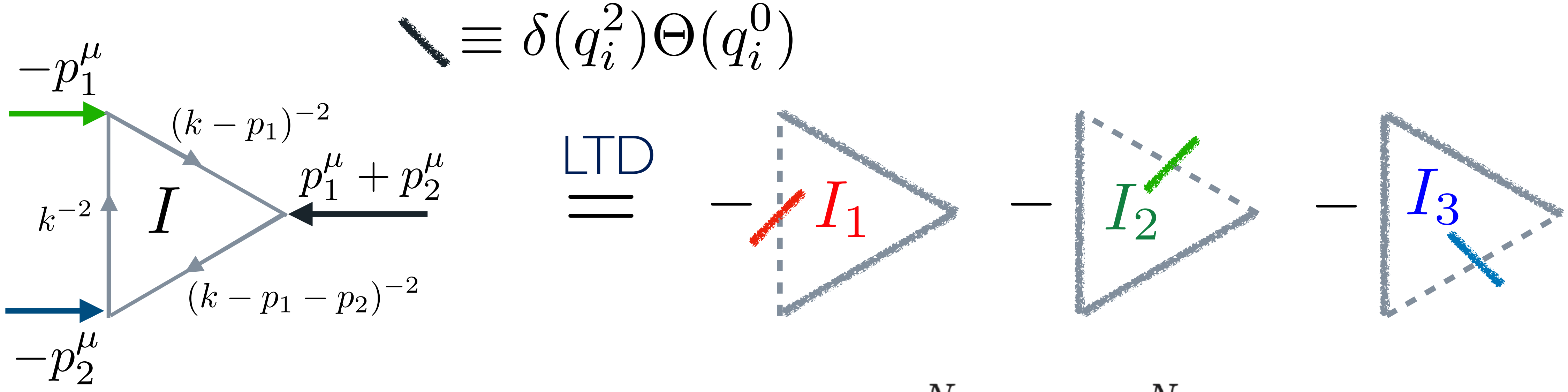


General 1-loop case:

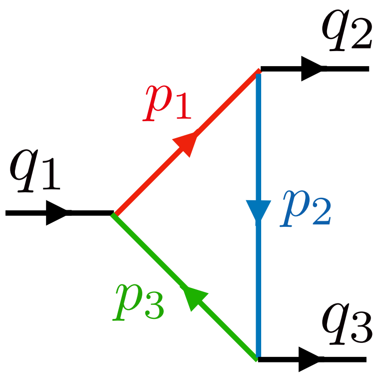
[Catani & al., arxiv:0804.3170]

$$I = -\int_q \sum_{i=1}^{N} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning, H-surfaces can be **algebraically canceled**:



$$= \frac{1}{2\pi i}\int_{\mathbb{R}} dk^0 \frac{1}{(k^0+p_1^0)^2-E_1^2}\frac{1}{(k^0+p_2^0)^2-E_2^2}\frac{1}{(k^0+p_3^0)^2-E_3^2} \overset{\text{LTD}}{=}$$

MULTI-LOOP TREE DUALITY: cLTD

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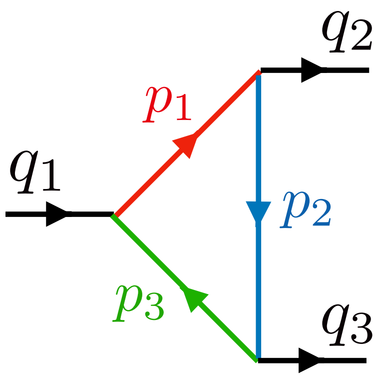

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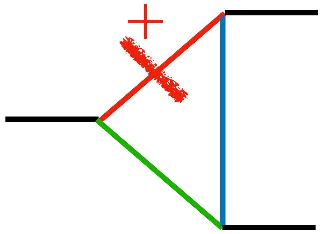

$$\frac{1}{2E_1} \qquad \frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

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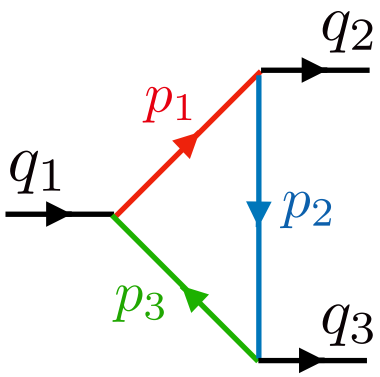
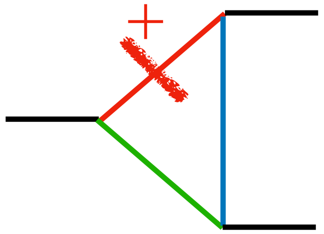
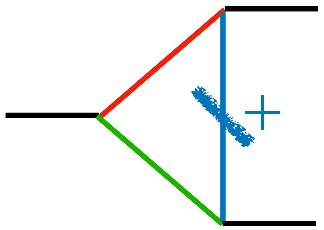


$$= \frac{1}{2\pi i}\int_{\mathbb{R}} dk^0 \, \frac{1}{(k^0+p_1^0)^2-E_1^2}\,\frac{1}{(k^0+p_2^0)^2-E_2^2}\,\frac{1}{(k^0+p_3^0)^2-E_3^2} \overset{\text{LTD}}{=}$$



$$\frac{1}{2E_1}\qquad\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\,\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$



$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\qquad\frac{1}{2E_2}\qquad\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

MULTI-LOOP TREE DUALITY: cLTD

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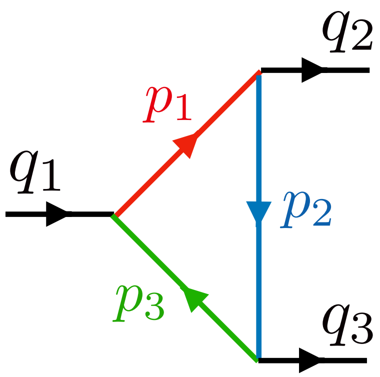
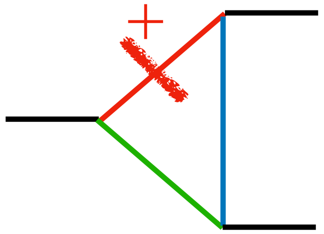
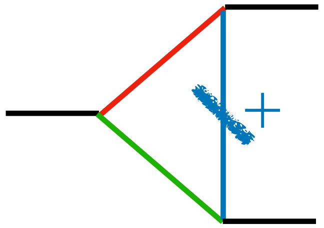


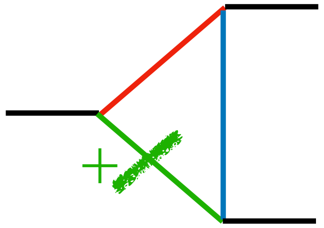
$$= \frac{1}{2\pi i}\int_{\mathbb{R}} dk^0 \, \frac{1}{(k^0+p_1^0)^2 - E_1^2} \, \frac{1}{(k^0+p_2^0)^2 - E_2^2} \, \frac{1}{(k^0+p_3^0)^2 - E_3^2} \overset{\text{LTD}}{=}$$



$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \quad \frac{1}{2E_2} \quad \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$



$$+ \frac{1}{(E_3 - p_3^0 + p_1^0)^2 - E_1^2} \quad \frac{1}{(E_3 - p_3^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{2E_3}$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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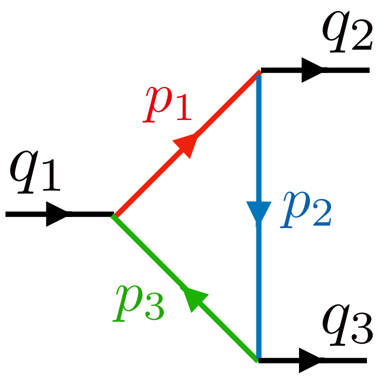

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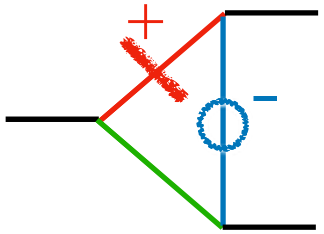
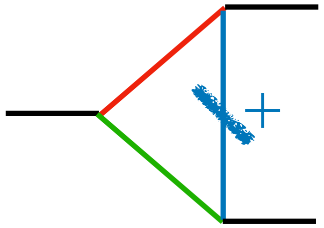
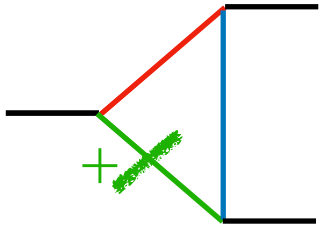


$$\frac{1}{2E_1}\,\boxed{\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}}\,\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$



$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\,\frac{1}{2E_2}\,\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

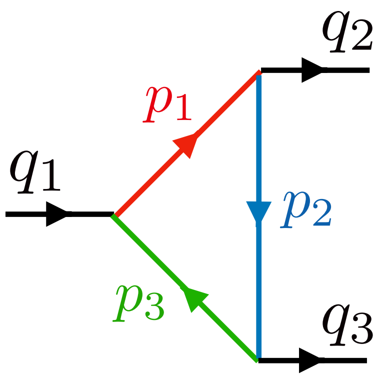


$$+\frac{1}{(E_3-p_3^0+p_1^0)^2-E_1^2}\,\frac{1}{(E_3-p_3^0+p_2^0)^2-E_2^2}\,\frac{1}{2E_3}$$

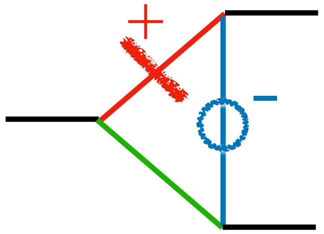
MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

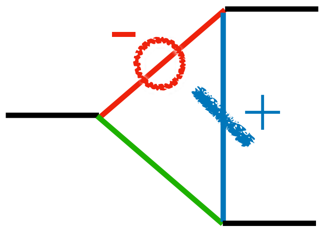
Using partial fractioning, H-surfaces can be **algebraically canceled**:



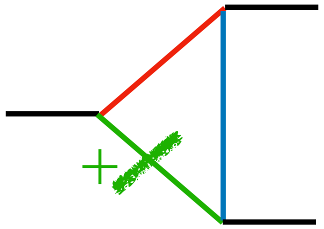
$$= \frac{1}{2\pi i}\int_{\mathbb{R}} dk^0 \frac{1}{(k^0+p_1^0)^2-E_1^2}\frac{1}{(k^0+p_2^0)^2-E_2^2}\frac{1}{(k^0+p_3^0)^2-E_3^2} \overset{\text{LTD}}{=}$$



$$\frac{1}{2E_1}\boxed{\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}}\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$



$$+\boxed{\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}}\frac{1}{2E_2}\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

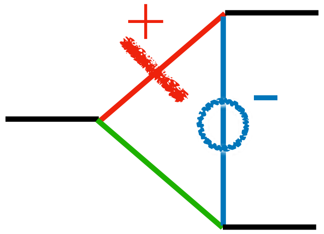


$$+\frac{1}{(E_3-p_3^0+p_1^0)^2-E_1^2}\frac{1}{(E_3-p_3^0+p_2^0)^2-E_2^2}\frac{1}{2E_3}$$

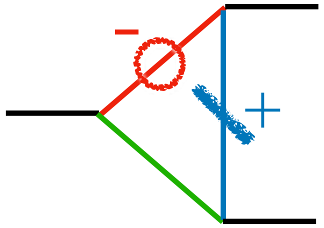
MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning , H-surfaces can be **algebraically canceled**: $\frac{1}{x^2-y^2} = \frac{1}{2y}\left(\frac{1}{x-y} - \frac{1}{x+y}\right)$



$$\frac{1}{2E_1} \quad \frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2} \quad \frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

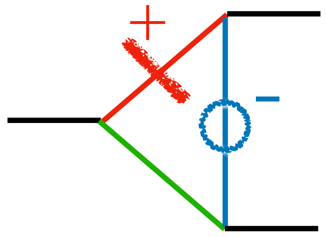


$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2} \quad \frac{1}{2E_2} \quad \frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

MULTI-LOOP TREE DUALITY: cLTD

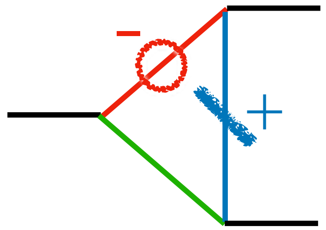
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning , H-surfaces can be **algebraically canceled**: $\frac{1}{x^2-y^2} = \frac{1}{2y}\left(\frac{1}{x-y} - \frac{1}{x+y}\right)$



$$\frac{1}{2E_1} \qquad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \qquad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

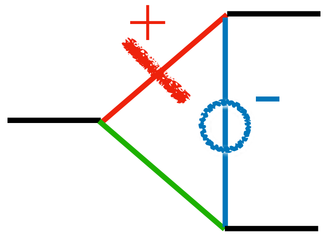


$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \qquad \frac{1}{2E_2} \qquad \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: CLTD

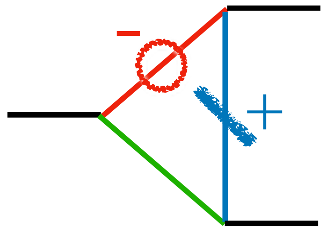
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning , H-surfaces can be **algebraically canceled**: $\frac{1}{x^2-y^2}=\frac{1}{2y}\left(\frac{1}{x-y}-\frac{1}{x+y}\right)$



$$\frac{1}{2E_1}\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_2}\left(\frac{1}{(E_1-p_1^0+p_2^0)-E_2}-\frac{1}{(E_1-p_1^0+p_2^0)+E_2}\right)$$



$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\frac{1}{2E_2}\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_1}\left(\frac{1}{(E_2-p_2^0+p_1^0)-E_1}-\frac{1}{(E_2-p_2^0+p_1^0)+E_1}\right)$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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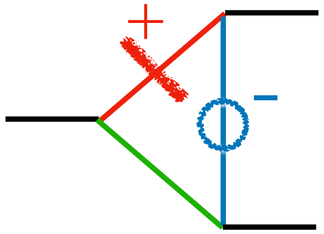
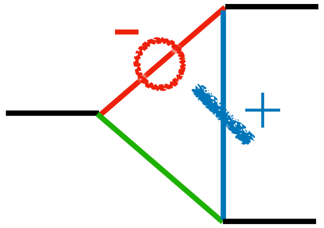


$$\frac{1}{2E_1}\;\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\;\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_2}\left(\frac{1}{(E_1-p_1^0+p_2^0)-E_2}-\frac{1}{(E_1-p_1^0+p_2^0)+E_2}\right)$$



$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\;\frac{1}{2E_2}\;\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_1}\left(\frac{1}{(E_2-p_2^0+p_1^0)-E_1}-\frac{1}{(E_2-p_2^0+p_1^0)+E_1}\right)$$

$$\supset\frac{1}{2E_1}\;\frac{1}{2E_2}\left(\frac{1}{E_1-p_1^0+p_2^0-E_2}\;\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}+\right.$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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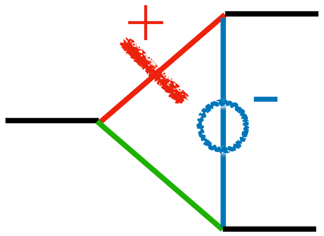
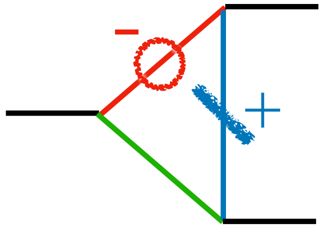


$$\frac{1}{2E_1}\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_2}\left(\frac{1}{(E_1-p_1^0+p_2^0)-E_2}-\frac{1}{(E_1-p_1^0+p_2^0)+E_2}\right)$$



$$+\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\frac{1}{2E_2}\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_1}\left(\frac{1}{(E_2-p_2^0+p_1^0)-E_1}-\frac{1}{(E_2-p_2^0+p_1^0)+E_1}\right)$$

$$\supset\frac{1}{2E_1}\frac{1}{2E_2}\left(\frac{1}{E_1-p_1^0+p_2^0-E_2}\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}+\frac{1}{E_2-p_2^0+p_1^0-E_1}\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}\right)$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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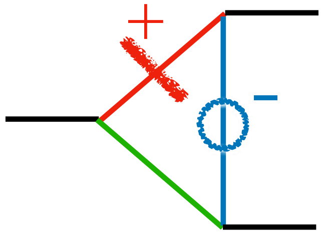
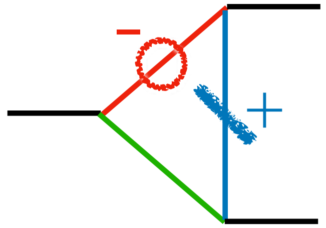


$$\frac{1}{2E_1}\;\frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2}\;\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_2}\left(\frac{1}{(E_1-p_1^0+p_2^0)-E_2}-\frac{1}{(E_1-p_1^0+p_2^0)+E_2}\right)$$



$$+\;\frac{1}{(E_2-p_2^0+p_1^0)^2-E_1^2}\;\frac{1}{2E_2}\;\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}$$

$$=\frac{1}{2E_1}\left(\frac{1}{(E_2-p_2^0+p_1^0)-E_1}-\frac{1}{(E_2-p_2^0+p_1^0)+E_1}\right)$$

$$\supset\frac{1}{2E_1}\;\frac{1}{2E_2}\left(\frac{1}{E_1-p_1^0+p_2^0-E_2}\;\frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}+\frac{1}{E_2-p_2^0+p_1^0-E_1}\;\frac{1}{(E_2-p_2^0+p_3^0)^2-E_3^2}\right)$$

$$=\frac{1}{2E_1}\;\frac{1}{2E_2}\;\frac{E_1+E_2-p_1-p_2+2p_3}{((E_1-p_1^0+p_3^0)^2-E_3^2)\,((E_2-p_2^0+p_3^0)^2-E_3^2)}$$

Manifestly **divergence-free** at :

$$E_2(\vec{k})=E_1(\vec{k})+p_2^0-p_1^0$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$\Big]$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Bigg[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(A)$$

$$\Bigg]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \, \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(A)$$

$$+ \frac{1}{-2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(-A)$$

$$\Big]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

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$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(A)$$

$$+ \frac{1}{-2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(-A)$$

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Theta(\Im[Ci-y])$$

$\Big]$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(A)$$

$$+ \frac{1}{-2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \Theta(-A)$$

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Theta(\Im[Ci-y])$$

$$+ \frac{1}{(-Ci-y-Ai)(-Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{-2Ci} \Theta(\Im[-Ci-y]) \quad \Big]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \, \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \overset{\equiv 1}{\Theta(\cancel{A})}$$

$$+ \frac{1}{-2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \overset{\equiv 0}{\Theta(\cancel{-A})}$$

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Theta(\Im[Ci-y])$$

$$+ \frac{1}{(-Ci-y-Ai)(-Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{-2Ci} \Theta(\Im[-Ci-y]) \quad \Big]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \overset{\equiv 1}{\cancel{\Theta(A)}}$$

$$+ \frac{1}{-2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)} \overset{\equiv 0}{\cancel{\Theta(-A)}}$$

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \overset{\equiv 1}{\cancel{\Theta(\Im[Ci-y])}}$$

$$+ \frac{1}{(-Ci-y-Ai)(-Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{-2Ci} \overset{\equiv 0}{\cancel{\Theta(\Im[-Ci-y])}} \quad \Big]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

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$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci}$$

$\Big]$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Bigg[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)}$$

(+1, 0, 0)

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci}$$

(0, 0,+1)

$$\Bigg]$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

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$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Bigg]$$

`(+1, 0, 0)`

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$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Big]$$

`(+1, 0, 0)`

`( 0, 0,+1)`

Integrate y then: $I = (-2\pi i)^2 \Big[$

$\Big]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$$\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)}$$

`(+1, 0, 0)`

$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \Big]$$

`( 0, 0,+1)`

Integrate y then: $I = (-2\pi i)^2 \Big[$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)}$$

`(+1,+1, 0)`

$\Big]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)}$$

(+1, 0, 0)

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Big]$$

(0, 0,+1)

Integrate y then: $I = (-2\pi i)^2 \Big[$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai+Bi-Ci)(Ai+Bi+Ci)}$$

$$+ \frac{1}{2Ai} \frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)} \frac{1}{2Ci} \Theta(C-A)$$

(+1,+1, 0)

(+1, 0,+1)

$$\Big]$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

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`(+1, 0, 0)`

$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Big]$$

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Integrate y then: $I = (-2\pi i)^2 \Big[$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai+Bi-Ci)(Ai+Bi+Ci)}$$

$$+ \frac{1}{2Ai} \frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)} \frac{1}{2Ci} \Theta(C-A)$$

`(+1,+1, 0)`

`(+1, 0,+1)`

$$\Big\downarrow$$

$$\sim \lim_{\delta \to 0} \Theta(\delta|\vec{k}| - \delta|\vec{l}|) = ???$$

$$\Big]$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \, \Big[$

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$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Big]$$

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(+1,+1, 0)

$$+ \frac{1}{2Ai} \frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)} \frac{1}{2Ci} \Theta(C-A)$$

(+1, 0,+1)

$$- \frac{1}{2Ai} \frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)} \frac{1}{2Ci} \Theta(C-A)$$

(+1, 0,+1)

$$\Big]$$

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`(+1,+1, 0)`

`(+1, 0,+1)`

$$- \frac{1}{2Ai} \frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)} \frac{1}{2Ci} \Theta(C-A)$$

$$- \frac{1}{-2Ai} \frac{1}{(Ai+Ci-Bi)(Ai+Ci+Bi)} \frac{1}{2Ci}$$

$$\Big]$$

`(+1, 0,+1)`

`(-1, 0,+1)`

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(x+y-Ci)(x+y+Ci)}$$

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$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai+Bi-Ci)(Ai+Bi+Ci)}$$

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$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

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`(+1, 0, 0)`

$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \Big]$$

`( 0, 0,+1)`

Integrate y then: $I = (-2\pi i)^2 \Big[$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)}$$

`(+1,+1, 0)`

$$\cancel{+ \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} O(C - A)}$$

~~`(+1, 0,+1)`~~

$$\cancel{- \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} O(C - A)}$$

~~`(+1, 0,+1)`~~

$$- \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci}$$

`(-1, 0,+1)`

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$$\cancel{+\frac{1}{2Ai}\frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)}\frac{1}{2Ci}\ O(C-A)}$$

~~(+1, 0,+1)~~

$$\cancel{-\frac{1}{2Ai}\frac{1}{(Ci-Ai-Bi)(Ci-Ai+Bi)}\frac{1}{2Ci}\ O(C-A)}$$

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$$-\frac{1}{-2Ai}\frac{1}{(Ai+Ci-Bi)(Ai+Ci+Bi)}\frac{1}{2Ci}$$

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Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \Big[$

$$\frac{1}{2Ai} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{(Ai+y-Ci)(Ai+y+Ci)}$$

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$$+ \frac{1}{(Ci-y-Ai)(Ci-y+Ai)} \frac{1}{(y-Bi)(y+Bi)} \frac{1}{2Ci} \Big]$$

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Integrate y then: $I = (-2\pi i)^2 \Big[$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai+Bi-Ci)(Ai+Bi+Ci)}$$

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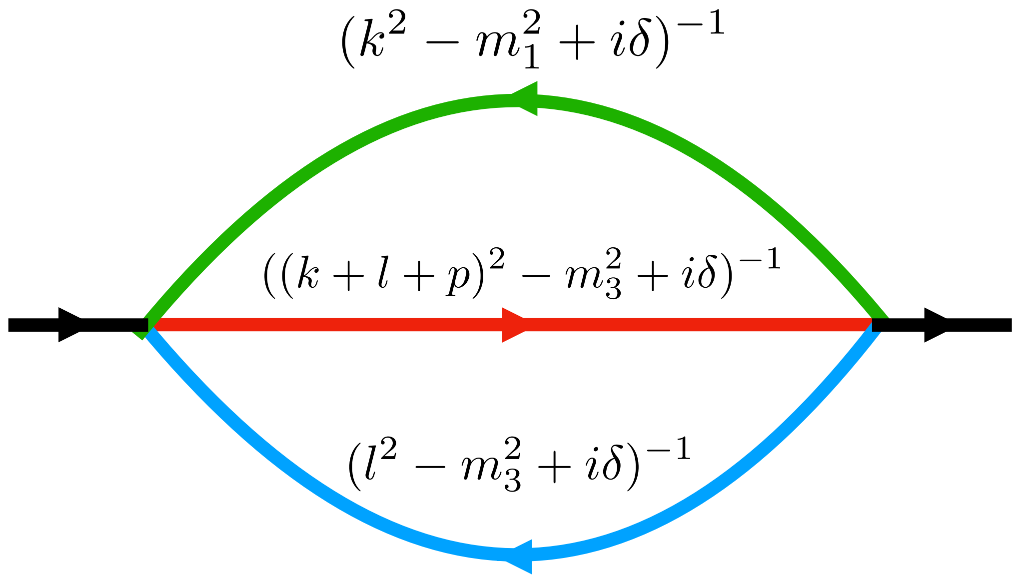
`( 0,+1,+1)`

The **signs** of the residues selected are **non-trivial** !

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Analogous sunrise case:



$$I = (2\pi i)^2 \Big[\quad \mathtt{R(+1,+1,\ 0)} + \mathtt{R(-1,\ 0,+1)} + \mathtt{R(\ 0,+1,+1)} \Big]$$

Now, what about beyond two-loop and arbitrary topologies ?

- Distributional identities: [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- Averaging procedure: [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- Iterative procedure: [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

We iteratively apply the residue theorem:

$$I = (-\mathrm{i})^n \int \prod_{j=1}^{n} \frac{\mathrm{d}^3\vec{k}_j}{(2\pi)^3} \sum_{\mathbf{b}\in\mathcal{B}} \mathrm{Res}_{\mathbf{b}}[f] \, , \qquad \mathrm{Res}_{\mathbf{b}}[f] = \frac{1}{\prod_{i\in\mathbf{b}} 2E_i} \frac{N}{\prod_{i\in\mathbf{e}\setminus\mathbf{b}} D_i} \Bigg|_{\{q_j^0=\sigma_j^{\mathbf{b}} E_j\}_{j\in\mathbf{b}}}$$

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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Key characteristics:

- A single residue per spanning-tree of the graph, with a common sign.

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MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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Key characteristics:

- ○ A single residue per spanning-tree of the graph, with a common sign.
- ○ No surviving Heaviside factors, all identically simplify to zero or one.
- ✓ Verified explicitly up to six loops.
- ✓ Also valid for complex momenta[†] and masses.

Algorithmic determination of the cut-structure $\boxed{\sigma_j^{\mathbf{b}}}$:

Given a: - loop momentum basis
- choice of complex contour closure (upper or lower)
- ordering in the integration of loop momenta energies

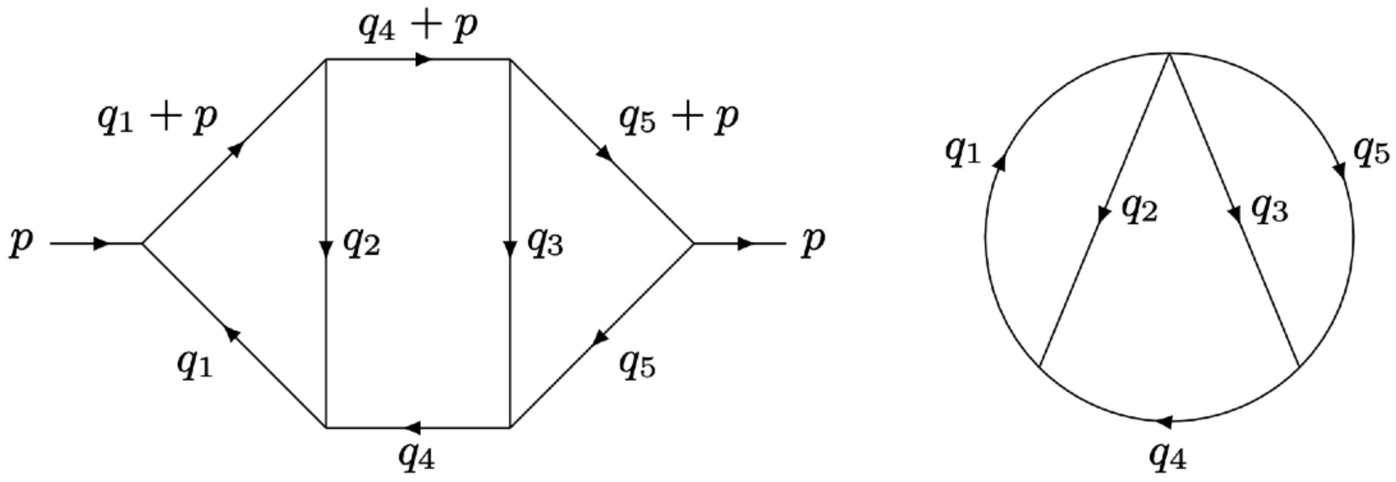
we can compute the cut structure signs for **any** multi-loop topology.

[†] Credits to S.Weinzierl

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.

(b) Reduced diagram.

$$\begin{aligned}
q_1 &= k_1 & \mathbf{s}_1 &= (\ \ 1,\ \ 0,\ \ 0) \\
q_2 &= \quad k_2 & \mathbf{s}_2 &= (\ \ 0,\ \ 1,\ \ 0) \\
q_3 &= \qquad k_3 & \mathbf{s}_3 &= (\ \ 0,\ \ 0,\ \ 1) \\
q_4 &= k_1 - k_2 & \mathbf{s}_4 &= (\ \ 1, -1,\ \ 0) \\
q_5 &= k_1 - k_2 - k_3 & \mathbf{s}_5 &= (\ \ 1, -1, -1)
\end{aligned}$$

(c) Momenta and signatures.

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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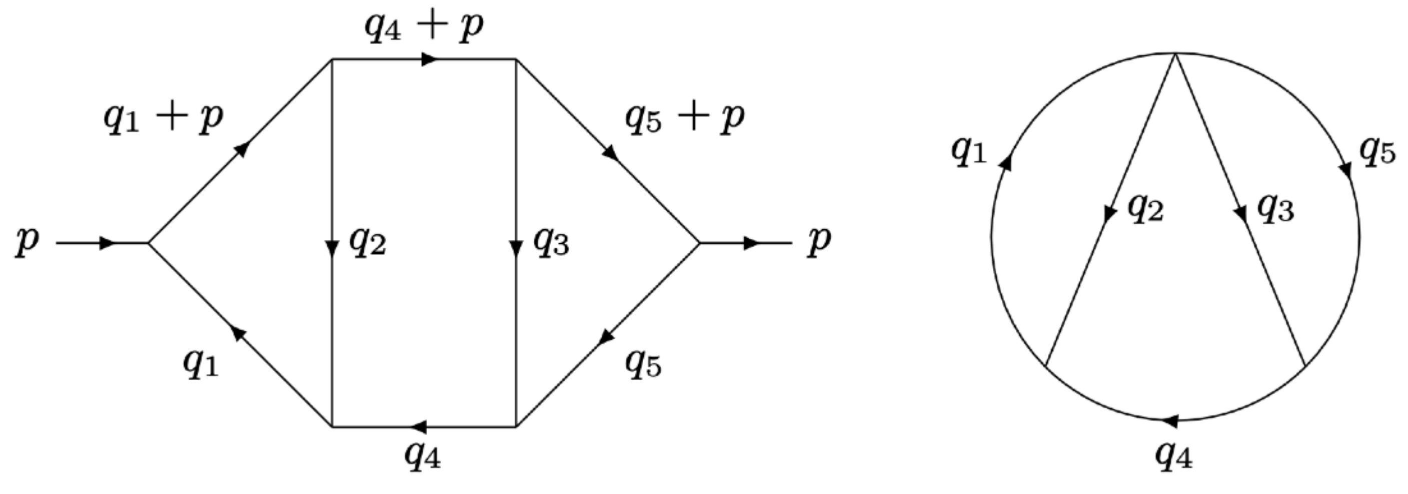

$$\begin{aligned}
q_1 &= k_1 & \mathbf{s}_1 &= (\ \ 1,\ \ \ 0,\ \ \ 0)\\
q_2 &= \quad\ \ k_2 & \mathbf{s}_2 &= (\ \ 0,\ \ \ 1,\ \ \ 0)\\
q_3 &= \qquad\quad\ k_3 & \mathbf{s}_3 &= (\ \ 0,\ \ \ 0,\ \ \ 1)\\
q_4 &= k_1 - k_2 & \mathbf{s}_4 &= (\ \ 1, -1,\ \ \ 0)\\
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\end{aligned}$$

(a) Original diagram. (b) Reduced diagram. (c) Momenta and signatures.

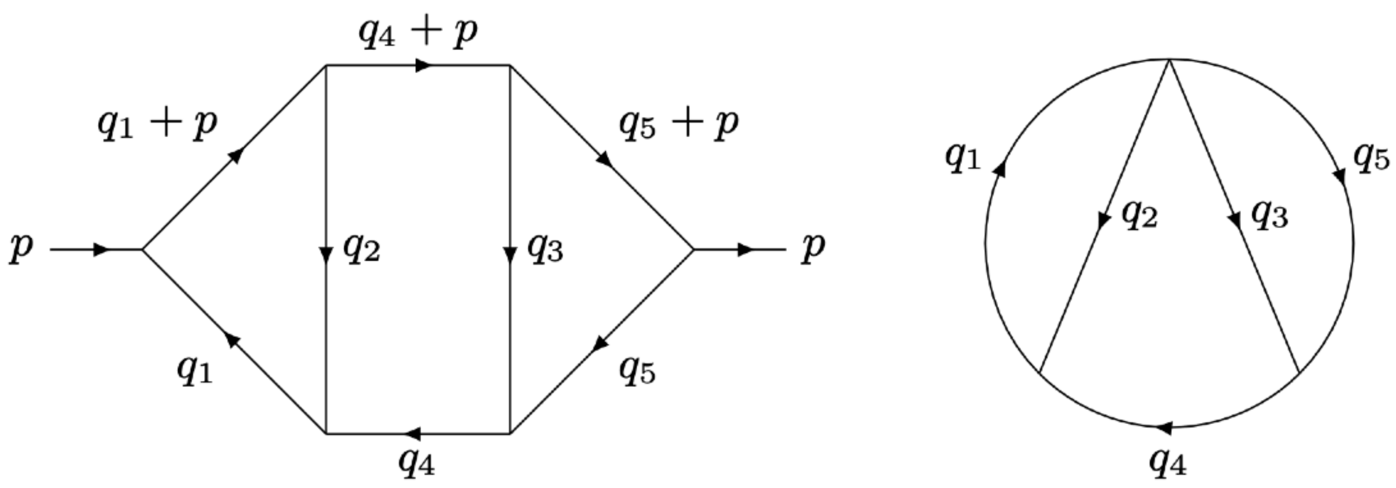
$$\sigma_j^{\mathbf{b}} \equiv \Big[$$

$$\begin{aligned}
\mathbf{\Sigma}_1 &= (\ \ 1,\ \ \ 1,\ \ \ 1,\ \ \ 0,\ \ \ 0),\\
\mathbf{\Sigma}_2 &= (\ \ 1, -1,\ \ \ 0,\ \ \ 0, -1),\\
\mathbf{\Sigma}_3 &= (\ \ 1,\ \ \ 0,\ \ \ 1, -1,\ \ \ 0),\\
\mathbf{\Sigma}_4 &= (\ \ 1,\ \ \ 0,\ \ \ 1,\ \ \ 0, -1),\\
\mathbf{\Sigma}_5 &= (\ \ 1,\ \ \ 0,\ \ \ 0,\ \ \ 1, -1),\\
\mathbf{\Sigma}_6 &= (\ \ 0,\ \ \ 1,\ \ \ 1,\ \ \ 1,\ \ \ 0),\\
\mathbf{\Sigma}_7 &= (\ \ 0,\ \ \ 1,\ \ \ 1,\ \ \ 0,\ \ \ 1),\\
\mathbf{\Sigma}_8 &= (\ \ 0,\ \ \ 1,\ \ \ 0,\ \ \ 1, -1)\Big]
\end{aligned}$$

Multi-Loop Tree Duality

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.

(b) Reduced diagram.

$$\begin{array}{llll}
q_1 = k_1 & & & \mathbf{s}_1 = (\ 1,\ \ 0,\ \ 0) \\
q_2 = & k_2 & & \mathbf{s}_2 = (\ 0,\ \ 1,\ \ 0) \\
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q_4 = k_1 - k_2 & & & \mathbf{s}_4 = (\ 1, -1,\ \ 0) \\
q_5 = k_1 - k_2 - k_3 & & & \mathbf{s}_5 = (\ 1, -1, -1)
\end{array}$$

(c) Momenta and signatures.

$$\sigma_j^{\mathbf{b}} \equiv \left[\begin{array}{l}
\boldsymbol{\Sigma}_1 = (\ \underline{1},\ \ \underline{1},\ \ \underline{1},\ \ 0,\ \ 0), \\
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\boldsymbol{\Sigma}_3 = (\ 1,\ \ 0,\ \ 1, -1,\ \ 0), \\
\boldsymbol{\Sigma}_4 = (\ 1,\ \ 0,\ \ 1,\ \ 0, -1), \\
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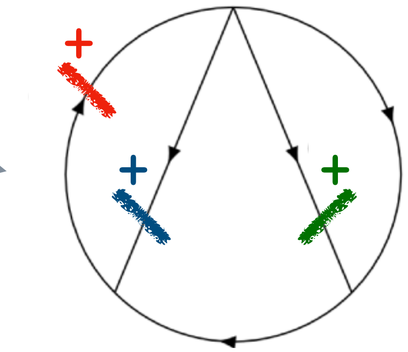
\boldsymbol{\Sigma}_7 = (\ 0,\ \ 1,\ \ 1,\ \ 0,\ \ 1), \\
\boldsymbol{\Sigma}_8 = (\ 0,\ \ 1,\ \ 0,\ \ 1, -1)
\end{array} \right]$$



Multi-Loop Tree Duality

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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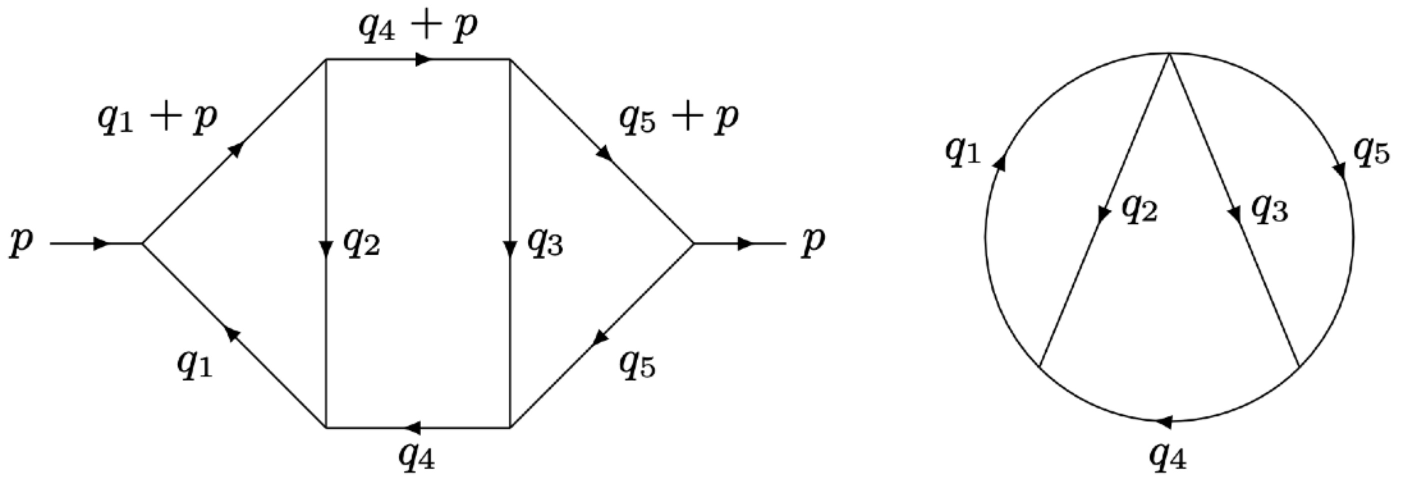

$$\begin{array}{ll}
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q_3 = \qquad\qquad k_3 & \mathbf{s}_3 = (\;0,\;\;\;0,\;\;\;1) \\
q_4 = k_1 - k_2 & \mathbf{s}_4 = (\;1,-1,\;\;\;0) \\
q_5 = k_1 - k_2 - k_3 & \mathbf{s}_5 = (\;1,-1,-1)
\end{array}$$

(a) Original diagram. (b) Reduced diagram. (c) Momenta and signatures.

$$\sigma_j^{\mathbf{b}} \equiv \Bigg[\begin{array}{l}
\boldsymbol{\Sigma}_1 = (\;\underline{1},\;\;\;\underline{1},\;\;\;\underline{1},\;\;\;0,\;\;\;0), \\
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\boldsymbol{\Sigma}_3 = (\;1,\;\;\;0,\;\;\;1,-1,\;\;\;0), \\
\boldsymbol{\Sigma}_4 = (\;1,\;\;\;0,\;\;\;1,\;\;\;0,-1), \\
\boldsymbol{\Sigma}_5 = (\;1,\;\;\;0,\;\;\;0,\;\;\;1,-1), \\
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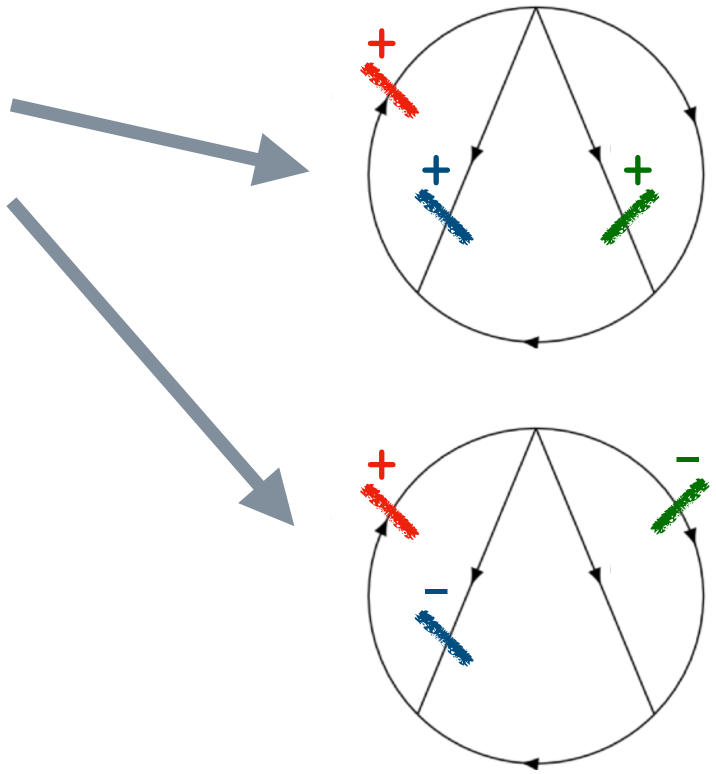
\end{array} \Bigg]$$



MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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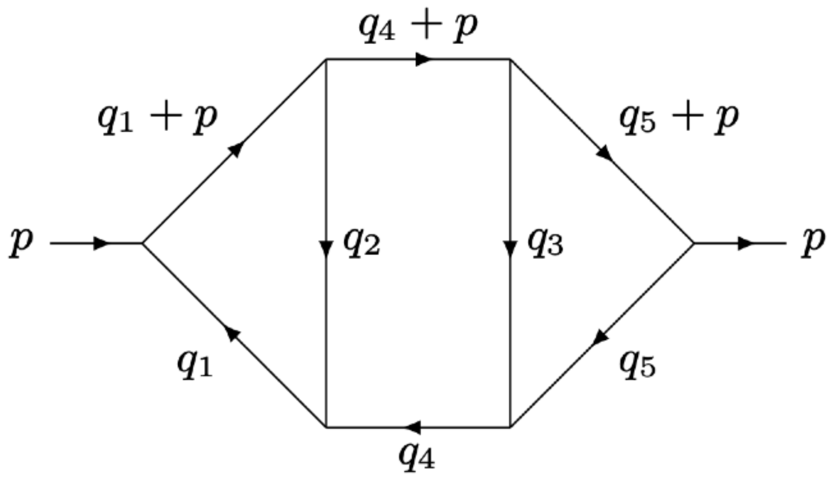
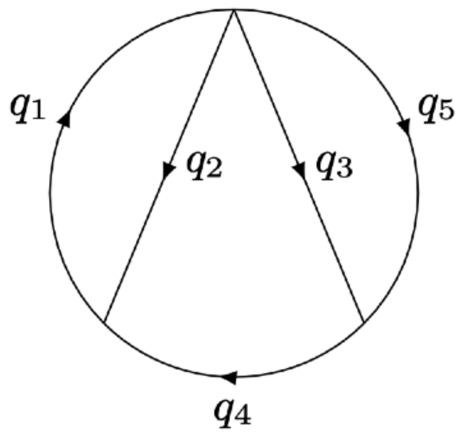


(a) Original diagram.



(b) Reduced diagram.

$$
\begin{array}{ll}
q_1 = k_1 & \mathbf{s}_1 = (\ 1,\ 0,\ 0) \\
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$$

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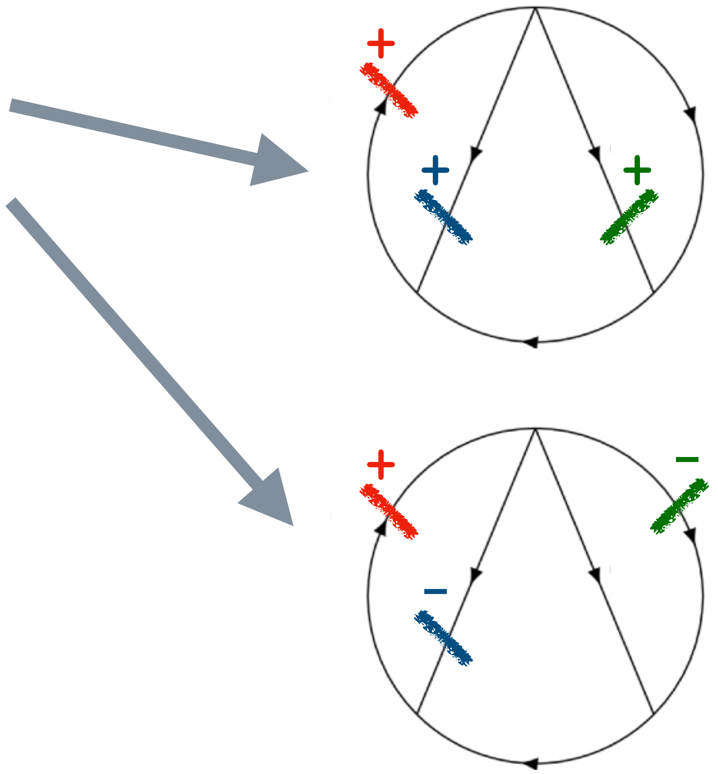
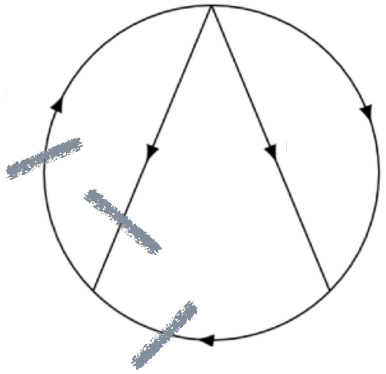
\mathbf{\Sigma}_2 = (\ \underline{1}, \underline{-1},\ 0,\ 0, \underline{-1}), \\
\mathbf{\Sigma}_3 = (\ 1,\ 0,\ 1, -1,\ 0), \\
\mathbf{\Sigma}_4 = (\ 1,\ 0,\ 1,\ 0, -1), \\
\mathbf{\Sigma}_5 = (\ 1,\ 0,\ 0,\ 1, -1), \\
\mathbf{\Sigma}_6 = (\ 0,\ 1,\ 1,\ 1,\ 0), \\
\mathbf{\Sigma}_7 = (\ 0,\ 1,\ 1,\ 0,\ 1), \\
\mathbf{\Sigma}_8 = (\ 0,\ 1,\ 0,\ 1, -1)
\end{array}
\Big]
$$

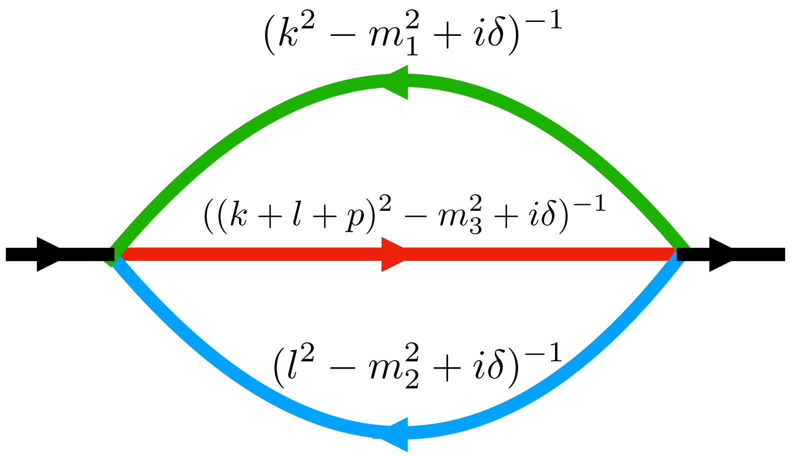


Ex. of an invalid spanning tree:



MULTI-LOOP TREE DUALITY: SINGULARITIES

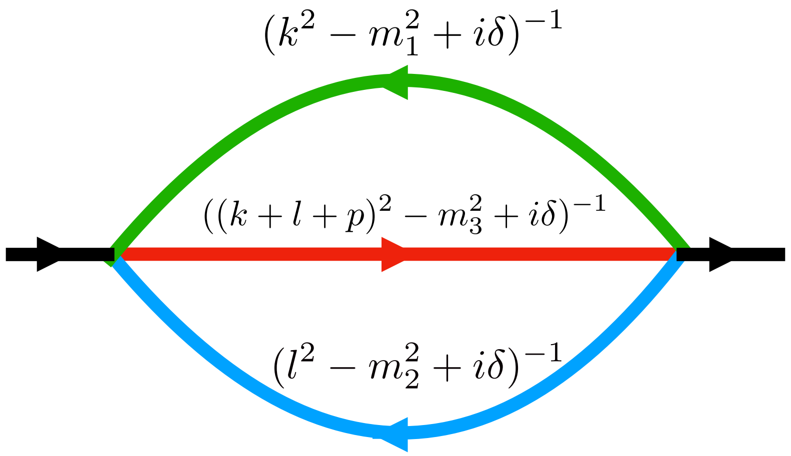
The sunrise again, with more details:



$$\begin{aligned} I = (2\pi i)^2 \Big[& \quad \mathtt{R(+1,+1,\ 0)} \\ & + \mathtt{R(-1,\ 0,+1)} \\ & + \mathtt{R(\ 0,+1,+1)} \quad \Big] \end{aligned}$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:

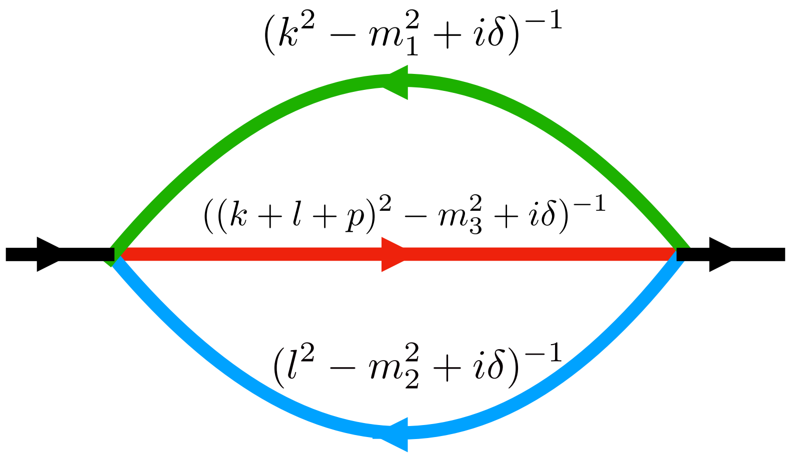


$$I = (2\pi i)^2 \Big[\quad \mathtt{R(+1,+1,\ 0)} \\ + \ \mathtt{R(-1,\ 0,+1)} \\ + \ \mathtt{R(\ 0,+1,+1)} \quad \Big]$$

$$\mathtt{R(+1,+1,\ 0)} \ \underset{l^0 = \sqrt{\vec{l}^2 + m_1^2 - i\delta}}{\overset{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}}{=}} (-i)^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



$$I = (2\pi i)^2 \Big[\; \mathtt{R(+1,+1,\ 0)} + \mathtt{R(-1,\ 0,+1)} + \mathtt{R(\ 0,+1,+1)} \; \Big]$$

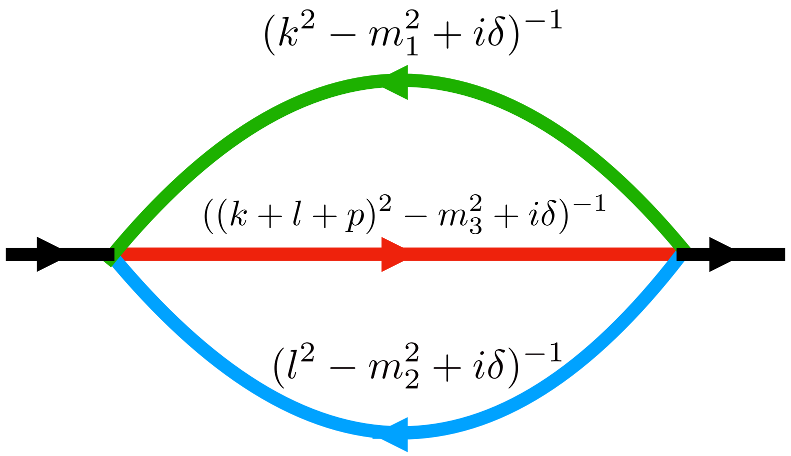
$$\mathtt{R(+1,+1,\ 0)} \underset{l^0 = \sqrt{\vec{l}^2 + m_1^2 - i\delta}}{\overset{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}}{=}} (-i)^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

$$\Delta_3^{\pm}(\vec{k}, \vec{l}) = +k^0 + l^0 + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

$$= +\sqrt{\vec{k}^2 + m_2^2 - i\delta} + \sqrt{\vec{l}^2 + m_2^2 - i\delta} + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



$$I = (2\pi i)^2 \Big[\quad \mathtt{R(+1,+1,\ 0)} + \mathtt{R(-1,\ 0,+1)} + \mathtt{R(\ 0,+1,+1)} \quad \Big]$$

$$\mathtt{R(+1,+1,\ 0)} \overset{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}}{\underset{l^0 = \sqrt{\vec{l}^2 + m_1^2 - i\delta}}{=}} (-i)^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

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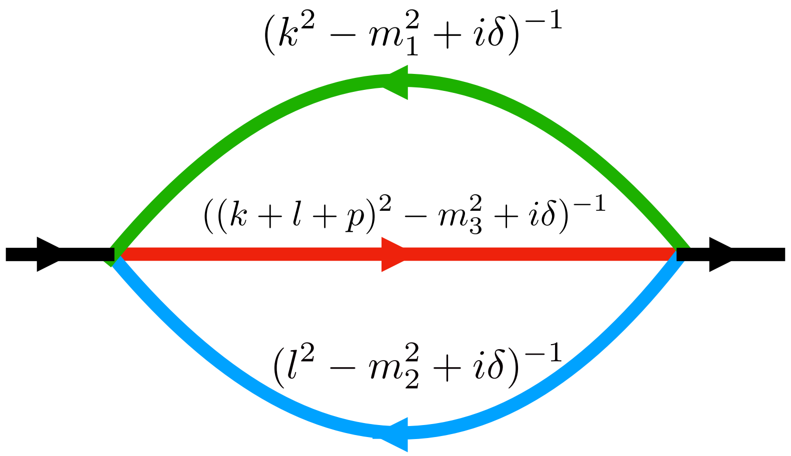
identical signs : bounded ellipsoid-like surface requiring deformation: E-surfaces

otherwise : unbounded hyperboloid-like surface dual-cancelling: H-surfaces

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



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$$\Delta_3^\pm(\vec{k},\vec{l}) = +k^0 + l^0 + p^0 \pm \sqrt{(\vec{k}+\vec{l}+\vec{p})^2+m_3^2-i\delta}$$

$$= +\sqrt{\vec{k}^2+m_2^2-i\delta} + \sqrt{\vec{l}^2+m_2^2-i\delta} + p^0 \pm \sqrt{(\vec{k}+\vec{l}+\vec{p})^2+m_3^2-i\delta}$$

identical signs : bounded ellipsoid-like surface requiring deformation: E-surfaces

otherwise : unbounded hyperboloid-like surface dual-cancelling: H-surfaces

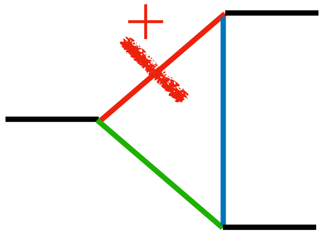
For Euclidean kinematics ($p_i^2 < 0$) : (on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2+m_i^2-i\delta}$)

E-surface has no solution → no non-integrable singularity → no deformation needed

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**



$$\frac{1}{2E_1} \qquad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \qquad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: CLTD

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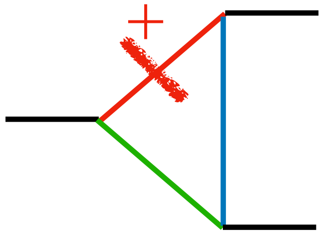

$$\frac{1}{2E_1} \boxed{\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2}} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$\boxed{= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)}$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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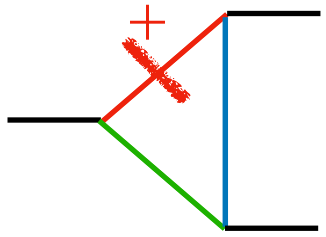
$$= \frac{1}{2E_3}\left(\frac{1}{E_1-p_1^0+p_3^0-E_3}-\frac{1}{E_1-p_1^0+p_3^0+E_3}\right)$$



$$\frac{1}{2E_1} \quad \frac{1}{(E_1-p_1^0+p_2^0)^2-E_2^2} \quad \frac{1}{(E_1-p_1^0+p_3^0)^2-E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1-p_1^0+p_2^0)-E_2}-\frac{1}{(E_1-p_1^0+p_2^0)+E_2}\right)$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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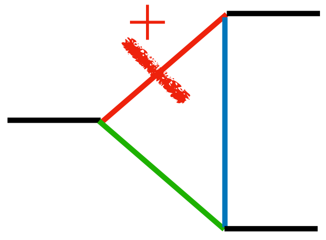


$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

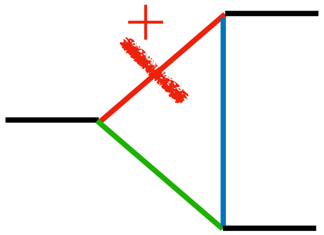
$$\supset \quad \frac{1}{2E_1}$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**

$$= \frac{1}{2E_3}\left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3}\right)$$



$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

$$\supset \quad \frac{1}{2E_1}\frac{1}{2E_2} \quad \frac{-1}{E_1 - p_1^0 + p_2^0 + E_2}$$

MULTI-LOOP TREE DUALITY: cLTD

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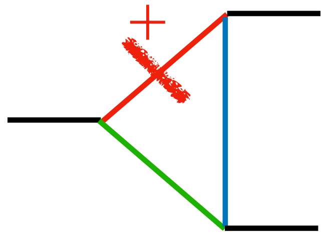


$$= \frac{1}{2E_3}\left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3}\right)$$

$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

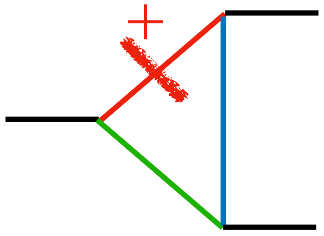
$$\supset \quad \frac{1}{2E_1}\frac{1}{2E_2}\frac{1}{2E_3}\frac{-1}{E_1 - p_1^0 + p_2^0 + E_2}\frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}$$

Multi-Loop Tree Duality: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**

$$= \frac{1}{2E_3}\left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3}\right)$$



$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

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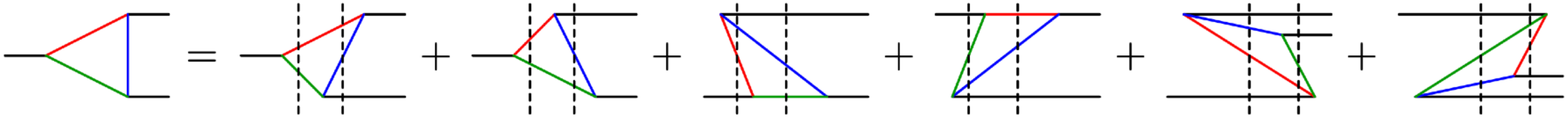
TOPT :

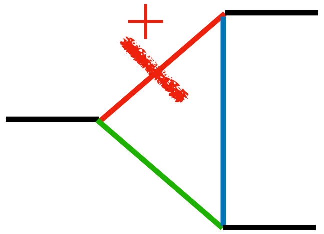


MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**

$$= \frac{1}{2E_3}\left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3}\right)$$

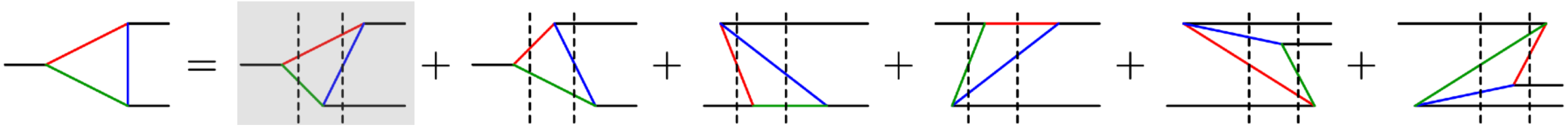


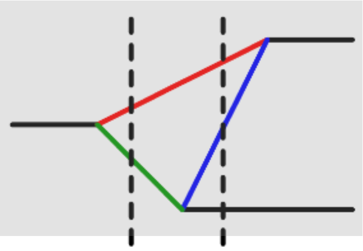
$$\frac{1}{2E_1} \qquad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \qquad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

$$\supset \quad \frac{1}{2E_1}\,\frac{1}{2E_2}\,\frac{1}{2E_3}\,\frac{-1}{E_1 - p_1^0 + p_2^0 + E_2}\,\frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}$$

TOPT :





$$= \frac{1}{2E_1}\,\frac{1}{2E_2}\,\frac{1}{2E_3}\,\frac{-1}{-q_1^0 + q_3^0 + (E_1 + E_2)}\,\frac{-1}{-q_1^0 + (E_1 + E_3)}$$

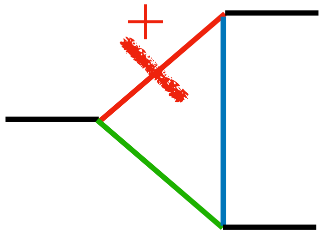
Identical !

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**

$$= \frac{1}{2E_3}\left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3}\right)$$

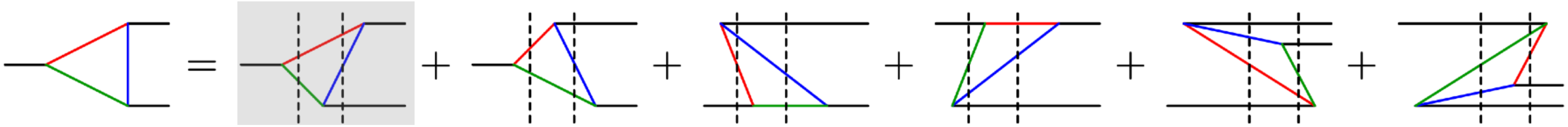


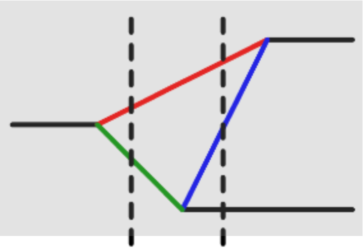
$$\frac{1}{2E_1} \quad \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \quad \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2}\left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2}\right)$$

$$\supset \quad \frac{1}{2E_1}\,\frac{1}{2E_2}\,\frac{1}{2E_3}\,\frac{-1}{E_1 - p_1^0 + p_2^0 + E_2}\,\frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}$$

TOPT :





$$= \frac{1}{2E_1}\,\frac{1}{2E_2}\,\frac{1}{2E_3}\,\frac{-1}{-q_1^0 + q_3^0 + (E_1 + E_2)}\,\frac{-1}{-q_1^0 + (E_1 + E_3)}$$

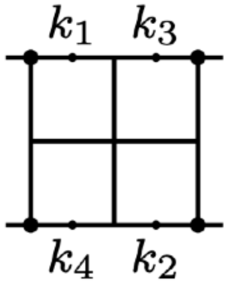
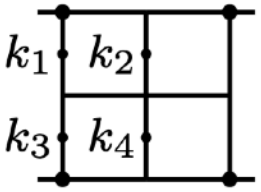
Identical !

In general however, **no one-to-one** correspondence. **cLTD** has fewer terms.

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

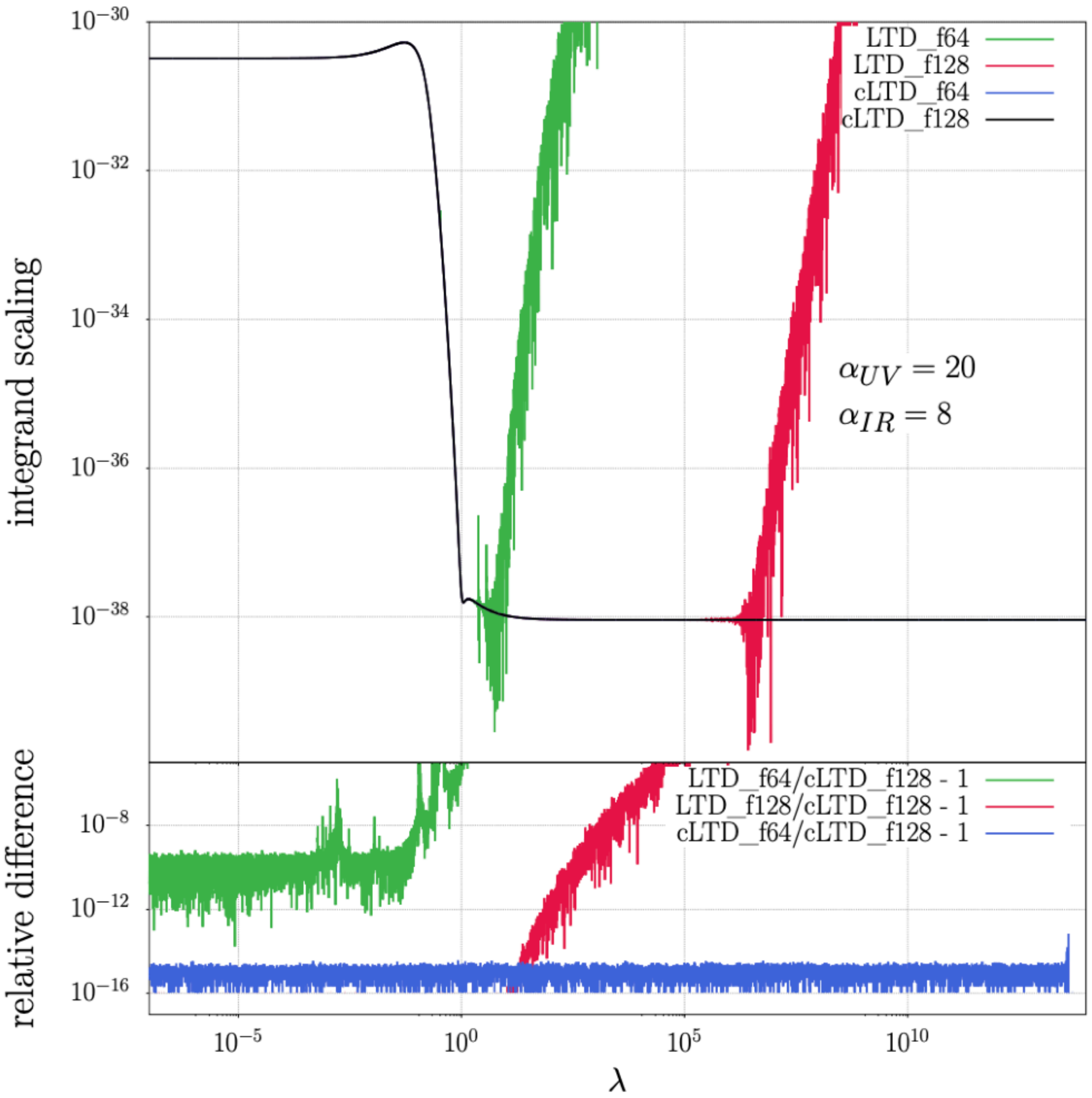
We give a completely general and **algorithmic** construction of **cLTD**:

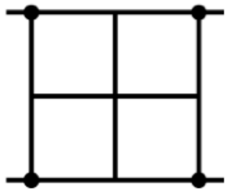
Loop momentum basis	Input		$n_{\mathrm{opt}}=0$		$n_{\mathrm{opt}}=1000$			
	signatures σ	n_{prop}	N_+	$N_\times$	$t^{(\mathrm{gen})}_{[\mathrm{min}]}$	N_+	$N_\times$	$t^{(\mathrm{run})}_{[\mu\mathrm{s}]}$
	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$	22’851	159’964	10.2	3’515	3’520	7.5
	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$	85’535	598’752	154.7	21’351	20’944	87

MULTI-LOOP TREE DUALITY: cLTD

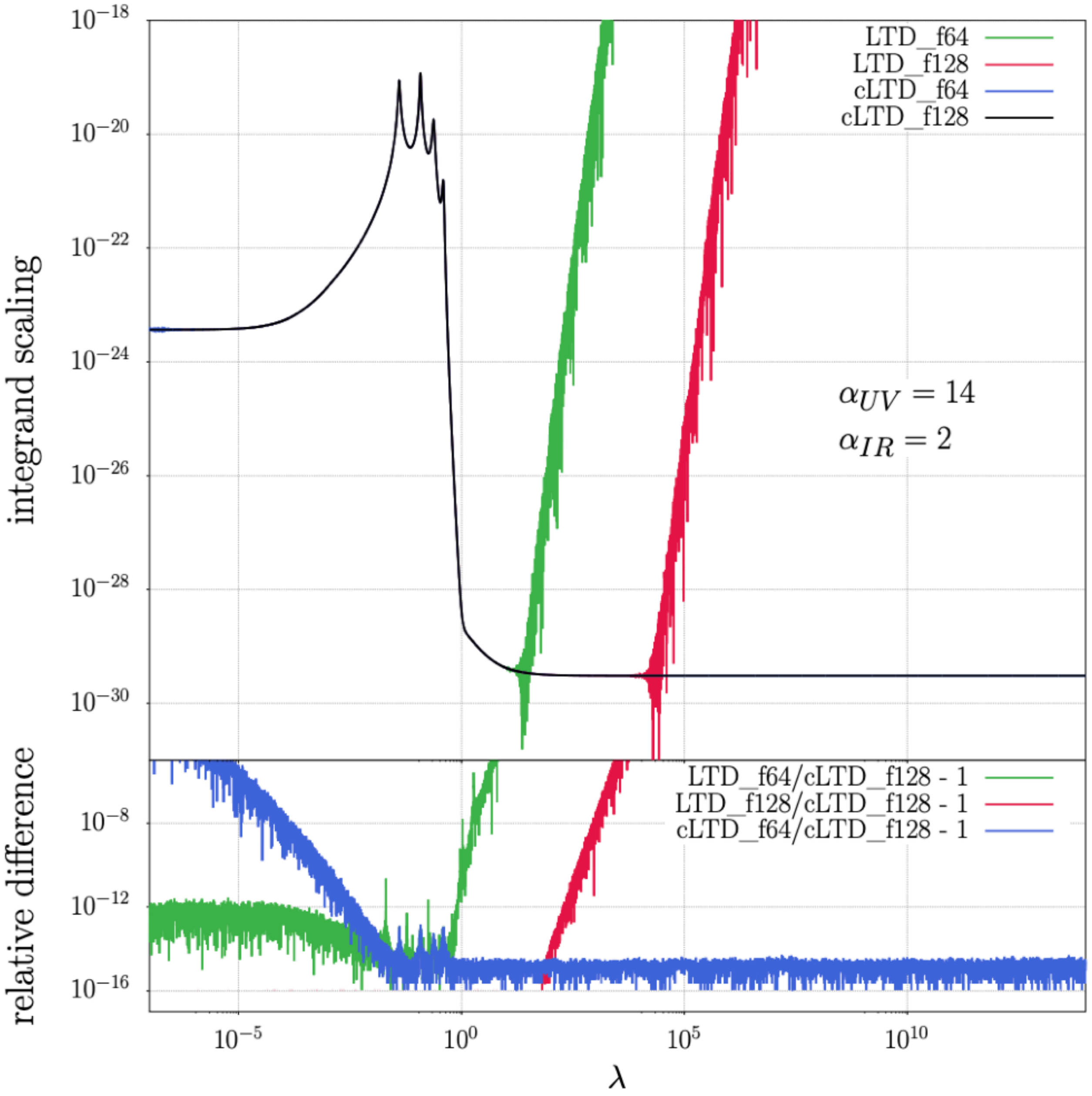
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

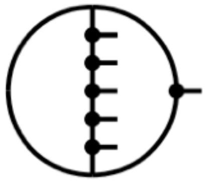
cLTD completely cures **numerical stability** issue in the **UV region**:



(a) 

Stability of the 2x2 fishnet 4-loop integral with a constant numerator set to 1.

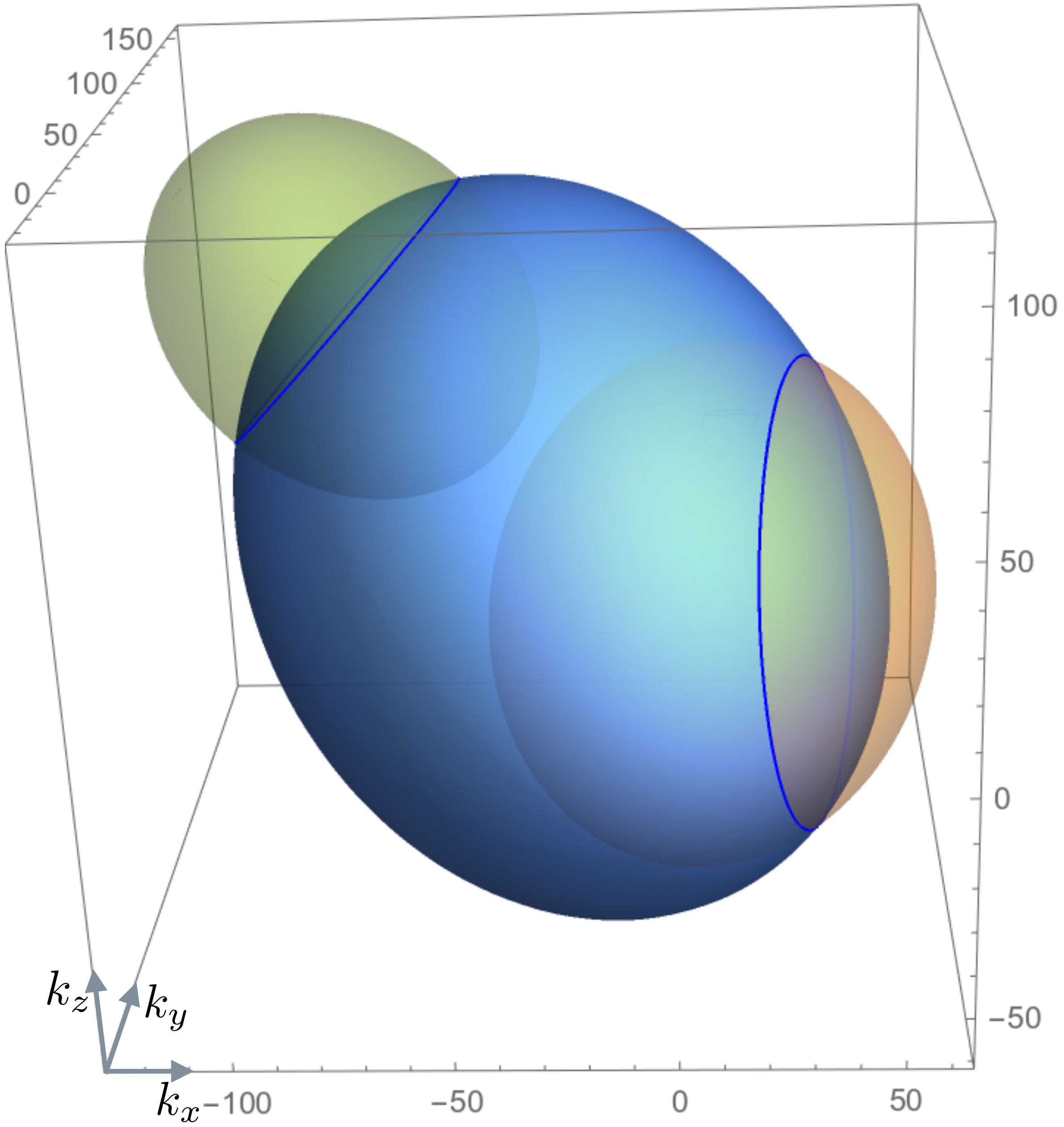


(b) 

Stability of a 2-loop 6-point integral with the rank-2 numerator $(k_1 + k_2) \cdot p_1 + k_1 \cdot k_2$.

THRESHOLD REGULARISATION

SINGULAR SURFACES - 2D ELLIPSOIDS



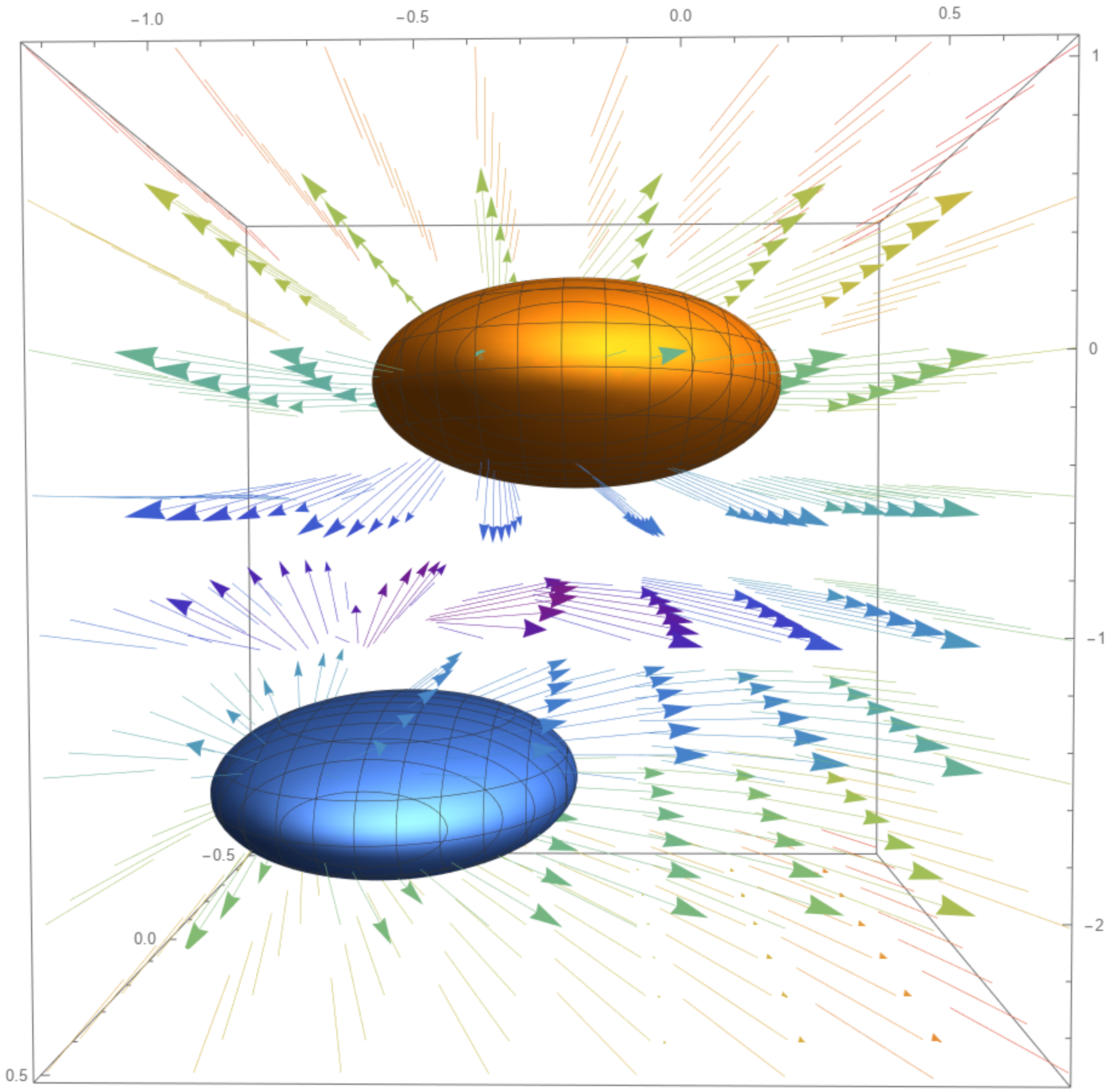
- General **one-loop** ellipsoid **expression**:

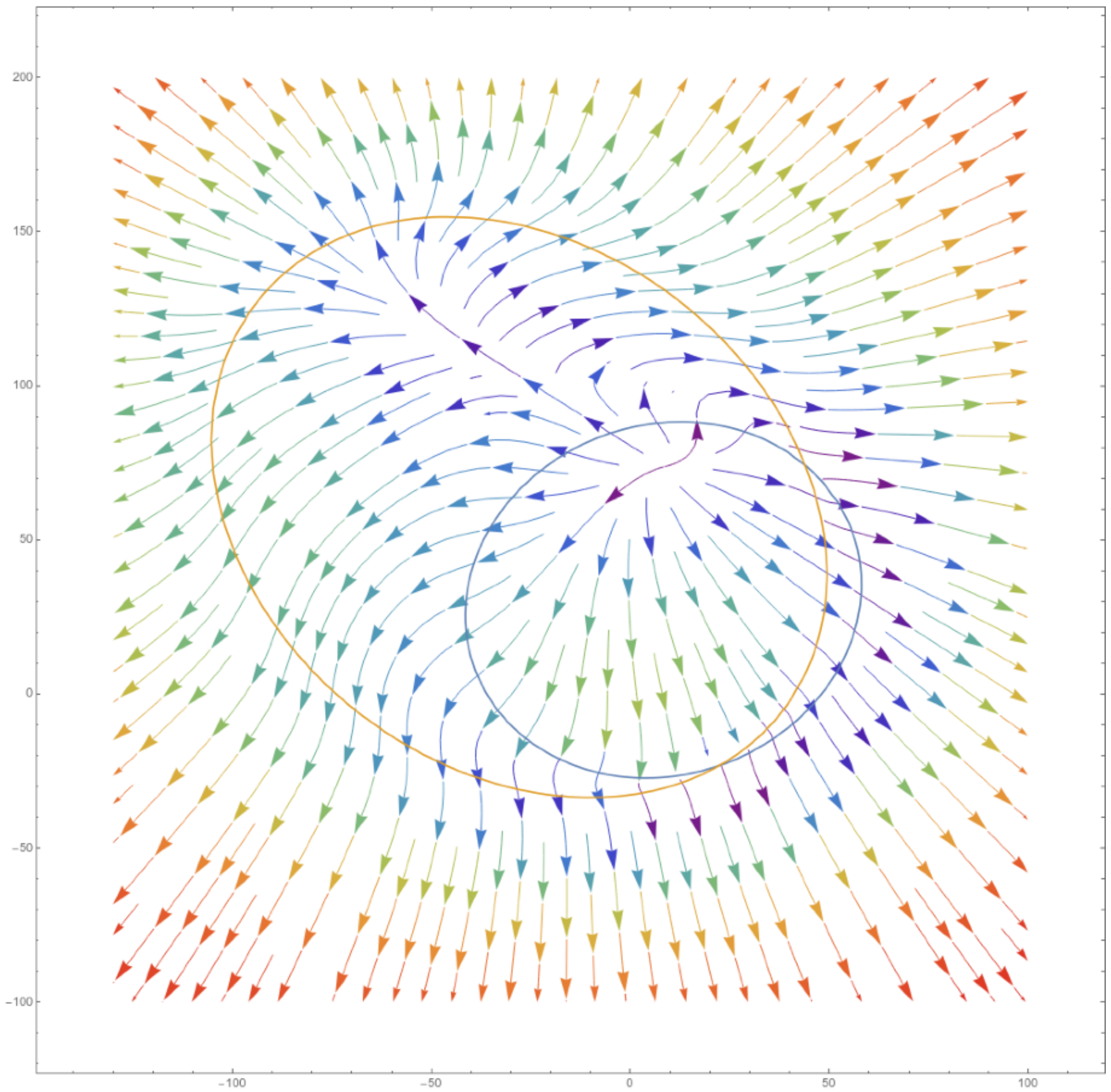
$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k}+\vec{p_i}\right)^2 + m_i^2 - i\delta} \\ + \sqrt{\left(\vec{k}+\vec{p_j}\right)^2 + m_j^2 - i\delta - p_i^0 + p_j^0}$$

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

Deformation: $\vec{k} \rightarrow \vec{k} - i\vec{\kappa}$

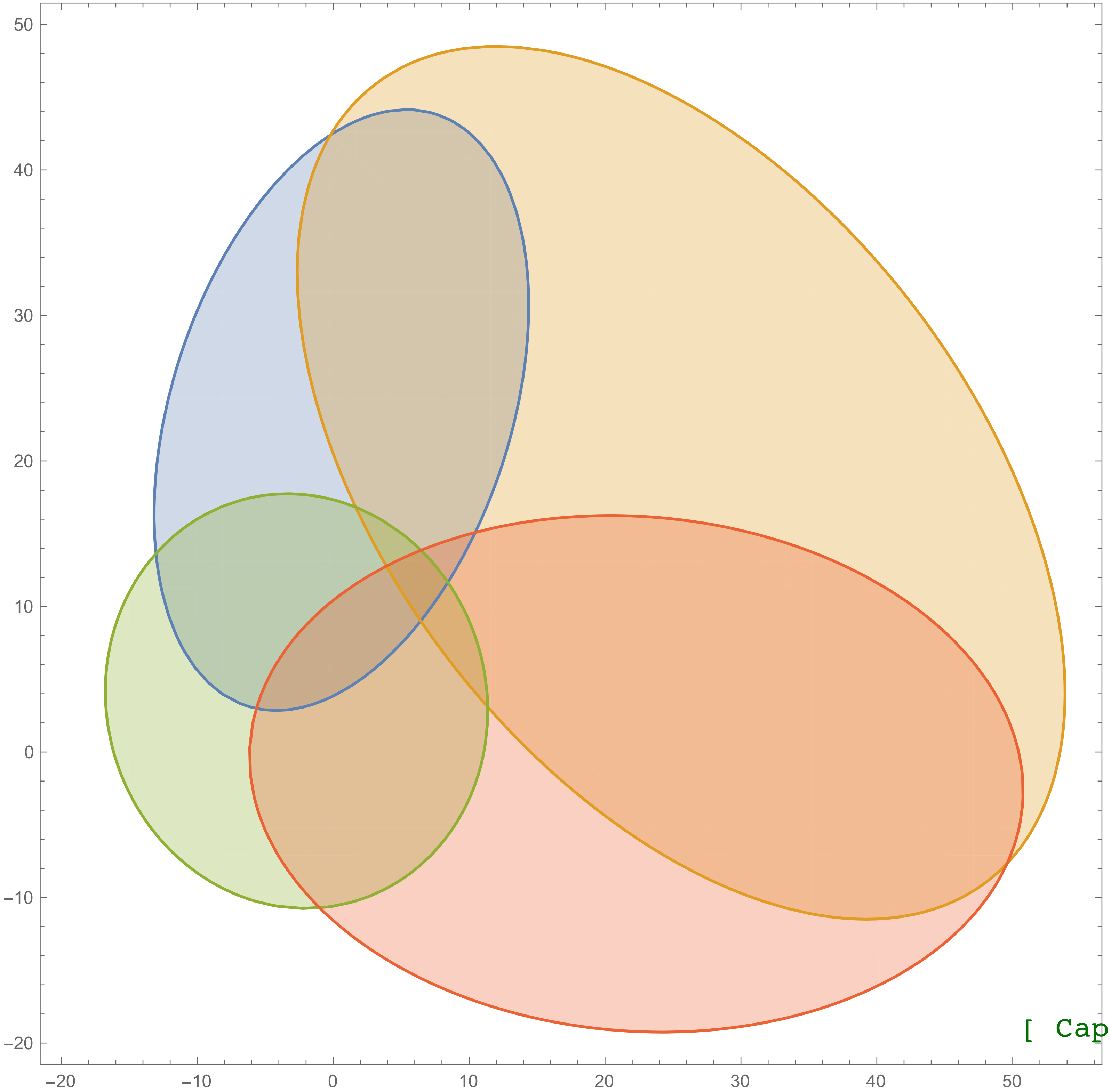
Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$





DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

How to construct such a field? For example for this case:



[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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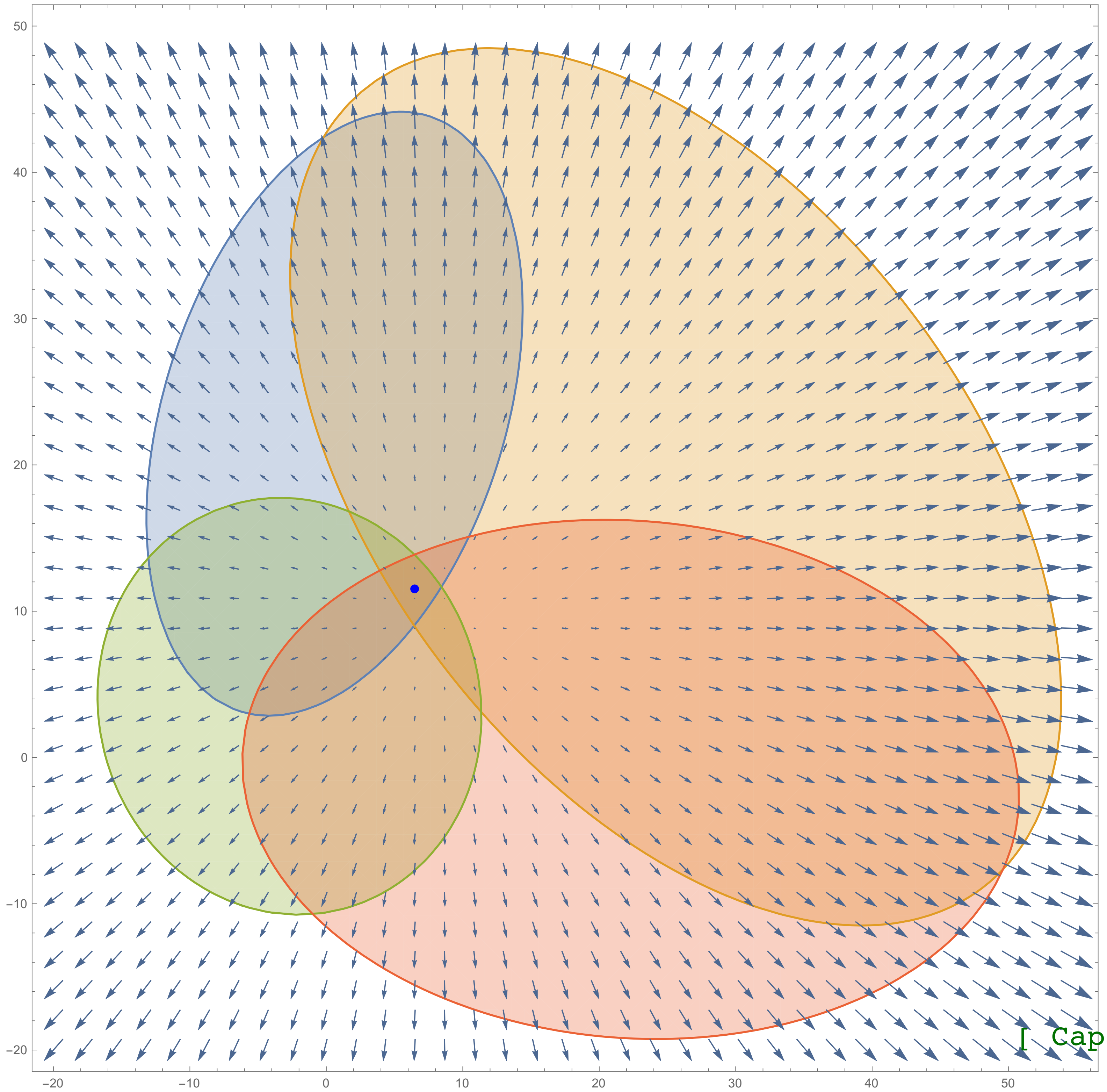
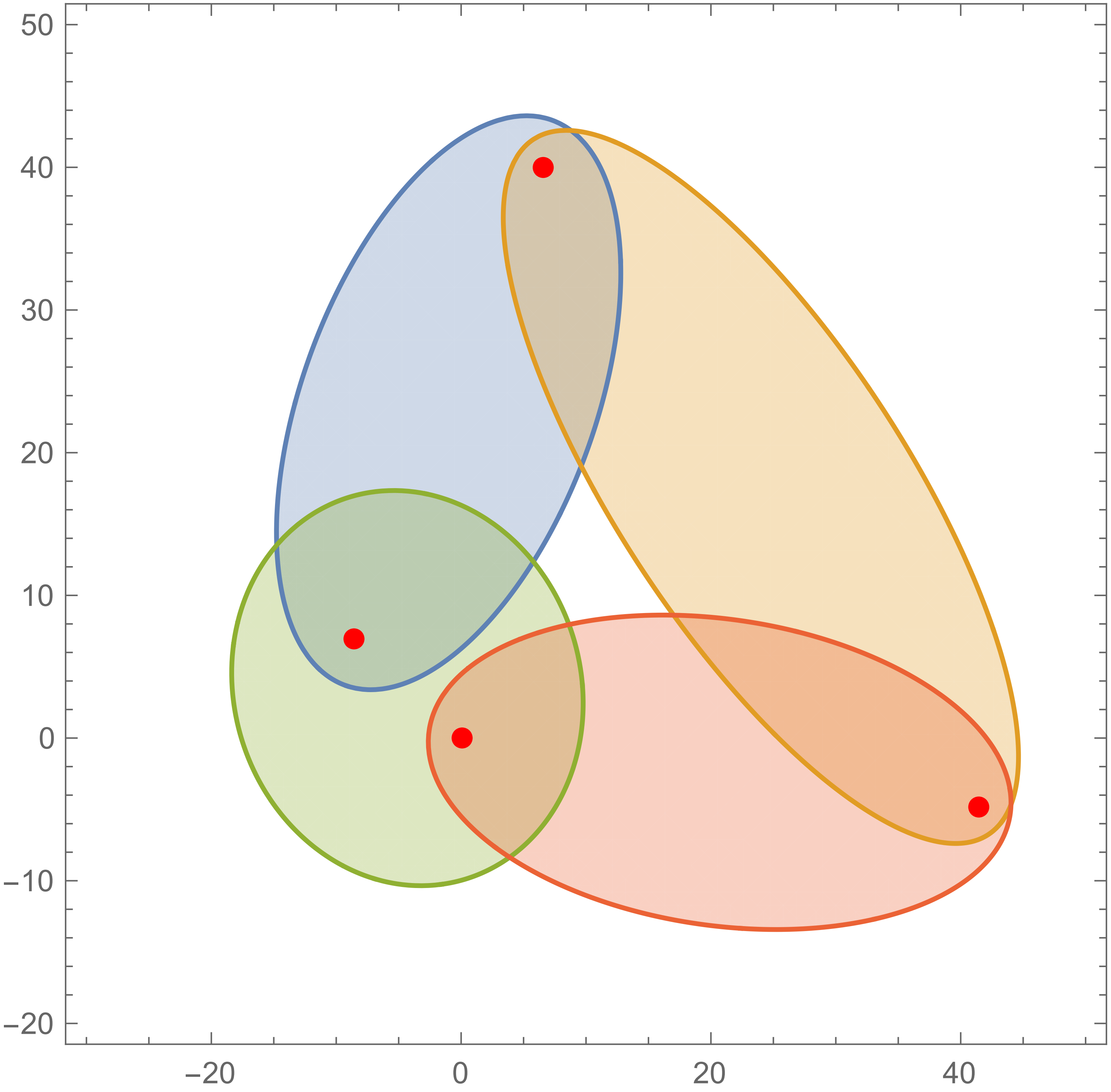


A **radial field** centered in the inside of all ellipsoids!

`[ Capatti, VH, Kermanschah, Pelloni, Ru`
`[ arxiv:1906.06138 ]`

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



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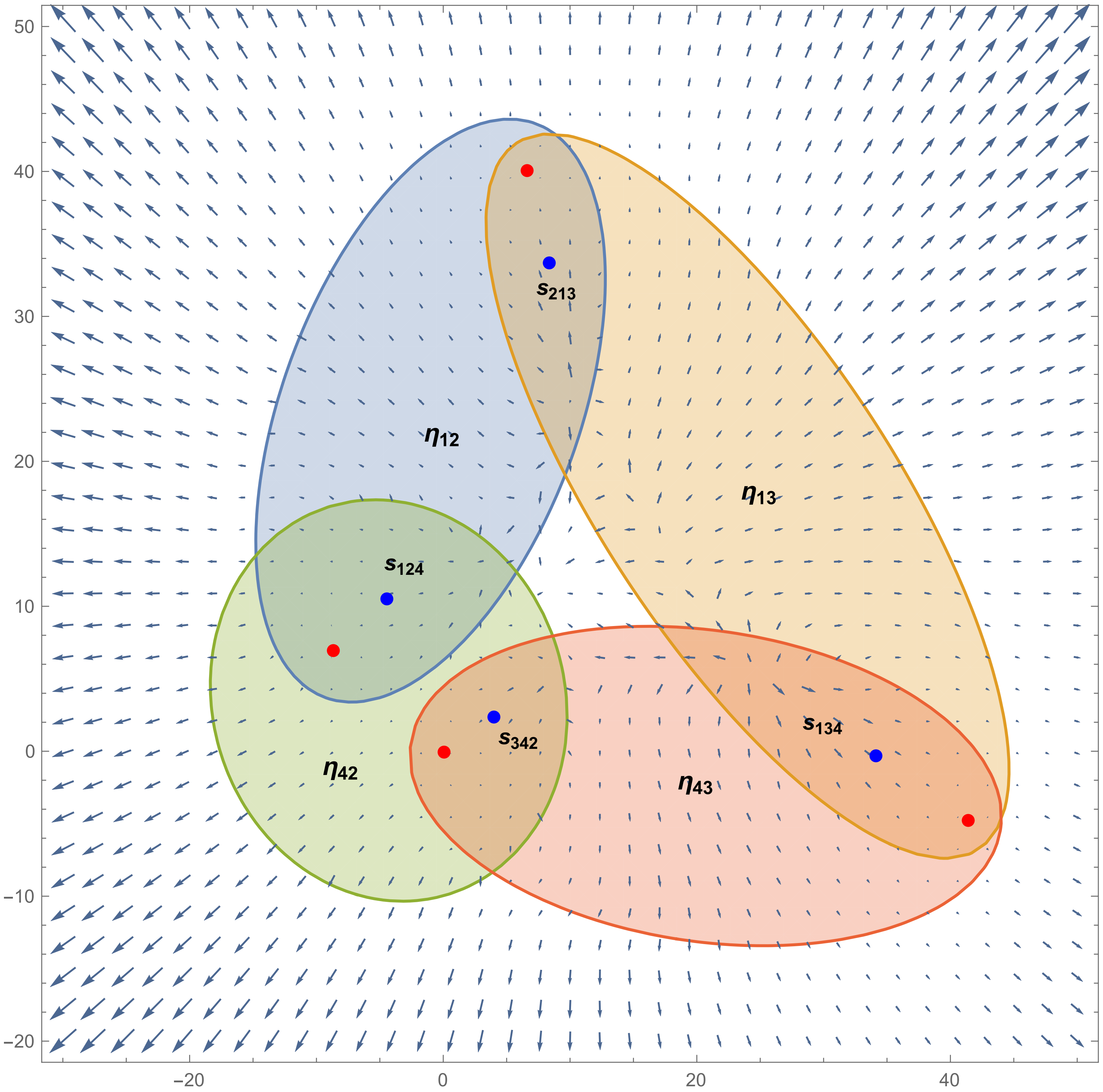

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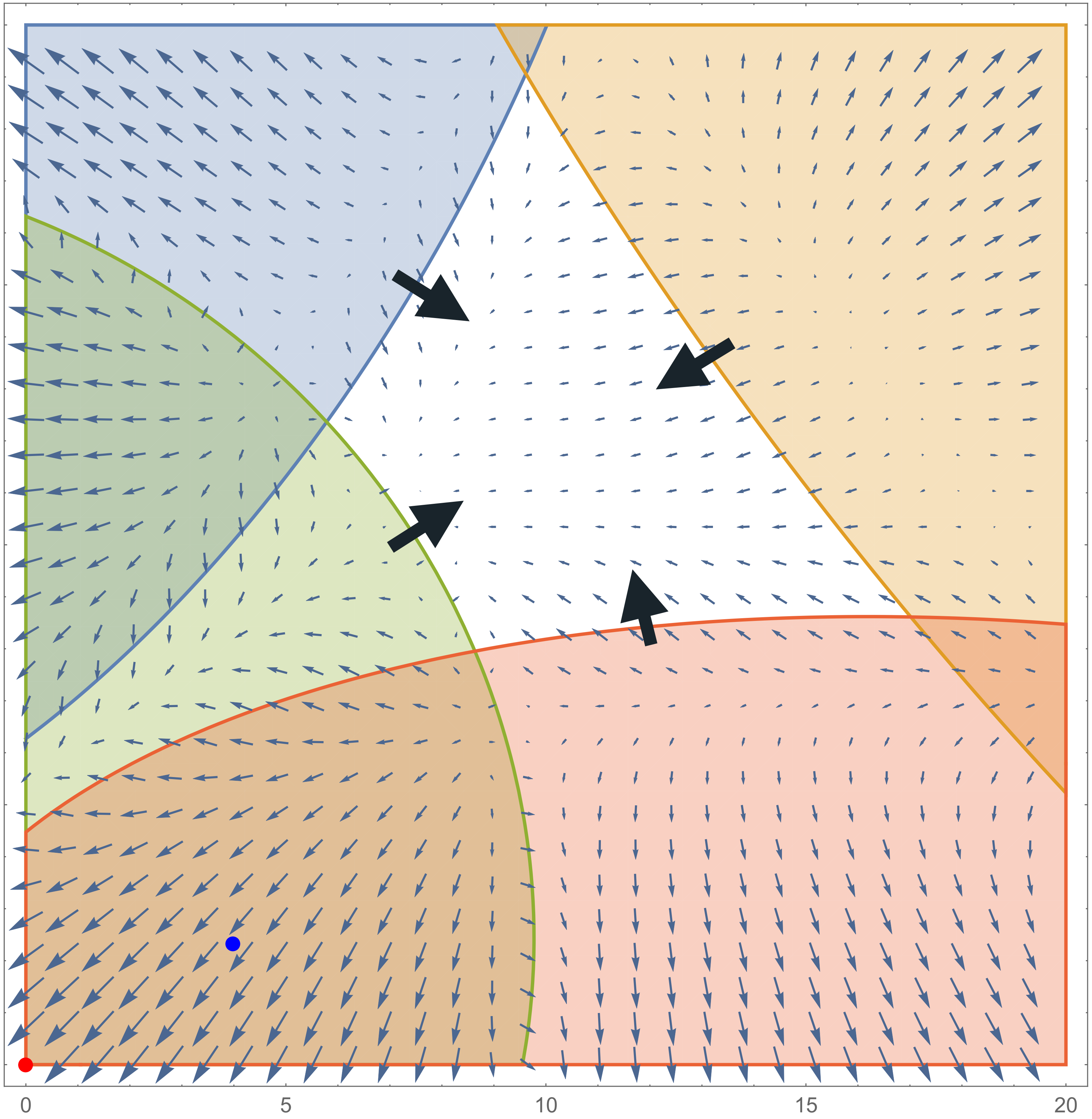

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0} \overset{p_1^\mu = (2,\vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

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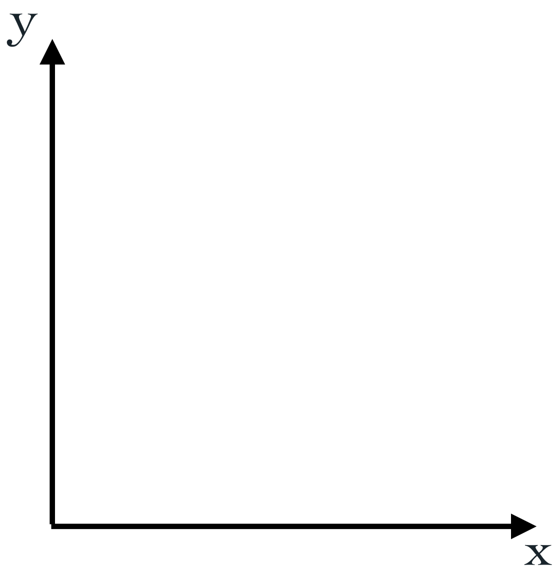

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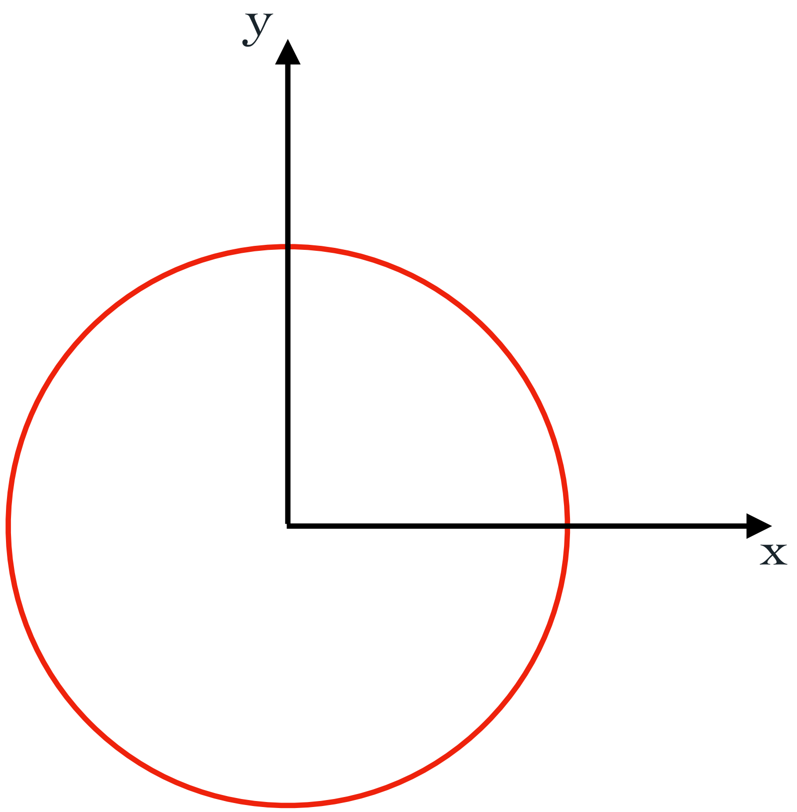

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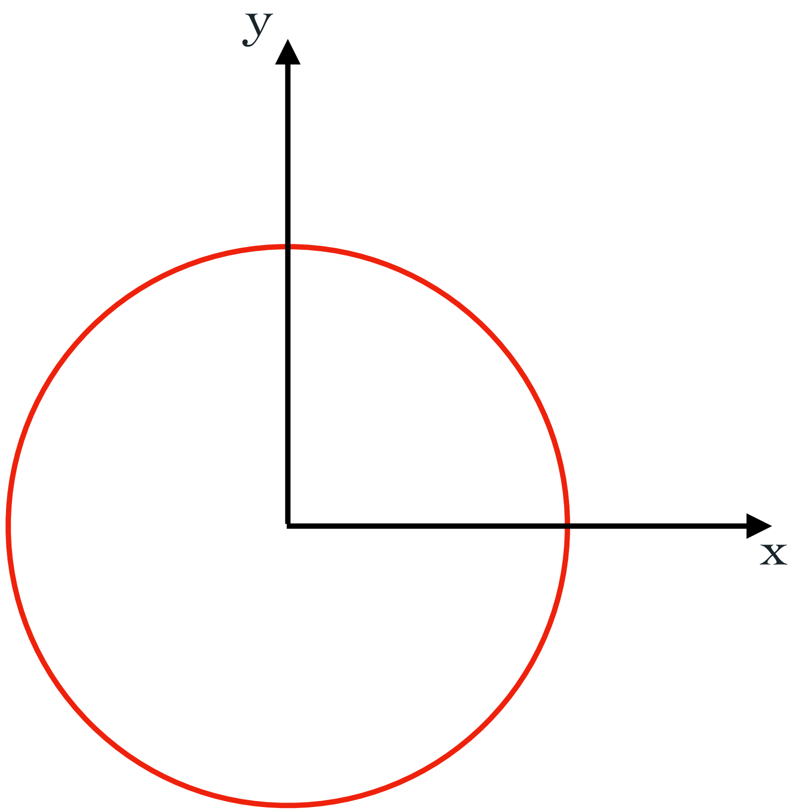

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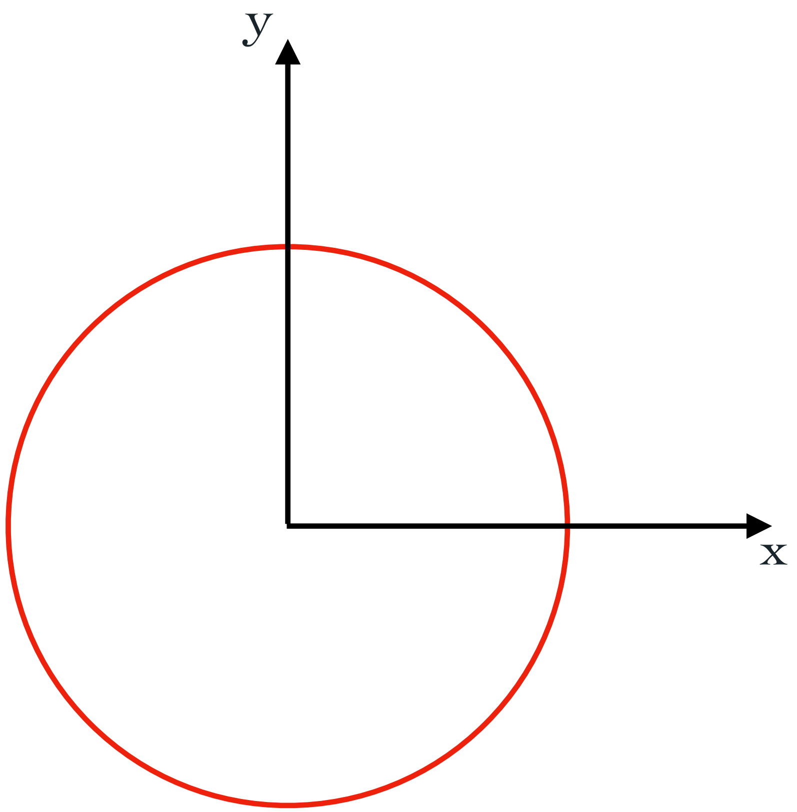
Local threshold counterterm

$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right]$$



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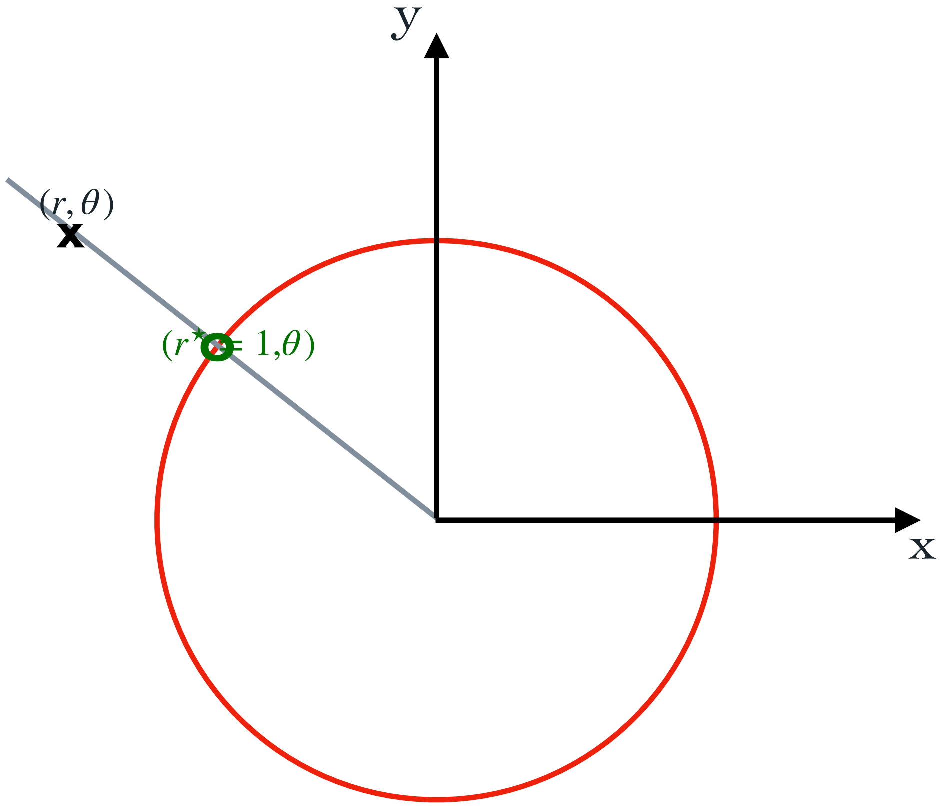

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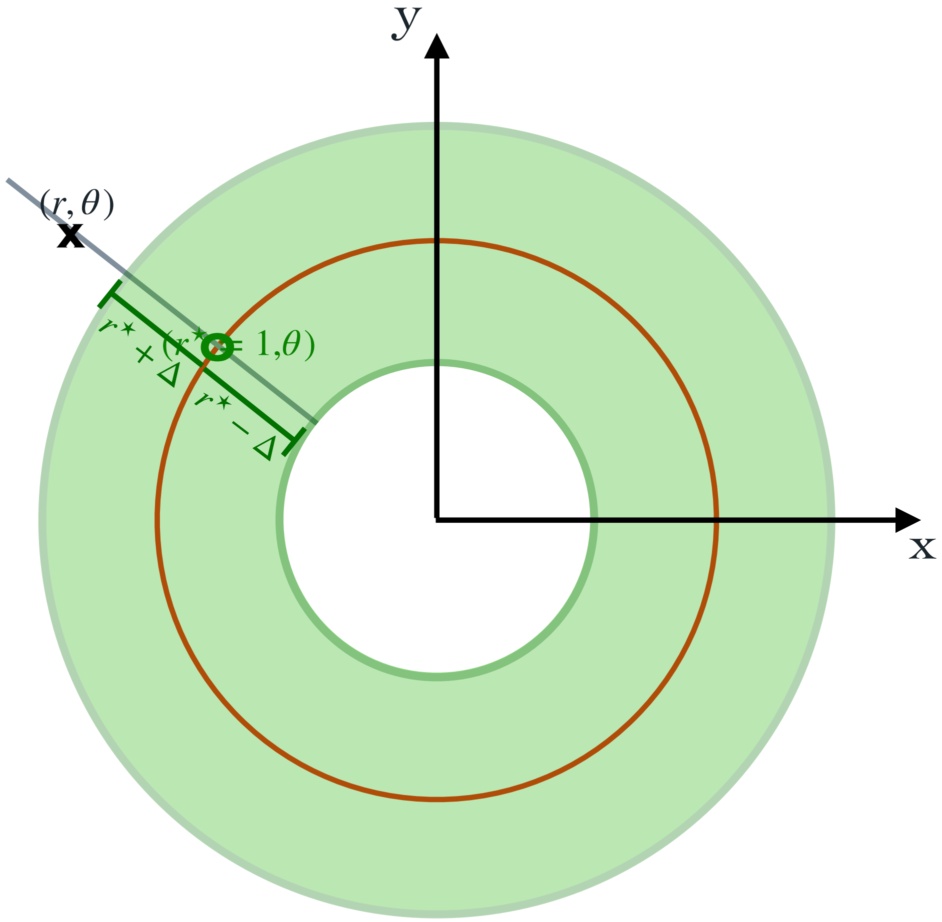

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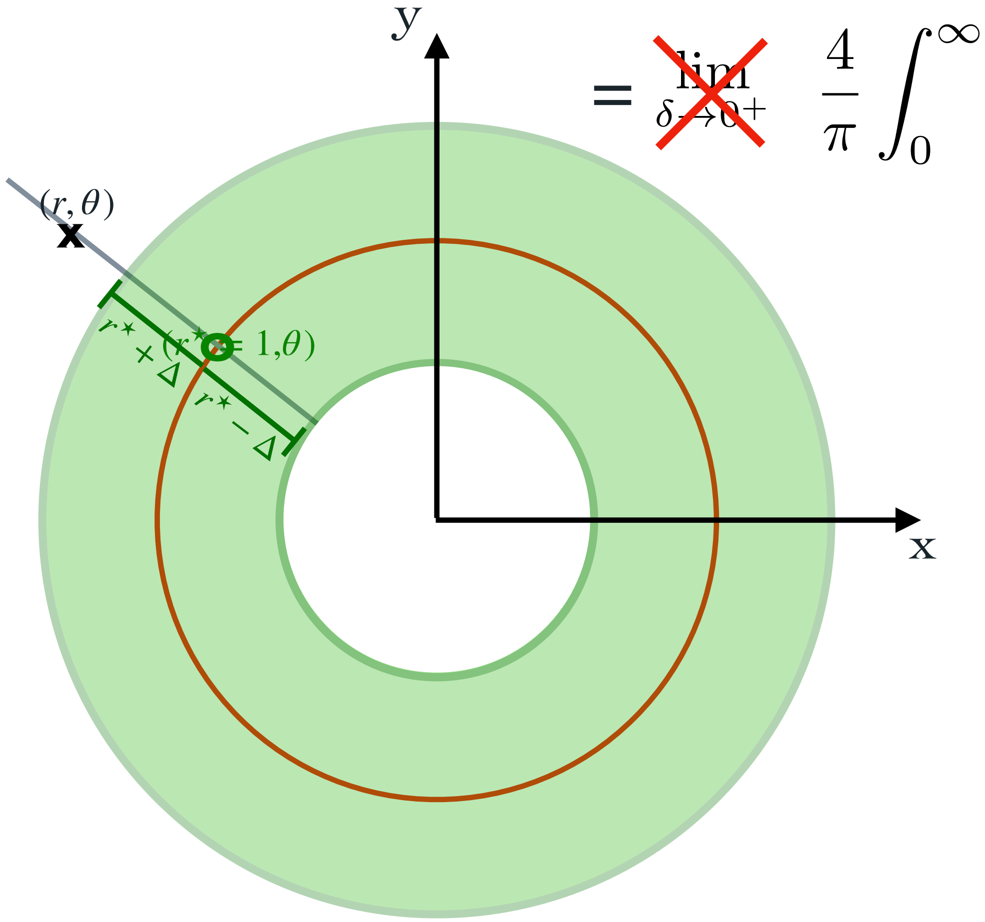

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$\delta = 0$ can be taken safely !

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Local threshold counterterm

Integrated threshold counterterm

$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right] + \frac{4}{\pi}\int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}$$

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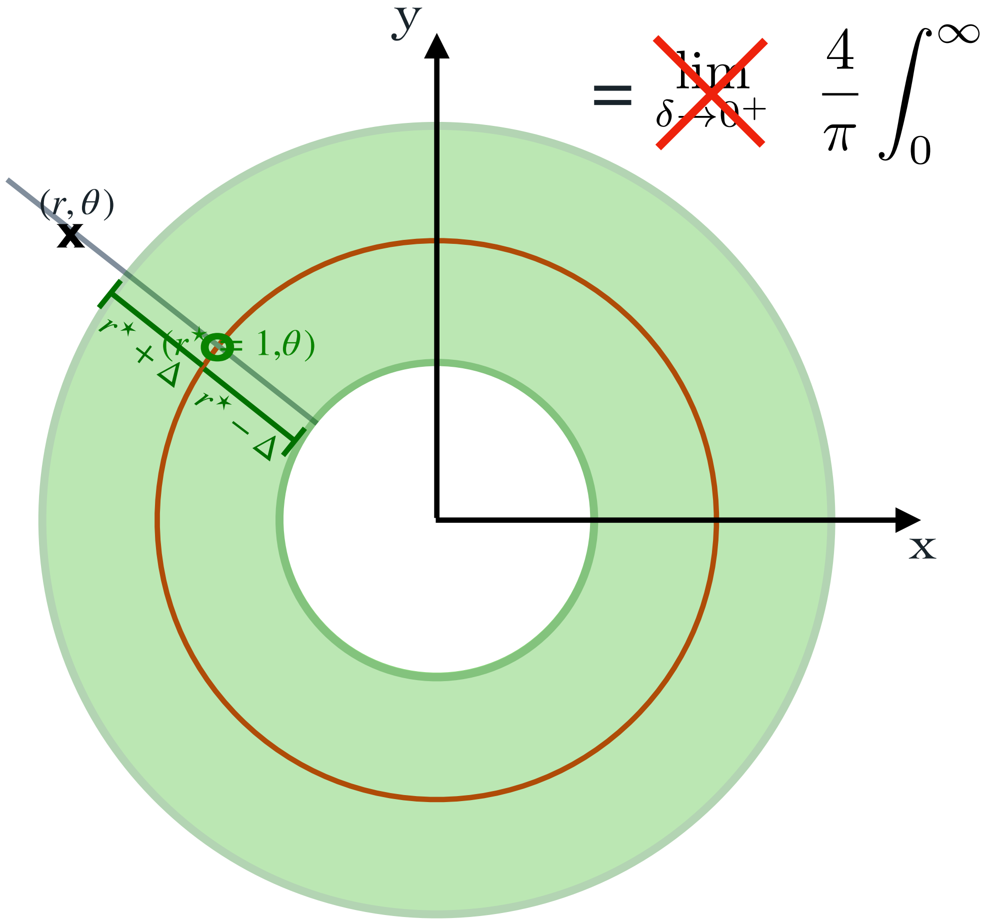

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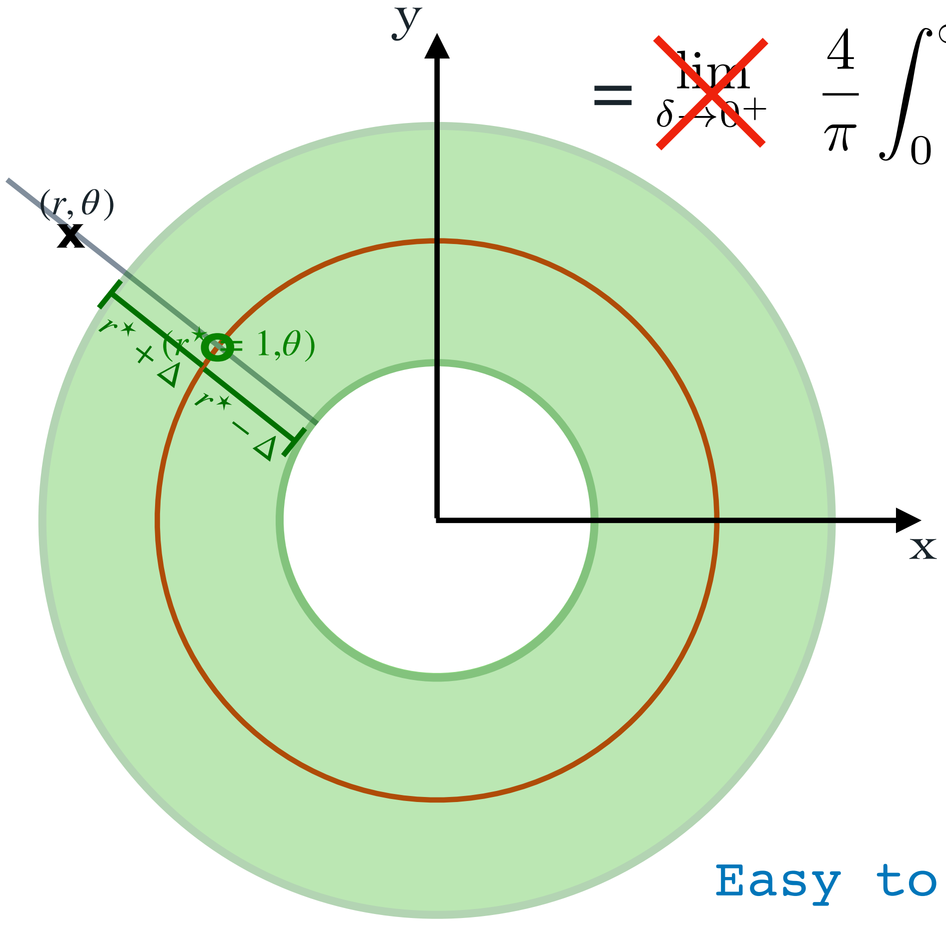

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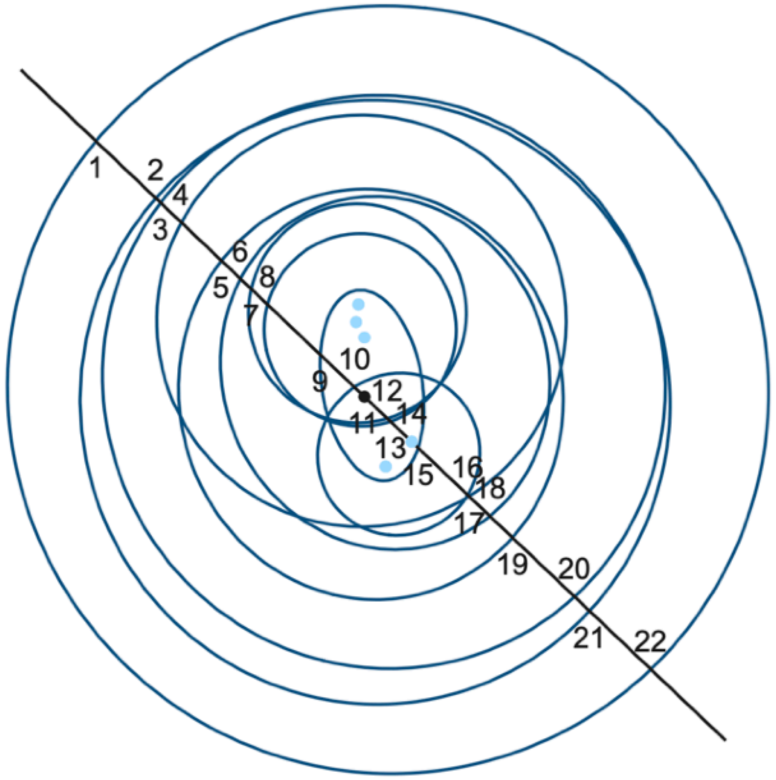
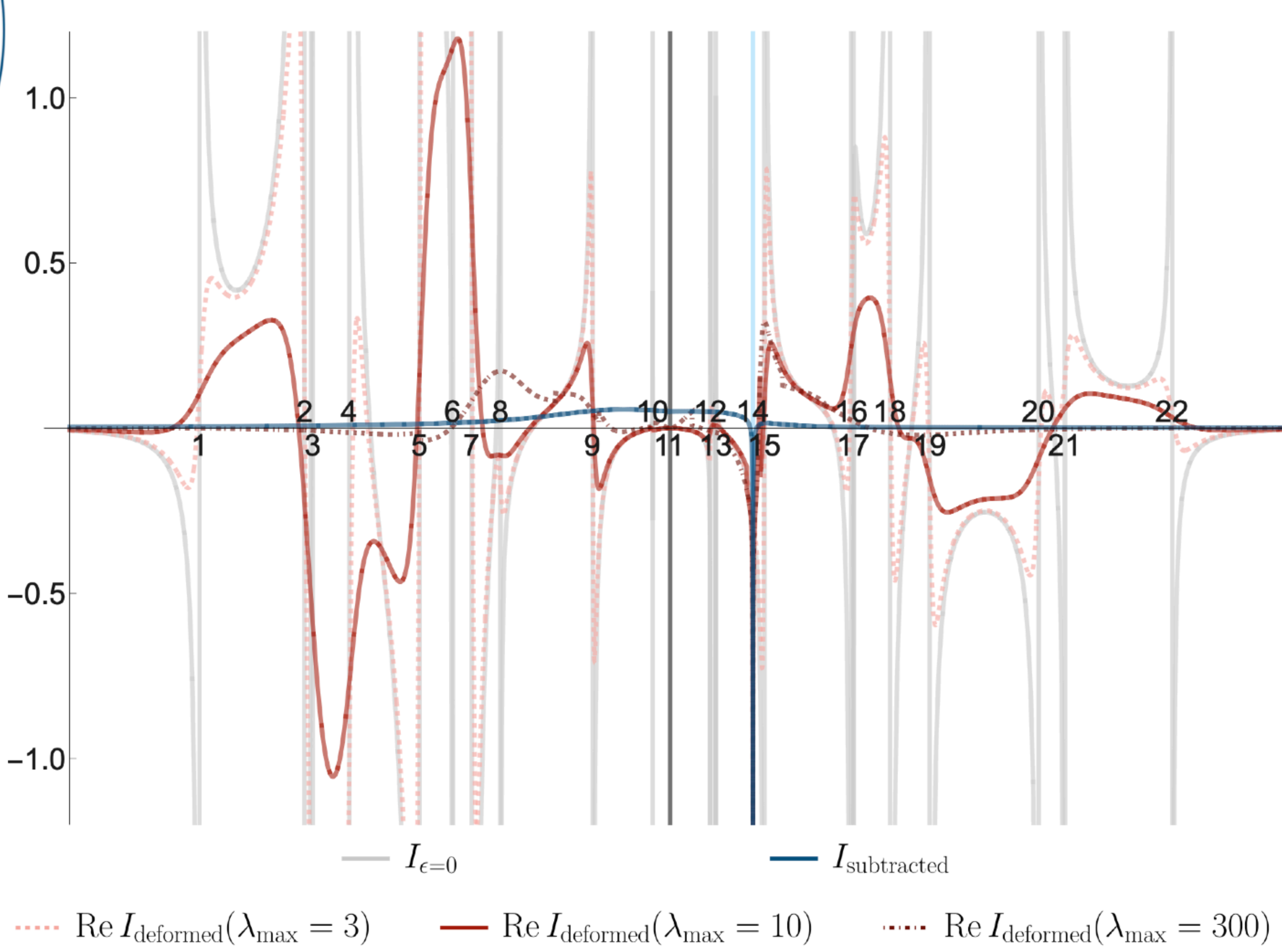
$$+ \lim_{\delta\to 0^+}\int_{1-\Delta}^{1+\Delta} dr\frac{4}{\pi}\frac{1}{2}\frac{1}{r-1\pm i\delta} \overset{\Delta\leq 1}{=} \underbrace{\frac{2}{\pi}\mathrm{PV}\left[\frac{1}{r-1\pm i\delta}\right]}_{0}\boxed{\mp 2\,\mathrm{i}}$$

Easy to compute since Principal Value is zero by construction

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]





UV SUBTRACTION

LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$$\mathrm{R}(\Gamma) = \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

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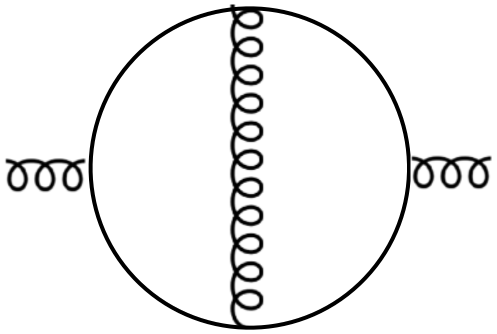

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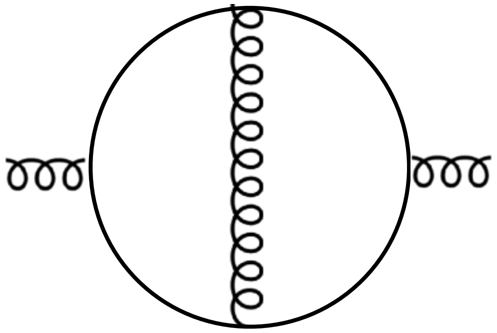

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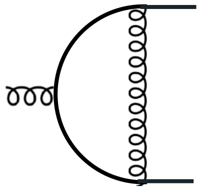
UV subgraphs :

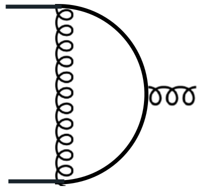
$$\text{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$$

$$\text{dod}\left(\gamma_4\right) = 2$$

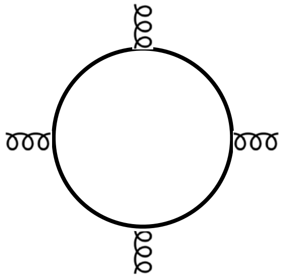
γ_1 =



γ_2 =



γ_3 =



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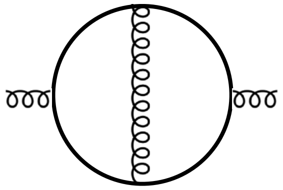

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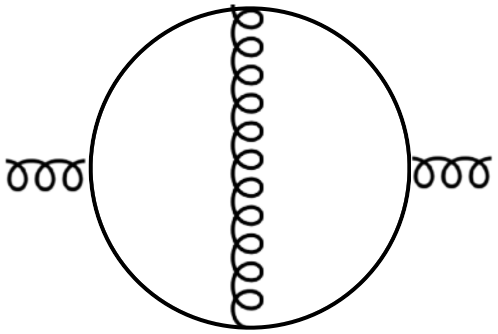
)

$$= \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

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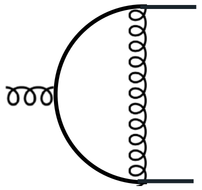

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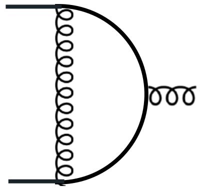

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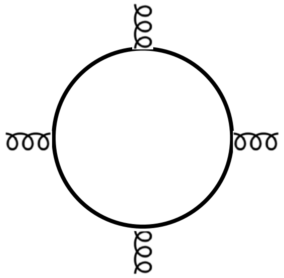

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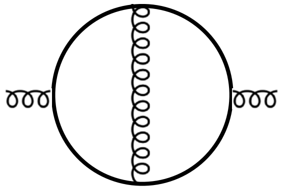
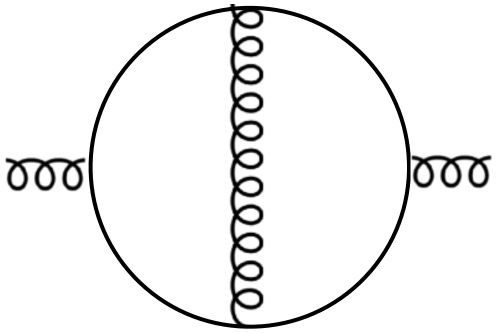


$$R(\Gamma) = \Gamma - K(\gamma_1)*\Gamma\setminus\gamma_1 - K(\gamma_2)*\Gamma\setminus\gamma_2 - K(\gamma_3)*\Gamma\setminus\gamma_3 - K(\gamma_4)*\Gamma\setminus\gamma_4$$

$$+ K(K(\gamma_1)*\Gamma\setminus\gamma_1) + K(K(\gamma_2)*\Gamma\setminus\gamma_2) + K(K(\gamma_3)*\Gamma\setminus\gamma_3)$$

Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

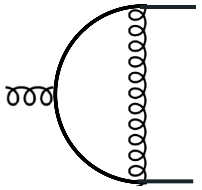


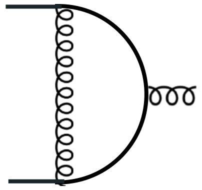
$$\mathrm{R}\left(\Gamma = \right) = \sum_{S\in W(\Gamma)} \Gamma\setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma\setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

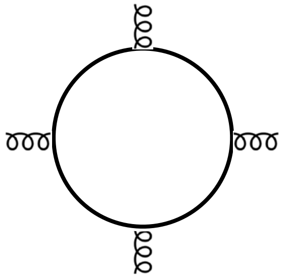
UV subgraphs :

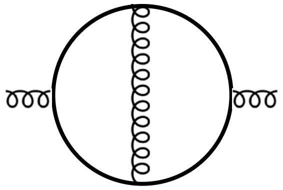
$$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$$

$$\mathrm{dod}\left(\gamma_4\right) = 2$$

$\gamma_1 =$ 

$\gamma_2 =$ 

$\gamma_3 =$ 

$\gamma_4 =$ 

$$\mathrm{R}(\Gamma) = \Gamma - \mathrm{K}(\gamma_1)*\Gamma\backslash\gamma_1 - \mathrm{K}(\gamma_2)*\Gamma\backslash\gamma_2 - \mathrm{K}(\gamma_3)*\Gamma\backslash\gamma_3 - \mathrm{K}(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+ \mathrm{K}(\mathrm{K}(\gamma_1)*\Gamma\backslash\gamma_1) + \mathrm{K}(\mathrm{K}(\gamma_2)*\Gamma\backslash\gamma_2) + \mathrm{K}(\mathrm{K}(\gamma_3)*\Gamma\backslash\gamma_3)$$

What is the operator K(γ) ? Anything we want ! so long as it:

- Locally cancels UV divergences of γ, even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme $(\overline{\mathrm{MS}} + \mathrm{OS})$
- Minimal analytics: at most single-scale all-massive vacuum integrals

LOCAL RENORMALISATION OPERATOR K

Our solution: $\mathsf{K}(\gamma) := \mathsf{T}(\gamma)$

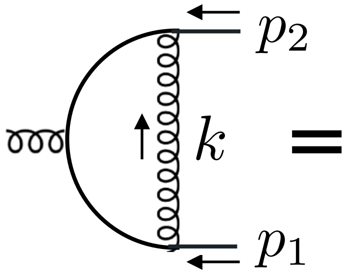
$\mathsf{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

LOCAL RENORMALISATION OPERATOR K

Our solution: K(γ) := T(γ)

T(γ) := **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

γ_1 =



$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

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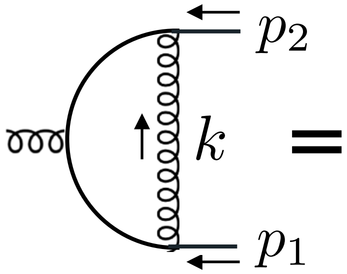
$\gamma_1 =$



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$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

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Our solution: $K(\gamma) := T(\gamma)$

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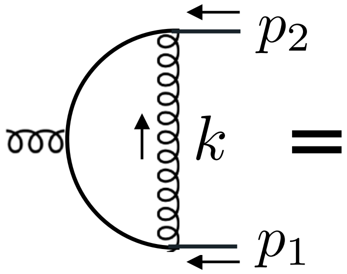

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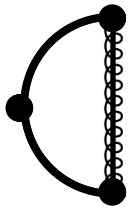
$$T(\gamma) = T_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \gamma^{\lambda} \bigg|_{\lambda=0} ,$$

$$T_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$

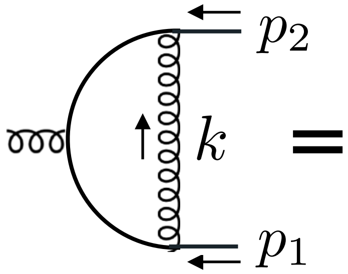


LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma) - [T](\gamma)$

$T(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

$\gamma_1 =$



$$= \frac{\mathcal{N}_{\gamma_1}(k,p_1,p_2,m)}{(\,(k-p_1)^2-m^2\,)(\,k^2\,)(\,(k+p_2)^2-m^2\,)}$$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k,\lambda p_1,\lambda p_2,\lambda m)}{(k-\lambda p_1)^2-m_{\mathrm{UV}}^2-\lambda^2(m^2-m_{\mathrm{UV}}^2))(k^2-m_{\mathrm{UV}}^2)((k+\lambda p_2)^2-m_{\mathrm{UV}}^2-\lambda^2(m^2-m_{\mathrm{UV}}^2)\,)}$$

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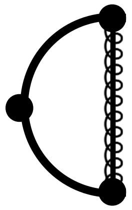
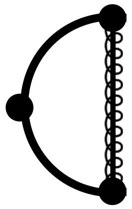


$[T](\gamma) :=$ **Integrated CT** , $[T](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon}\int d^{4-2\epsilon}k$



$$= \sum_{k=-\infty}^{+\infty}\alpha_k\epsilon^k$$

LOCAL RENORMALISATION OPERATOR K

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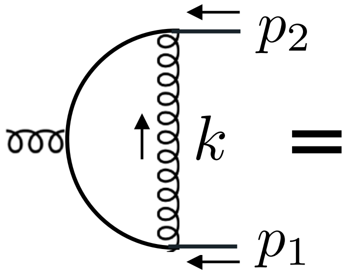
$T(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\text{dod}(\gamma)$

$$\gamma_1 =$$



$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

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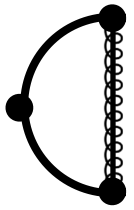

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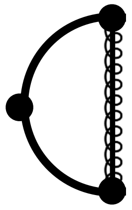
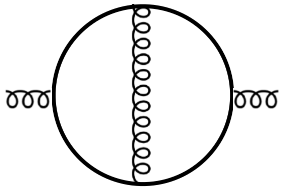


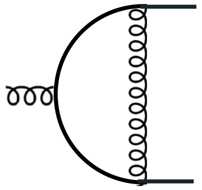
$$= \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$$

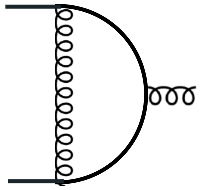
$\delta^{\text{X}}(\gamma) :=$ **Renormalisation CT**
in scheme X , $(-[T] + \delta^{\overline{\text{MS}}}) := \bar{\text{K}}$, $\bar{\text{K}}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$

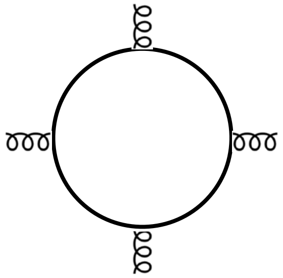
R-OPERATOR UNFOLDING

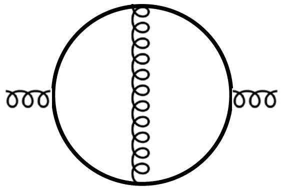
Γ = 

with UV subgraphs

γ_1 = 

γ_2 = 

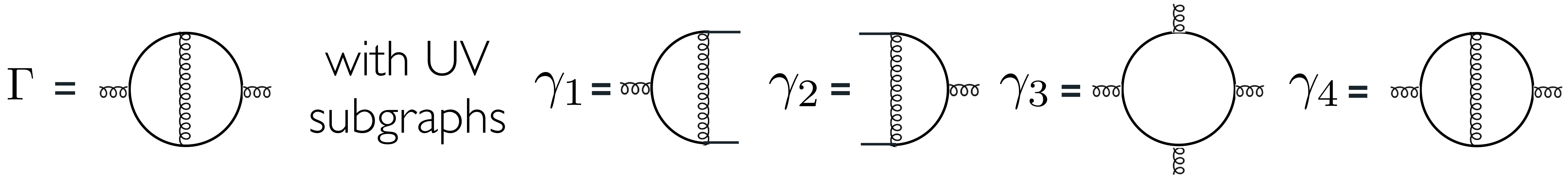
γ_3 = 

γ_4 = 

$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$

R-OPERATOR UNFOLDING



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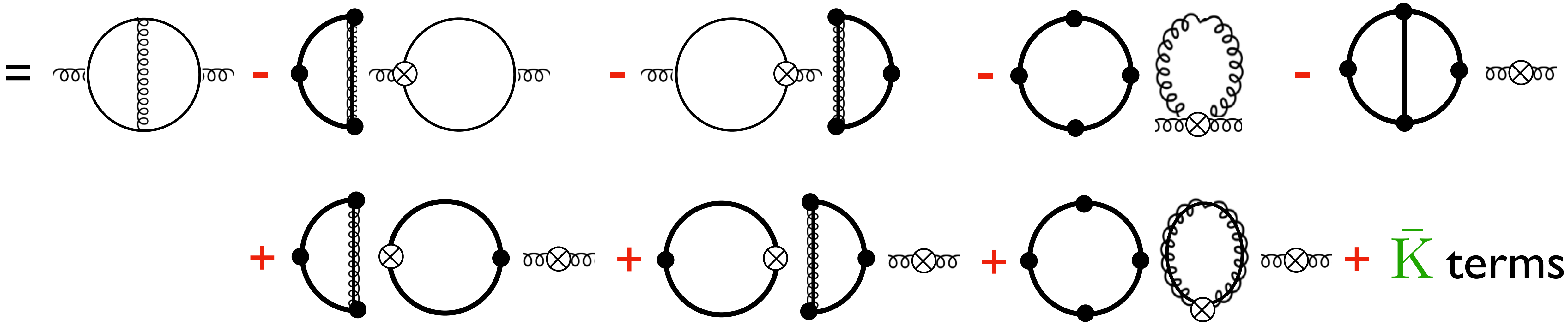
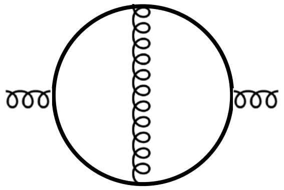
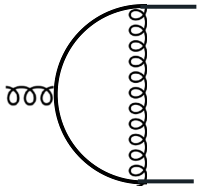
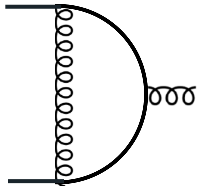
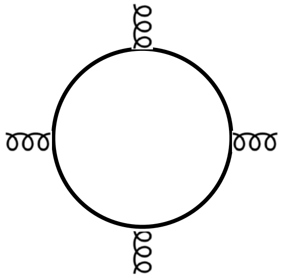
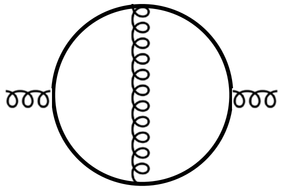
$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\text{ terms}$$



R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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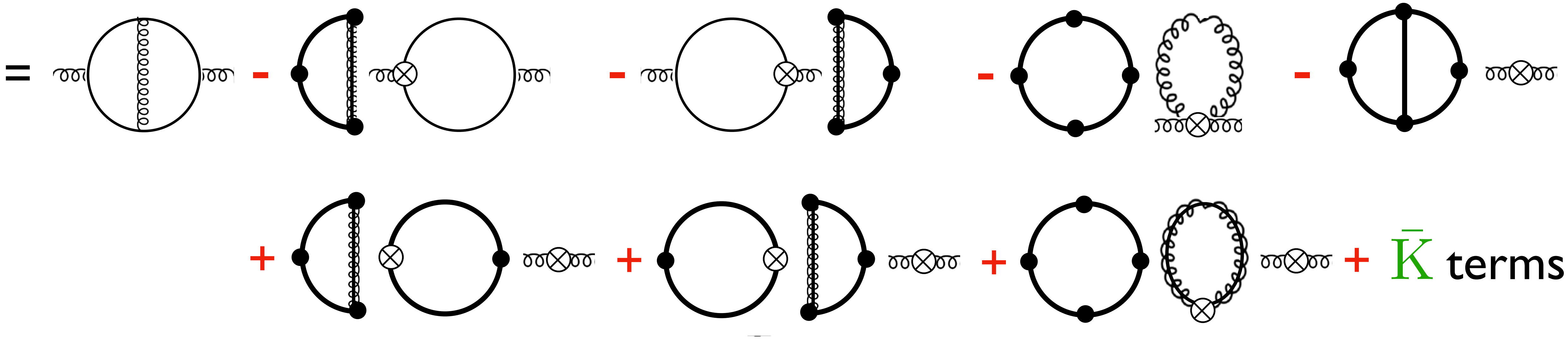
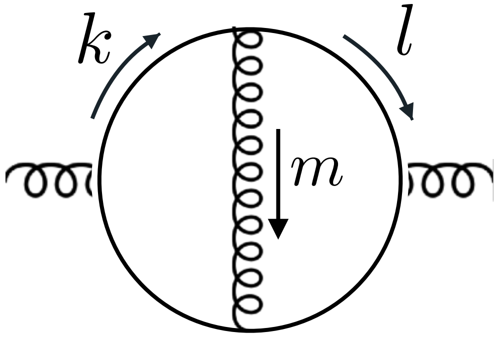


The four different types of UV limits are now **finite** !



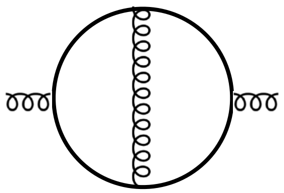
$k, m \to \infty$, l finite

$l, m \to \infty$, k finite

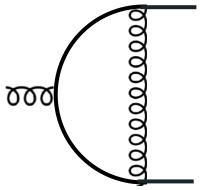
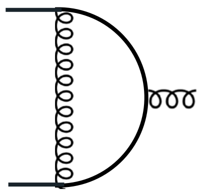
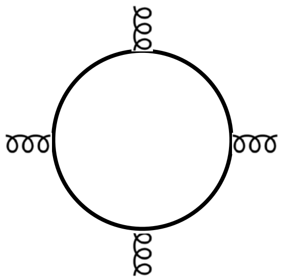
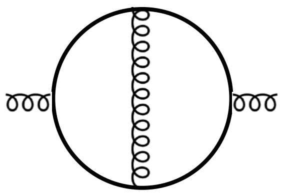
$k, l \to \infty$, m finite

$k, l, m \to \infty$

R-OPERATOR UNFOLDING

$\Gamma =$ 

with UV subgraphs

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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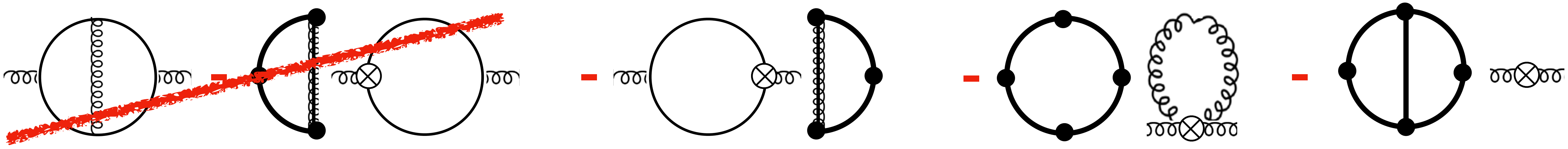
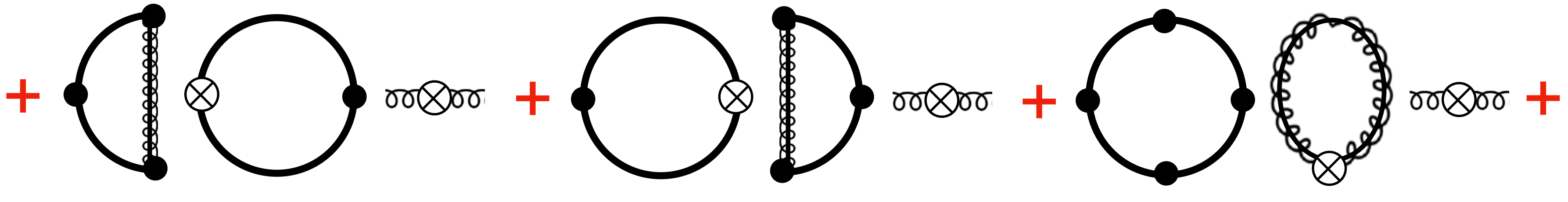
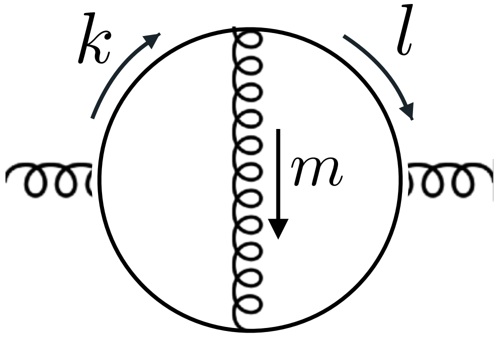
$=$ 



$+ \bar{K}$ terms

The four different types of UV limits are now **finite** !



$\boxed{k, m \to \infty \;, l \;\; \text{finite}}$

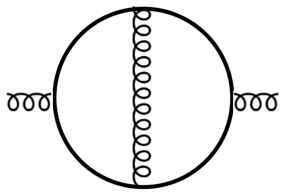
$l, m \to \infty \;, k \;\; \text{finite}$

$k, l \to \infty \;, m \; \text{finite}$

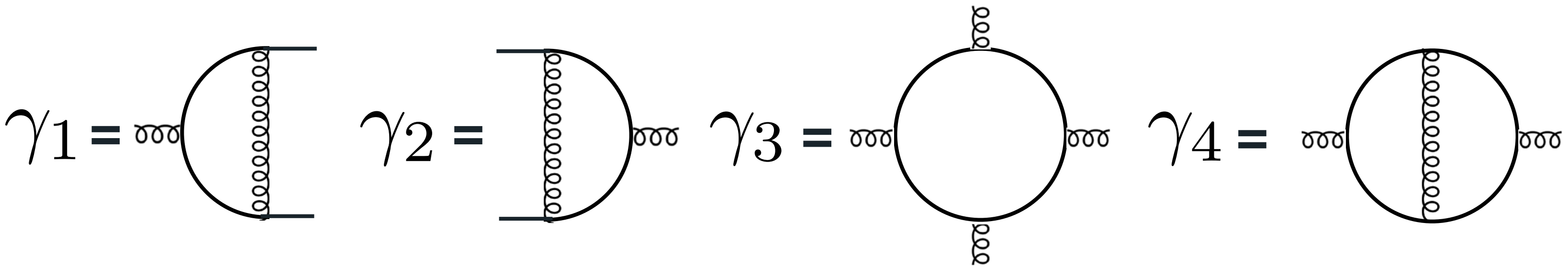
$k, l, m \to \infty$

R-OPERATOR UNFOLDING

$\Gamma =$



with UV subgraphs



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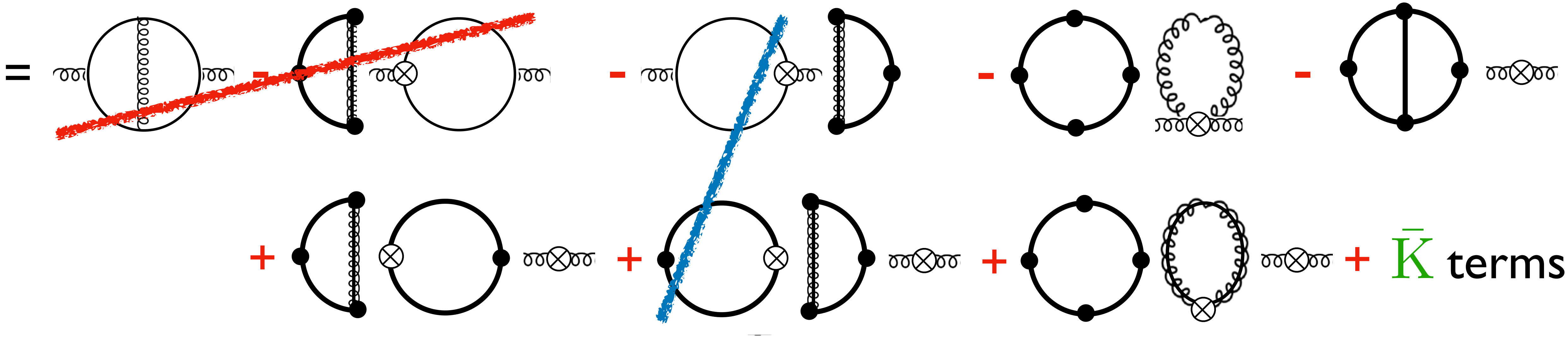
$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$



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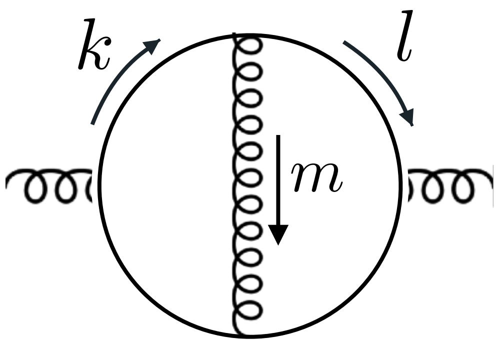

$\boxed{k, m \to \infty \, , l \text{ finite}}$

$l, m \to \infty \, , k$ finite

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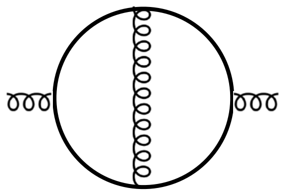
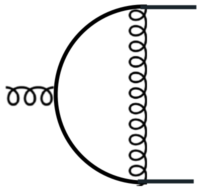
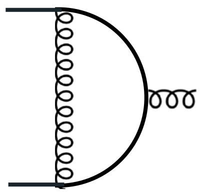
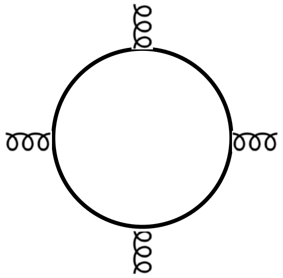
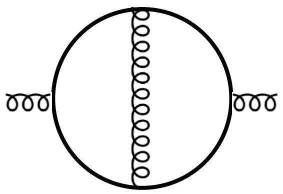
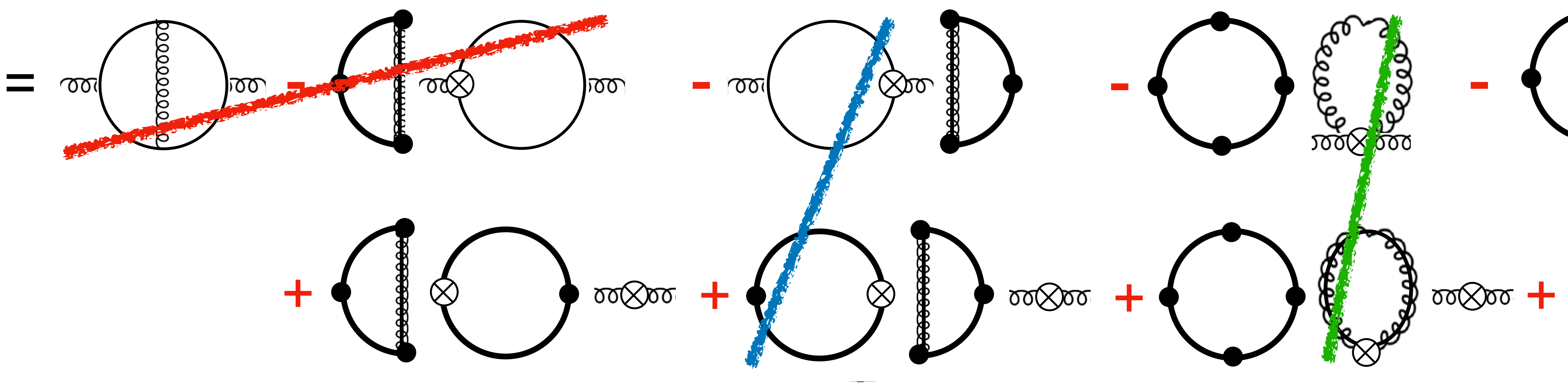
$k, l, m \to \infty$

R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

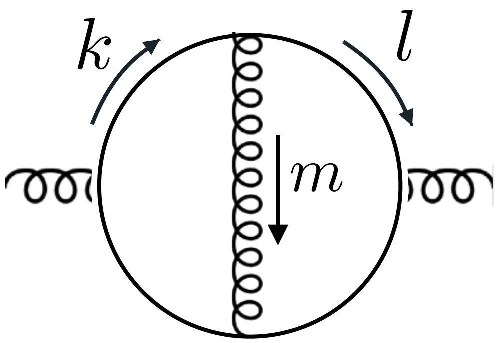
$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\text{ terms}$$



$+ \bar{K}$ **terms**

The four different types of UV limits are now **finite** !



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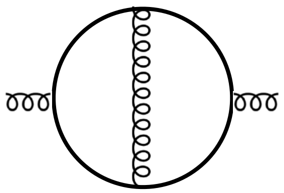
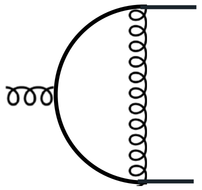
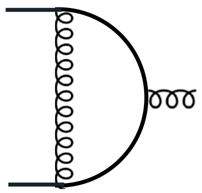
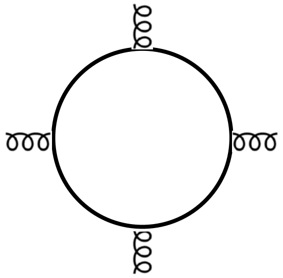
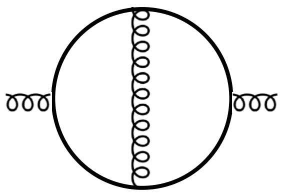
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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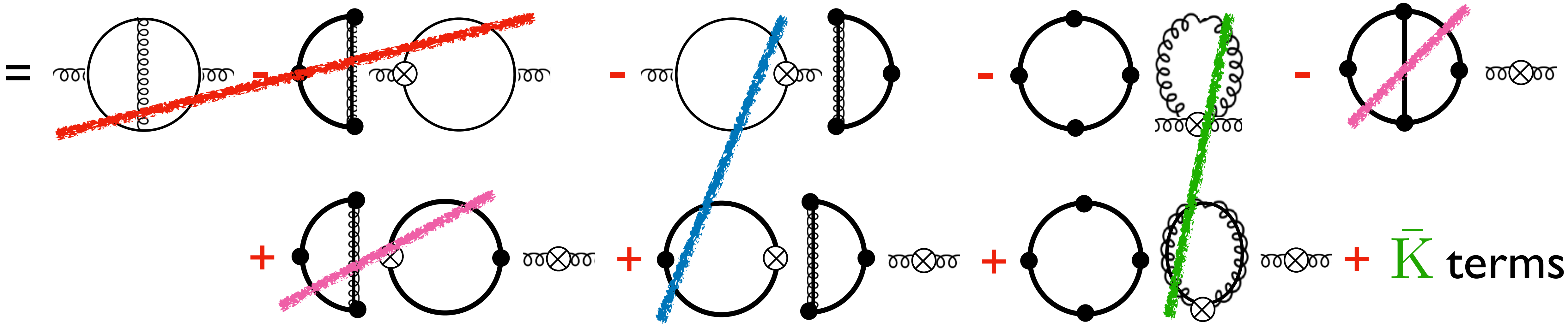

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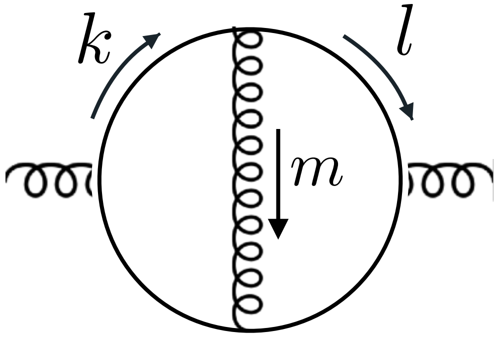

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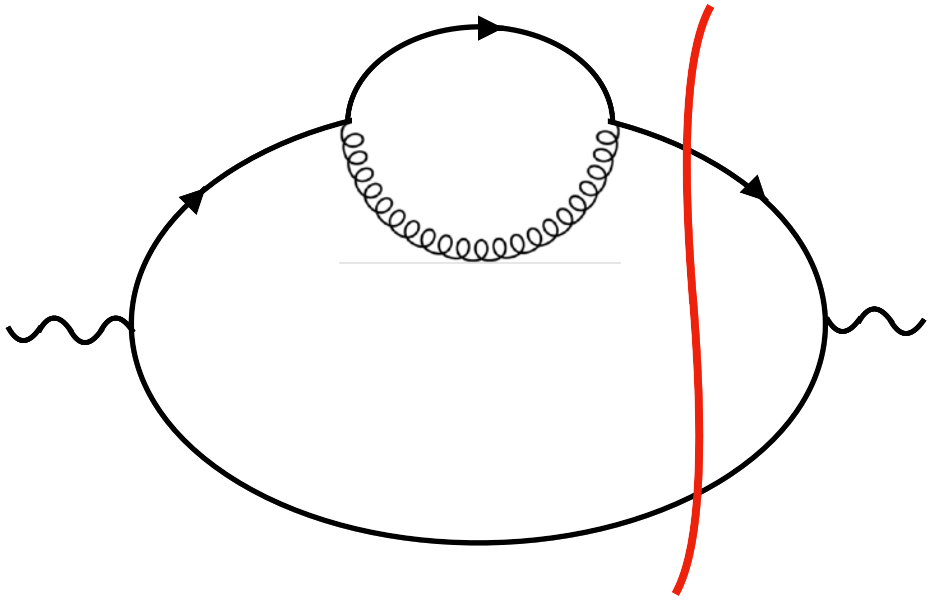
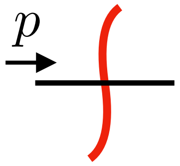
LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider ***truncated*** amplitudes only :



Traditional Cutkosky rule



$$= -2\pi \mathrm{i} \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

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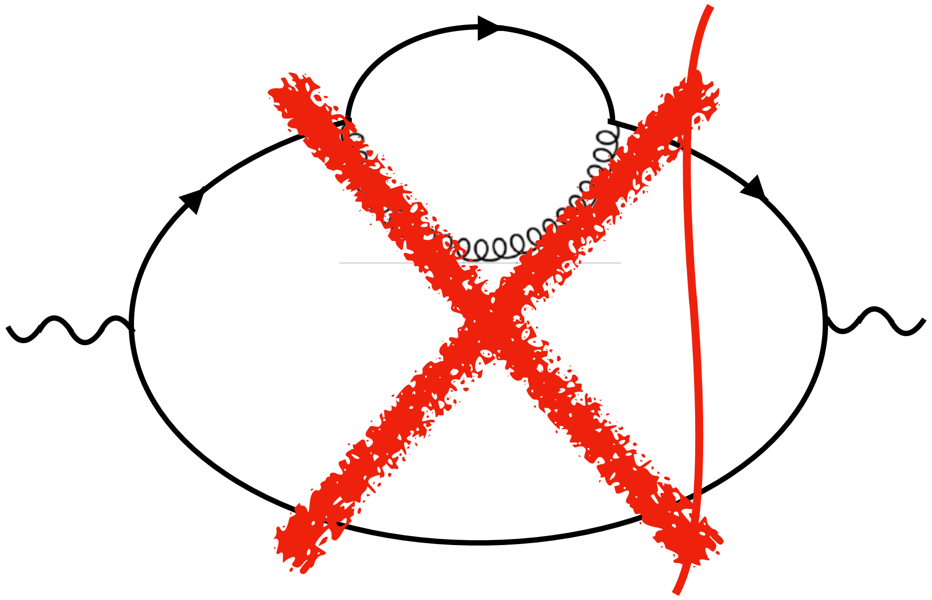

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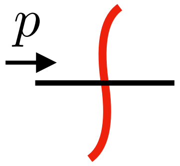

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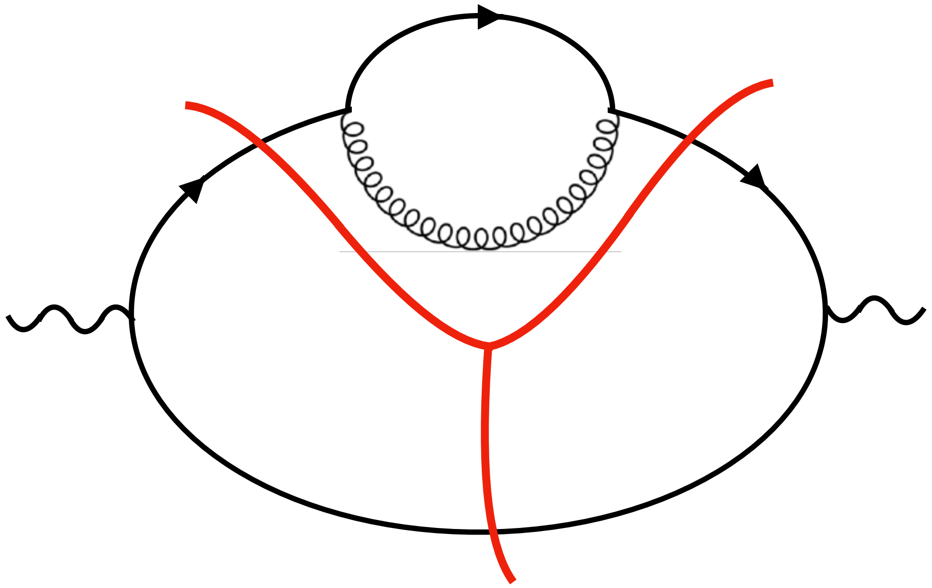
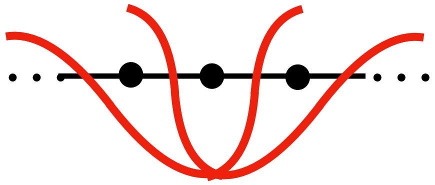
So consider this Cutkosky cut as a **higher-order residue** → **Generalised cutting rule**





$$= -2\pi i \frac{\delta^{(n)}\left[p^0 - E(\vec{p})\right]}{(p^0 + E(\vec{p}))^2}$$

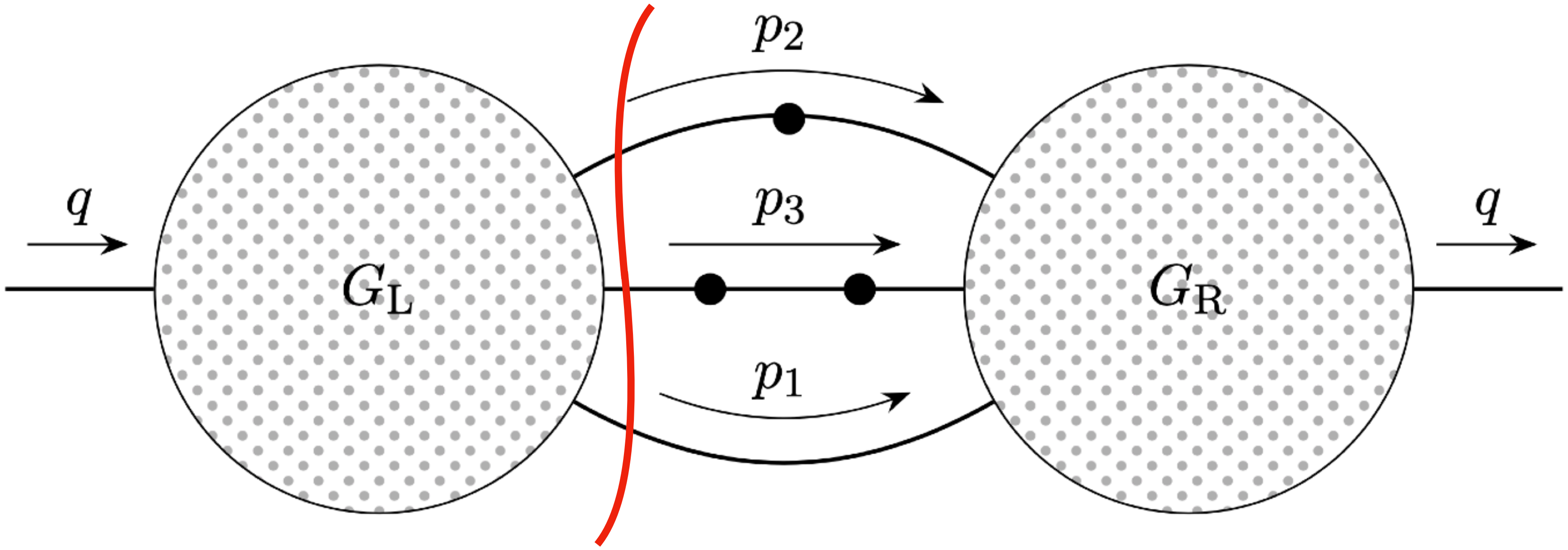
$n - \text{times}$

$$\int dx\, \delta^{(n+1)}[x] f(x) = \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=0}$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\ )]\delta^{(2)}[p_2^0 - E(\vec{p_2}\ )]\delta^{(3)}[p_3^0 - E(\vec{p_3}\ )]$$

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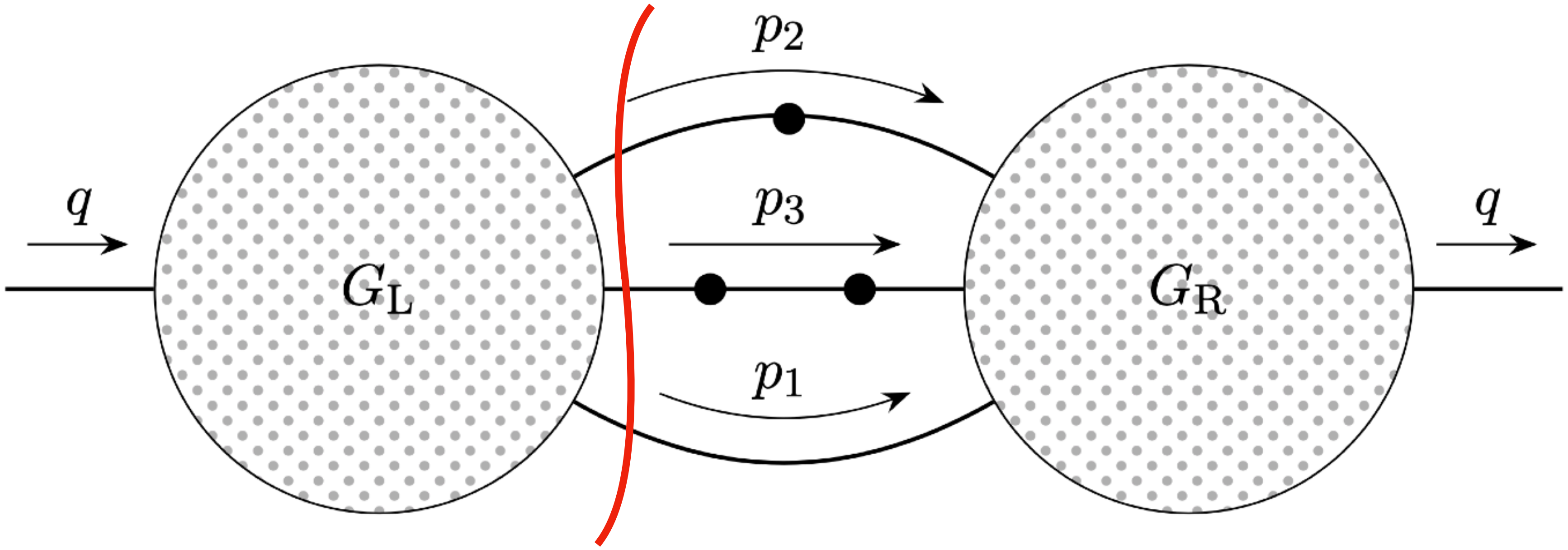

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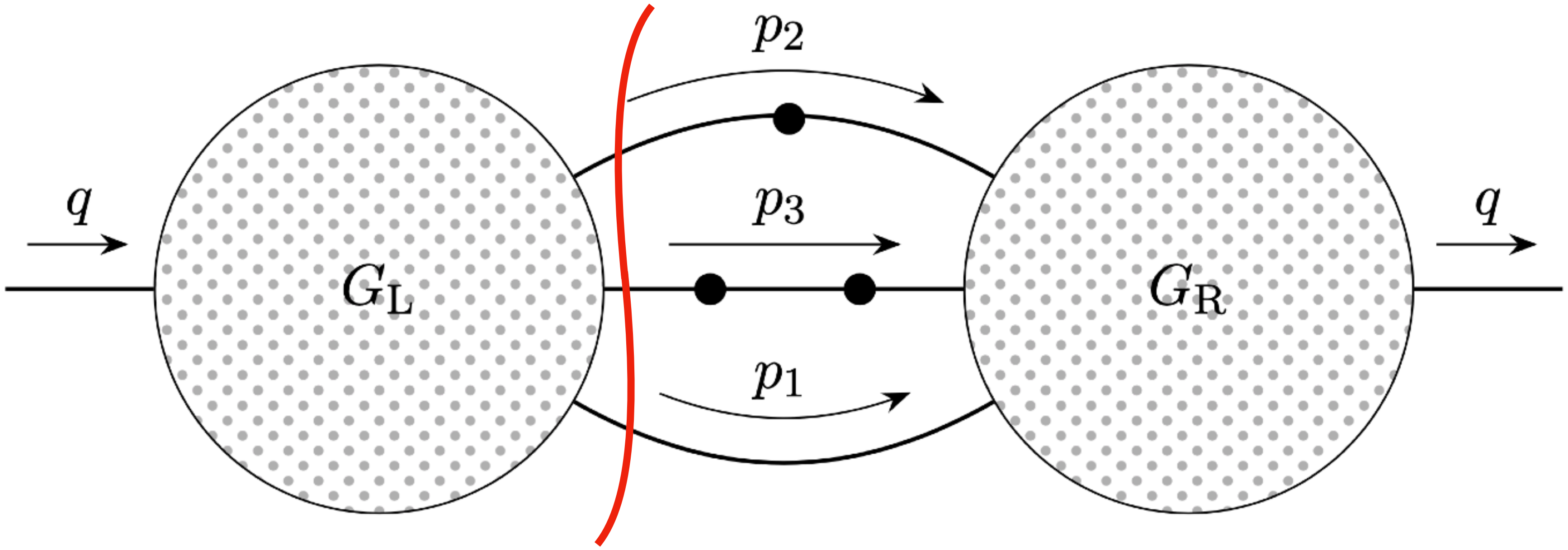

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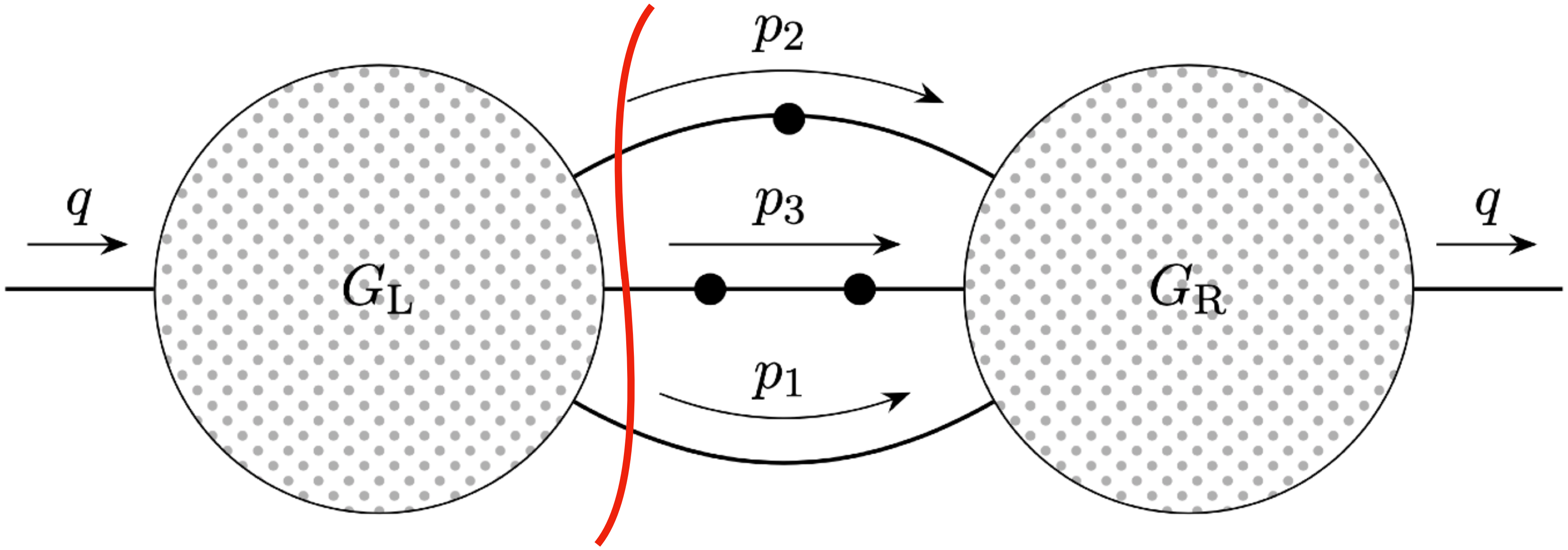

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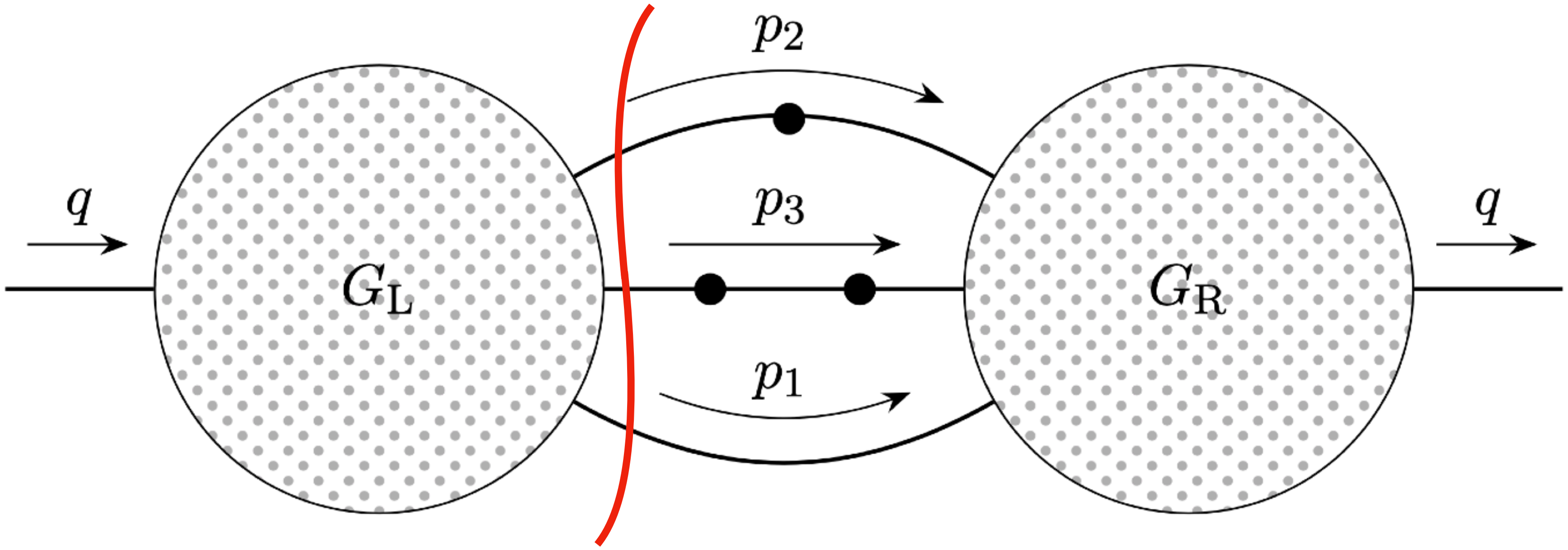

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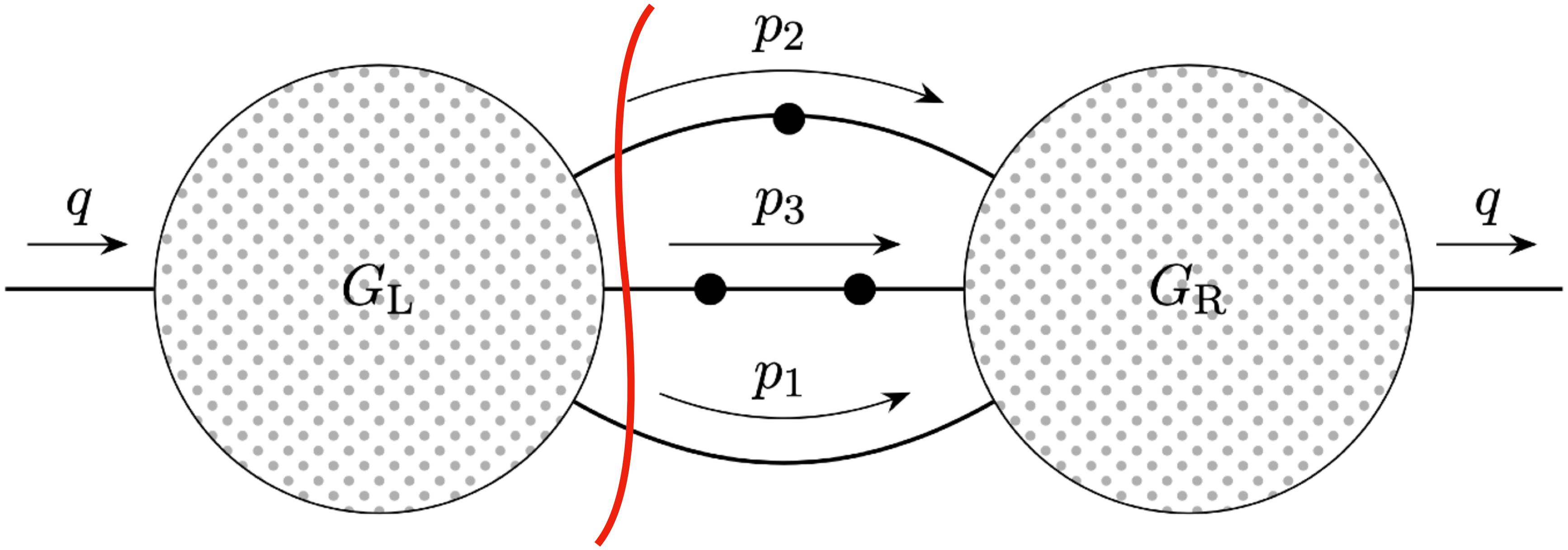

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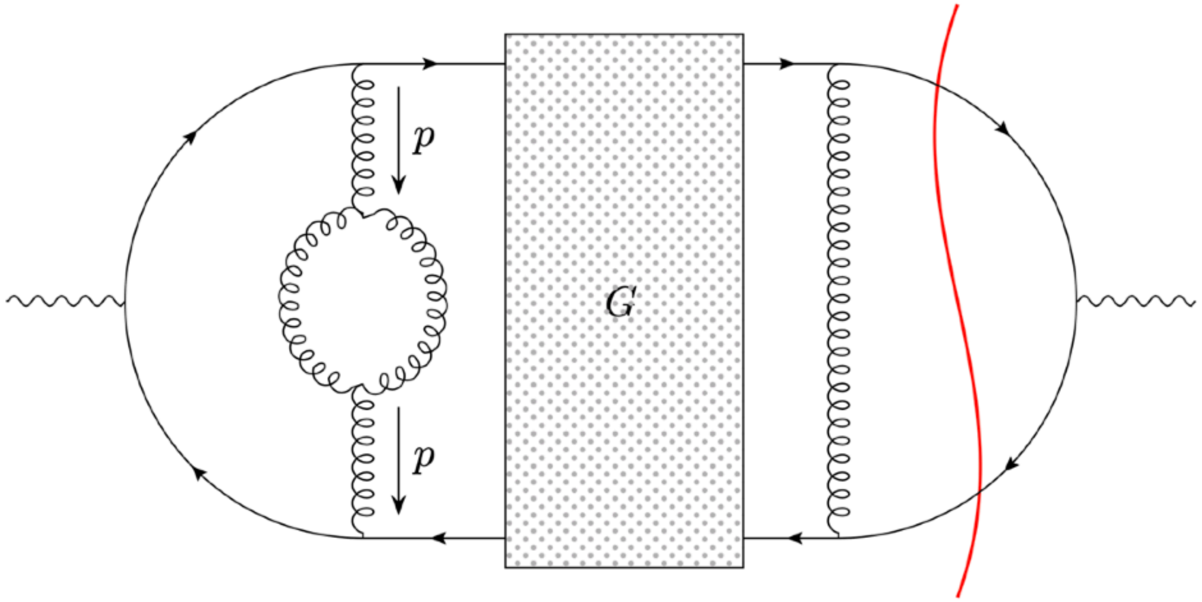
Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]



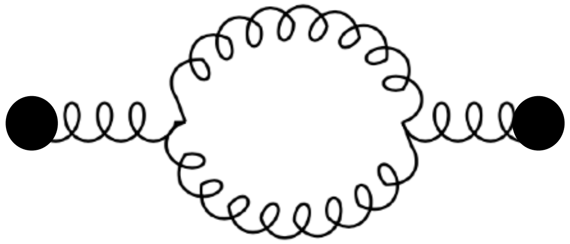
For $p = 0$

$$\propto \frac{1}{(p^2)^2}$$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have



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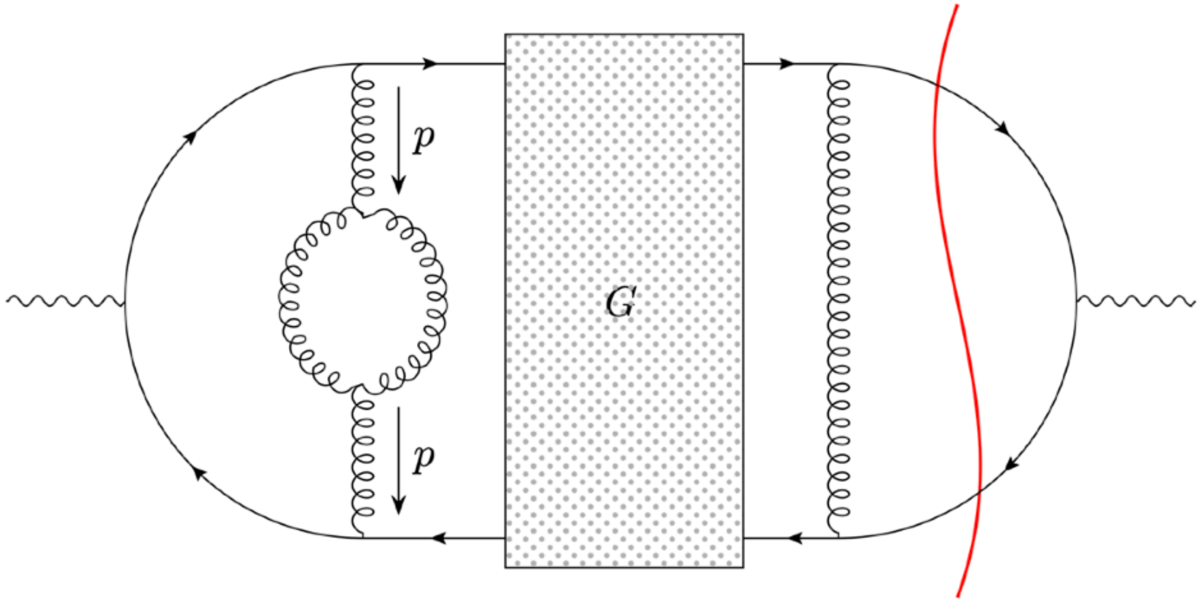

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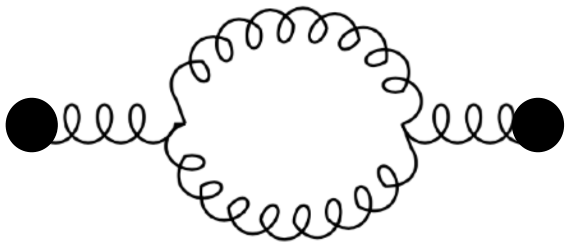
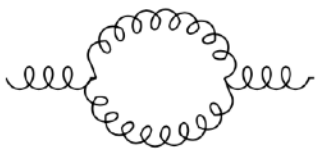
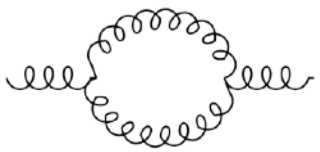


$\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$

but not at the local level; we must introduce **spurious soft counterterms** :



$-\tilde{T}_1\Big($



$\Big),\ \tilde{T}_{\text{soft\_dod}}(\gamma) = \sum_{j=0}^{\text{soft\_dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma(\lambda\, p)\Big|_{\lambda=0},\ [\tilde{T}] = 0$

COMBINED UV AND SPURIOUS IR FOREST

Eurekâ moment:

- Remarkably, we always have : $\mathrm{soft\_dod} = \mathrm{UV\_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\boxed{\hat{T}_{\mathrm{dod}} = T_{\mathrm{dod}} + \tilde{T}_{\mathrm{dod}-1} - T_{\mathrm{dod}}\tilde{T}_{\mathrm{dod}-1}}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

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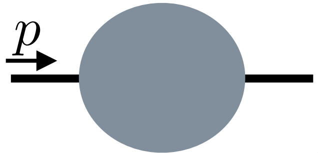
Novelty though: **automatic renormalisation** of fermion **masses in the OS scheme:**

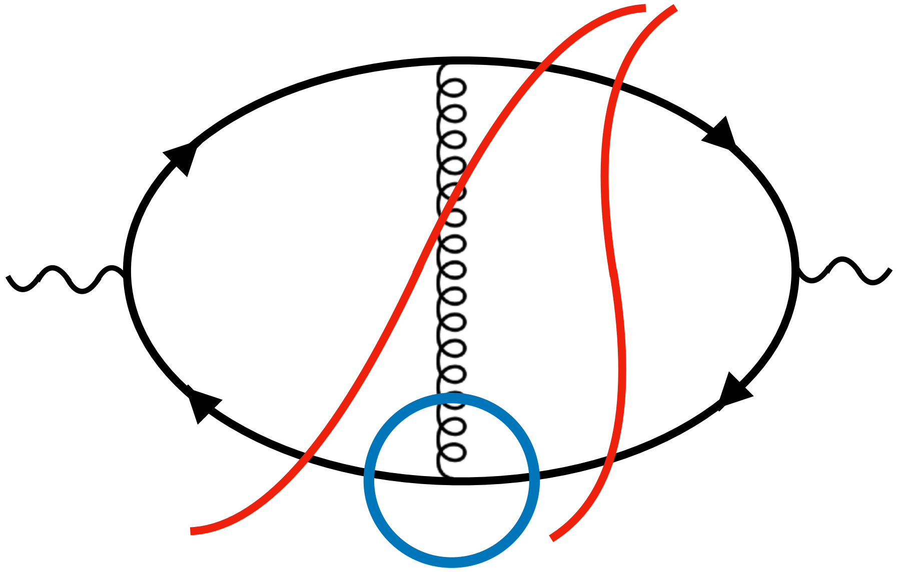
$T^{\mathrm{os}\pm}\Big(\Sigma =$ 

$\Big) = (1 \pm \gamma^0)\Sigma(p = \pm p^{\mathrm{os}})$, $\quad p^{\mathrm{os}} = (m, 0, 0, 0)$

$$\frac{1}{2}\left(\left[T^{\mathrm{os}+}(\Sigma)\right] + \left[T^{\mathrm{os}-}(\Sigma)\right]\right) = \delta m^{\mathrm{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

LOCAL UNITARITY



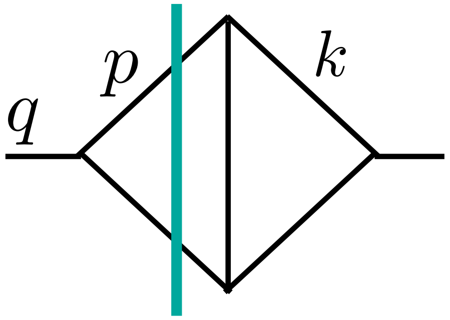
local

cancellation

LOCALITY UNITARITY

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)$$



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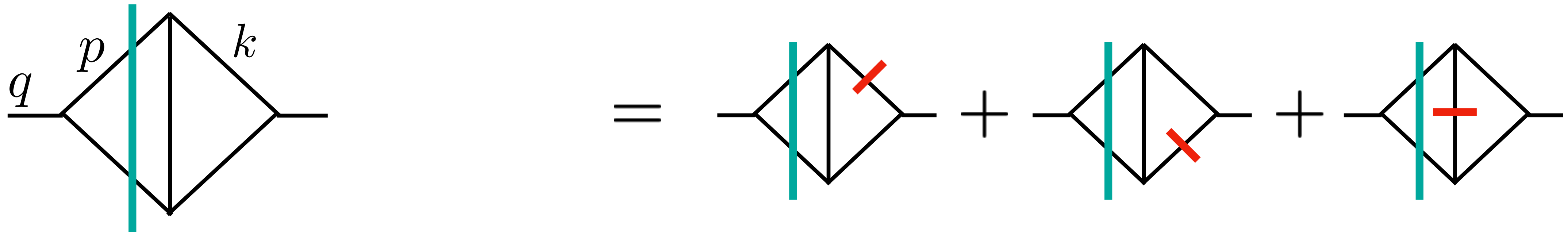

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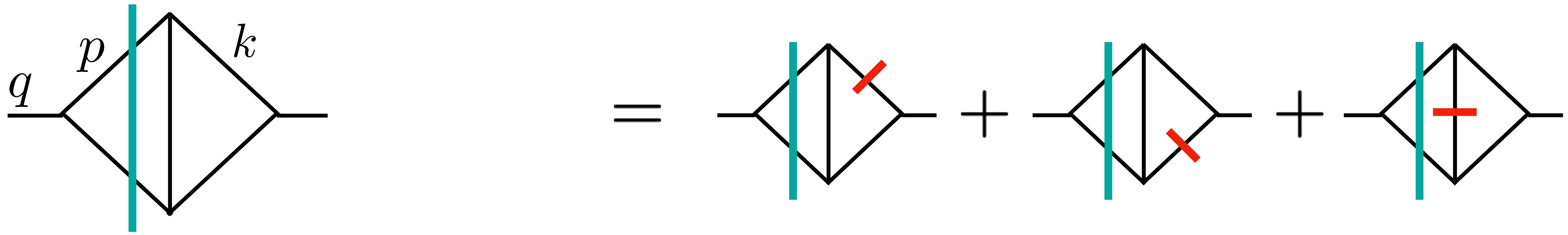
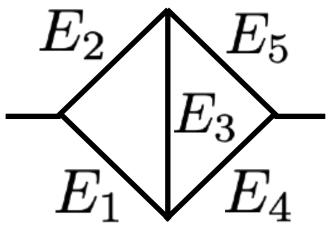
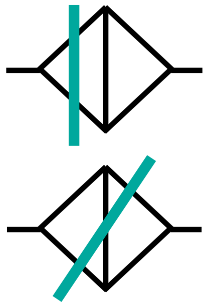


But the measure is **not yet fully aligned**:





$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\,\left(\delta(E_1+E_2-Q_0)f_{\mathrm{virt}}+\delta(E_1+E_3+E_5-Q_0)f_{\mathrm{real}}\right)$$

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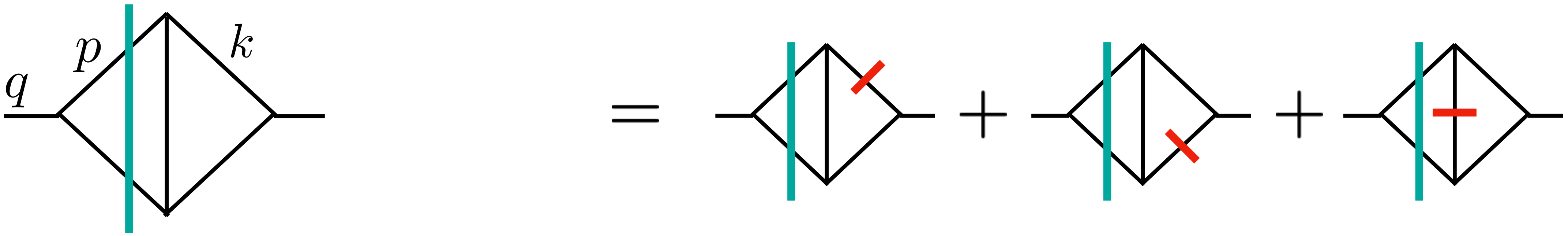
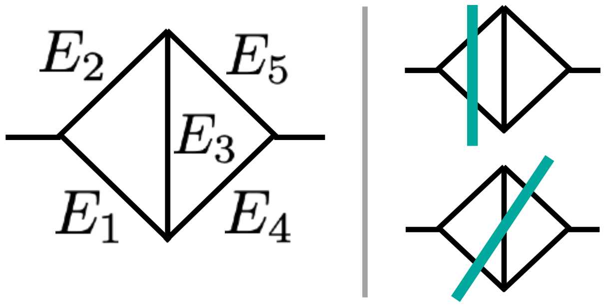
We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{d^3\vec{p}}{2|\vec{p}|} d^4k\, \delta(|\vec{p}| + |\vec{p}-\vec{q}| - Q_0) \quad \rightarrow \quad \frac{d^3\vec{p}}{2|\vec{p}|} \frac{d^3\vec{k}}{2|\vec{k}|} \delta(|\vec{p}| + |\vec{p}-\vec{q}| - Q_0)$$



But the measure is **not yet fully aligned**:



$$= \int d^3\vec{k} d^3\vec{p} \left(\delta\underbrace{(E_1 + E_2 - Q_0)}_{\eta_v(\vec{k},\vec{p})} f_{\text{virt}} + \delta\underbrace{(E_1 + E_3 + E_5 - Q_0)}_{\eta_r(\vec{k},\vec{p})} f_{\text{real}}\right)$$

(on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

CAUSAL FLOW

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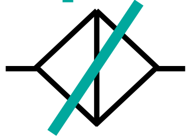

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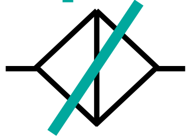

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A different perspective on the usual phase space mapping problem

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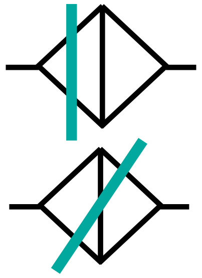

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Soper,
arXiv: 9804454 (1998)

Soper,
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ZC, Hirschi, Pelloni, Ruijl
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General FSR cancellations
For N to M N^kLO processes

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A toy example:

$$\int \mathrm{d}^3\vec{k}\,\delta(|\vec{k}| - Q_0)\,f(\vec{k})$$

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$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0)\, f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

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Solve all deltas in the common scaling variable. This completes the alignment of the measure!

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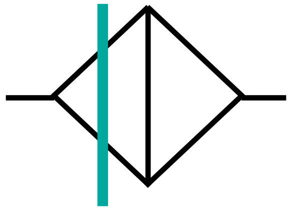
When applying this construction to LU we get:



$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\delta(E_1 + E_2 - Q_0) f_{\mathrm{virt}} = \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\, g_{\mathrm{v}}(t_{\mathrm{v}}^\star)$$

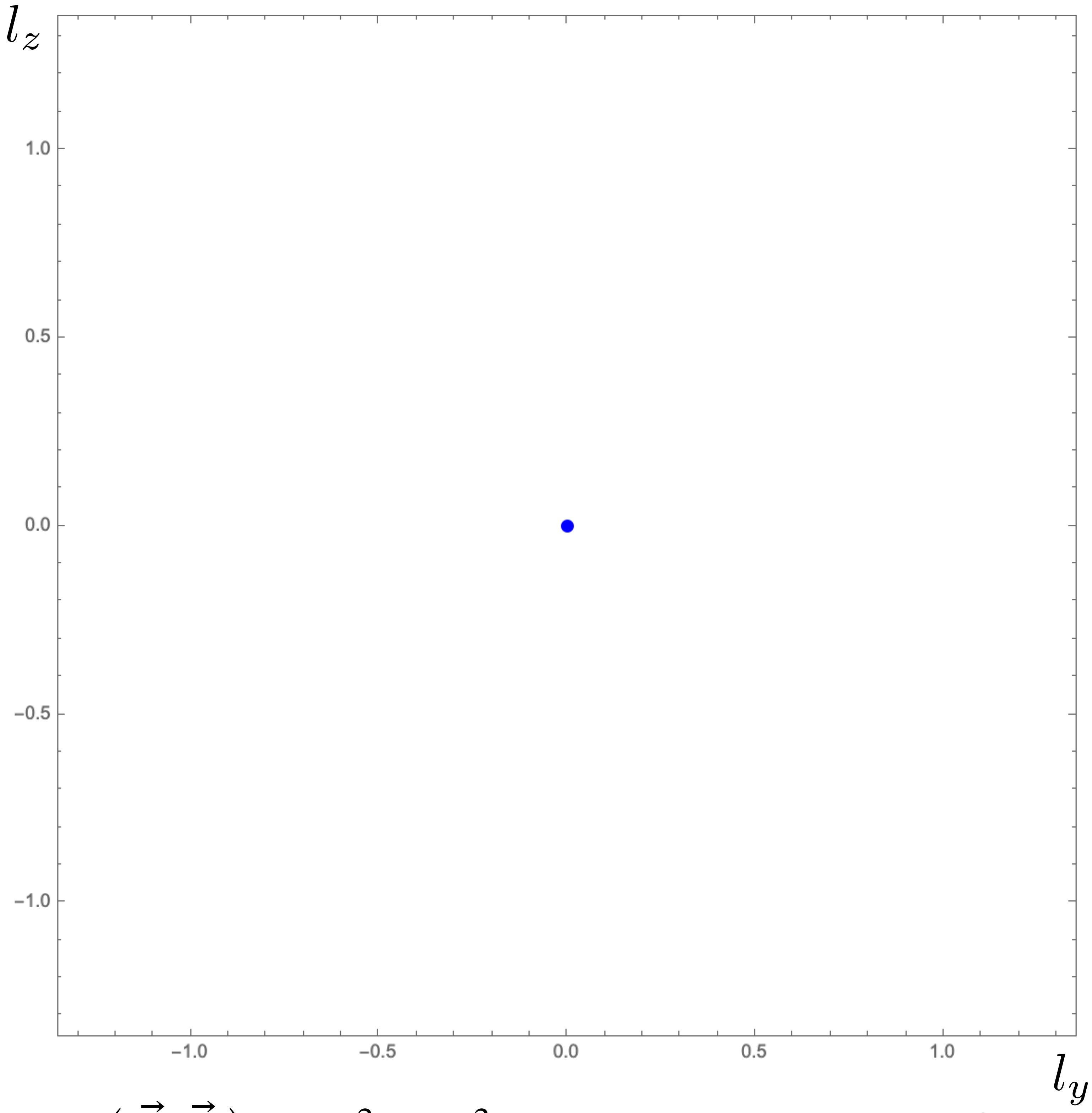
where $$t_{\mathrm{v}}^\star = t_{\mathrm{v}}^\star(\vec{k},\vec{p}) = \frac{Q_0}{E_1 + E_2}$$

$$(\vec{p},\vec{k}) \to \vec{\phi}(t,(\vec{p},\vec{k}))$$

“Causal flow” is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t\vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0,(\vec{k},\vec{l})) = (\vec{k},\vec{l}) \end{cases}$$

LOCALITY UNITARITY: VISUALISATION



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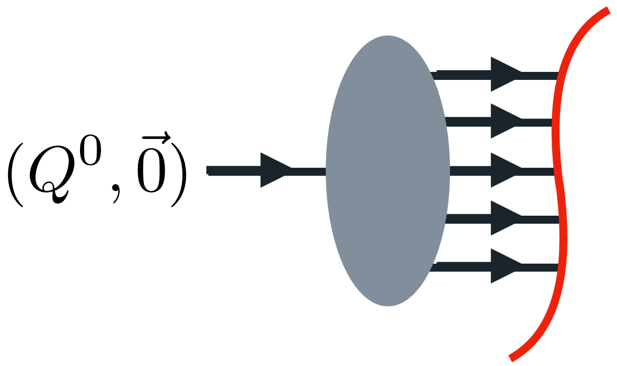

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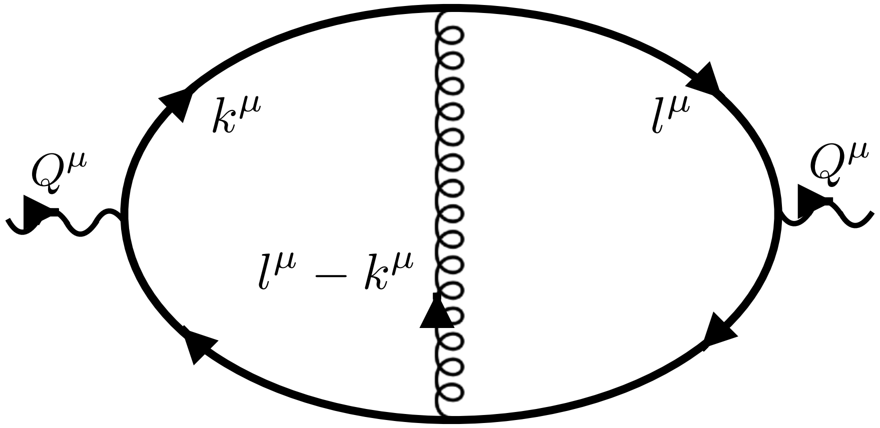

— = Cutkosky cut ≡ threshold

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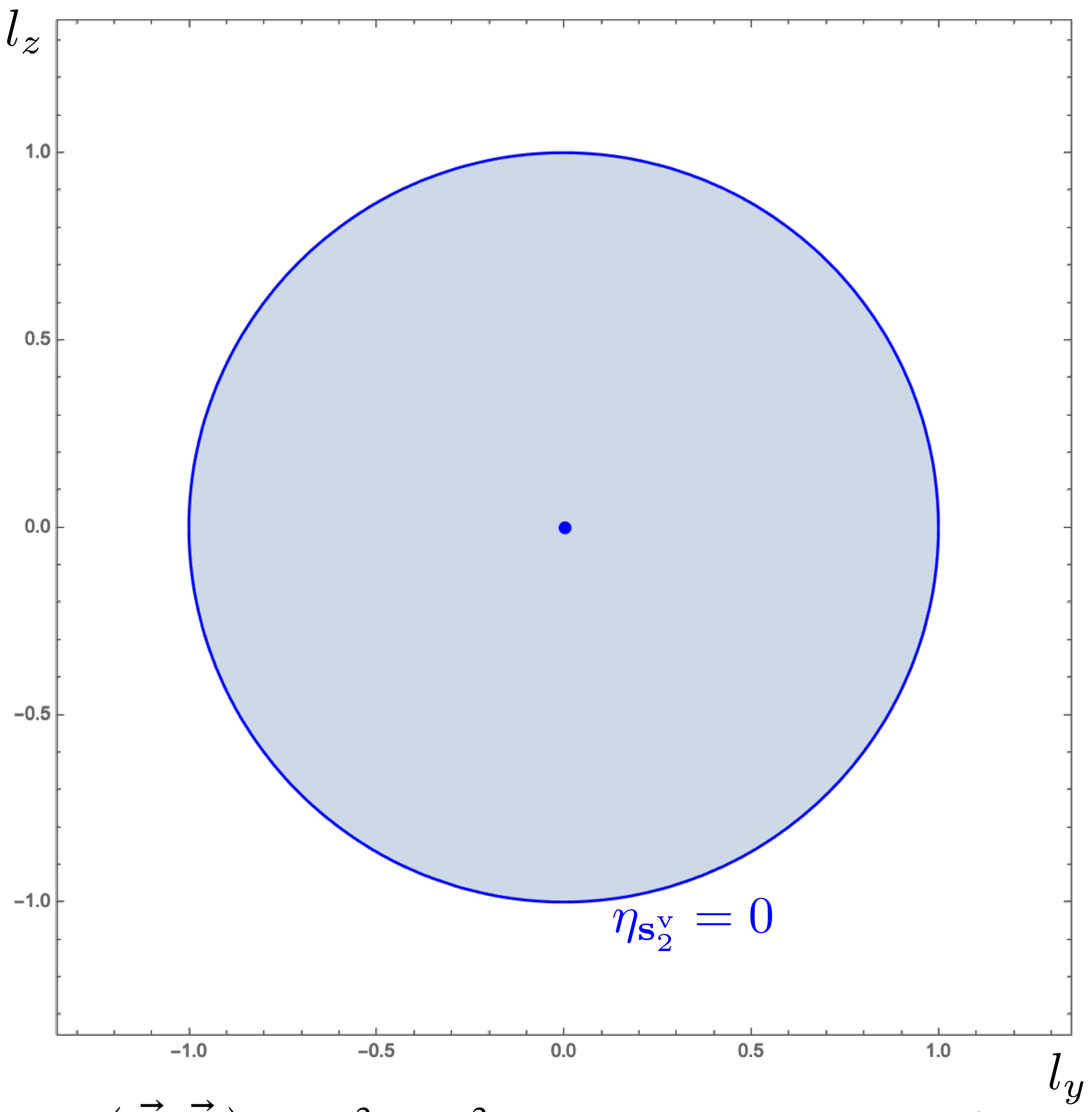

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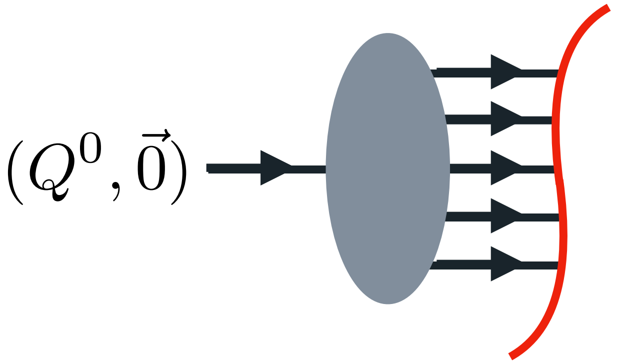

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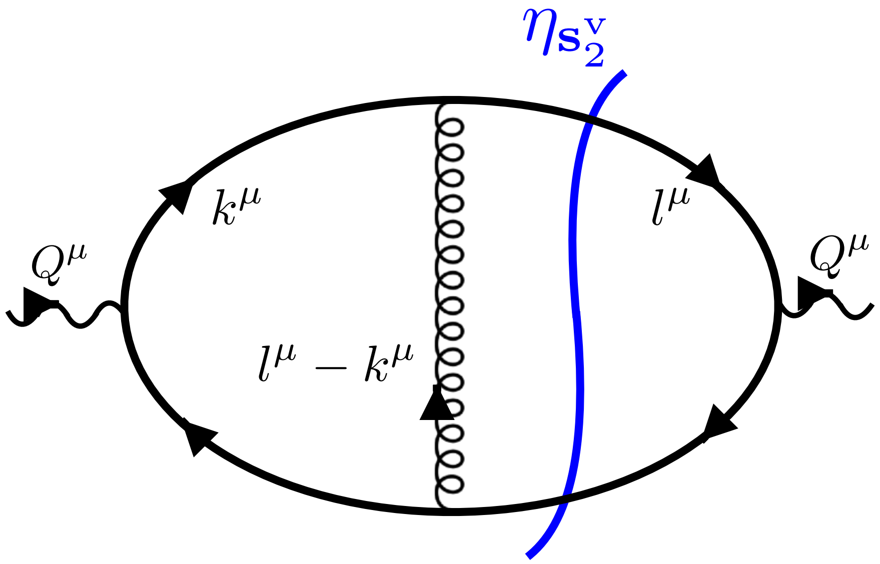

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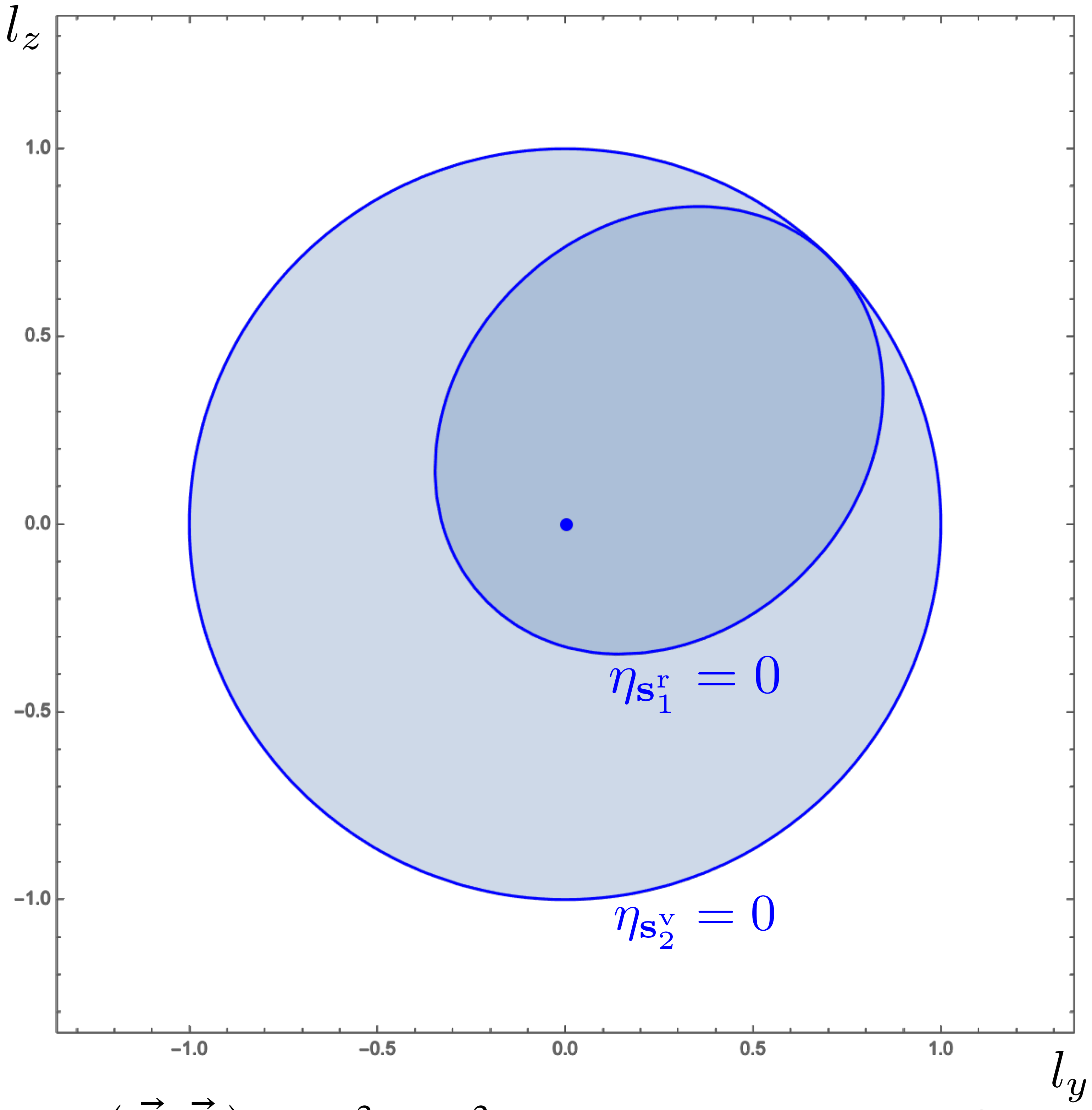

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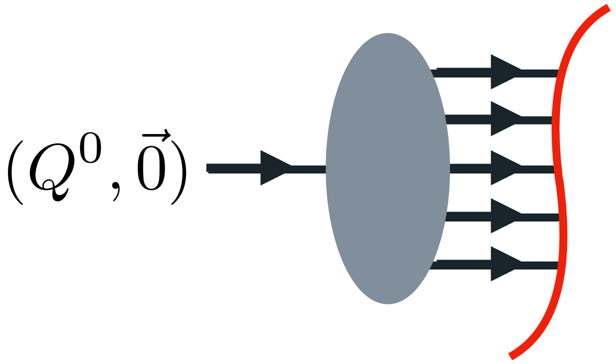

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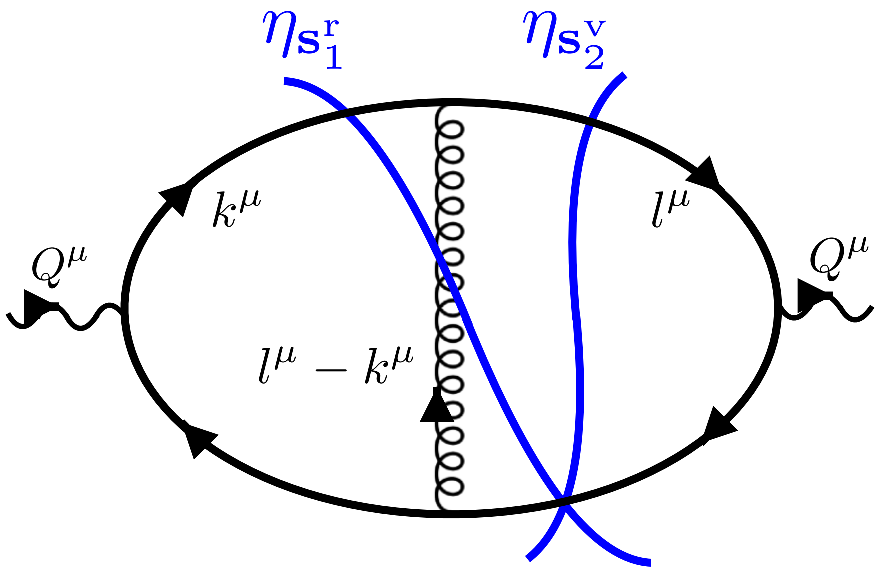

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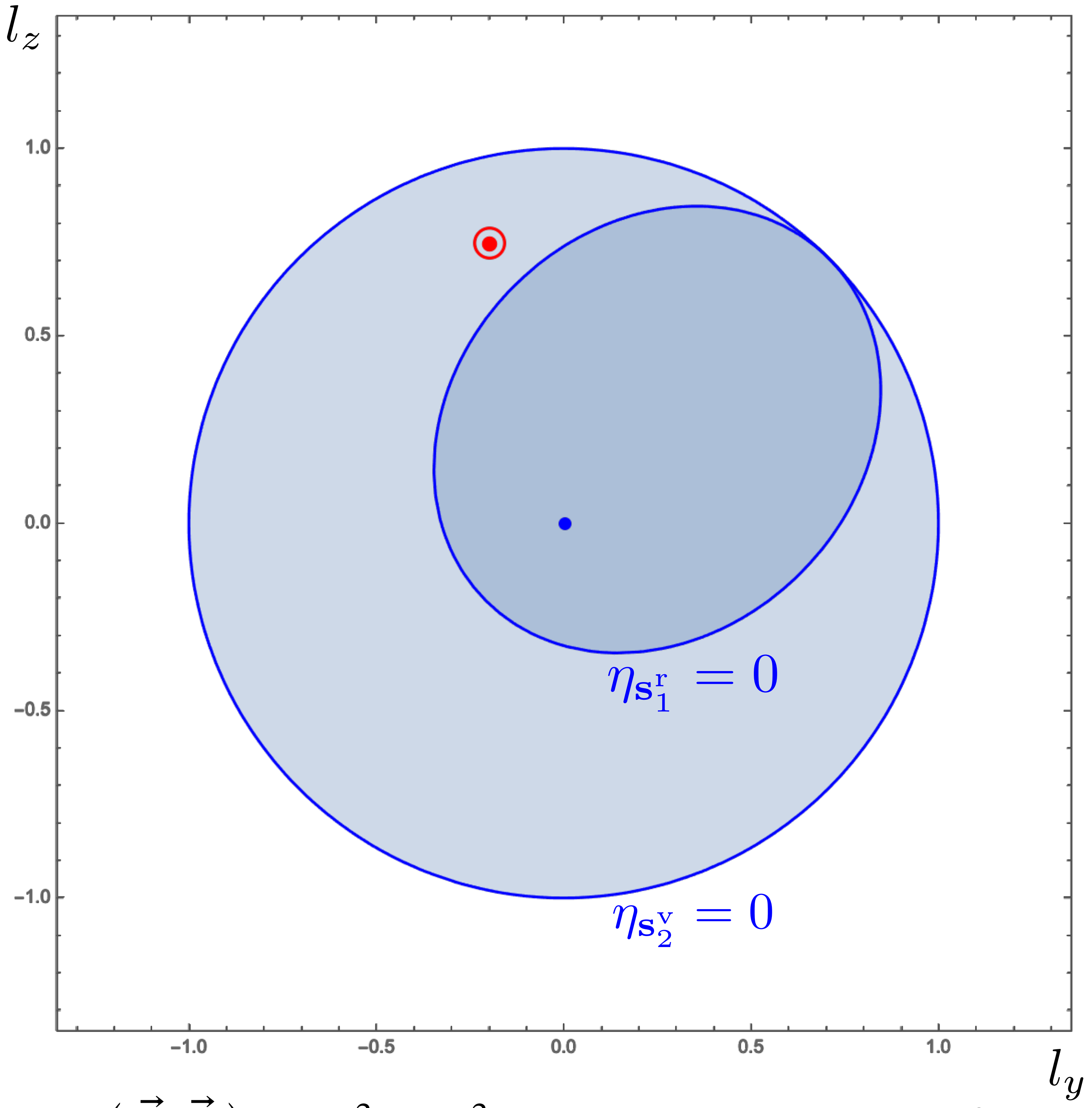

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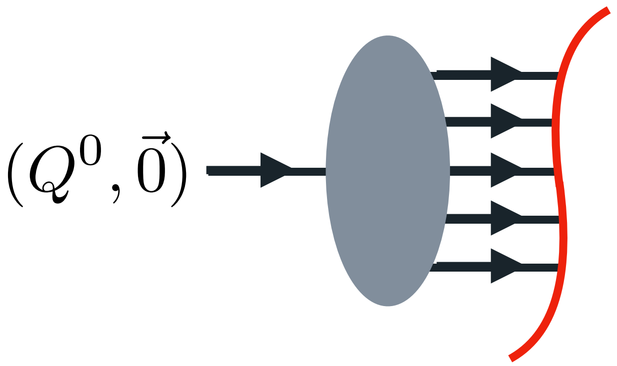

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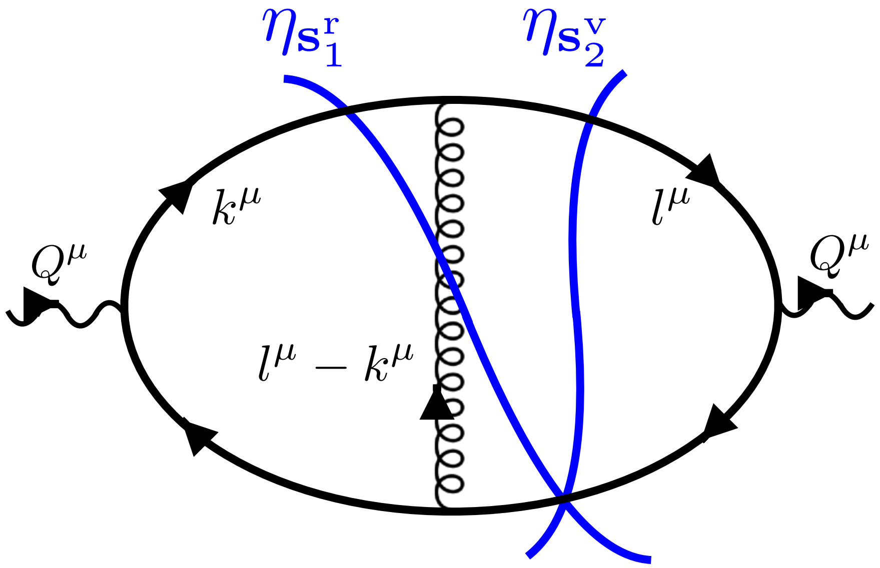

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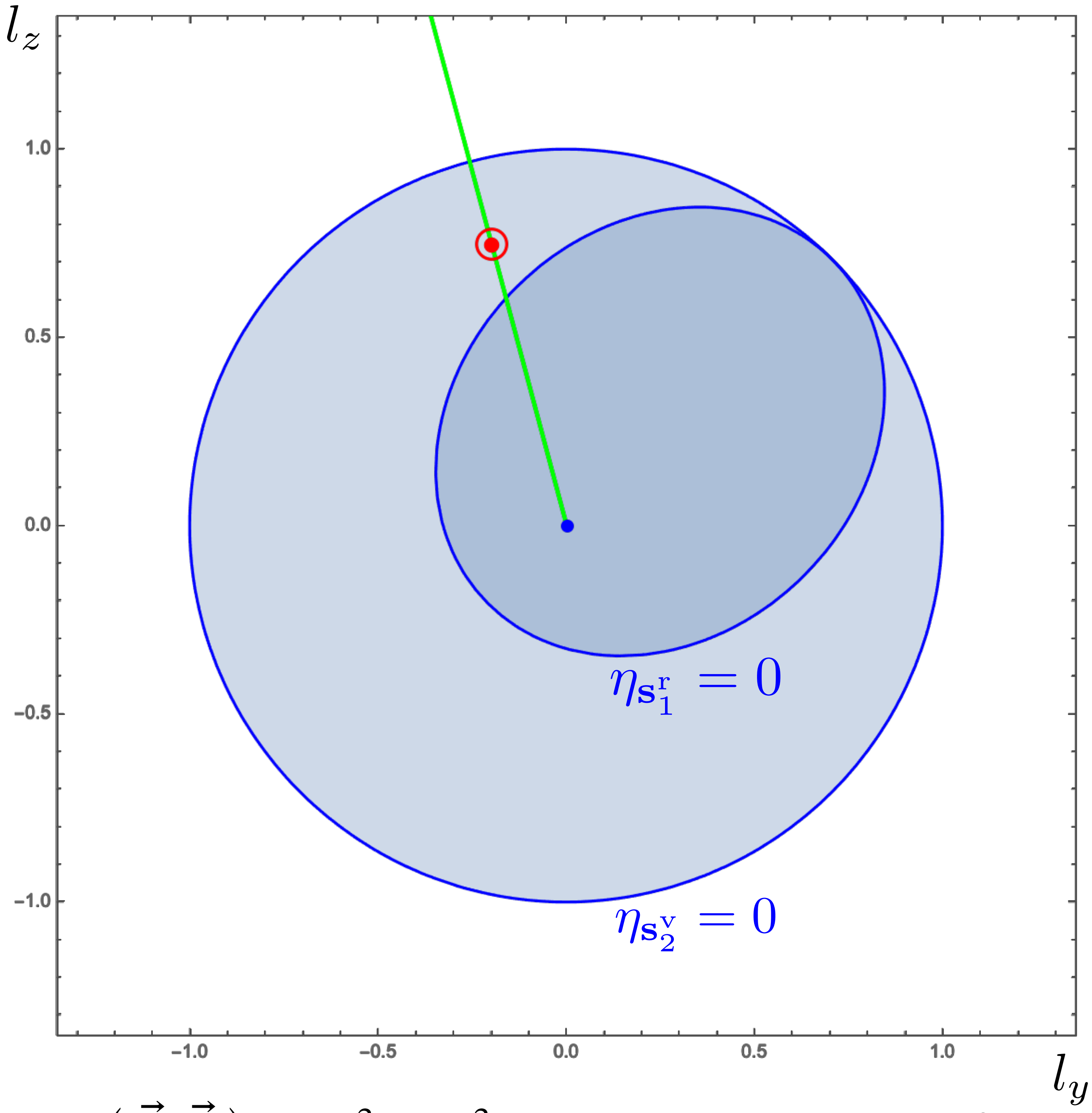

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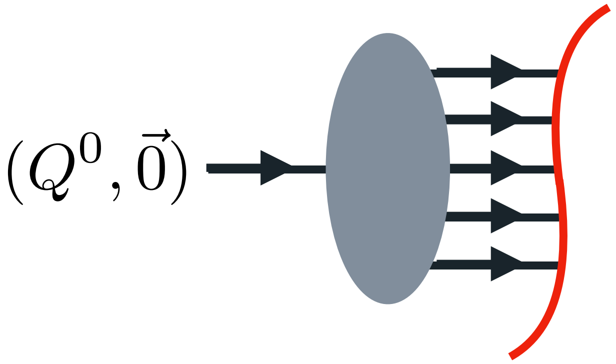

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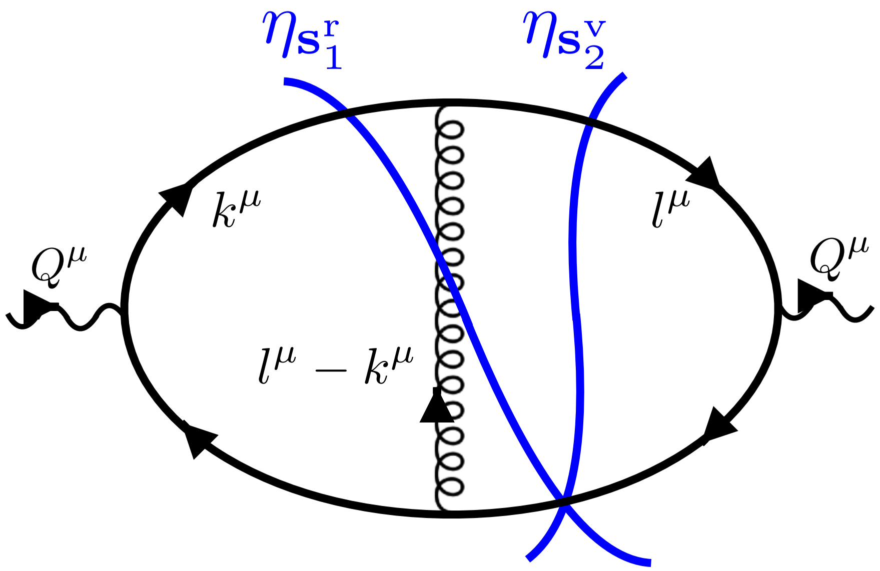

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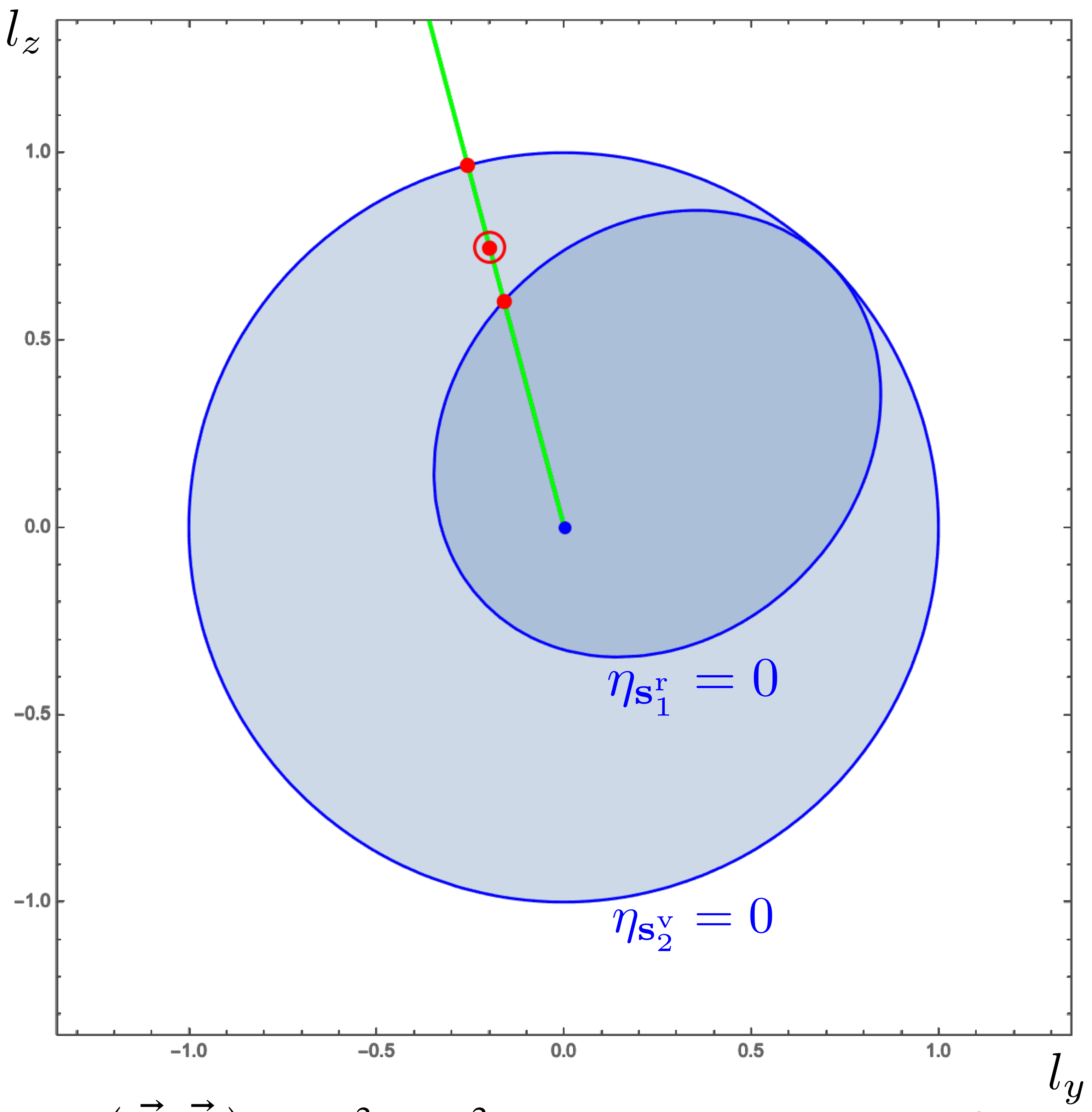

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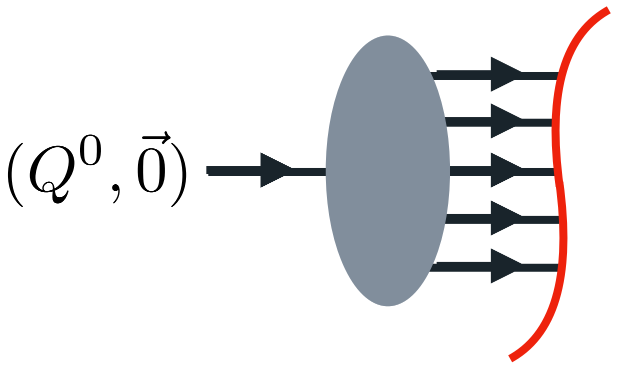

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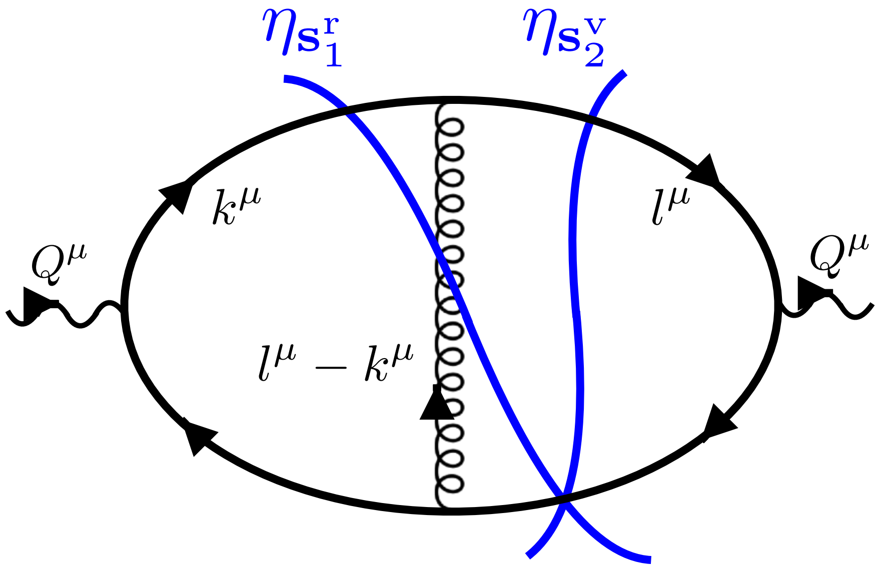

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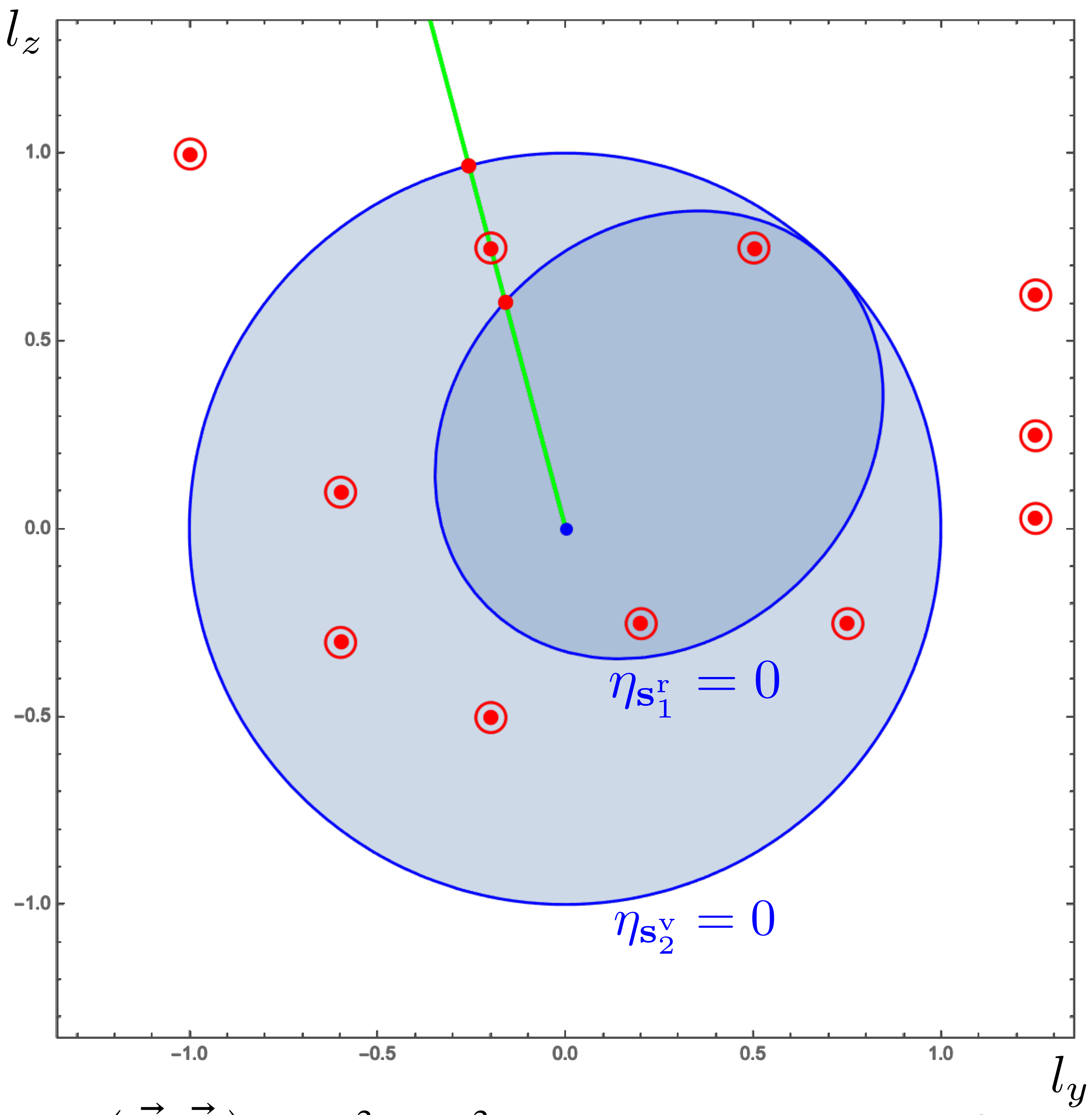

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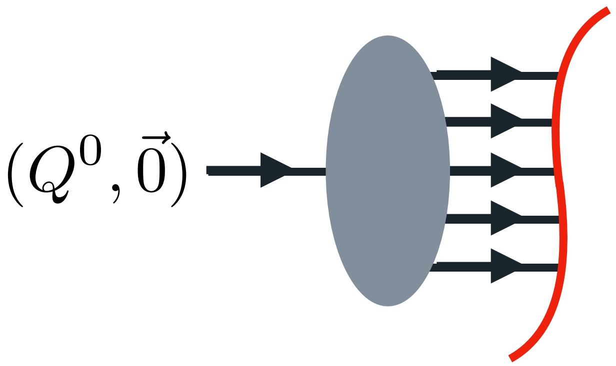

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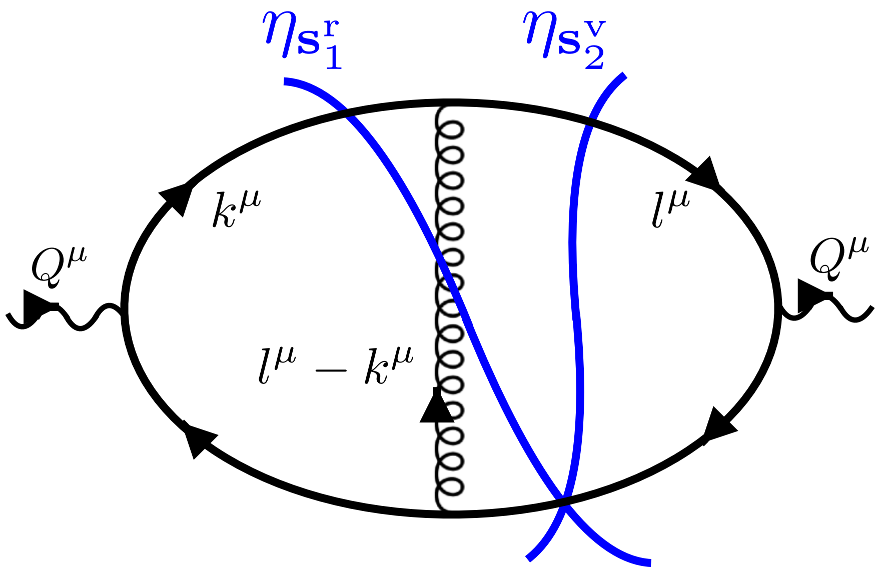

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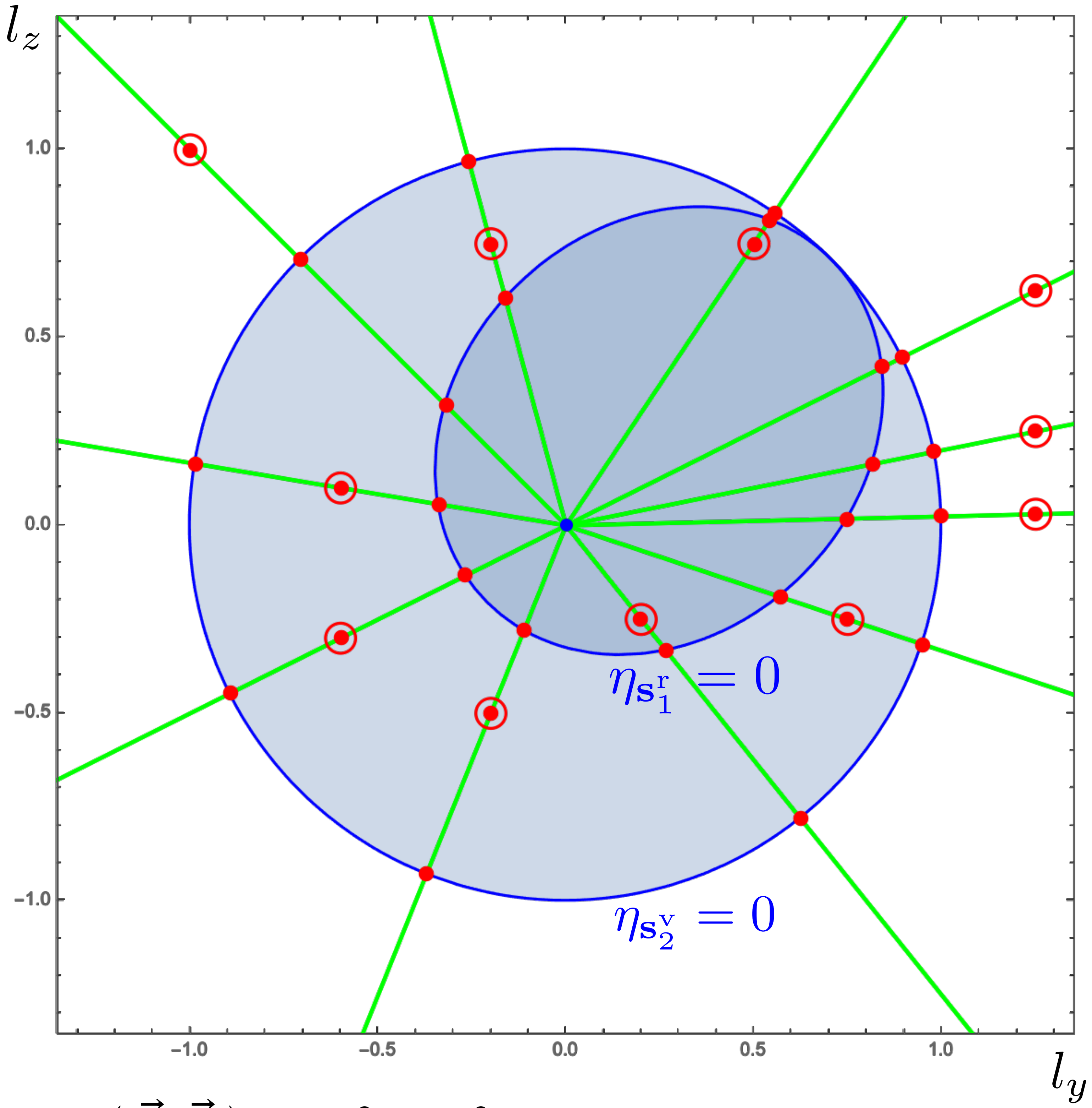

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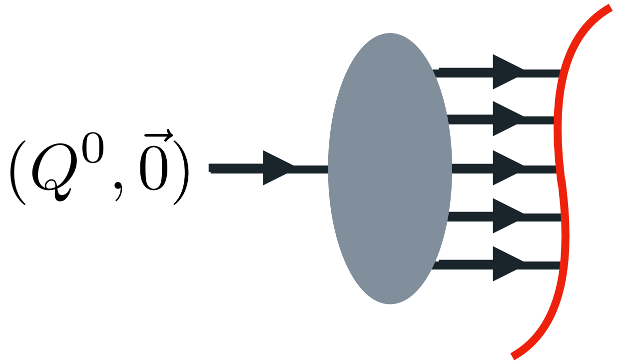

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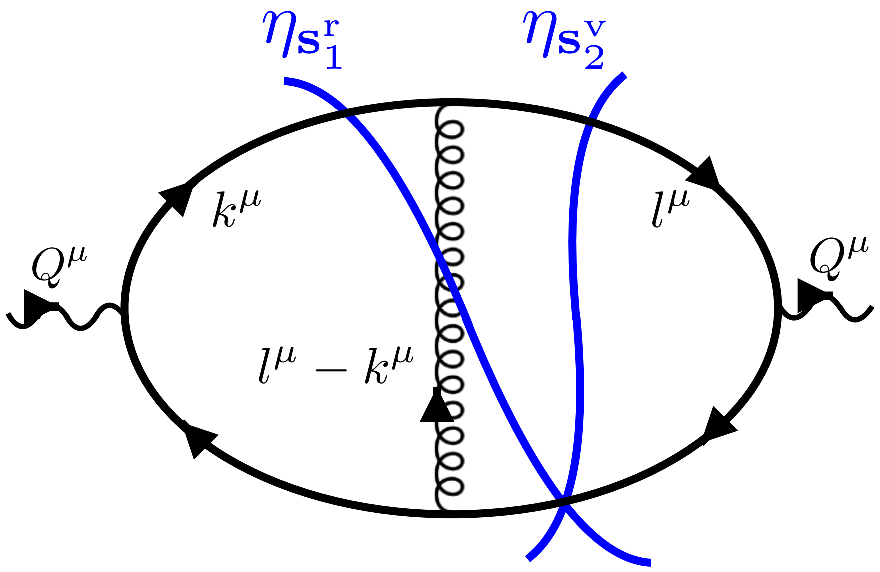

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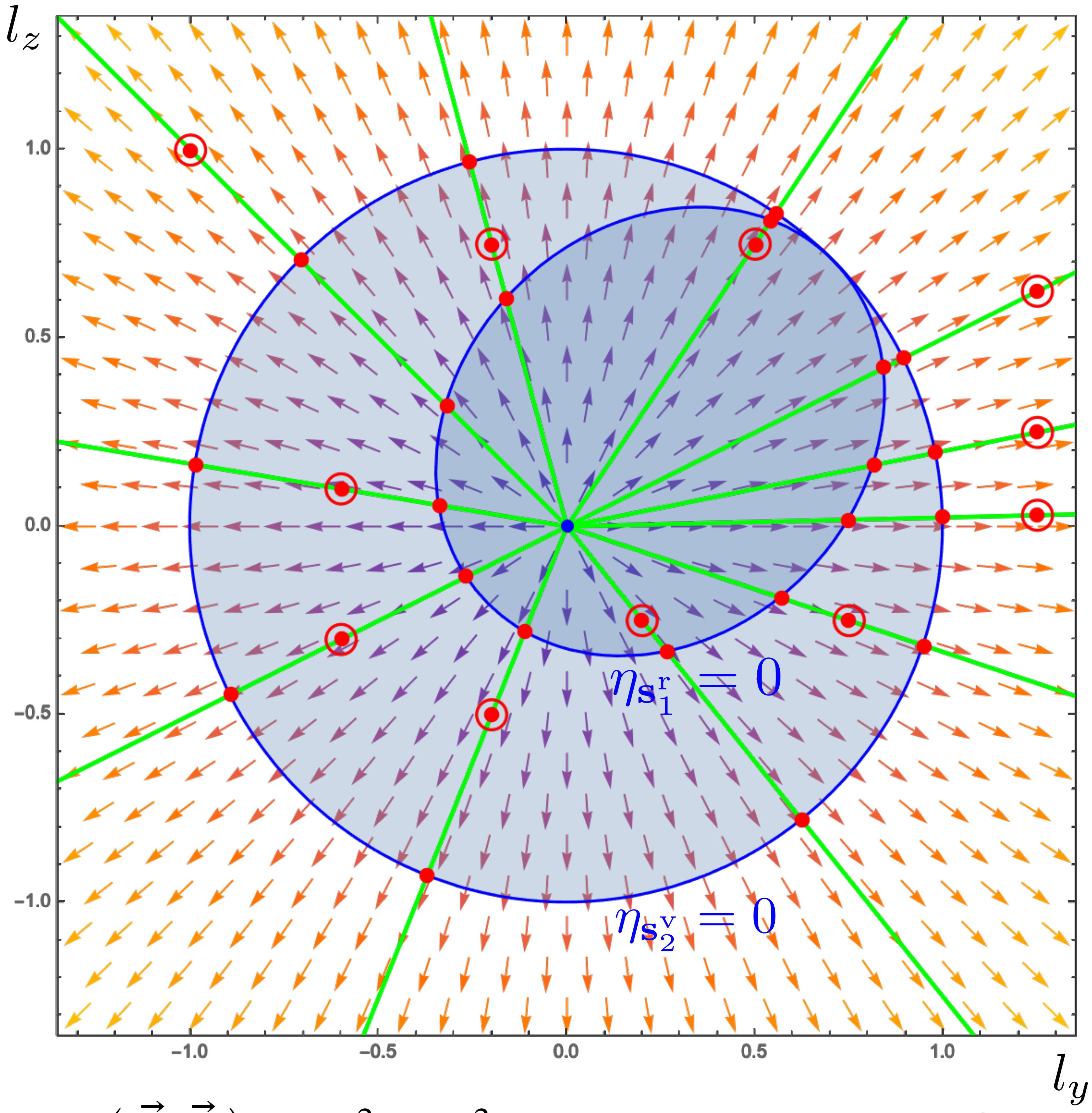

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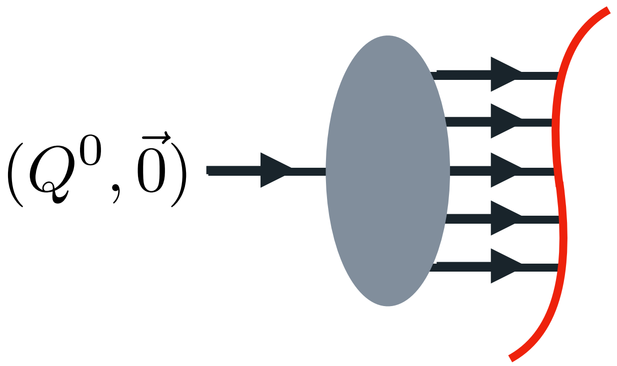

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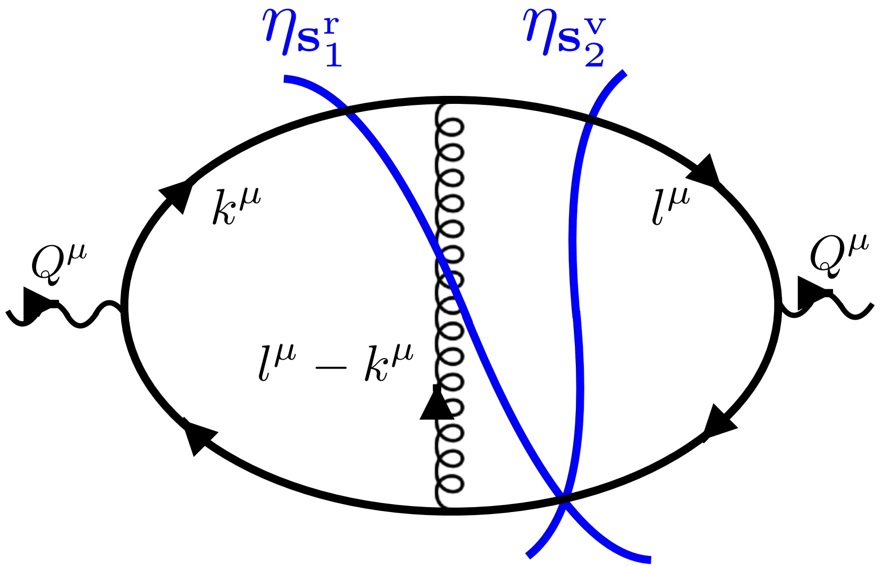

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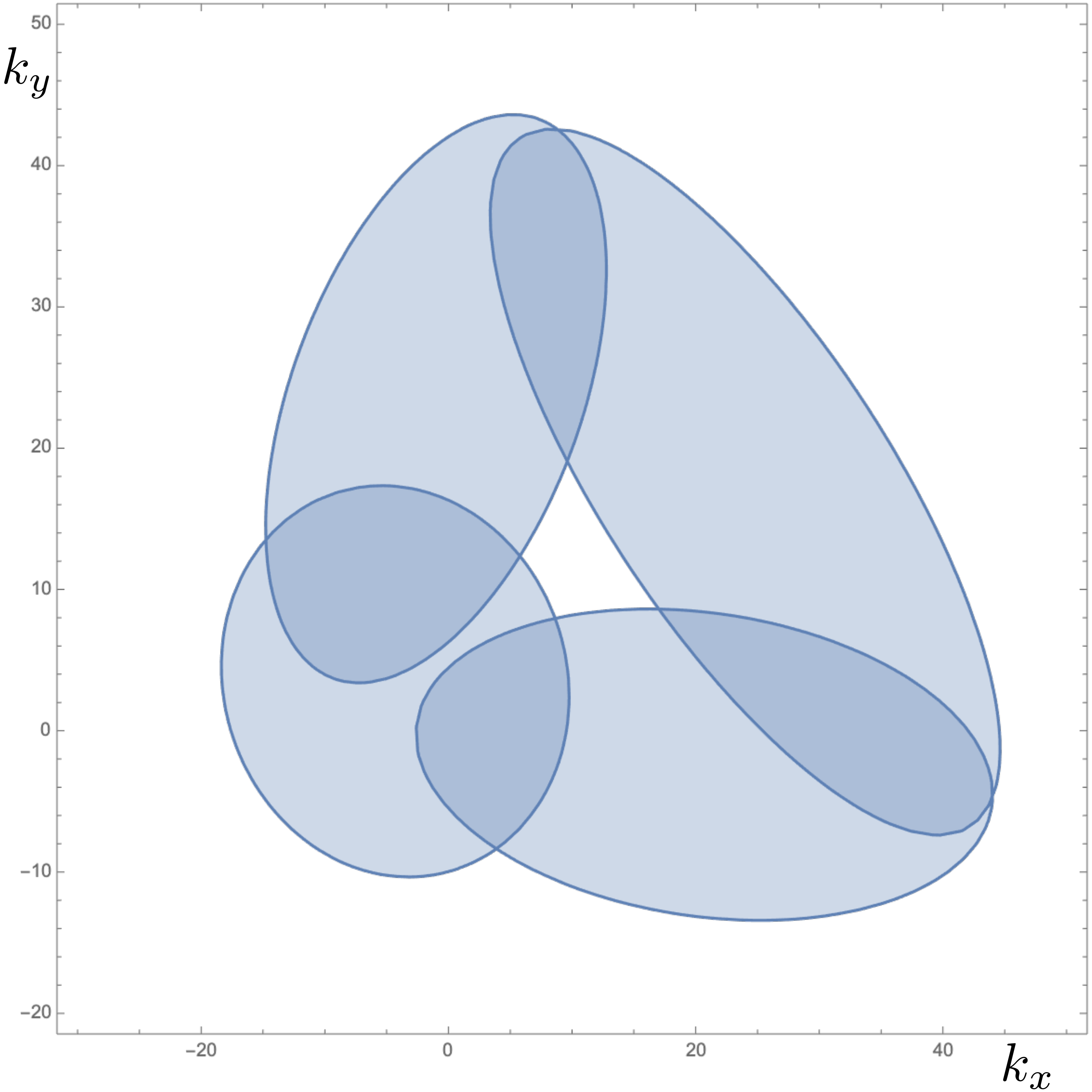
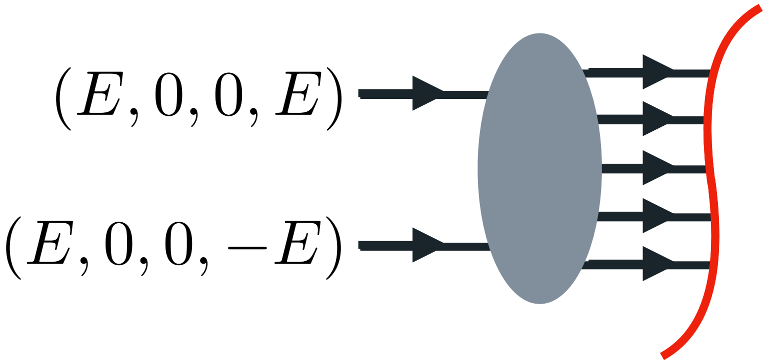
LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

The **rescaling** change of variables is however **not general** :





Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

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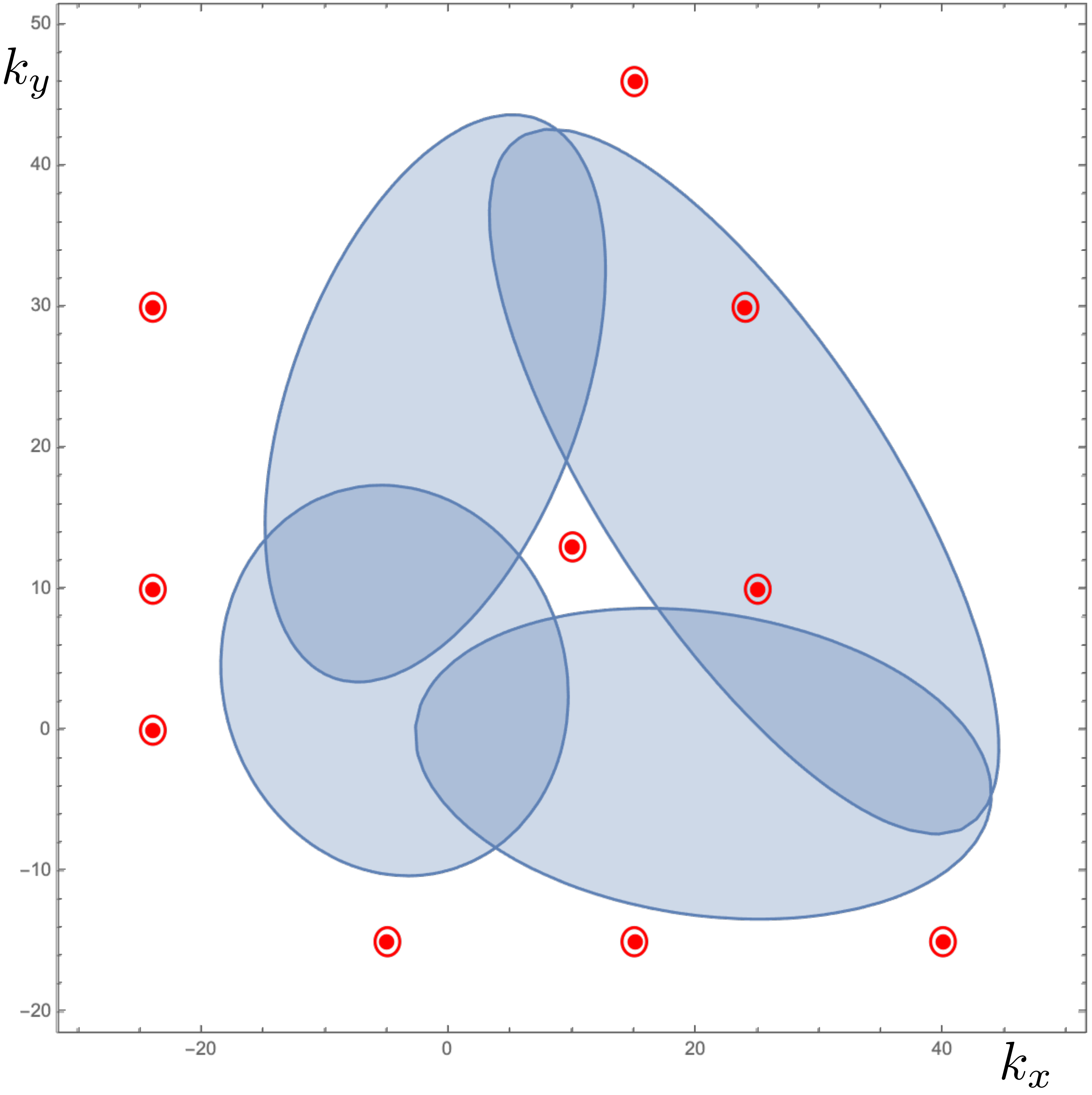

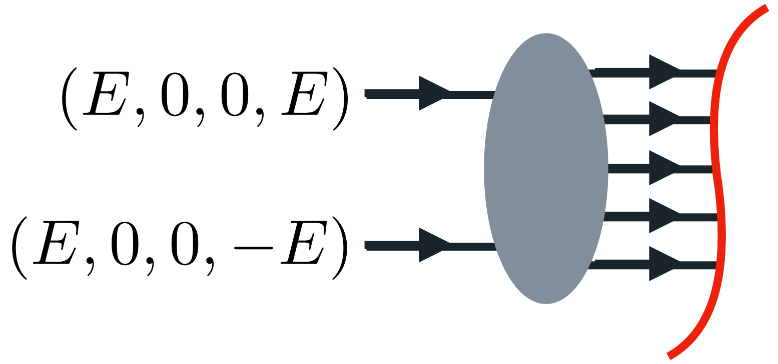

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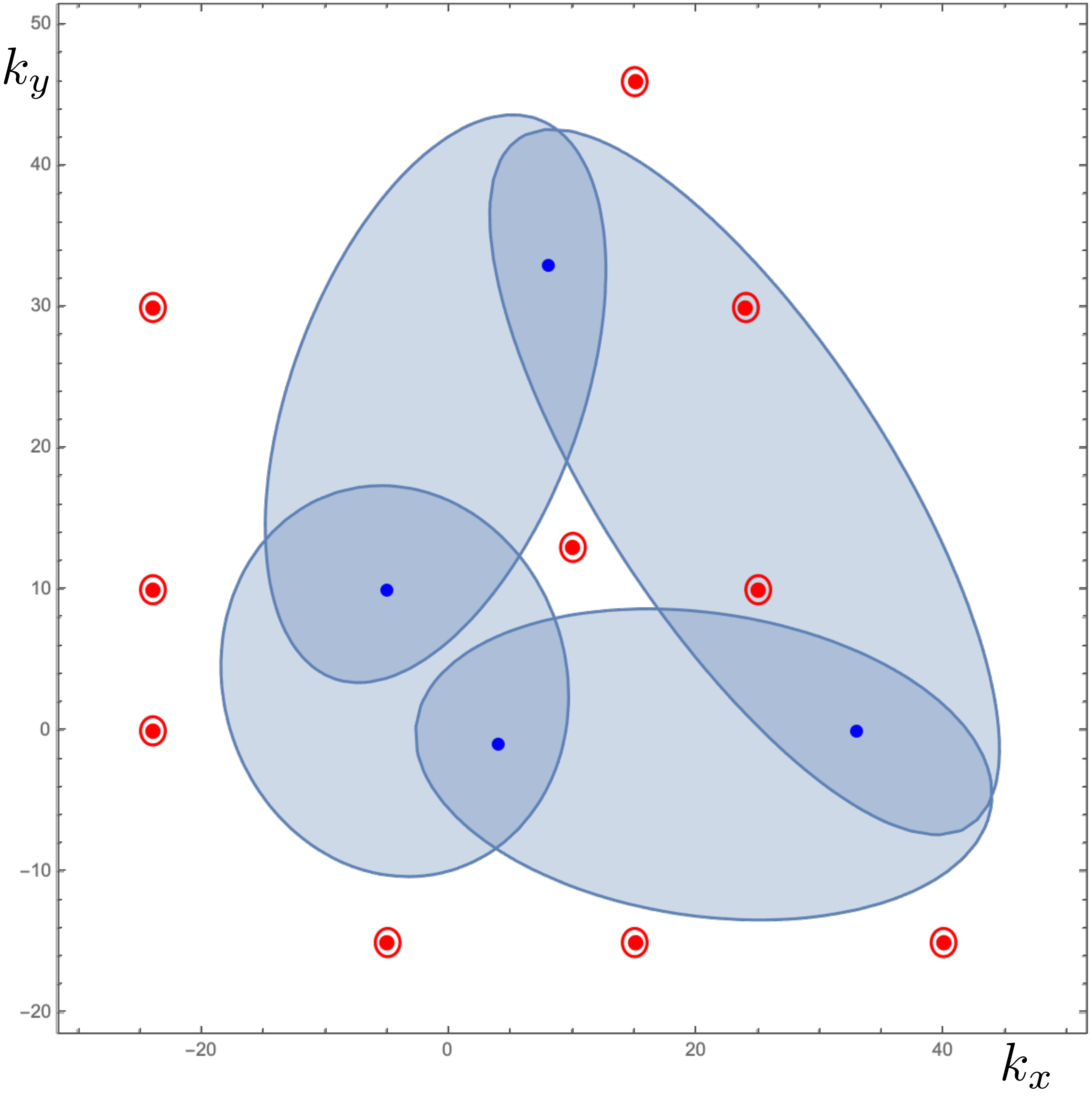

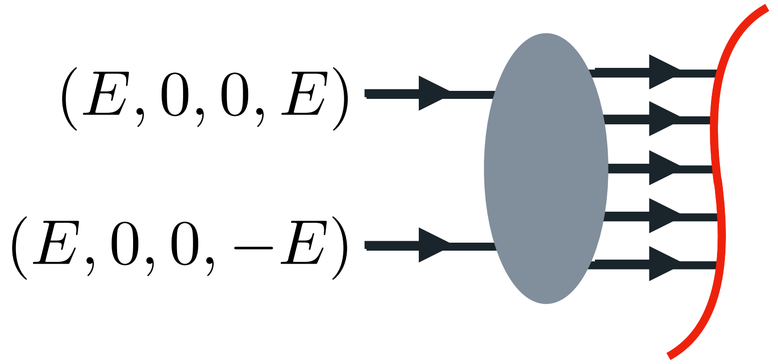

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[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY

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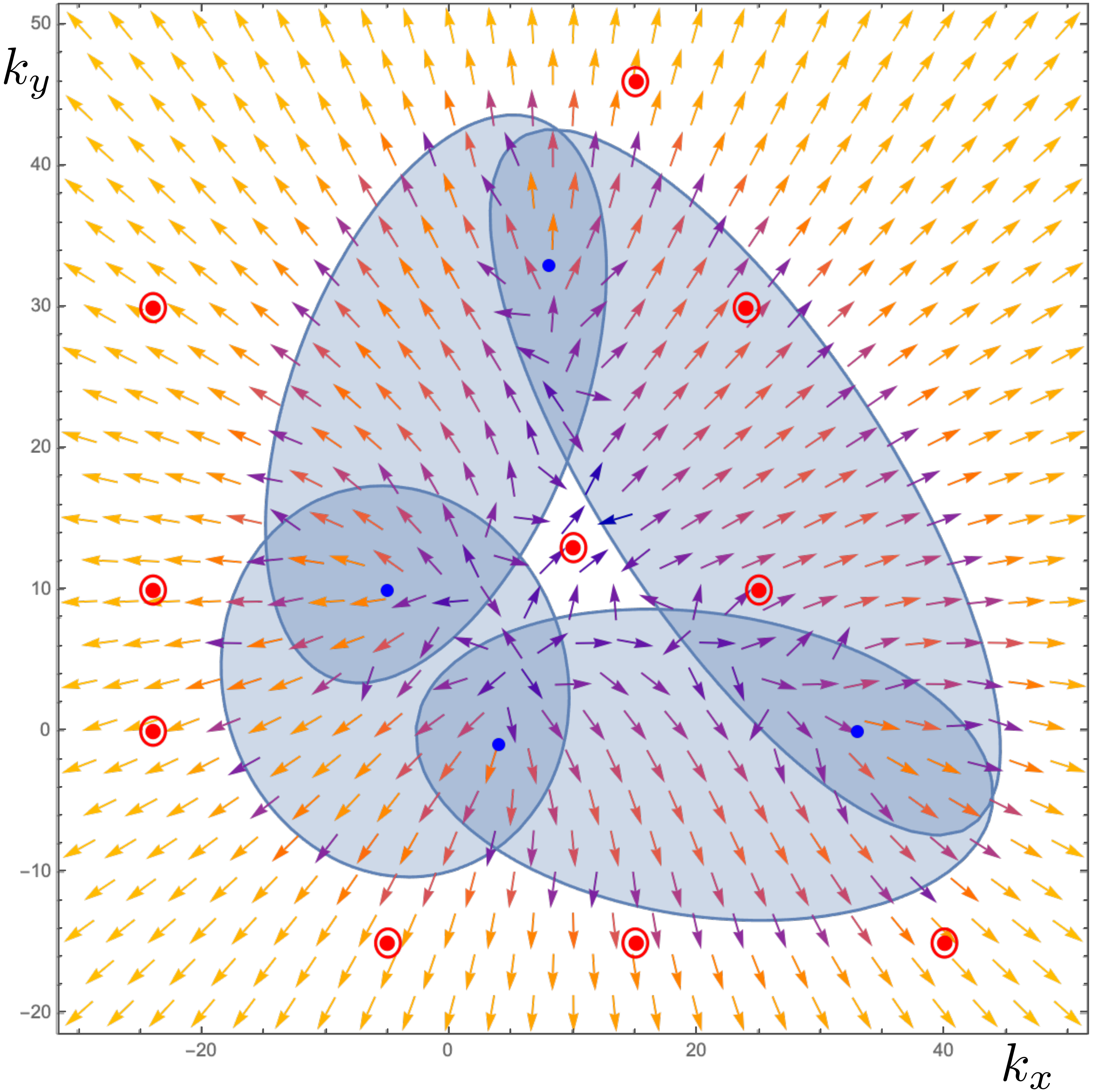

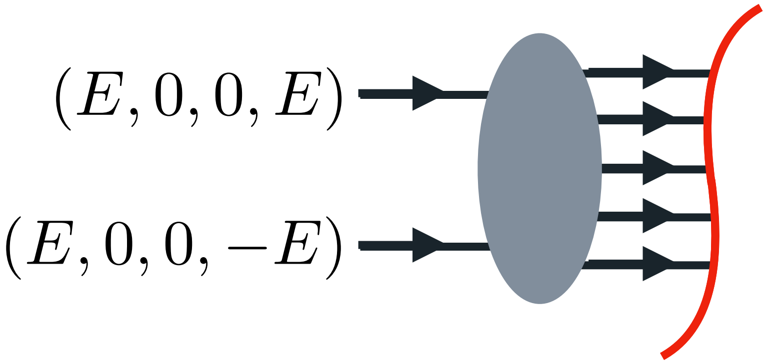

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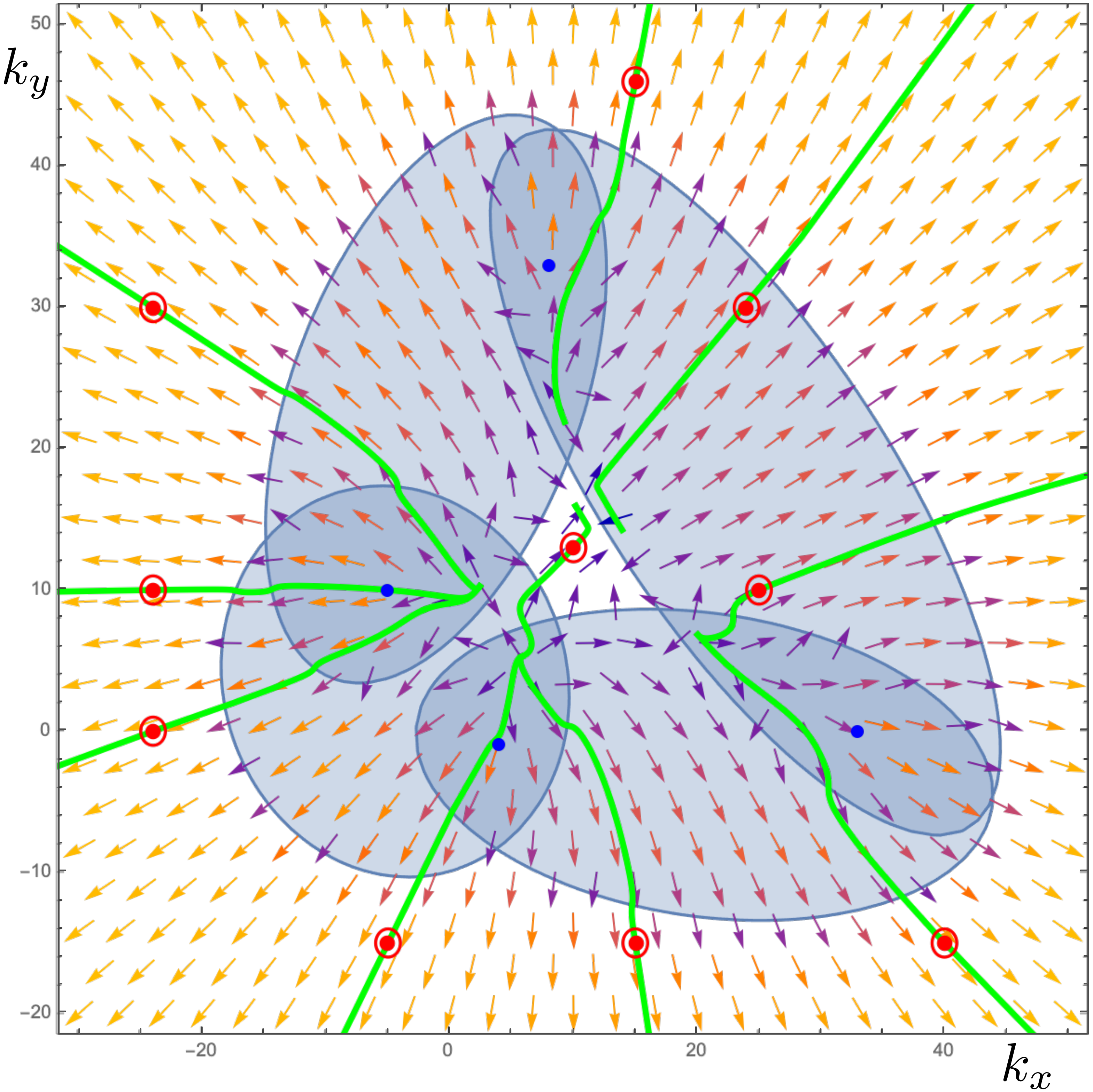

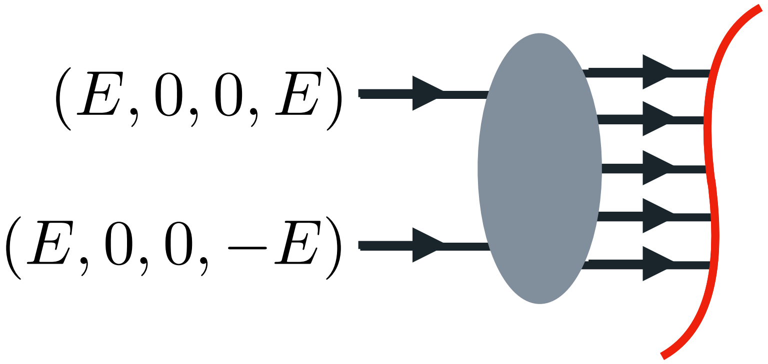

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In general, this **ODE** can be **solved numerically**.

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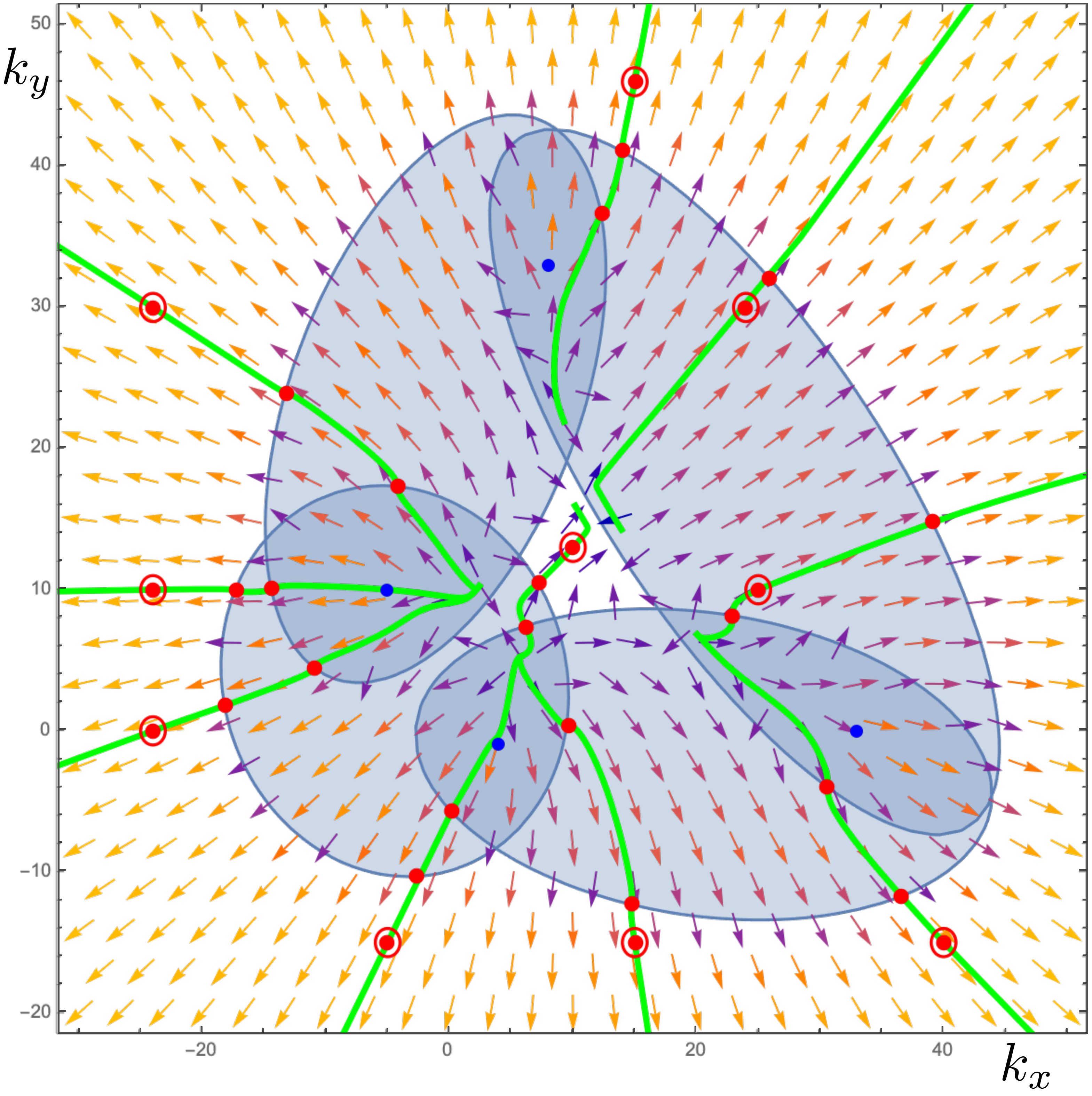

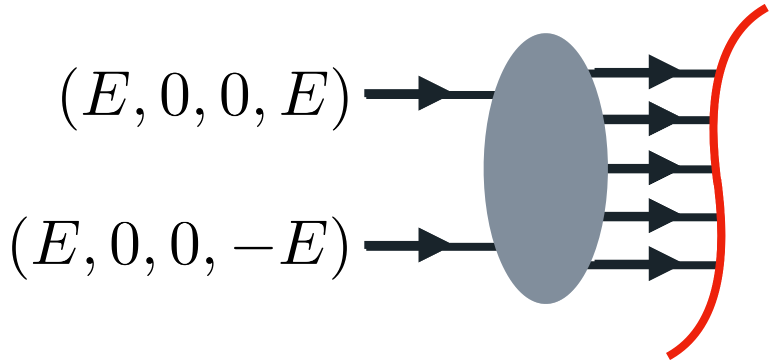

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[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068]   [Summary in proceedings, arXiv : 2110.15662]

Locality Unitarity: all-orders proof

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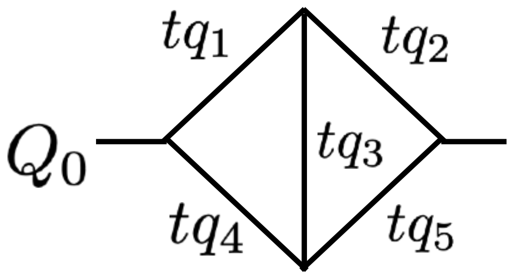

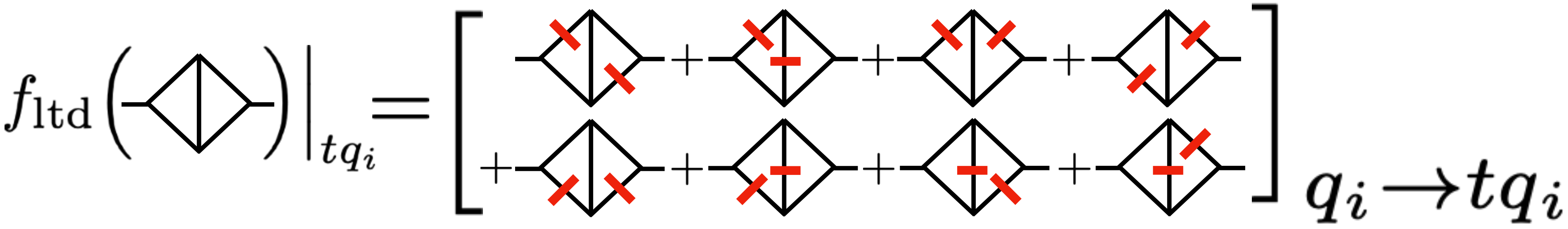

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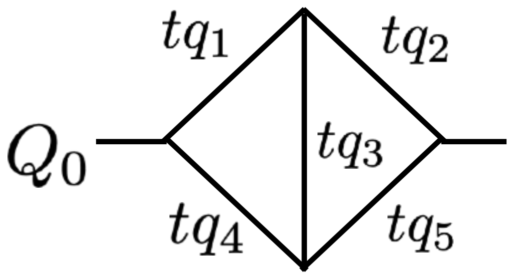
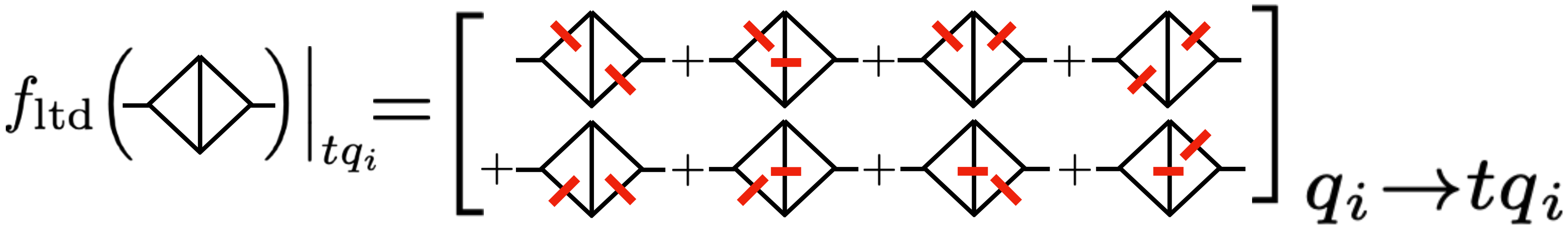
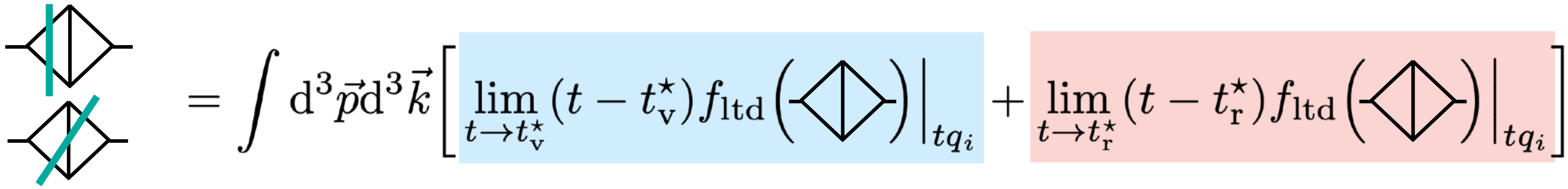




Then one can capture the **thresholds** of this forward-scattering graphs with



$g_{\rm v}$, $g_{\rm r}$ can be written as different limits of the same function!

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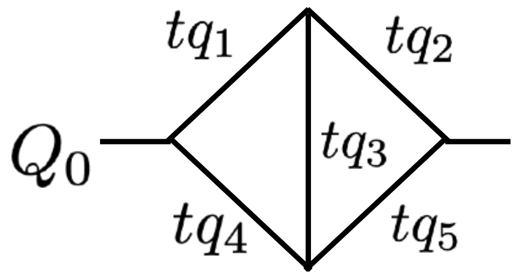

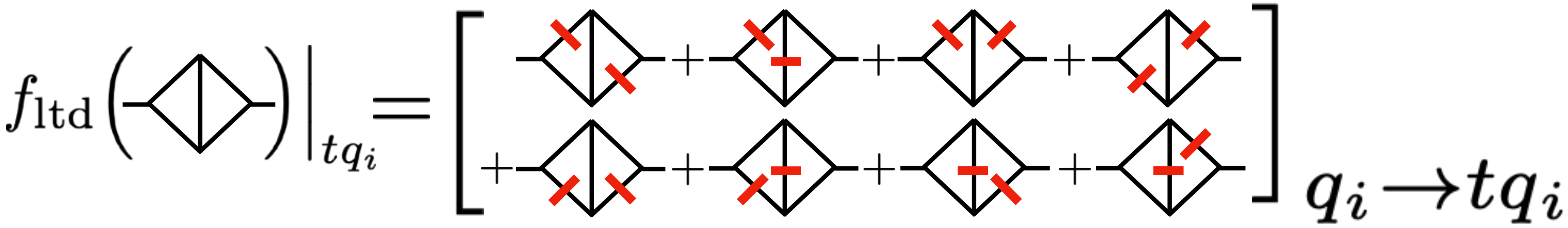

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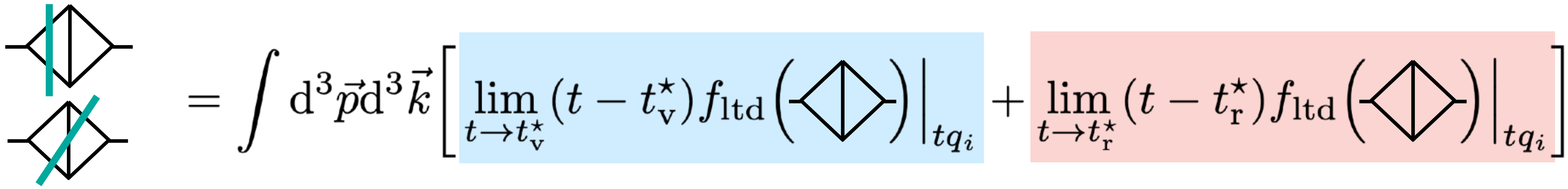

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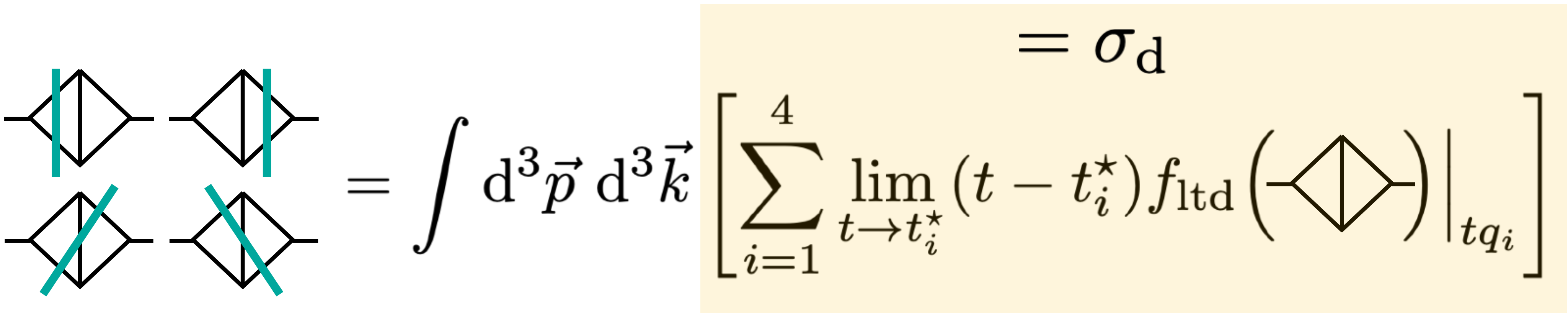
Solving delta in the scaling variable $\Longrightarrow$ **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$



LU representation

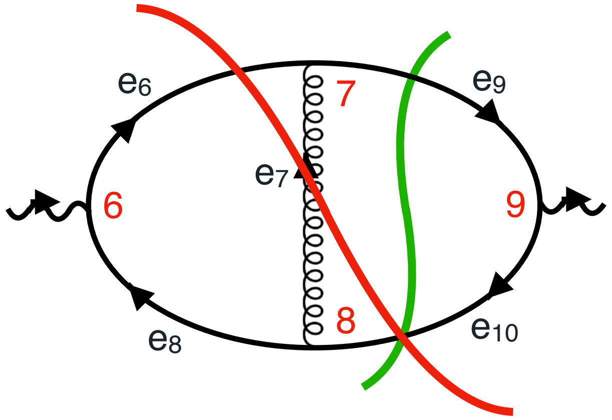
Cutkosky, but at the local level! We prove cancellations by studying the limit $t_r^\star \to t_{\rm v}^\star$

LOCALITY UNITARITY

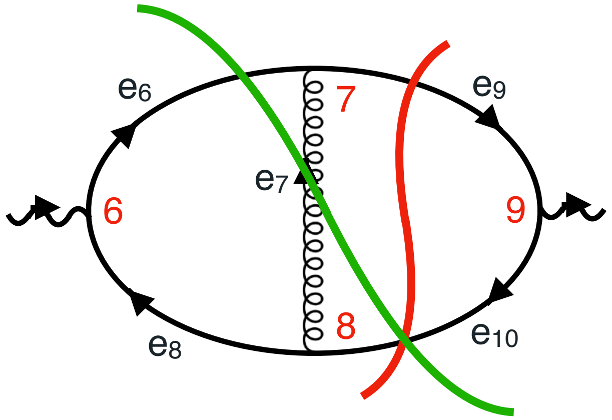
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

🟥 = Cutkosky cut 🟩 = threshold singularity



cancels



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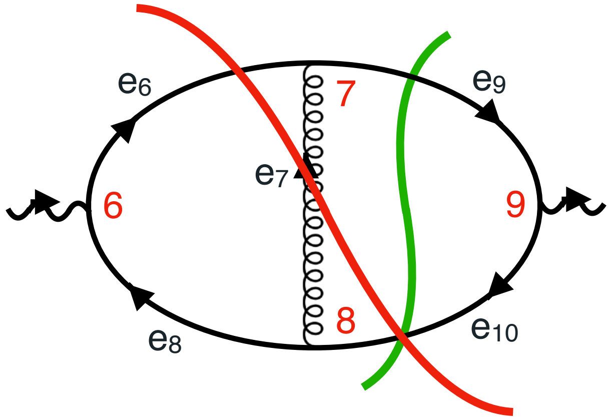

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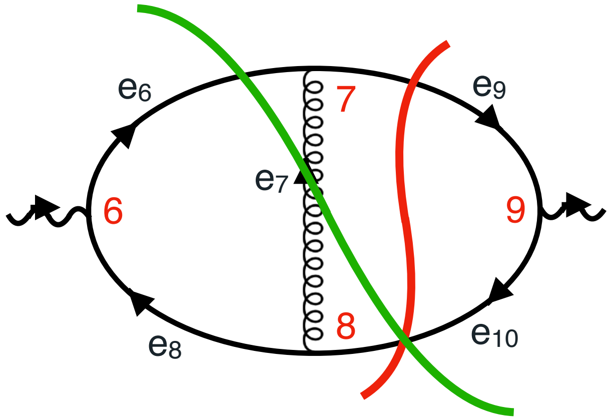

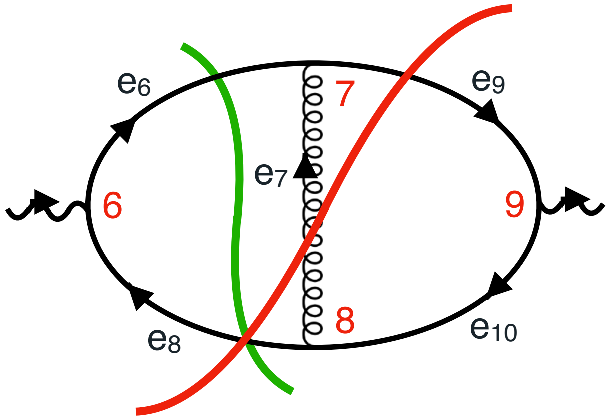

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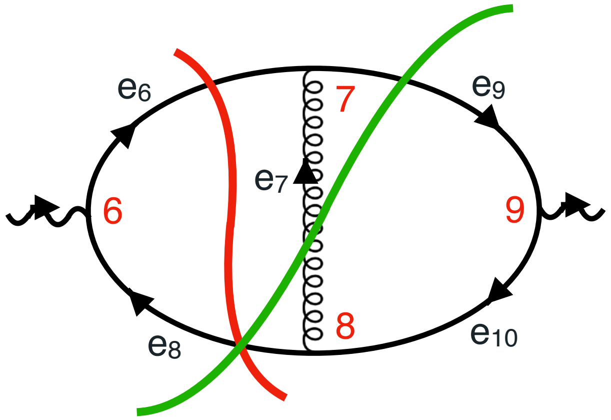

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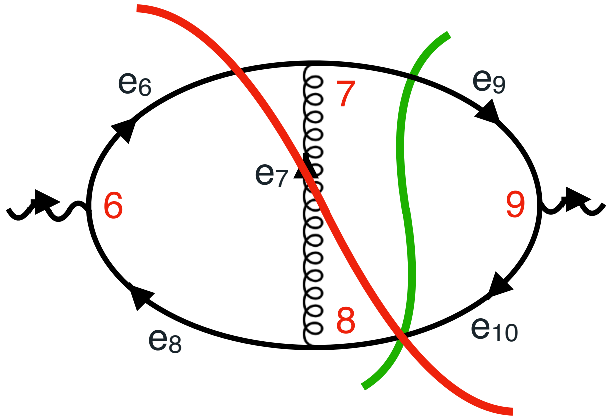

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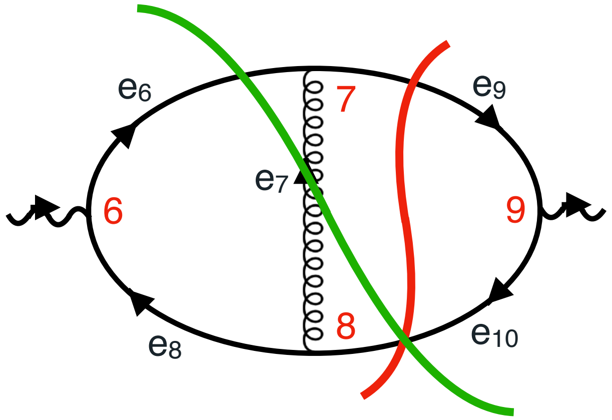

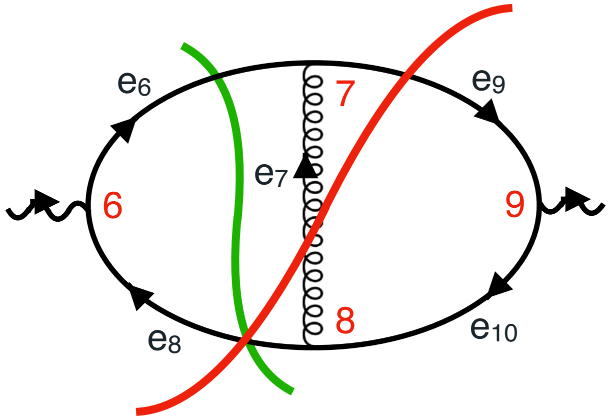

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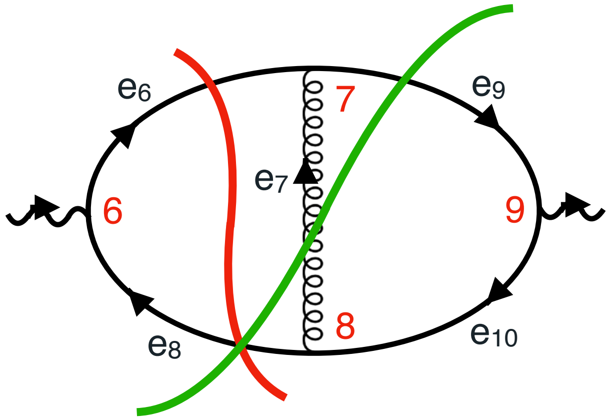
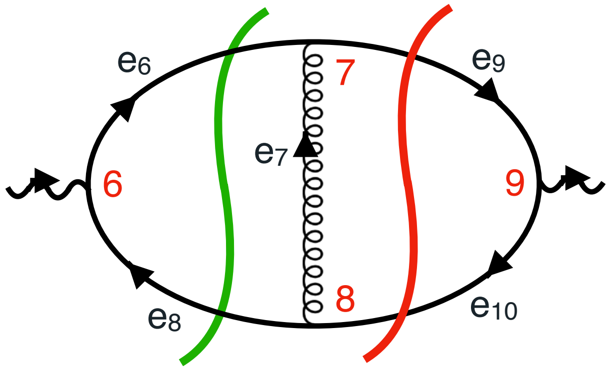


Even for **non-pinched singular threshold** ! (when $\mathcal{O}_{\mathbf{s}} \equiv 1$) :



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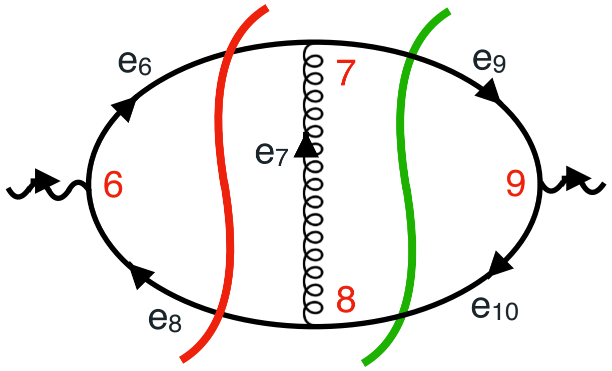

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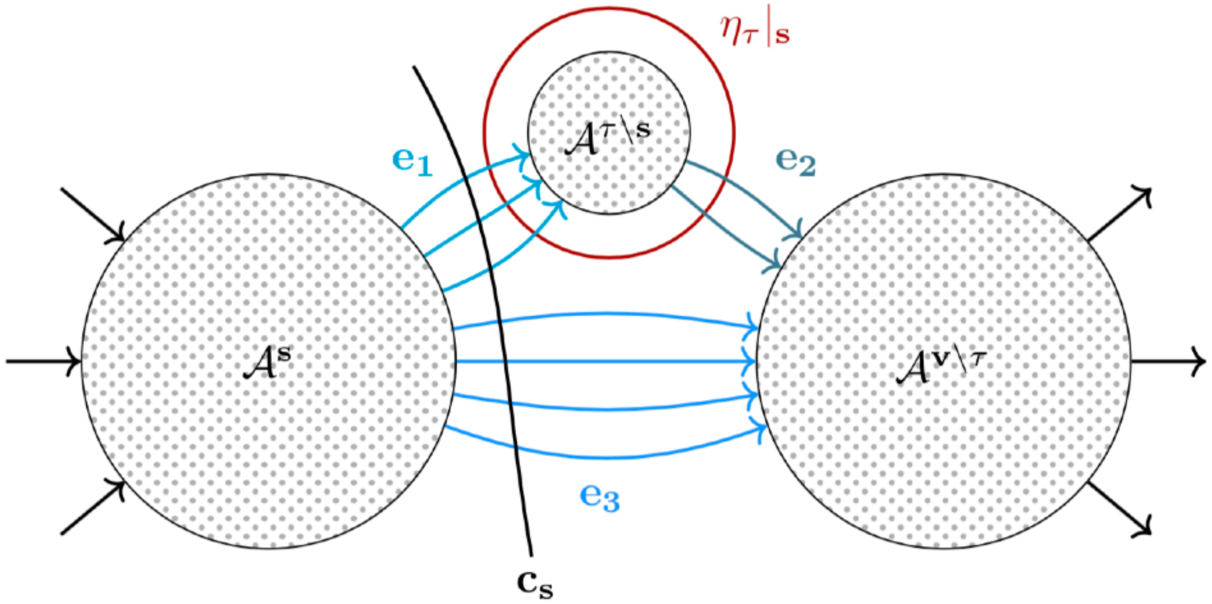
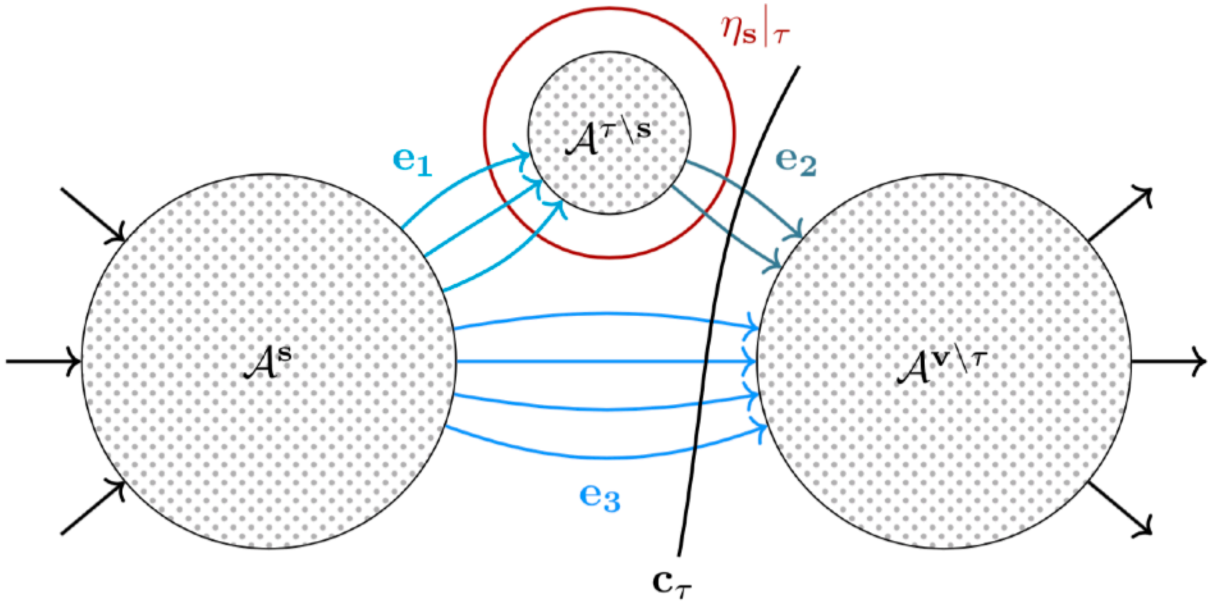
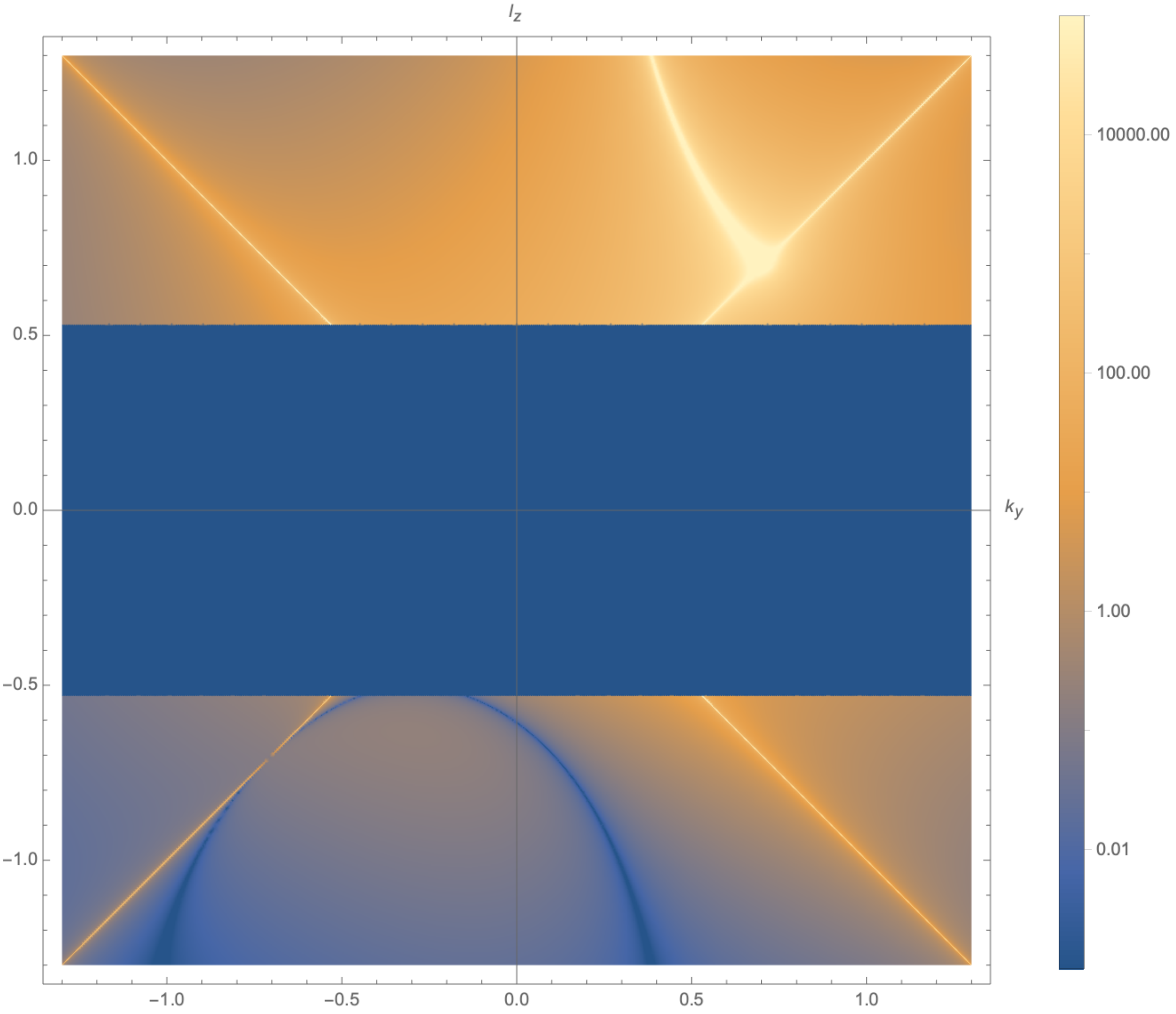


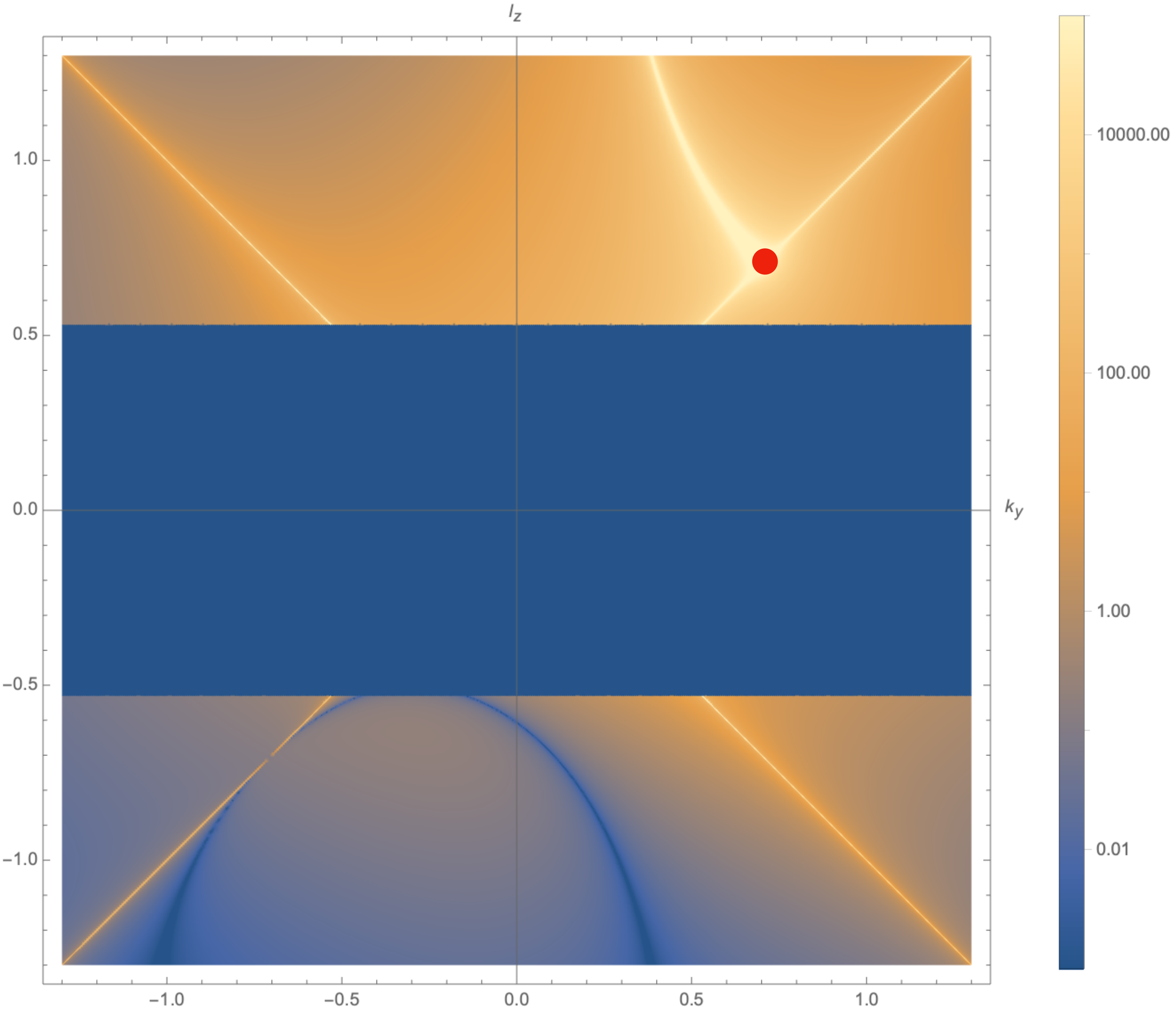


$I_{\mathbf{s}_1^{\mathrm{v}}}$



$I_{\mathbf{s}_1^{\mathrm{v}}}$

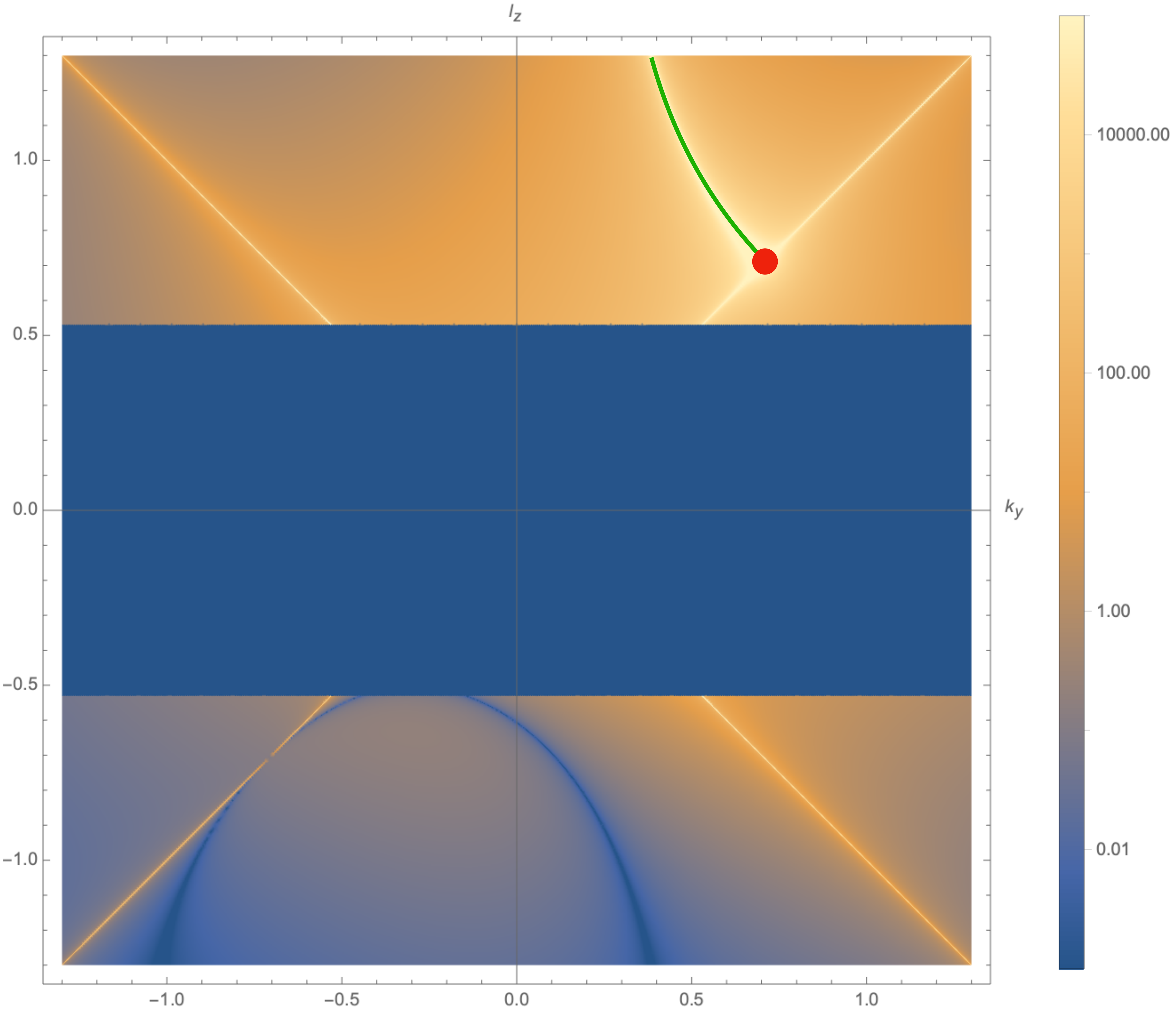
soft



$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

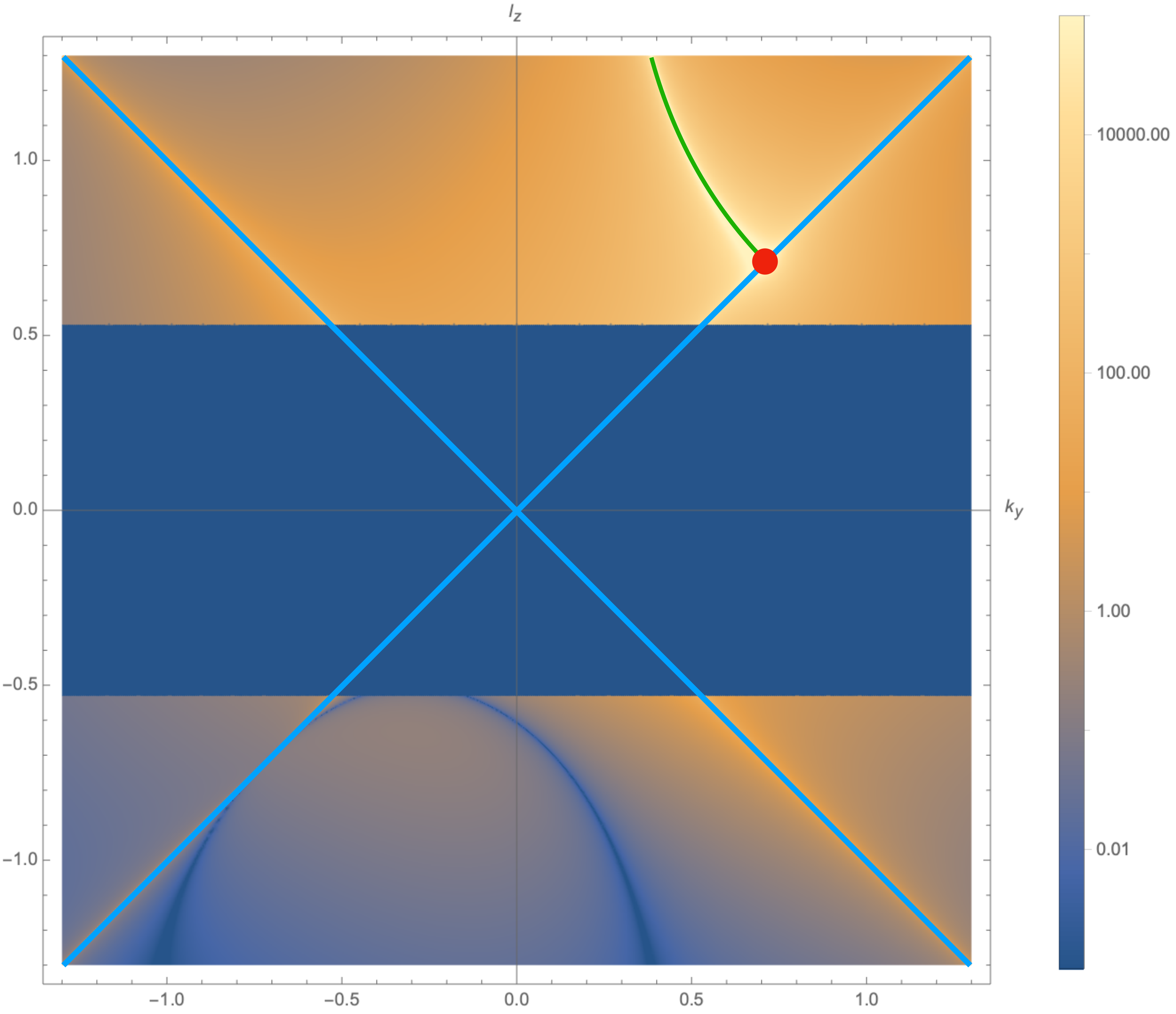


$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

non-pinch cancellation



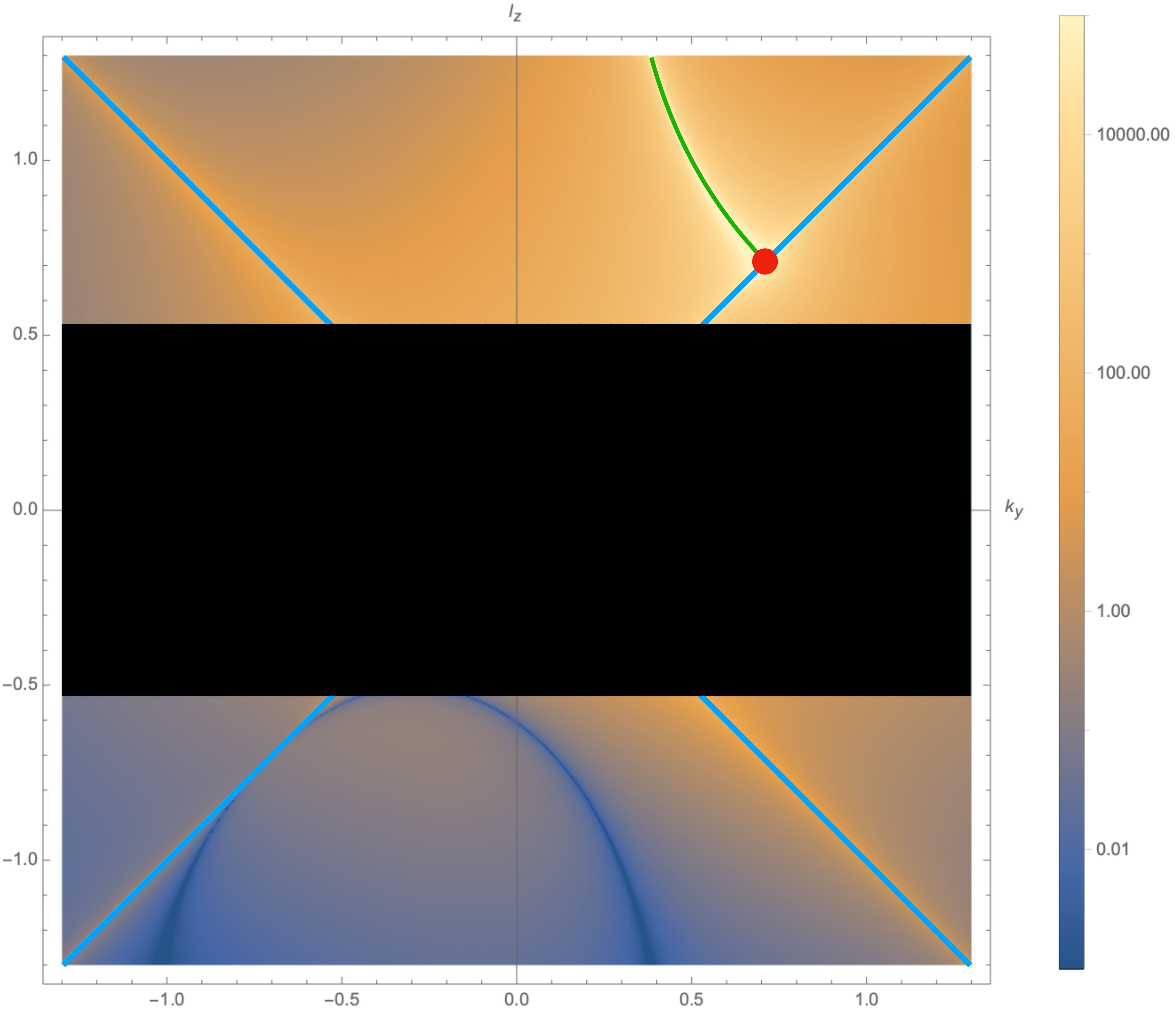
$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

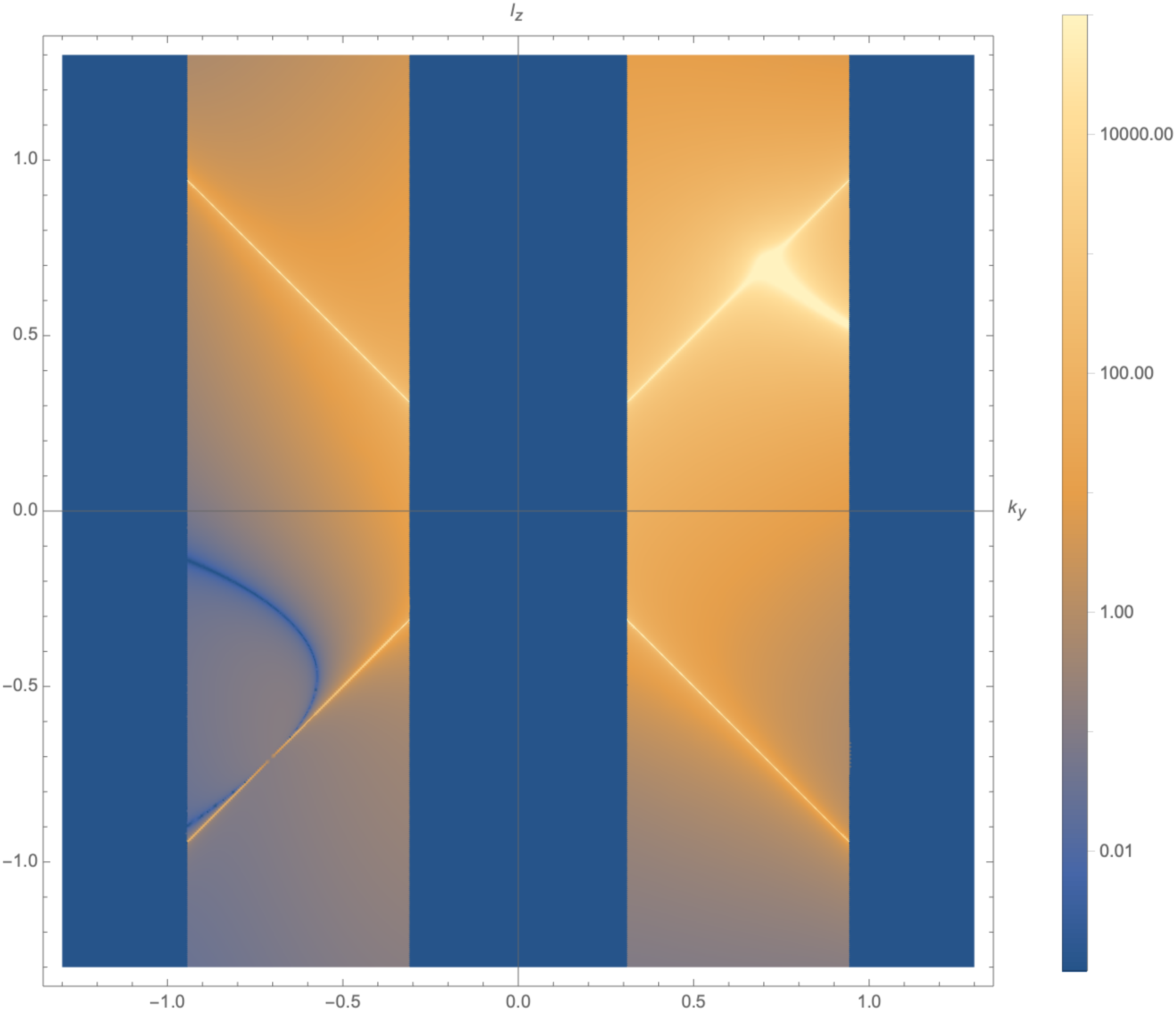
non-pinch cancellation

observable cuts

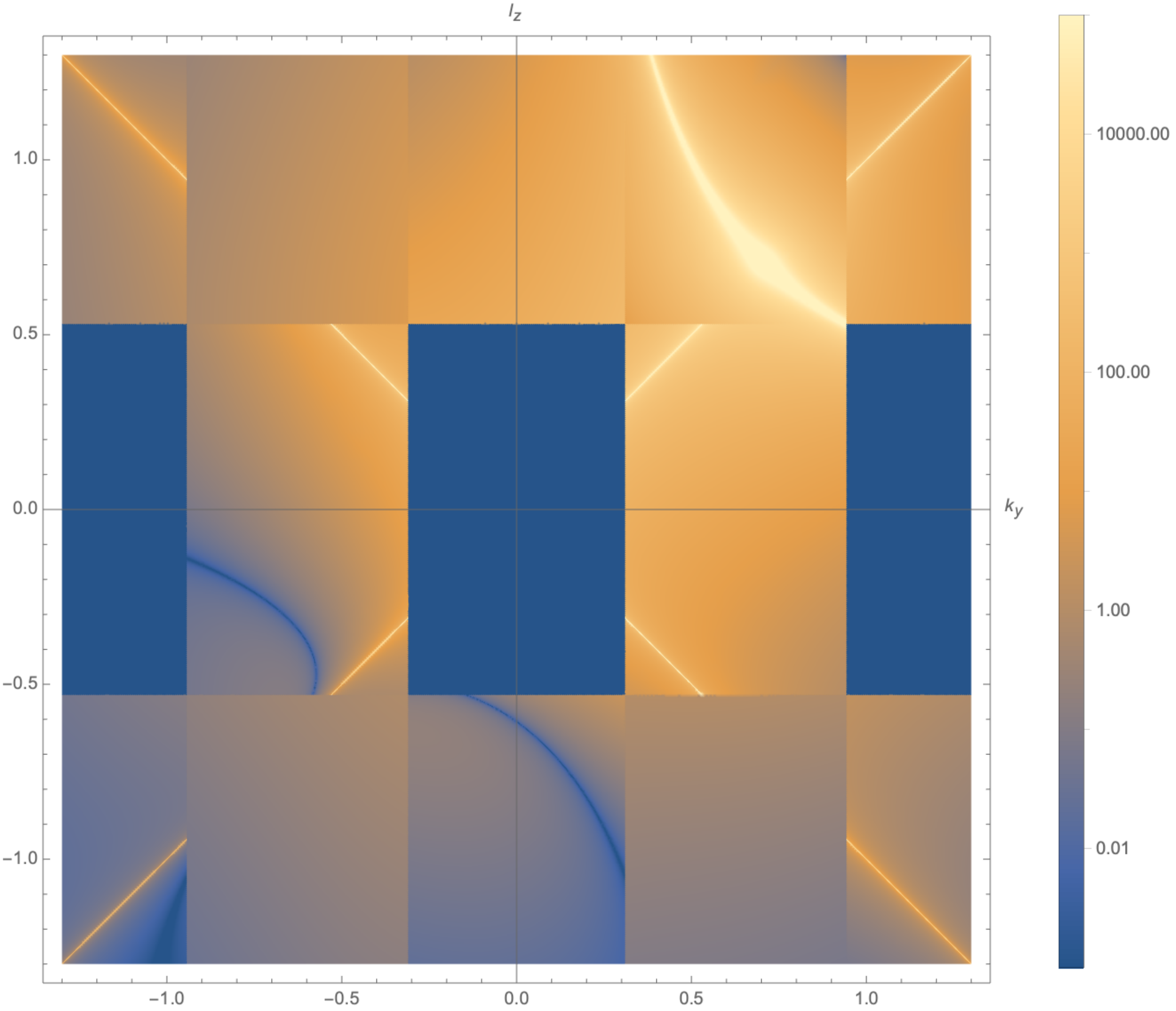


[redacted]

$I_{\mathbf{s}_2^{\mathrm{v}}}$



$$I_{\mathbf{s}_1^{\mathrm{v}}} + I_{\mathbf{s}_2^{\mathrm{v}}}$$



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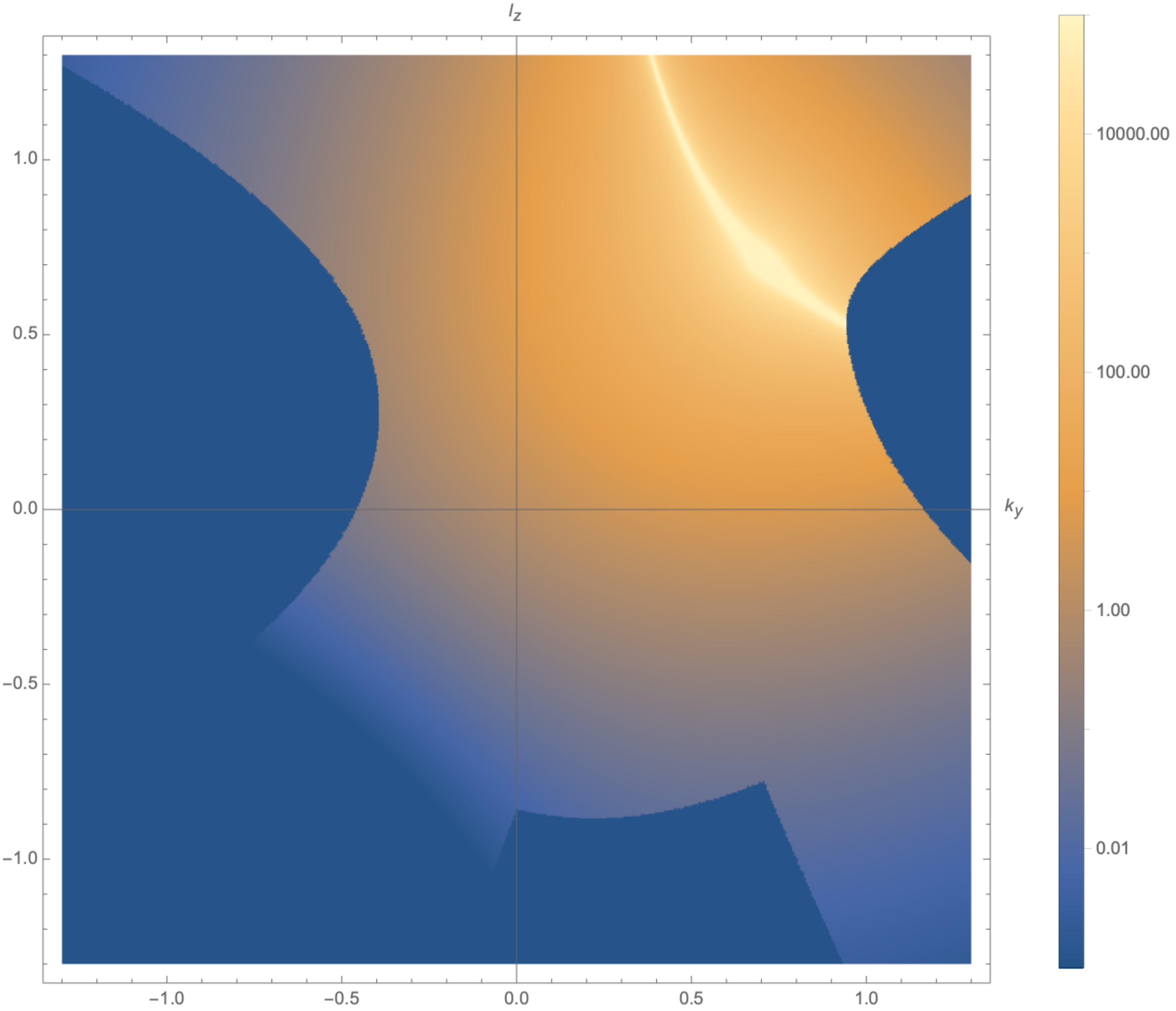
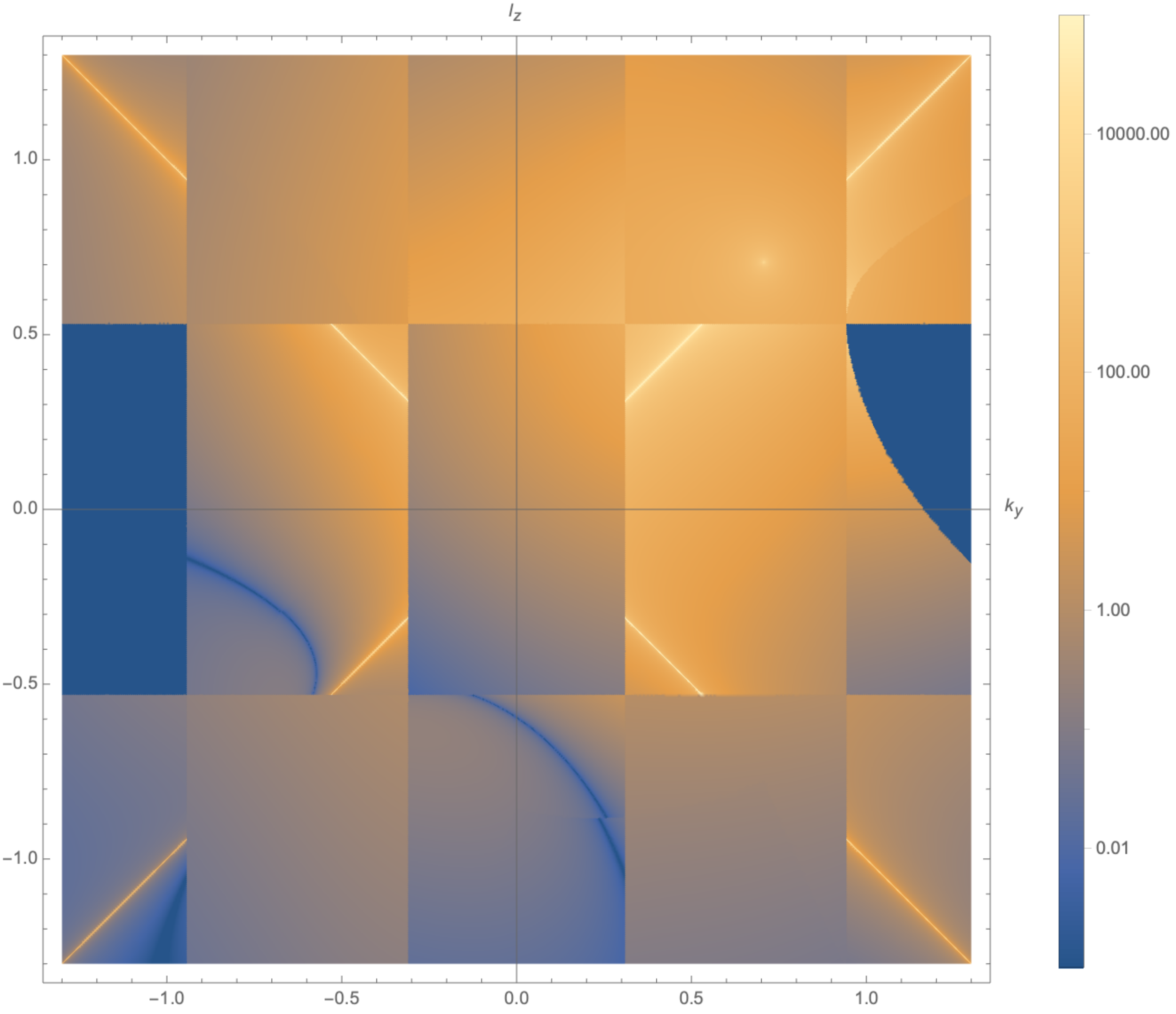
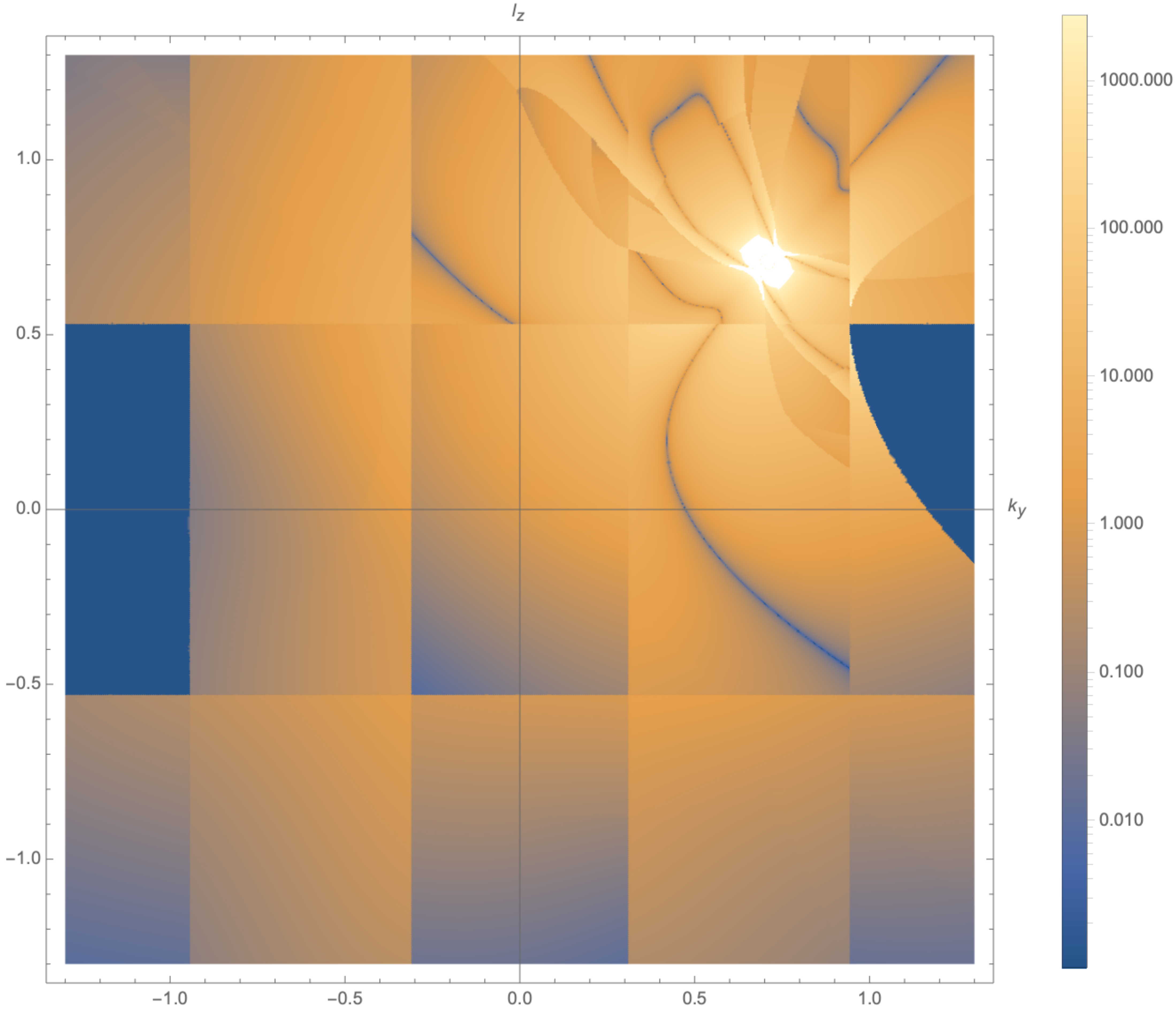


$I_{\sum}$

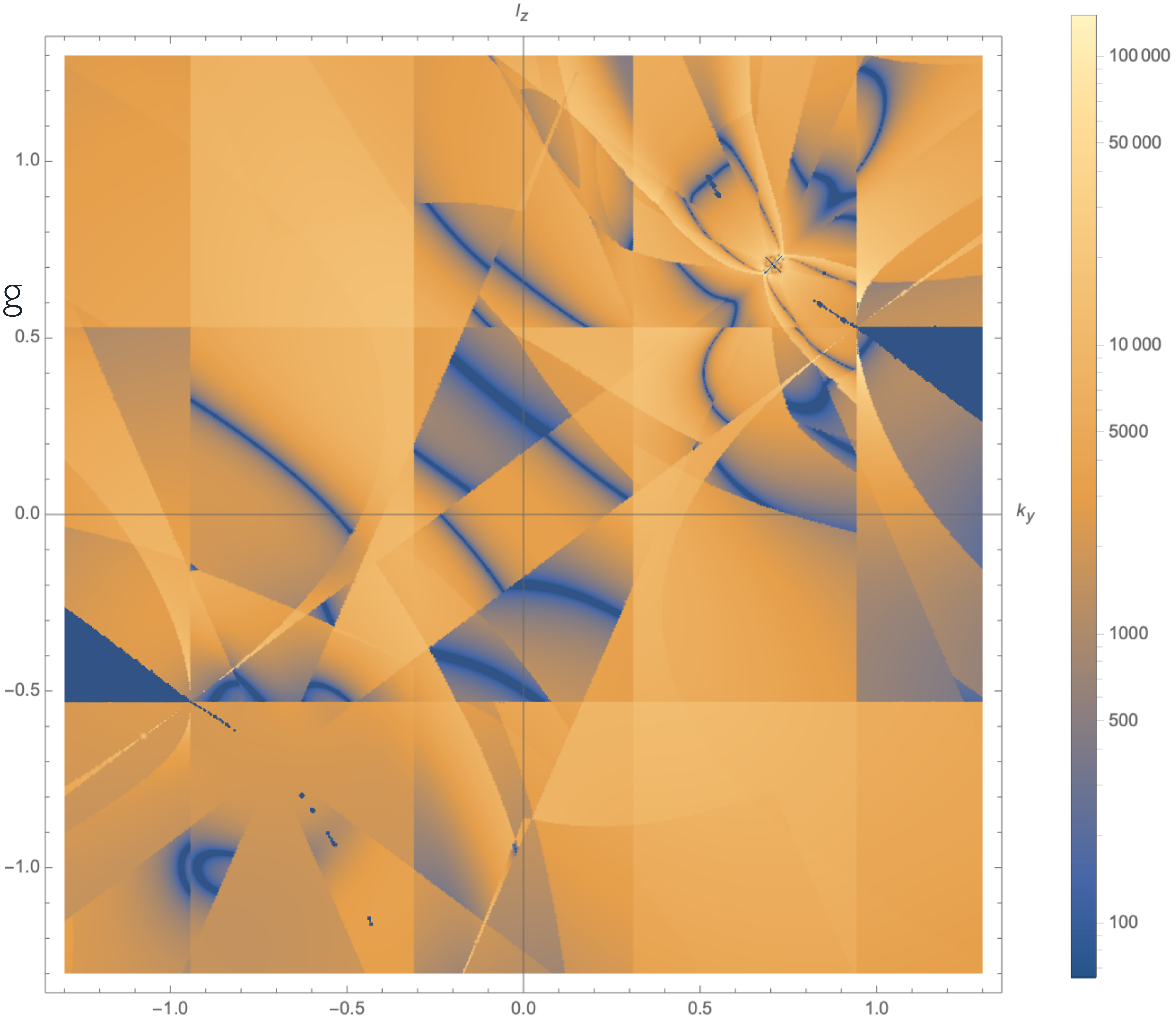


$\mathrm{Re}\left[I_{\Sigma}\right]$

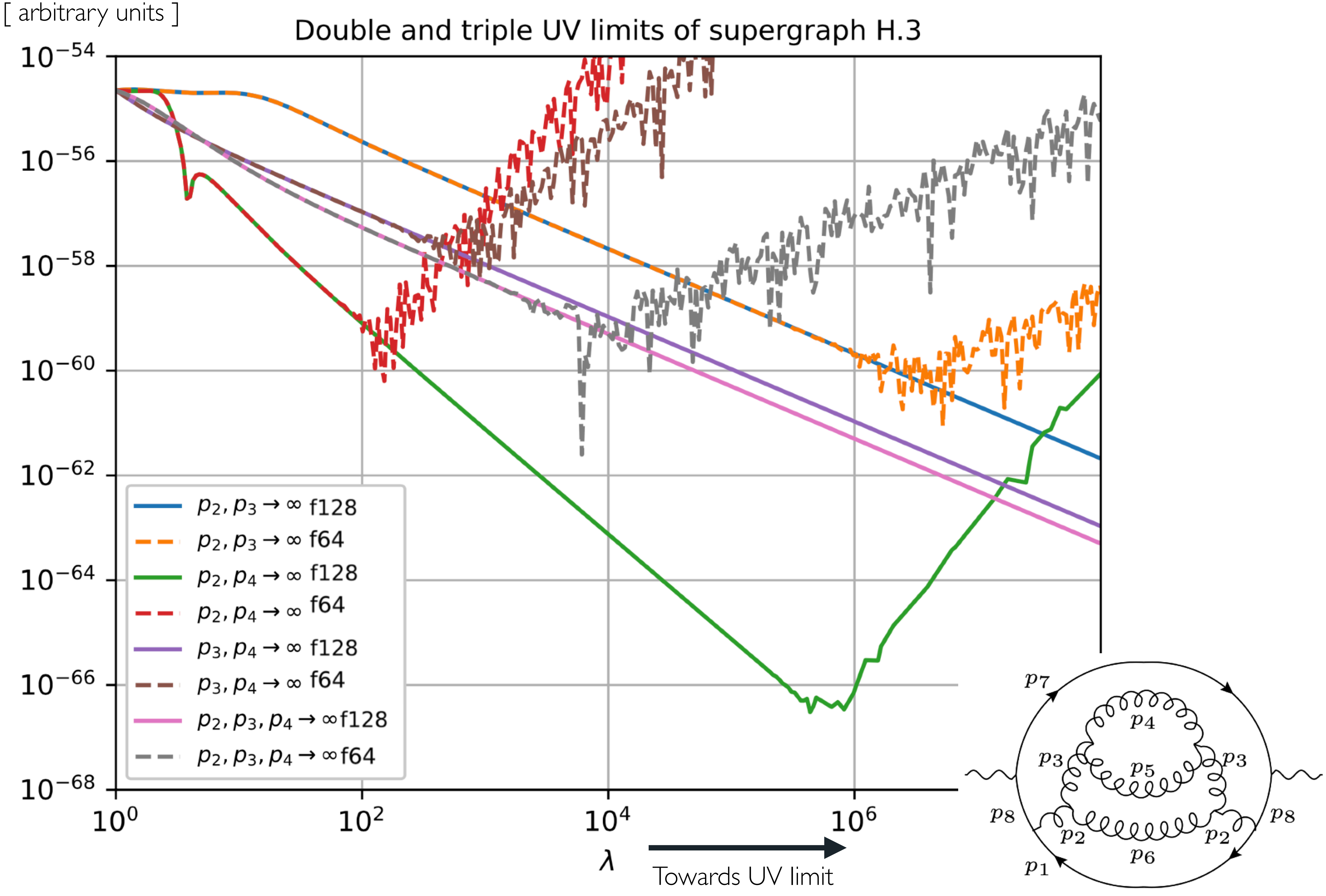
with deformation



$\mathrm{Re}\left[I_{\sum}\right]$ with deformation and multichanneling

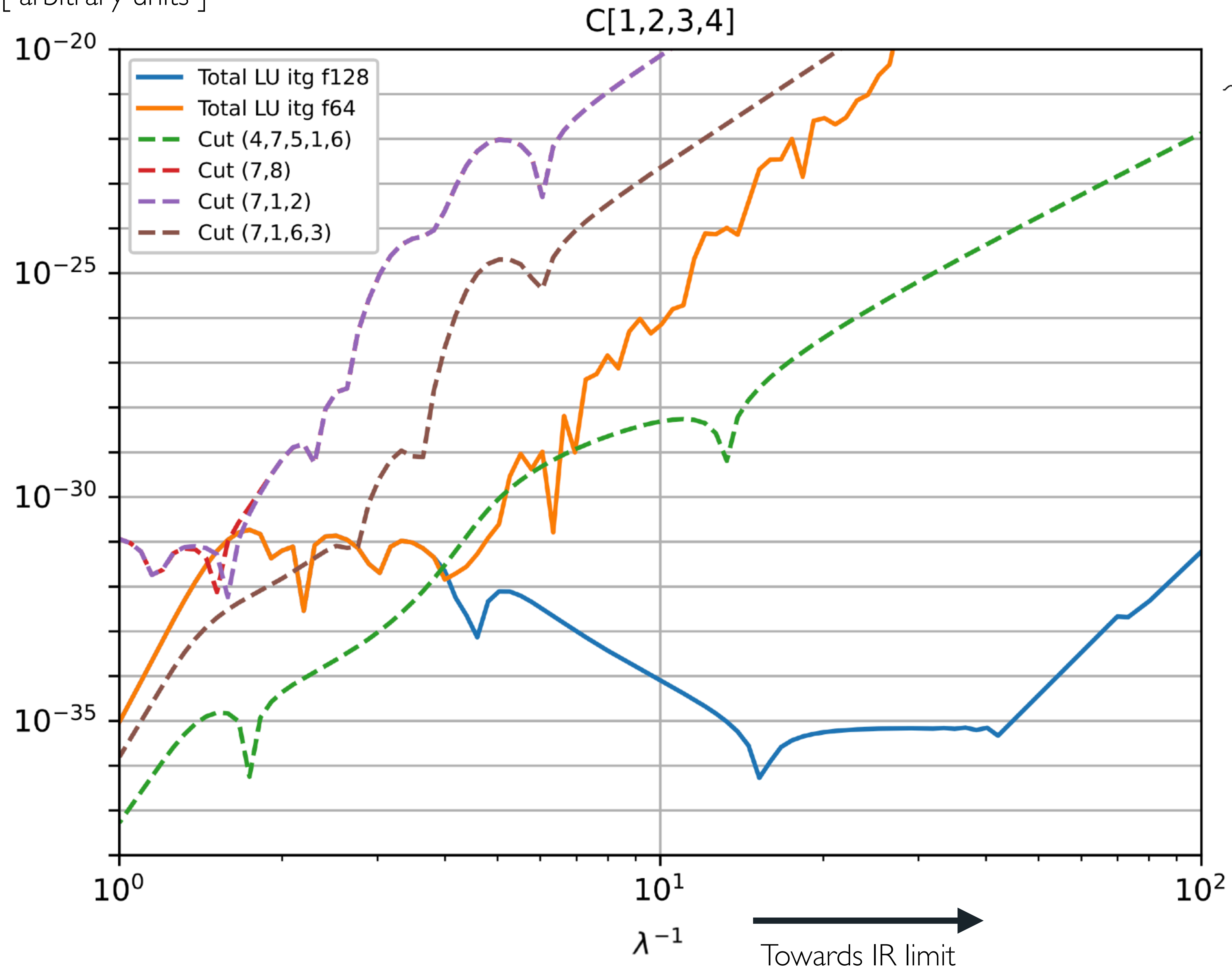


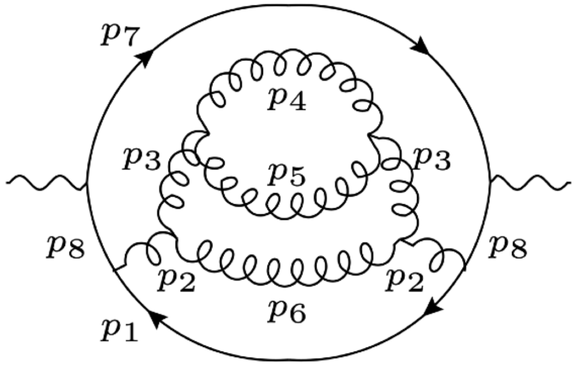
TESTING N3LO UV LIMITS



TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]





Towards IR limit

TESTING IR QUADRUPLE COLLINEAR LIMITS

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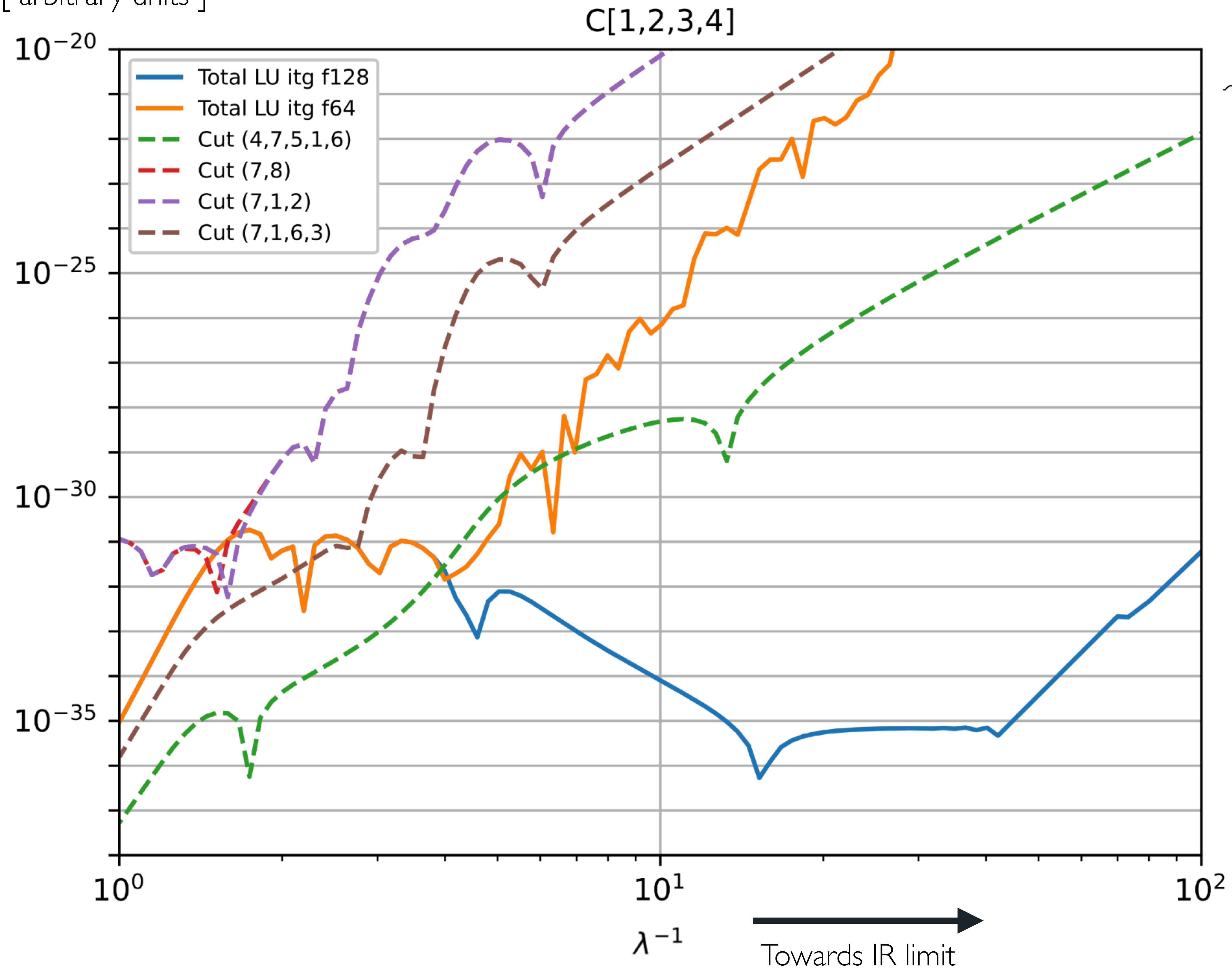

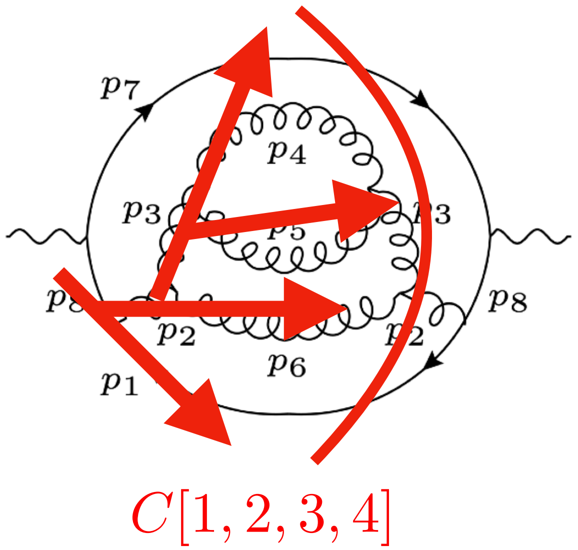

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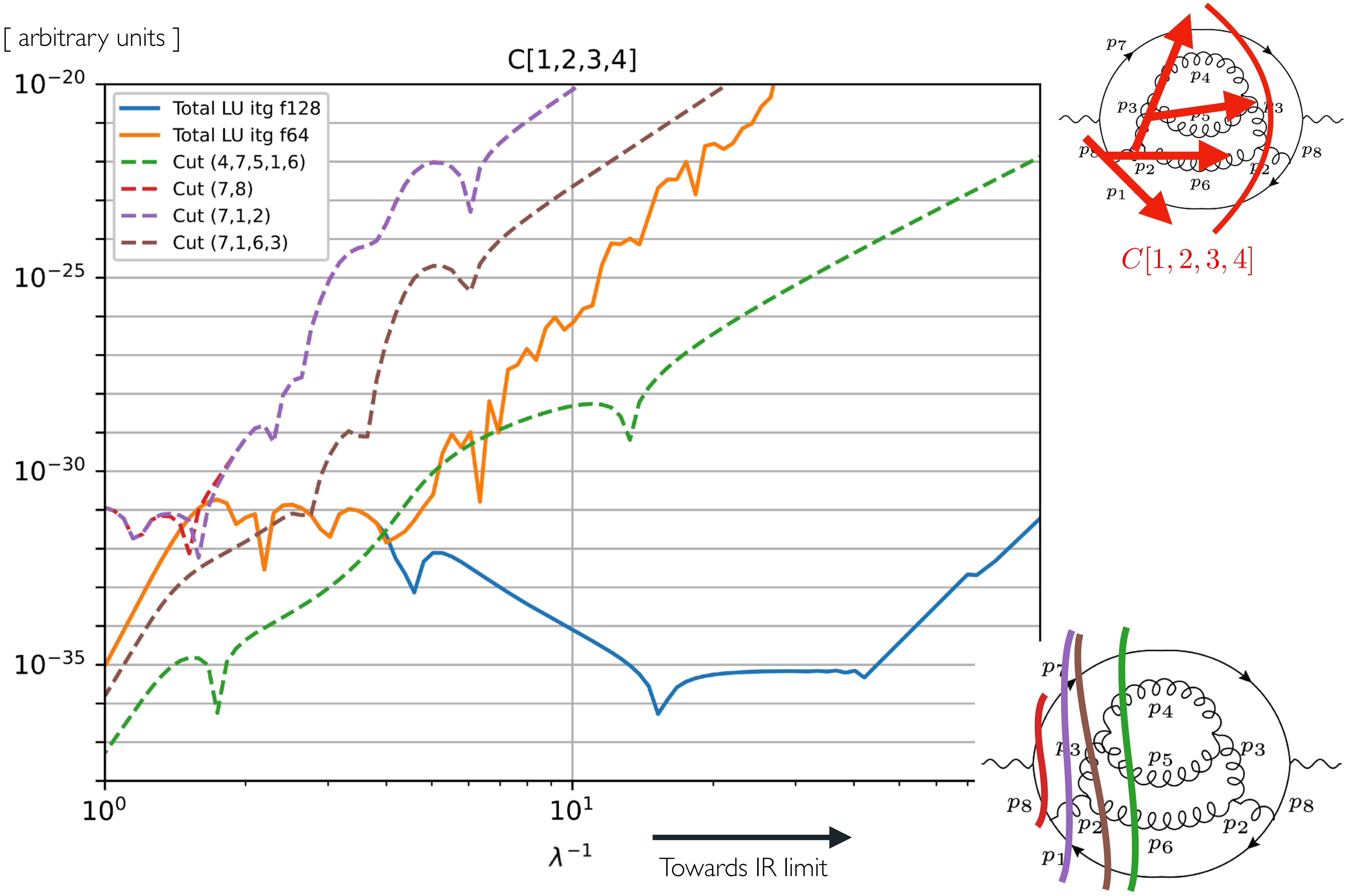

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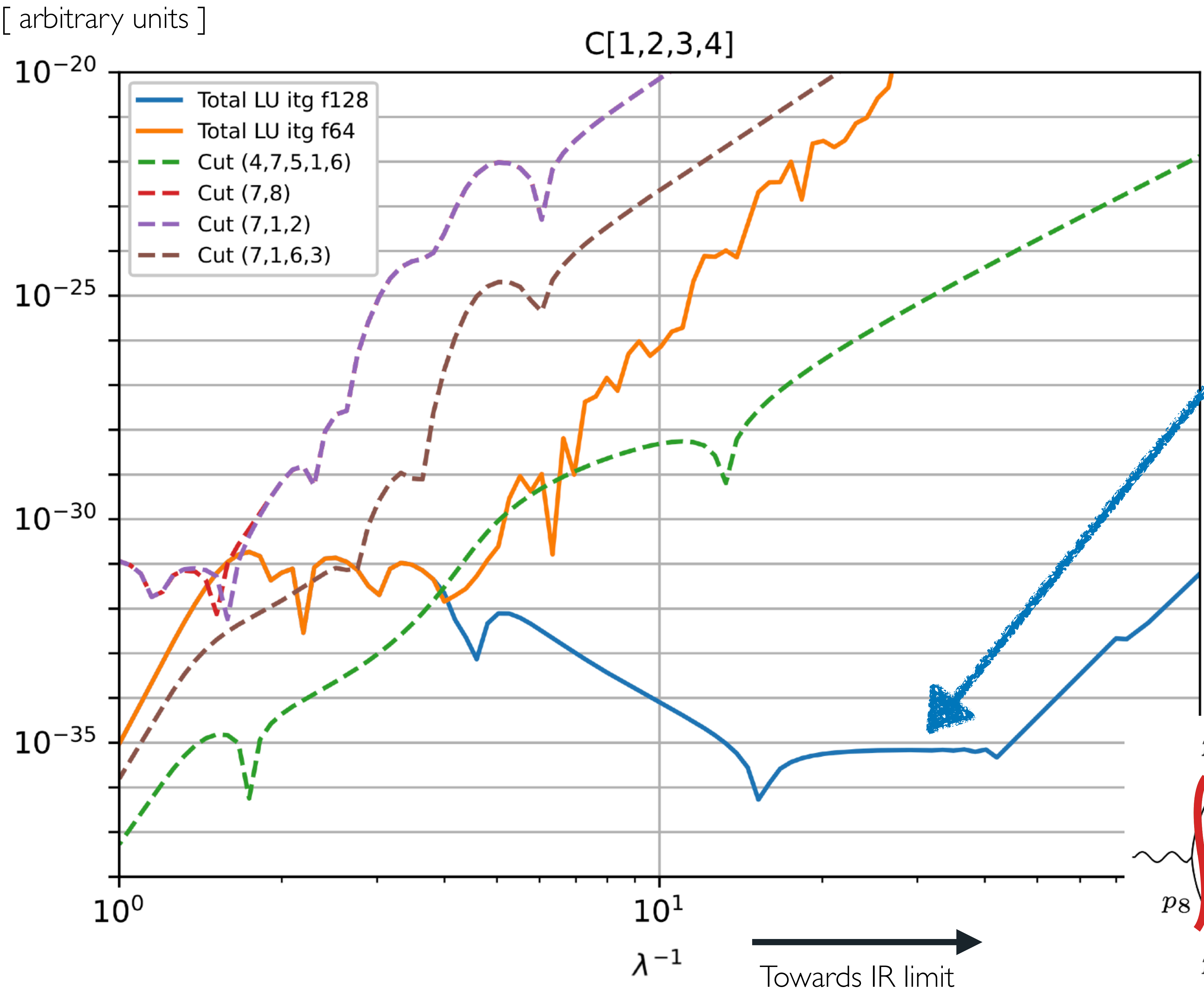

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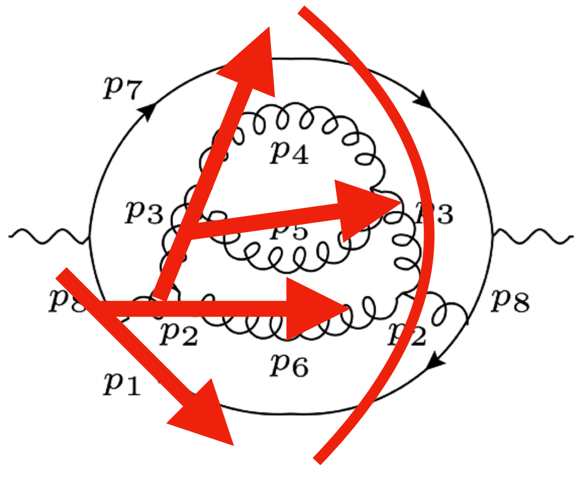
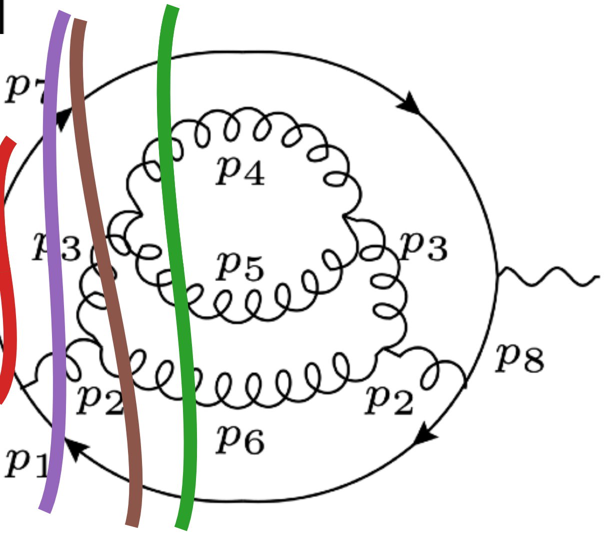
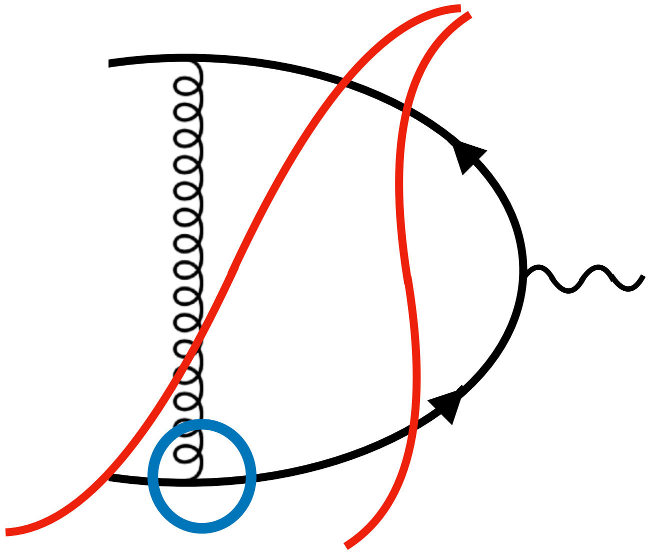


$C[1,2,3,4]$

LU integrand always **goes to a constant** on **collinear limits** without incl. *any* scaling of the measure !
No residual integrable singularity.



INITIAL-STATE SINGULARITIES



KLN CAN WORK FOR INITIAL-STATE !

INITIAL-STATE SINGULARITIES: IDEA

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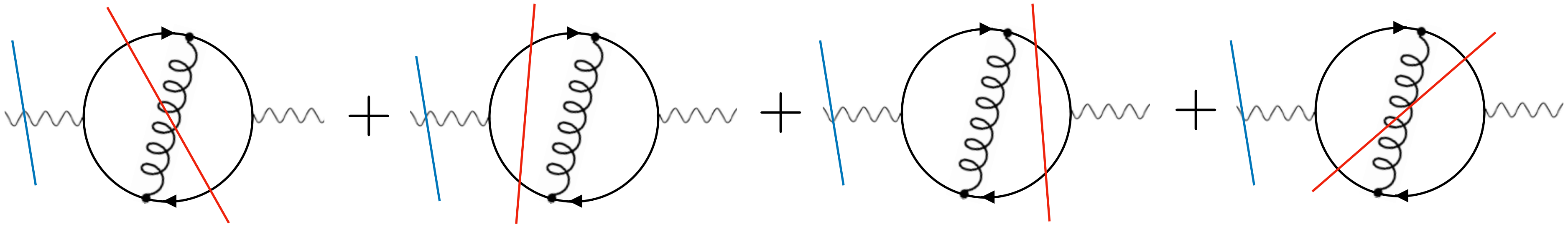
For $e^+e^- \rightarrow 2j$ @ $\mathbf{NLO}$ we used



(Including all degenerate configurations, higher final-state multiplicities)

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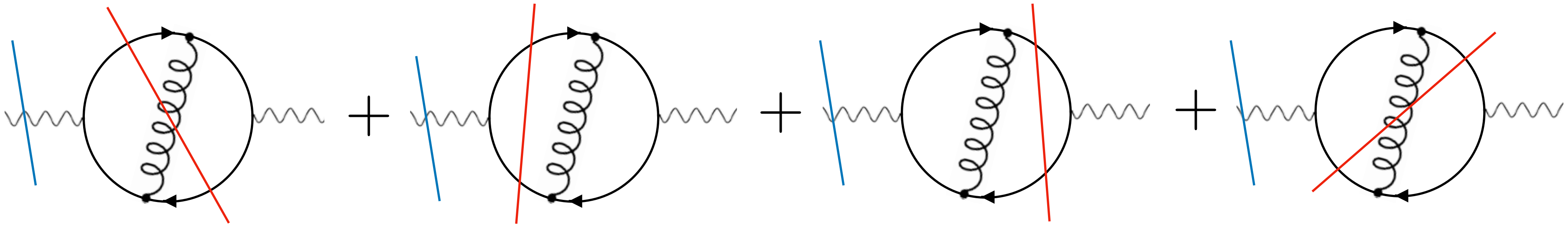

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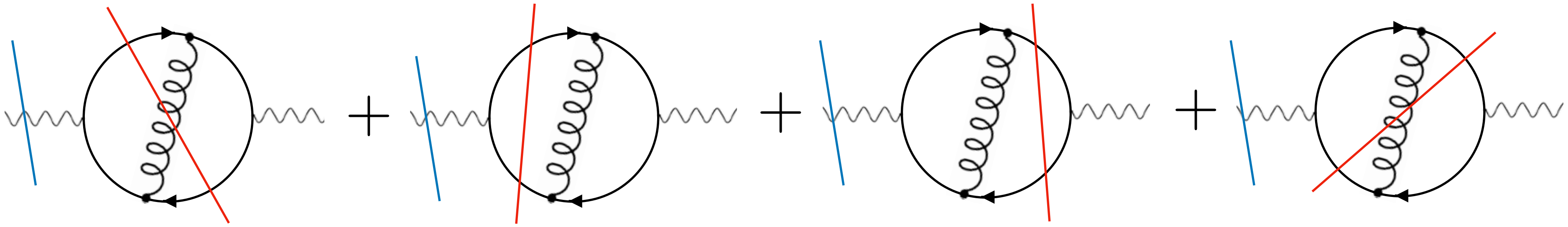

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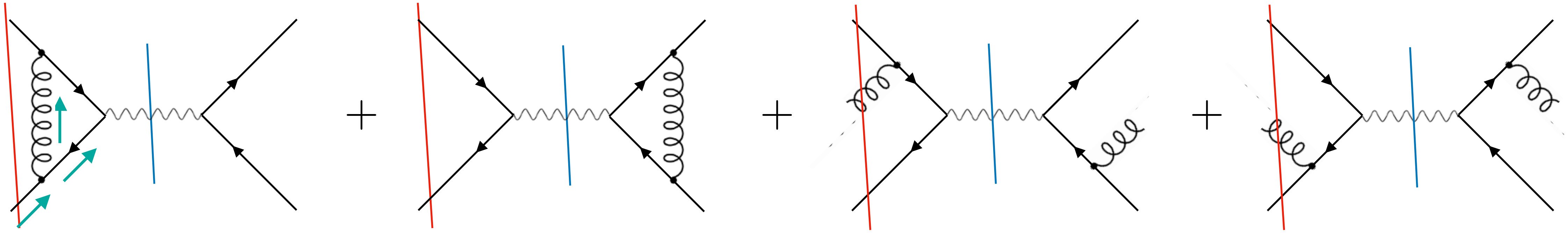

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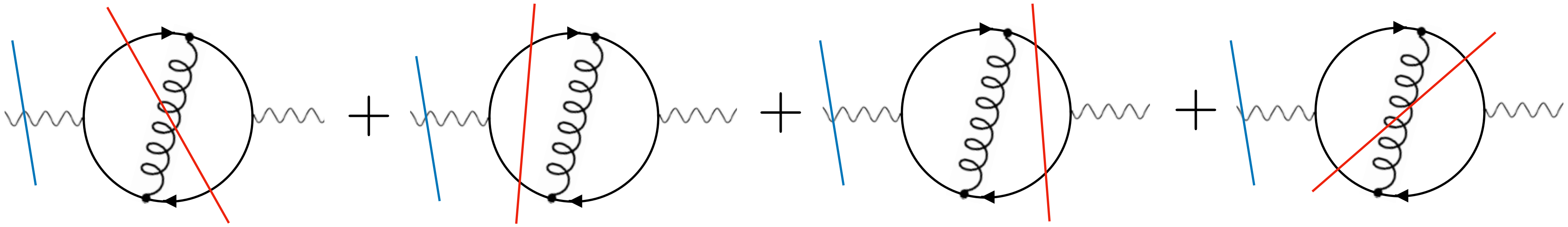

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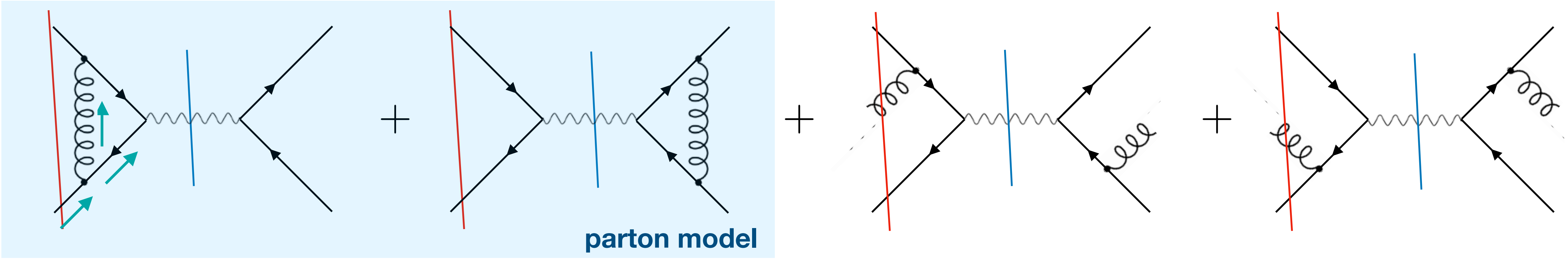

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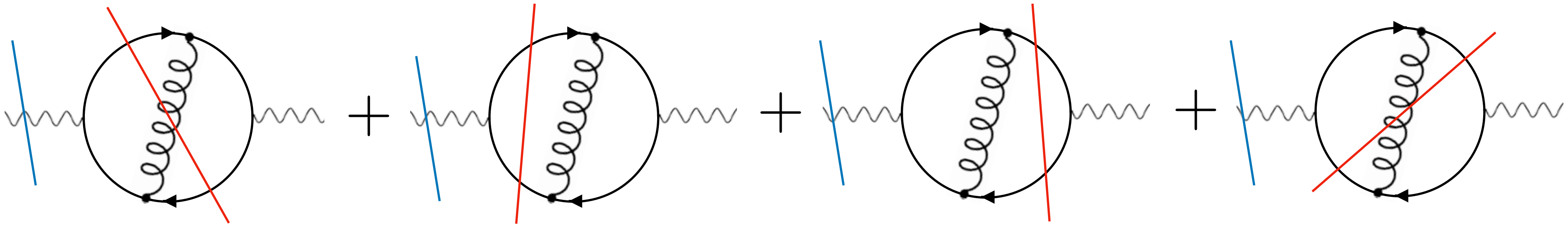
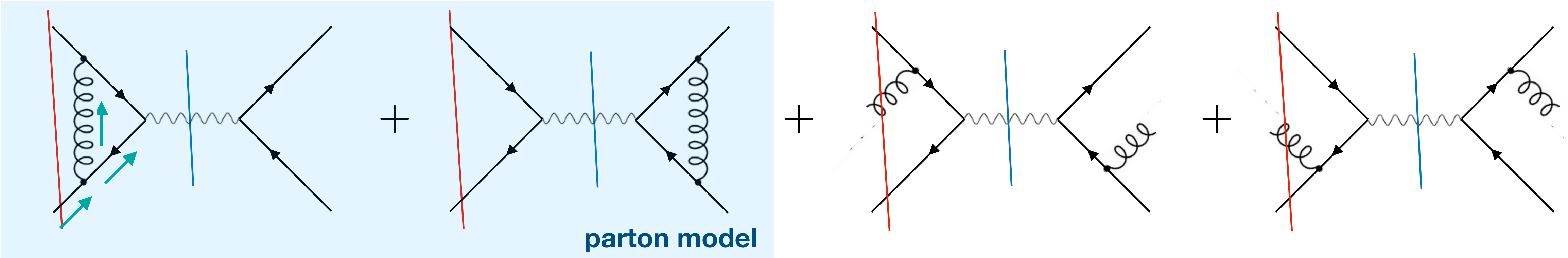


(Including all degenerate configurations, higher final-state multiplicities)

Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$



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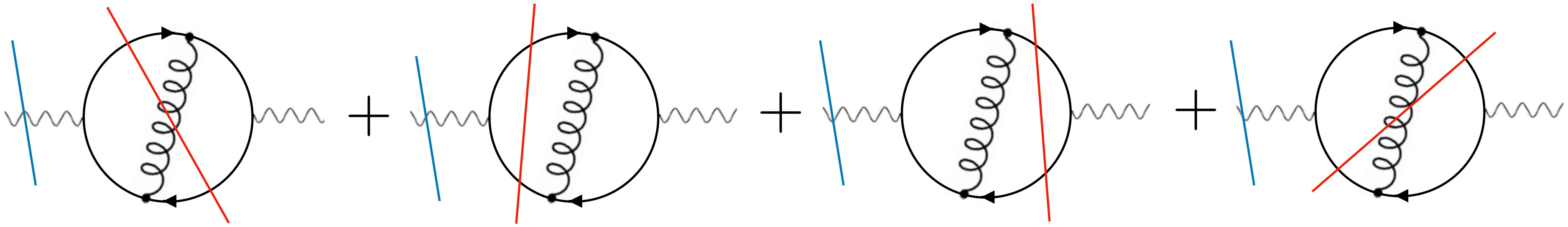

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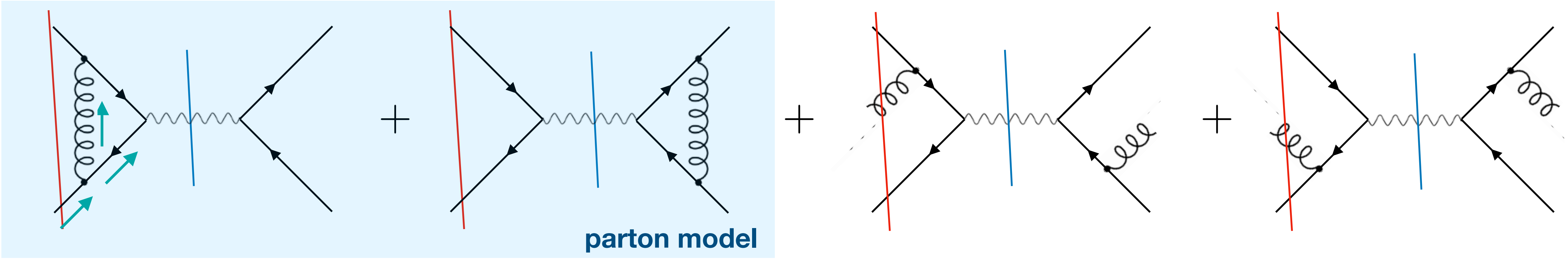

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Include degenerate initial states $\longrightarrow$ **Higher multiplicity initial states**

INITIAL-STATE SINGULARITIES: IDEA

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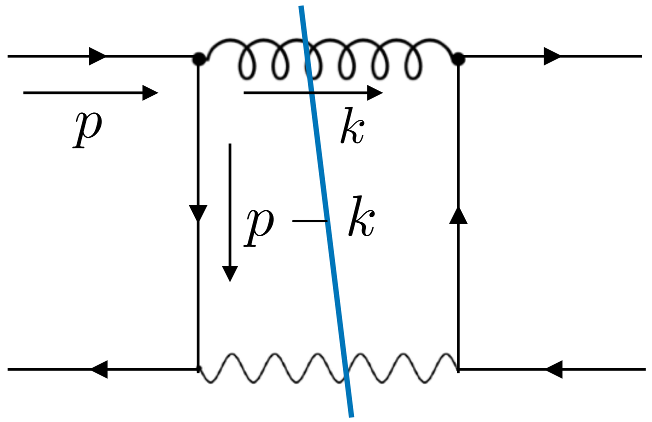

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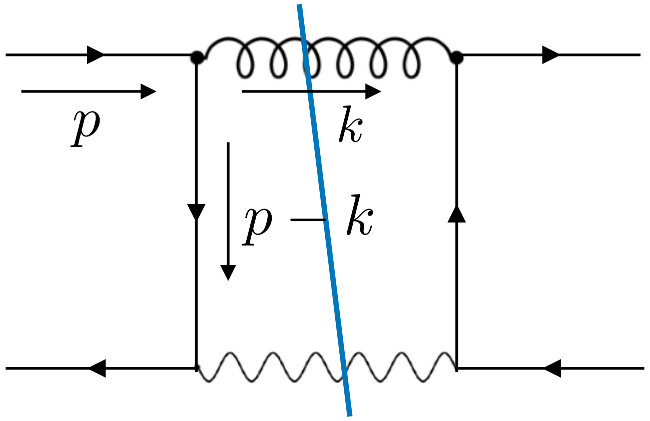

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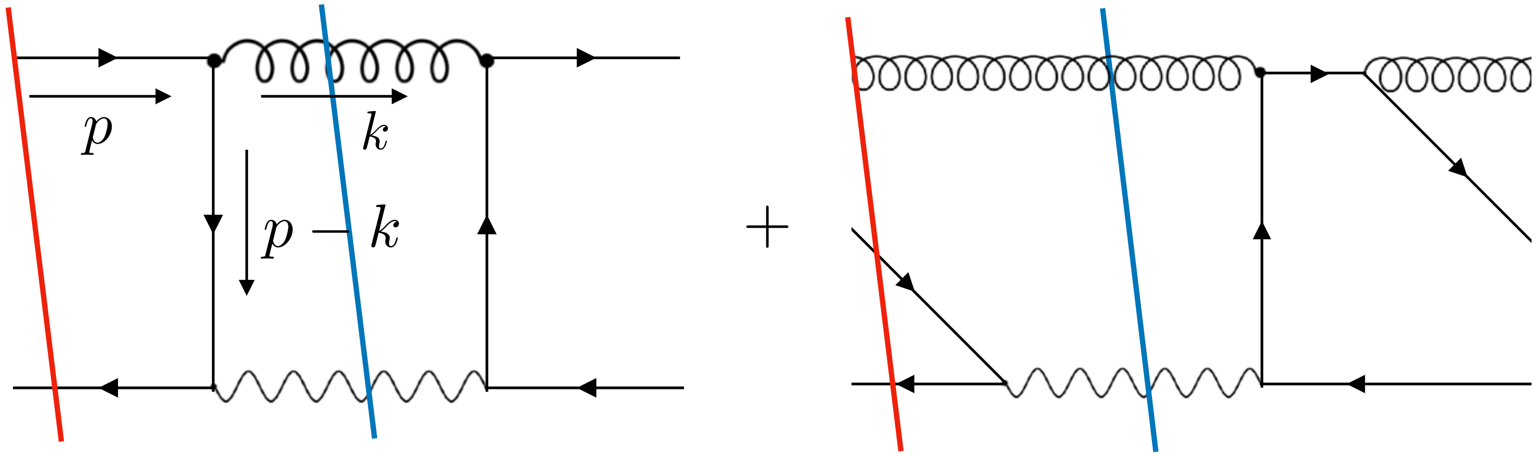

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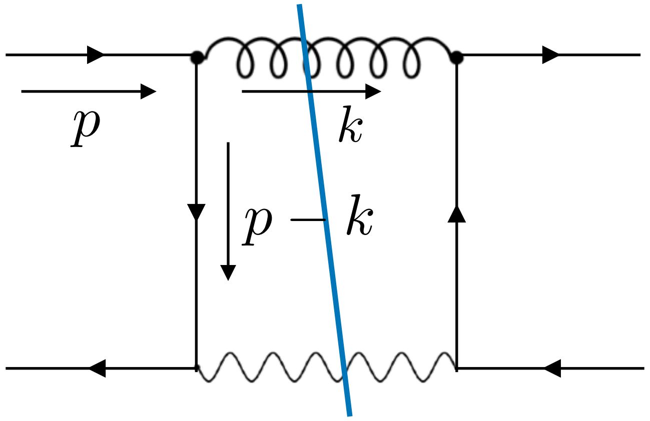

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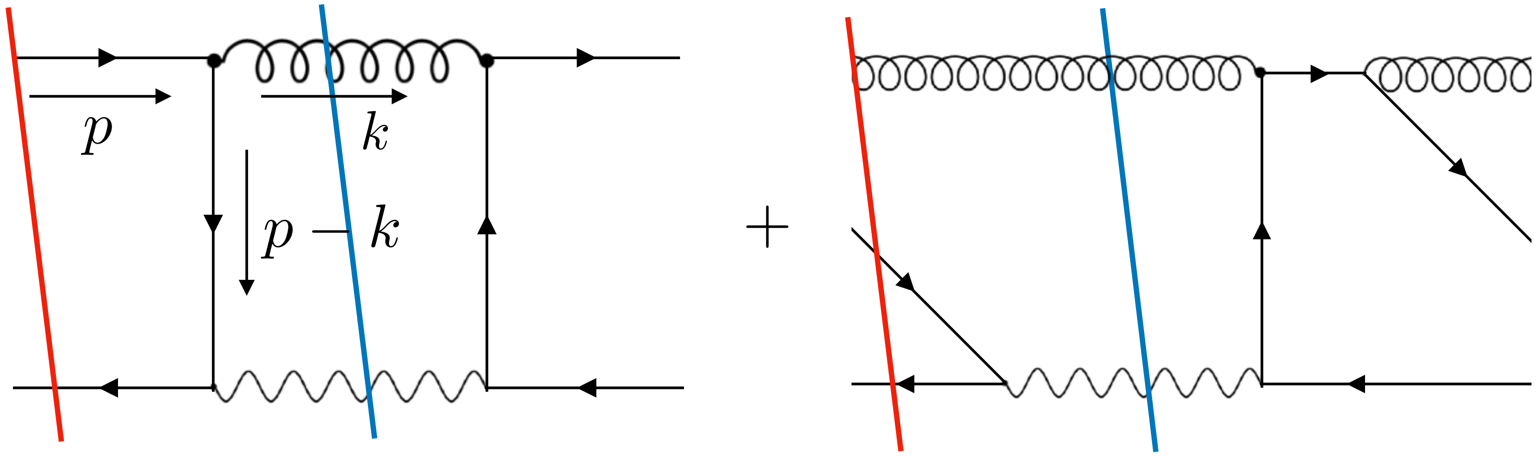

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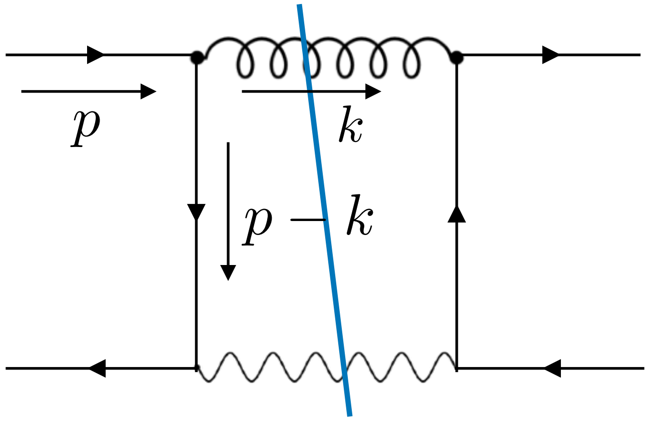

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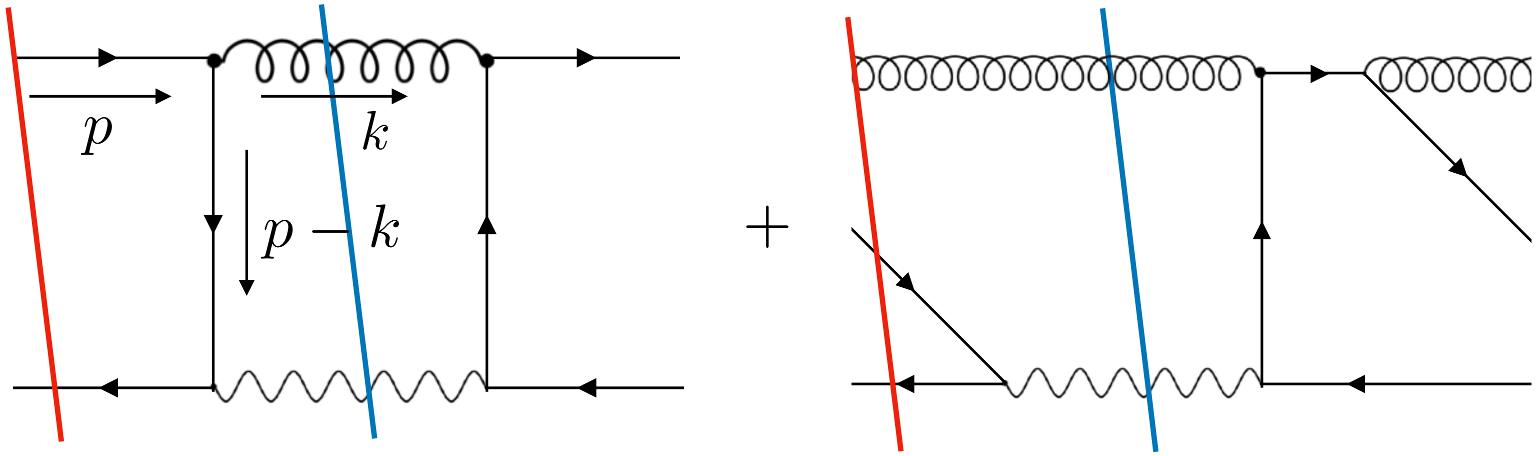

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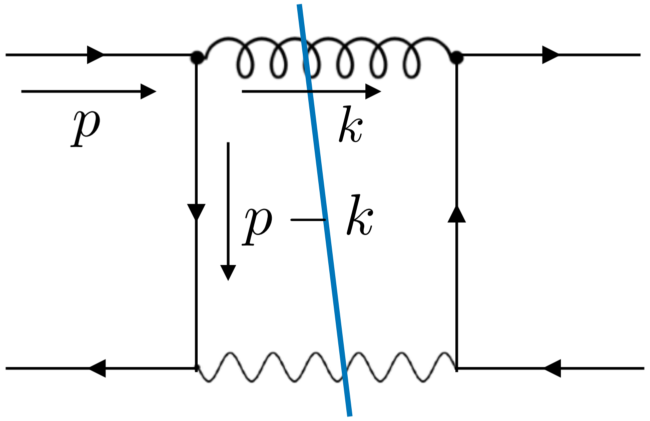

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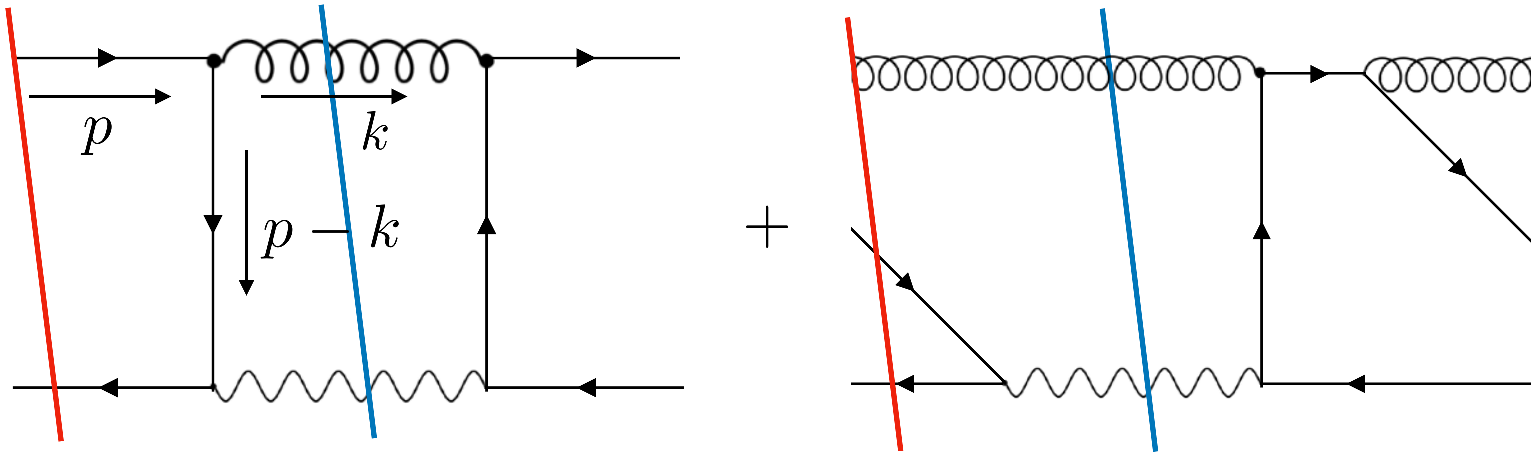

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But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

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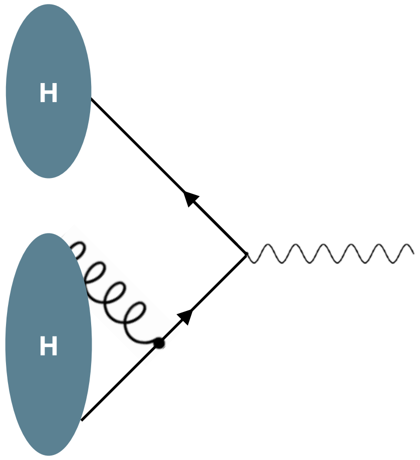
This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



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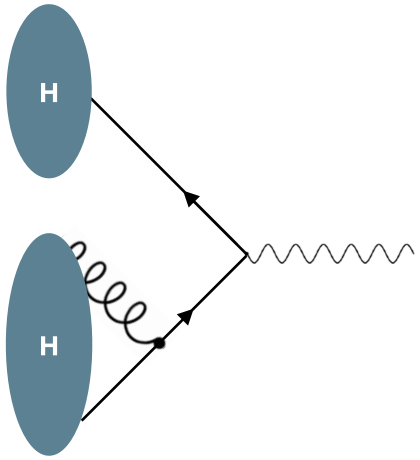
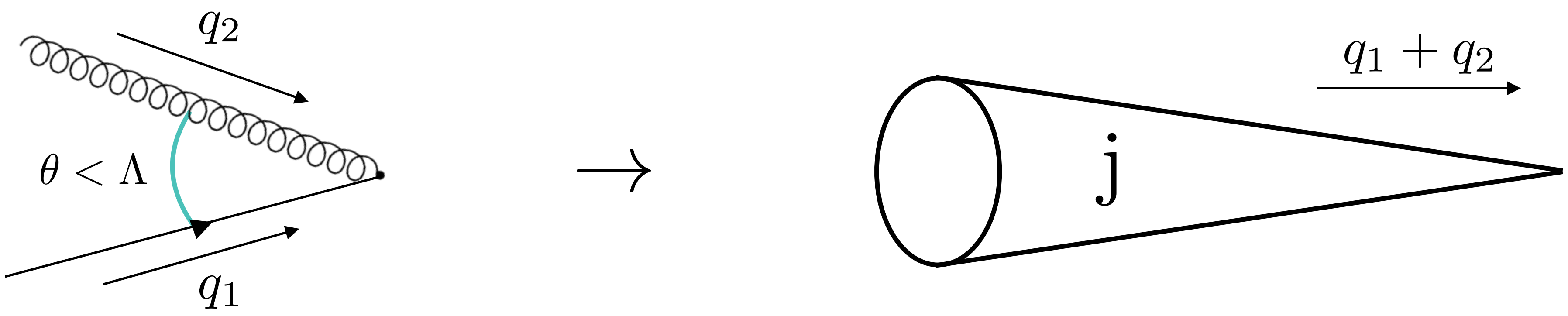


and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



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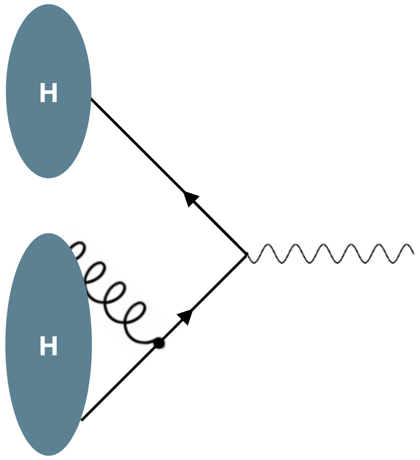

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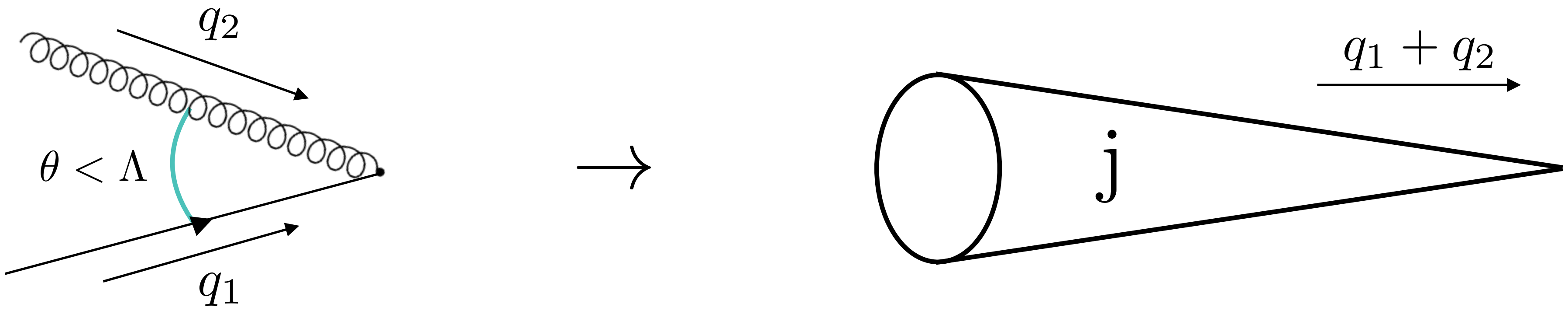

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Cluster initial states analogously to final states: symmetry initial-final state

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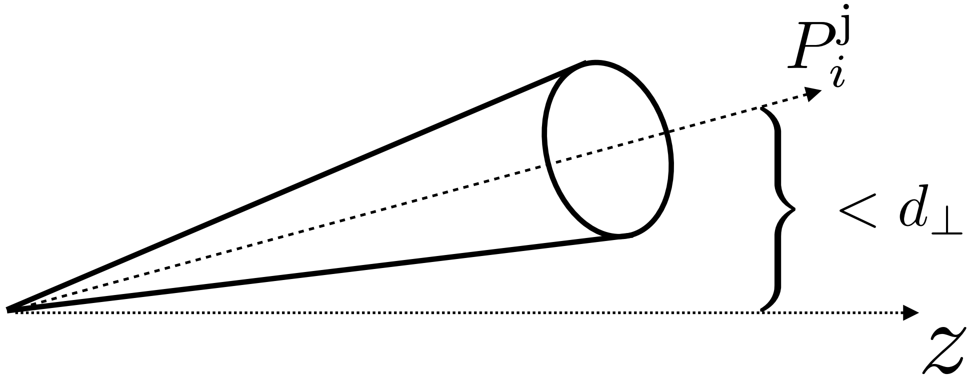

If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

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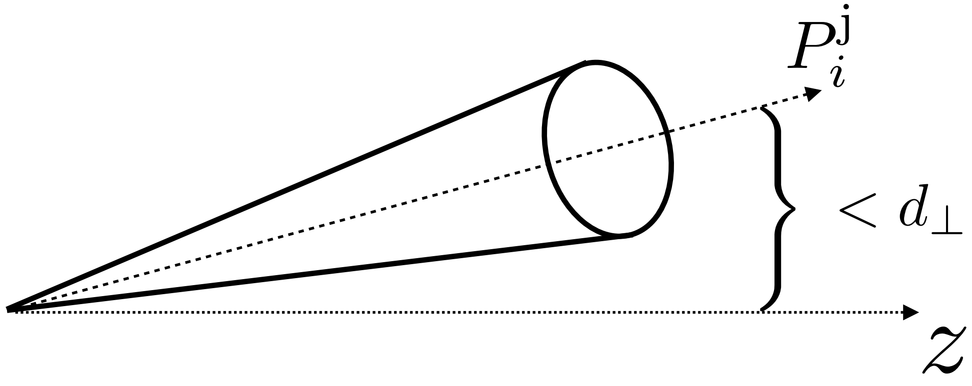

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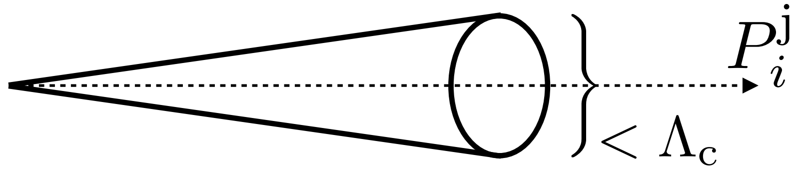

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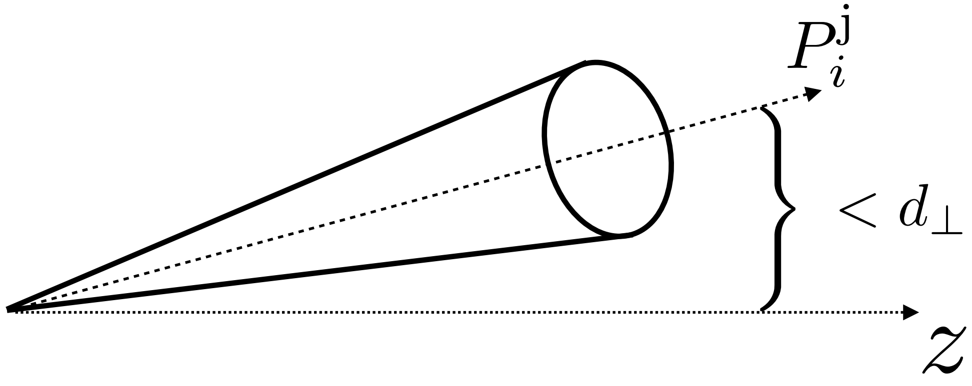

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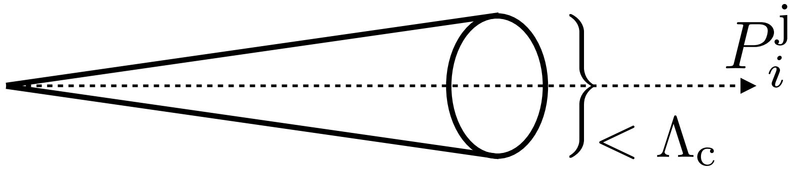

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Λ_{c} **is the equivalent of the factorisation scale!** $\approx \log(\Lambda_{\mathrm{c}})$

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- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?**

Banfi, Salam, Zanderighi, arXiv:0407286 (2004)

INITIAL-STATE SINGULARITIES: "PDFS"

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We started with a very generic formalism for scattering

$$\sigma(HH \to X + n\,\mathrm{j}) = \sum_{m} \int \left[\prod_{i=1}^{m} \mathrm{d}^3\vec{p}_i\right] f(p_1, ..., p_m) \; \frac{\mathrm{d}^m\sigma}{\mathrm{d}p_1 ... \mathrm{d}p_m}(p_1, ..., p_m \to X + n\mathrm{j})$$

Sum over number of initial state partons

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Cross-sections for m initial state partons

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And we "forced" the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \to X + n\,\mathrm{j}) = \int \mathrm{d}x_1\mathrm{d}x_2 \; f(x_1,\Lambda_\mathrm{c})\; f(x_2,\Lambda_\mathrm{c})\; \frac{\mathrm{d}^2\sigma_p}{\mathrm{d}x_1\mathrm{d}x_2}(2\,\mathrm{j} \to X + n\,\mathrm{j},\Lambda_\mathrm{c})$$

But we did not need to start from this factorised ansatz!

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Numerical example result for this finite sum of two interference diagrams:

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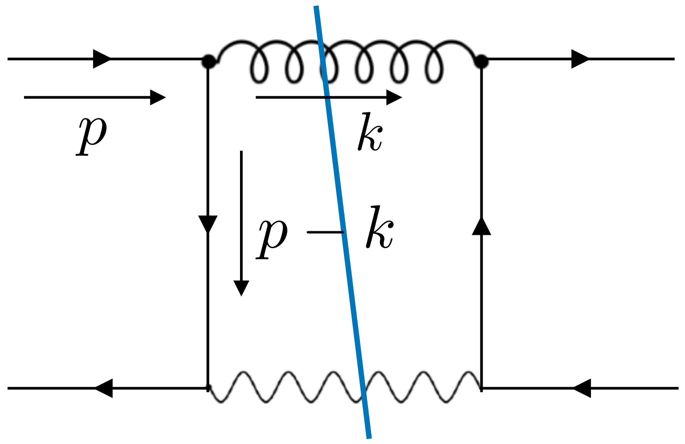

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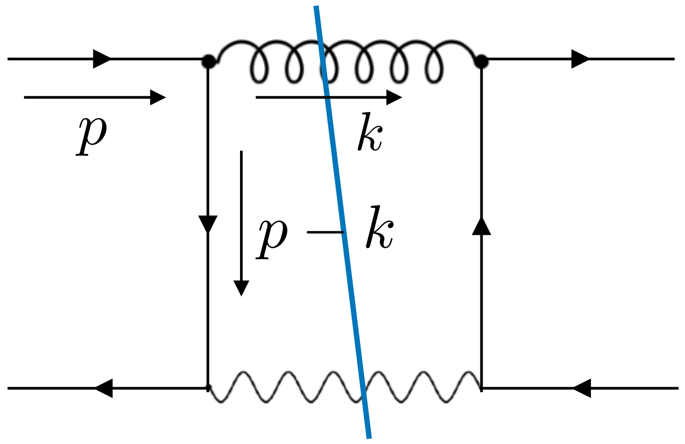

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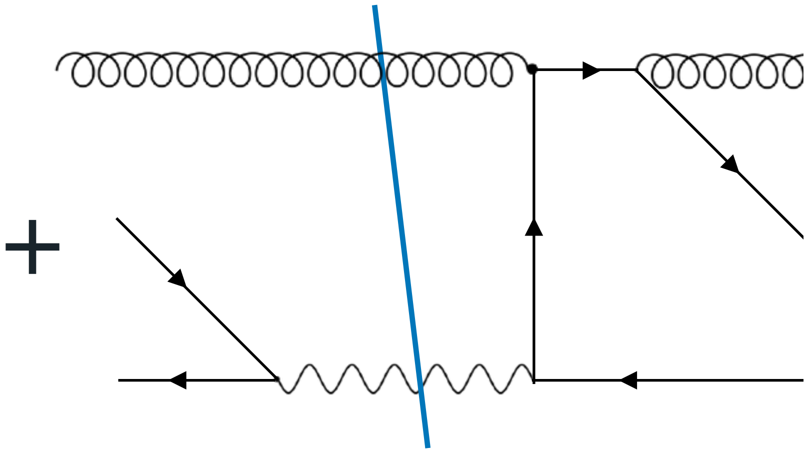

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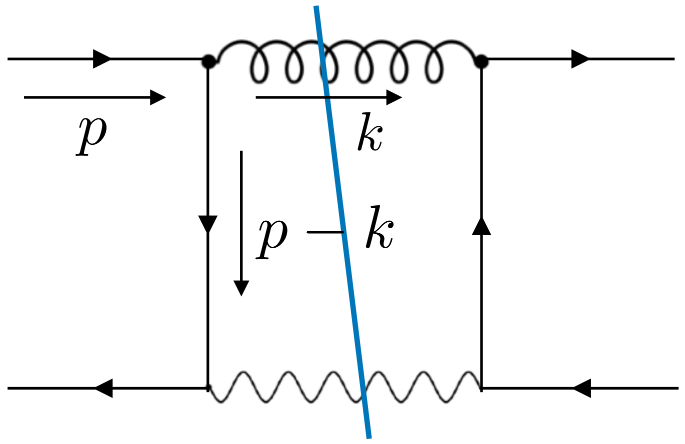

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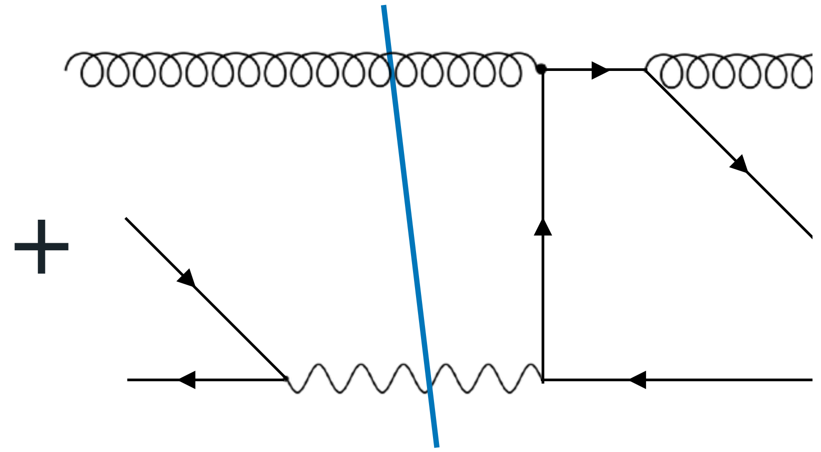

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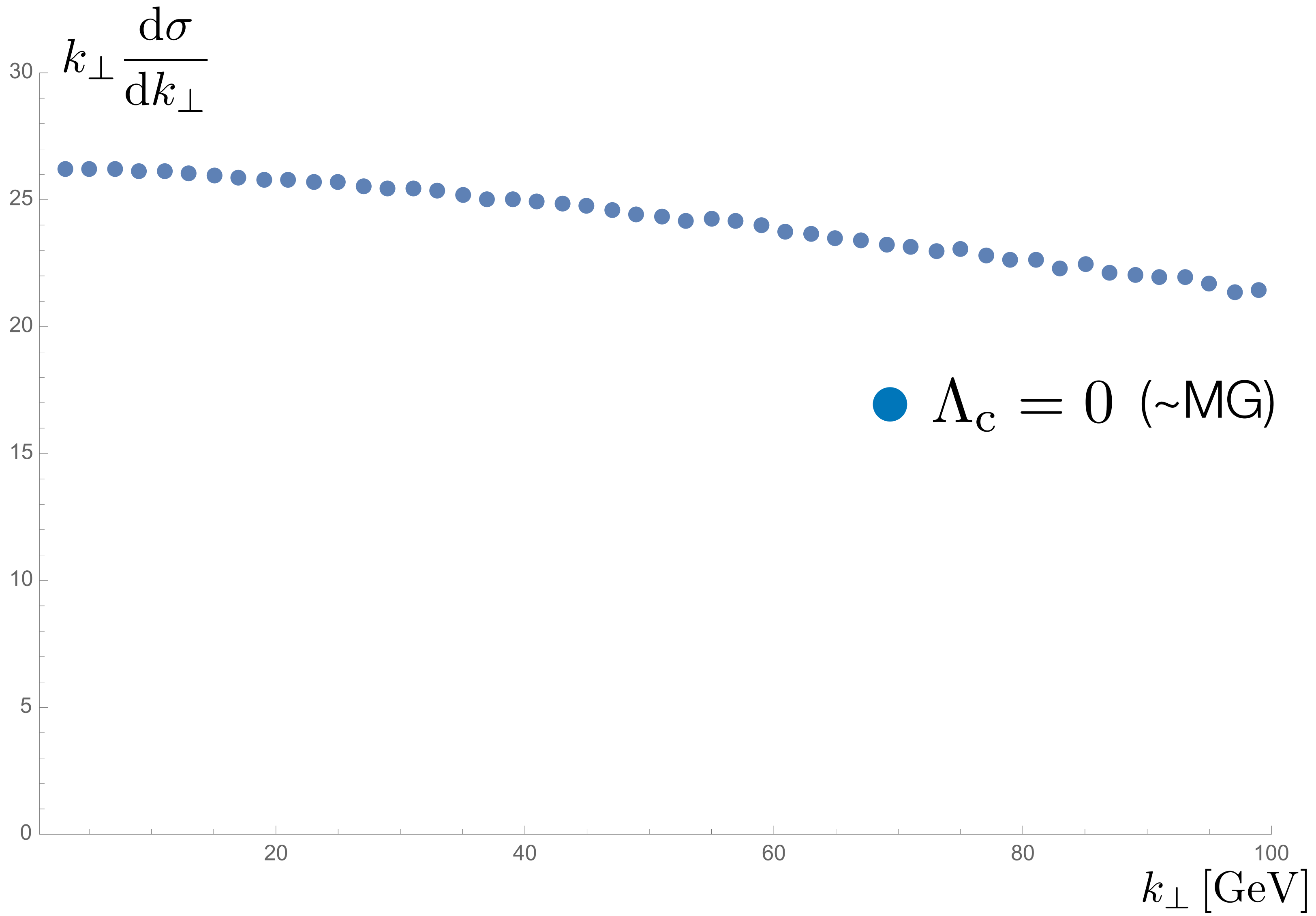

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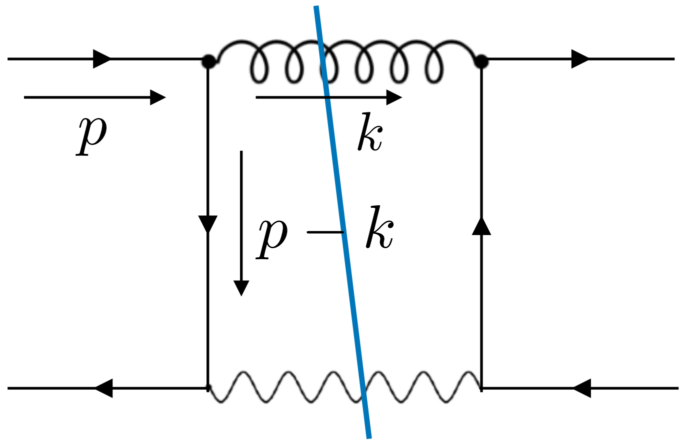

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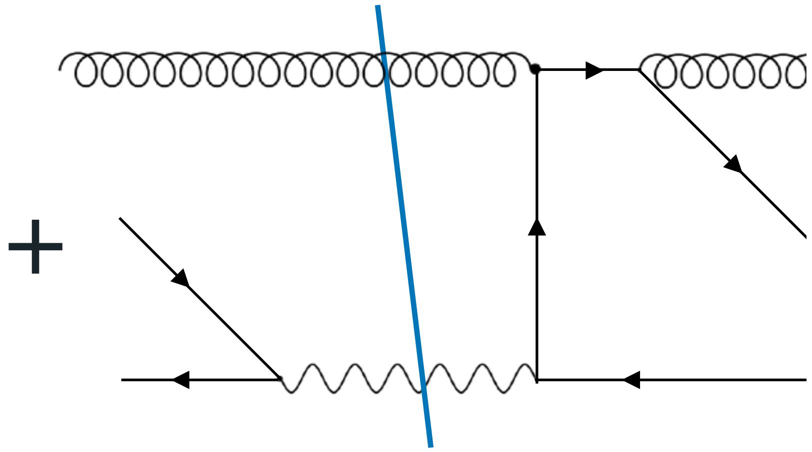

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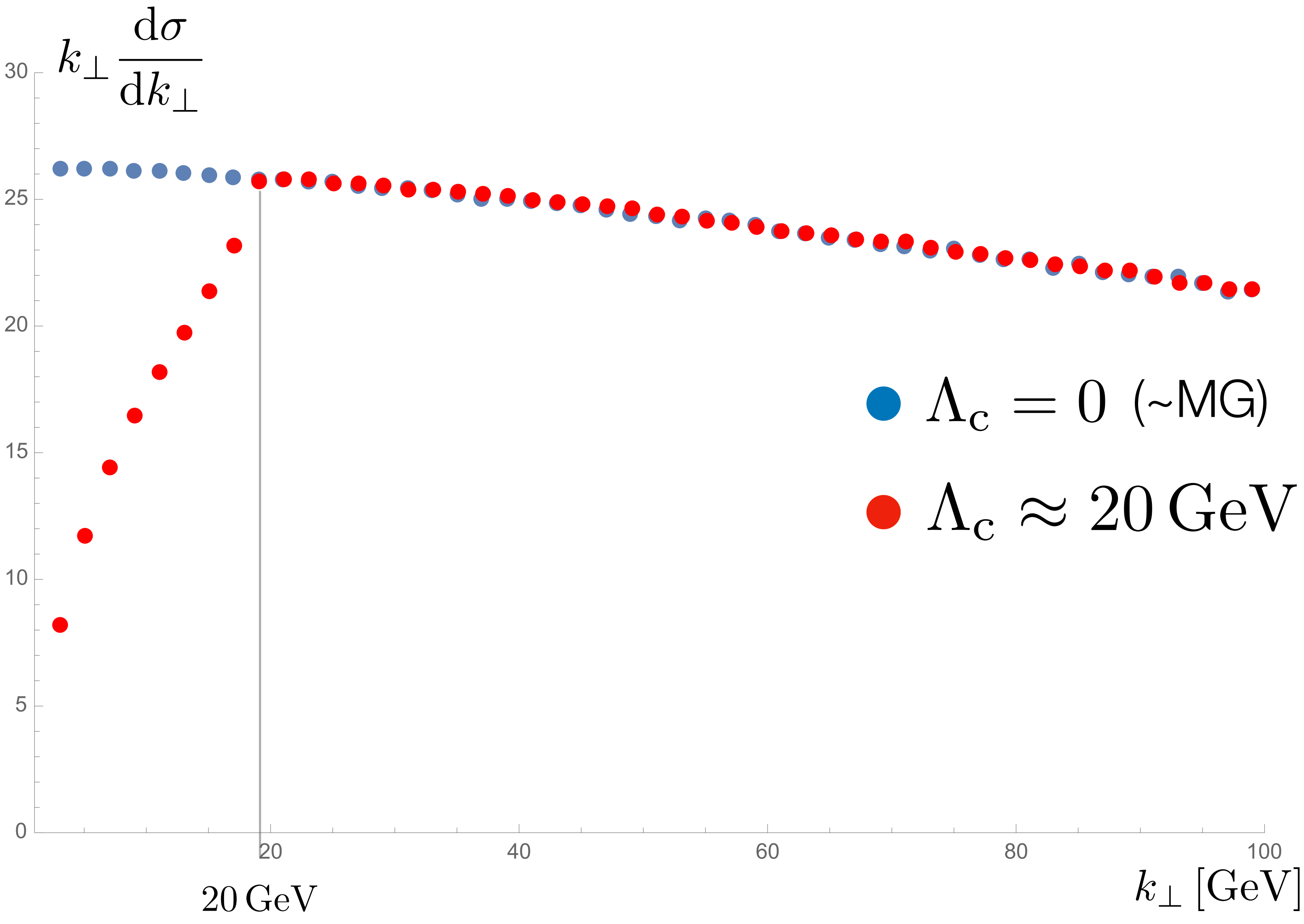

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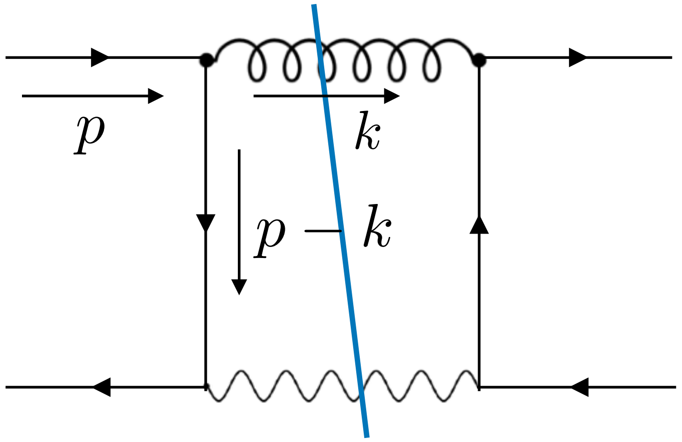

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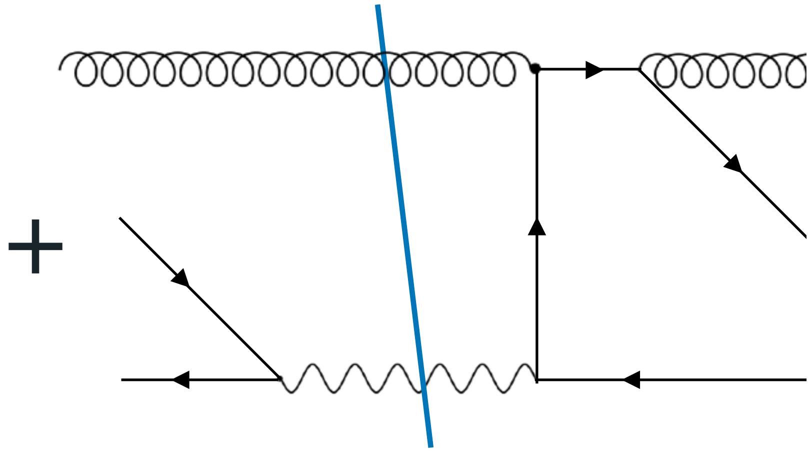

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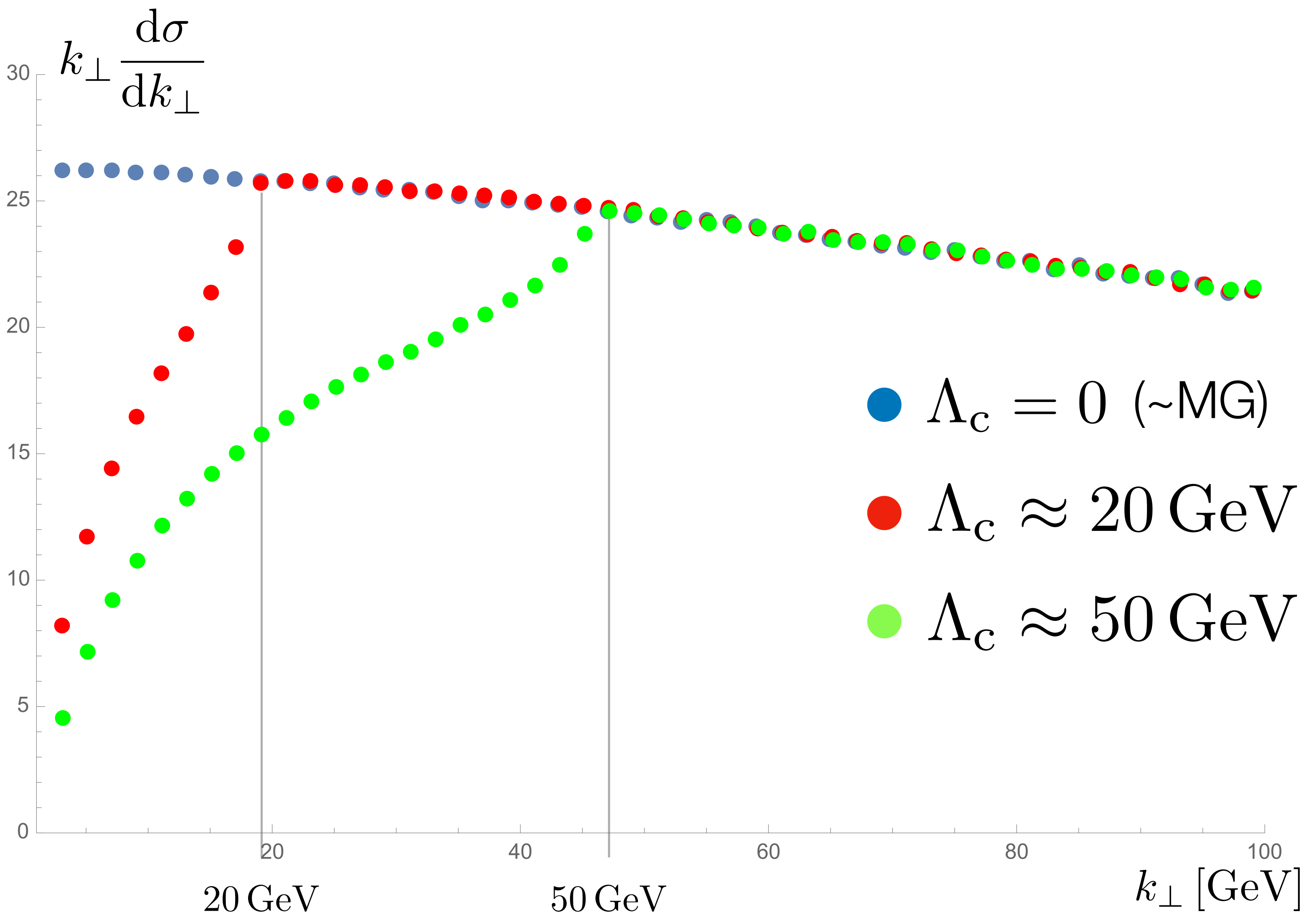

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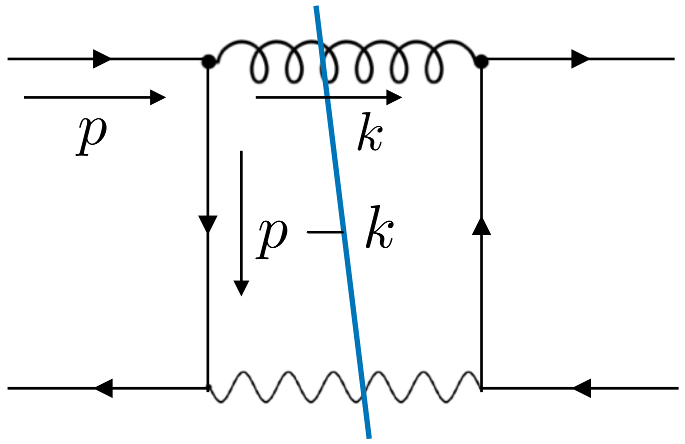

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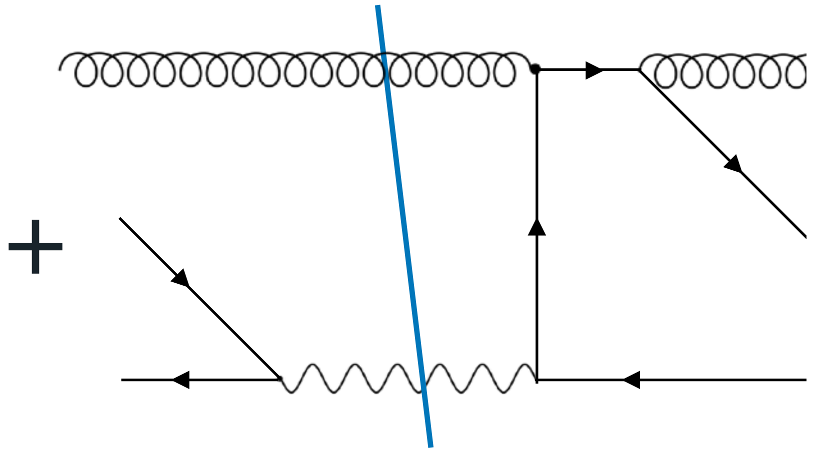

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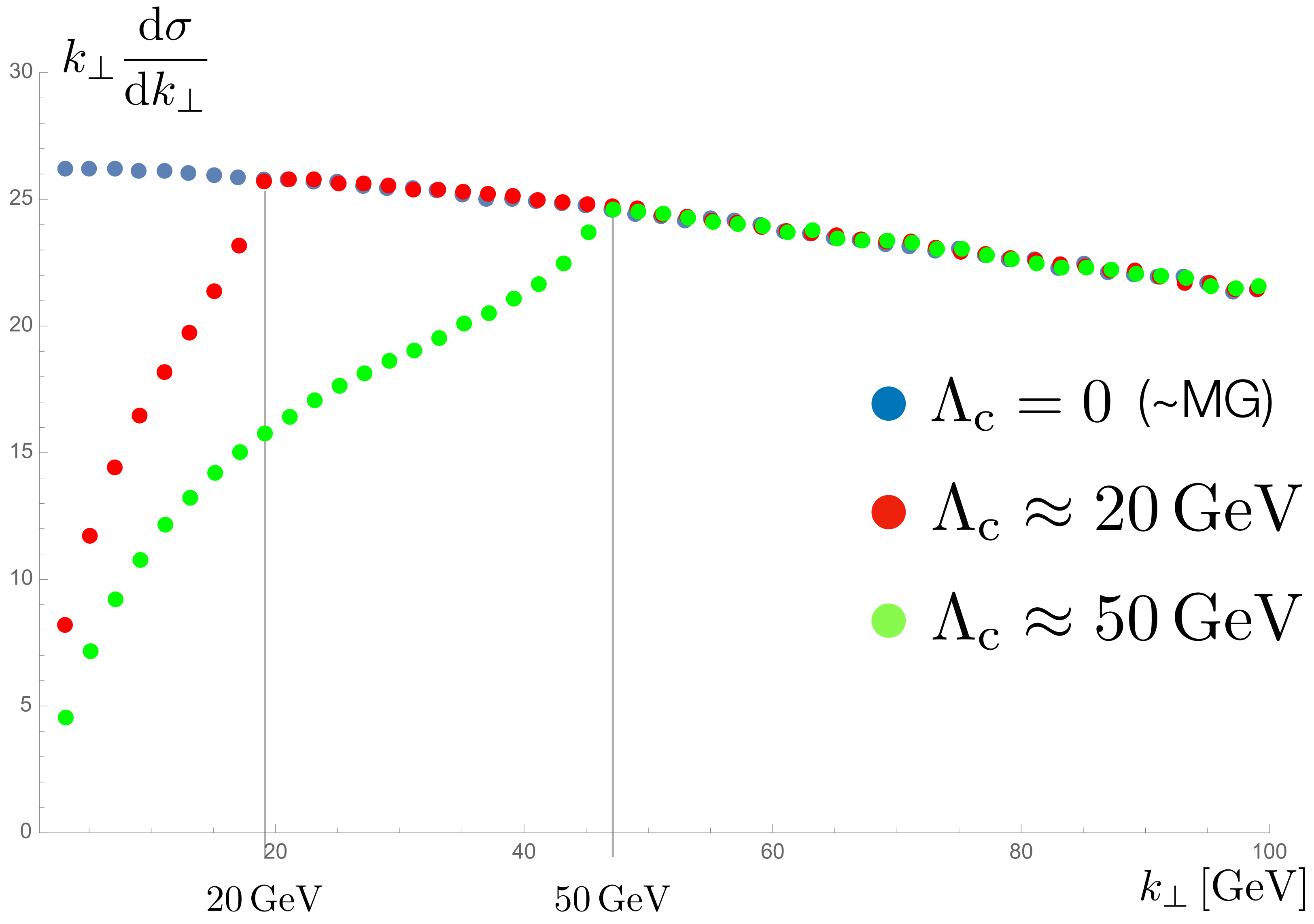

Note that : for $\Lambda_c > 50\,\mathrm{GeV}$
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

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Consider now the inclusive integral

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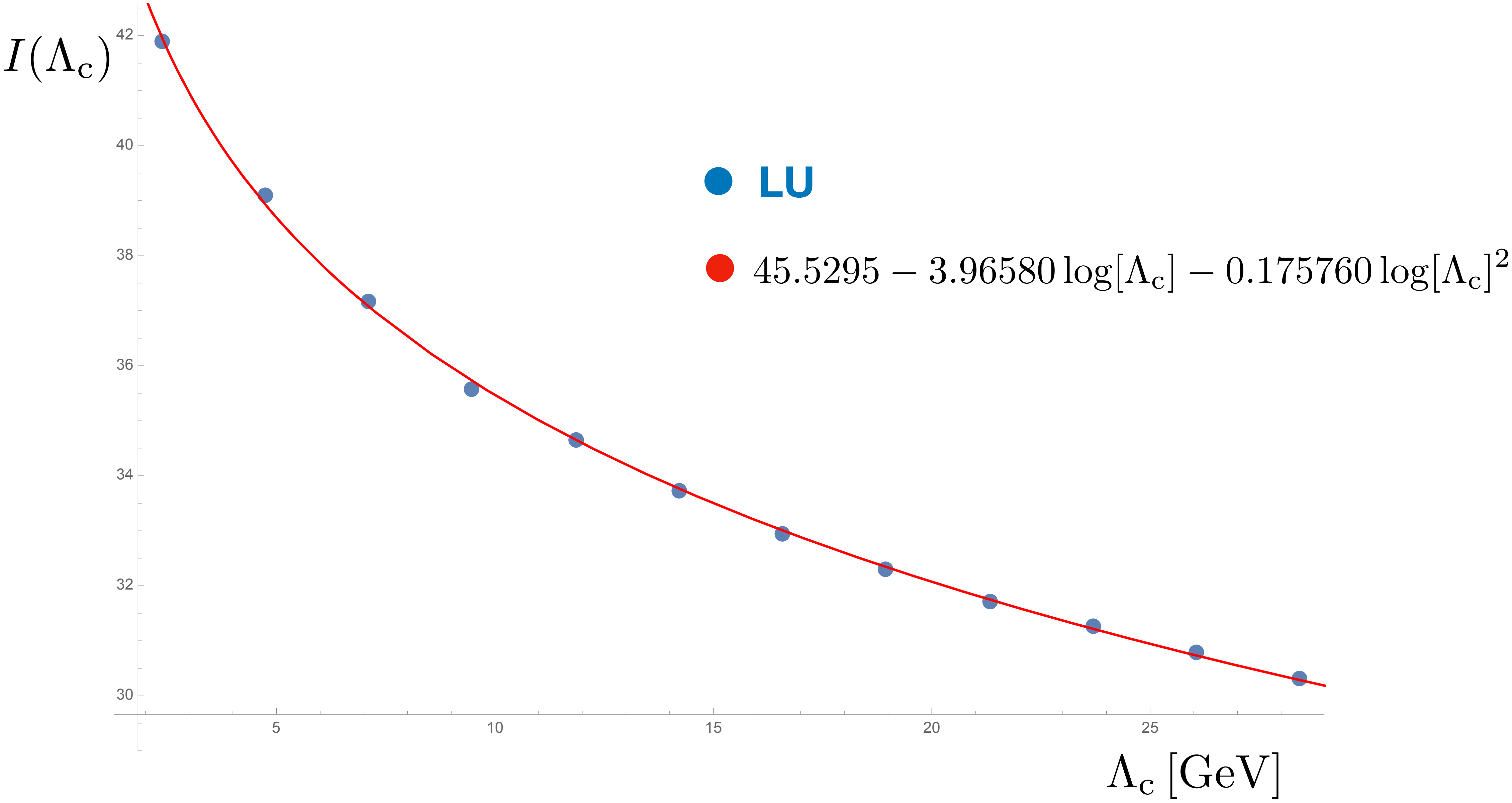
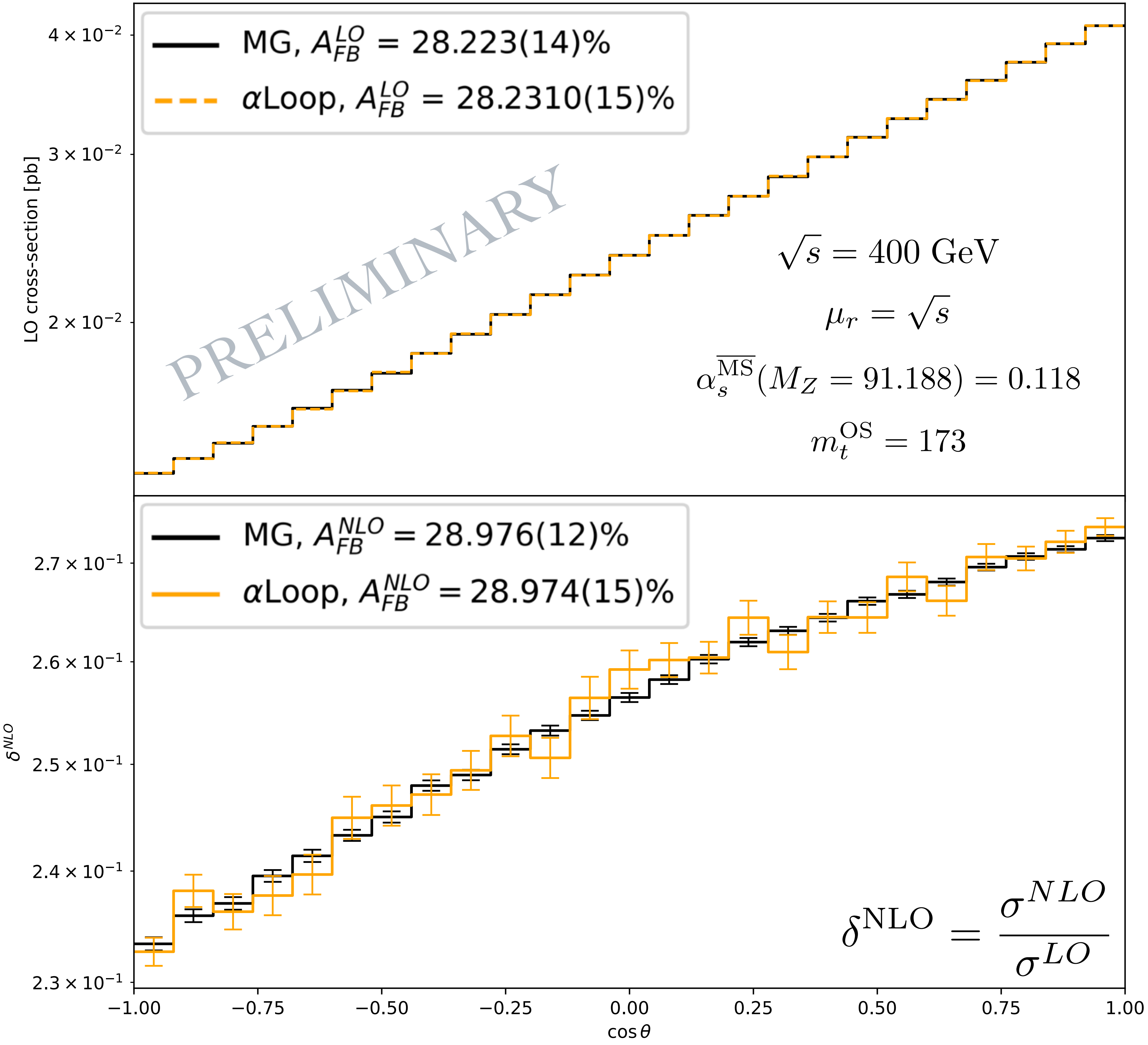


APPLICATIONS

EXAMPLE II : NLO AFB FOR $e^+e^- \to \gamma^\star/Z \to t\bar{t}$



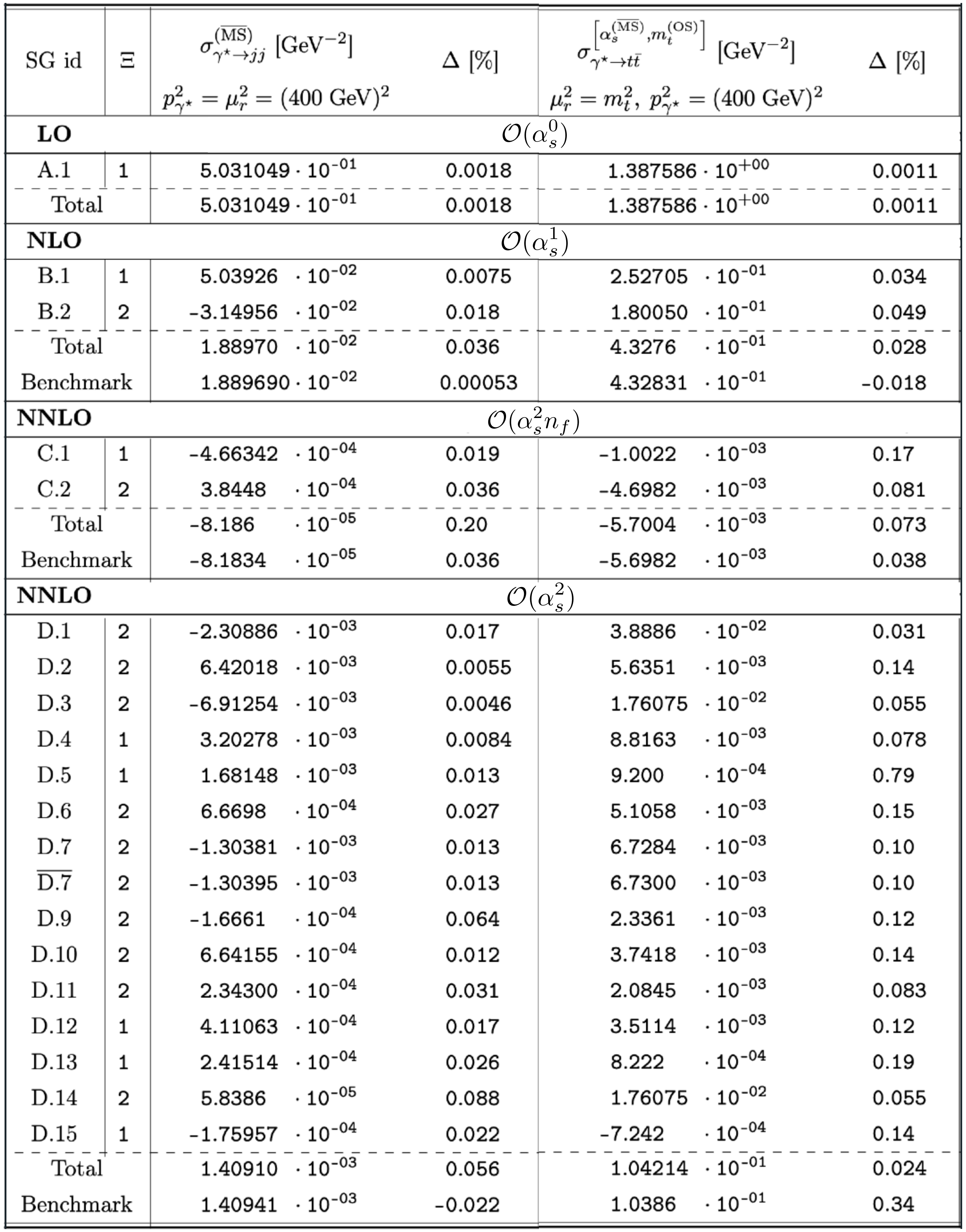
First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer

NUMERICAL RESULTS



SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO			$\mathcal{O}(\alpha_s^0)$		
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO			$\mathcal{O}(\alpha_s^1)$		
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO			$\mathcal{O}(\alpha_s^2 n_f)$		
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
Benchmark		$-8.1834 \cdot 10^{-05}$	0.036	$-5.6982 \cdot 10^{-03}$	0.038
NNLO			$\mathcal{O}(\alpha_s^2)$		
D.1	2	$-2.30886 \cdot 10^{-03}$	0.017	$3.8886 \cdot 10^{-02}$	0.031
D.2	2	$6.42018 \cdot 10^{-03}$	0.0055	$5.6351 \cdot 10^{-03}$	0.14
D.3	2	$-6.91254 \cdot 10^{-03}$	0.0046	$1.76075 \cdot 10^{-02}$	0.055
D.4	1	$3.20278 \cdot 10^{-03}$	0.0084	$8.8163 \cdot 10^{-03}$	0.078
D.5	1	$1.68148 \cdot 10^{-03}$	0.013	$9.200 \cdot 10^{-04}$	0.79
D.6	2	$6.6698 \cdot 10^{-04}$	0.027	$5.1058 \cdot 10^{-03}$	0.15
D.7	2	$-1.30381 \cdot 10^{-03}$	0.013	$6.7284 \cdot 10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395 \cdot 10^{-03}$	0.013	$6.7300 \cdot 10^{-03}$	0.10
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D.10	2	$6.64155 \cdot 10^{-04}$	0.012	$3.7418 \cdot 10^{-03}$	0.14
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D.12	1	$4.11063 \cdot 10^{-04}$	0.017	$3.5114 \cdot 10^{-03}$	0.12
D.13	1	$2.41514 \cdot 10^{-04}$	0.026	$8.222 \cdot 10^{-04}$	0.19
D.14	2	$5.8386 \cdot 10^{-05}$	0.088	$1.76075 \cdot 10^{-02}$	0.055
D.15	1	$-1.75957 \cdot 10^{-04}$	0.022	$-7.242 \cdot 10^{-04}$	0.14
Total		$1.40910 \cdot 10^{-03}$	0.056	$1.04214 \cdot 10^{-01}$	0.024
Benchmark		$1.40941 \cdot 10^{-03}$	-0.022	$1.0386 \cdot 10^{-01}$	0.34

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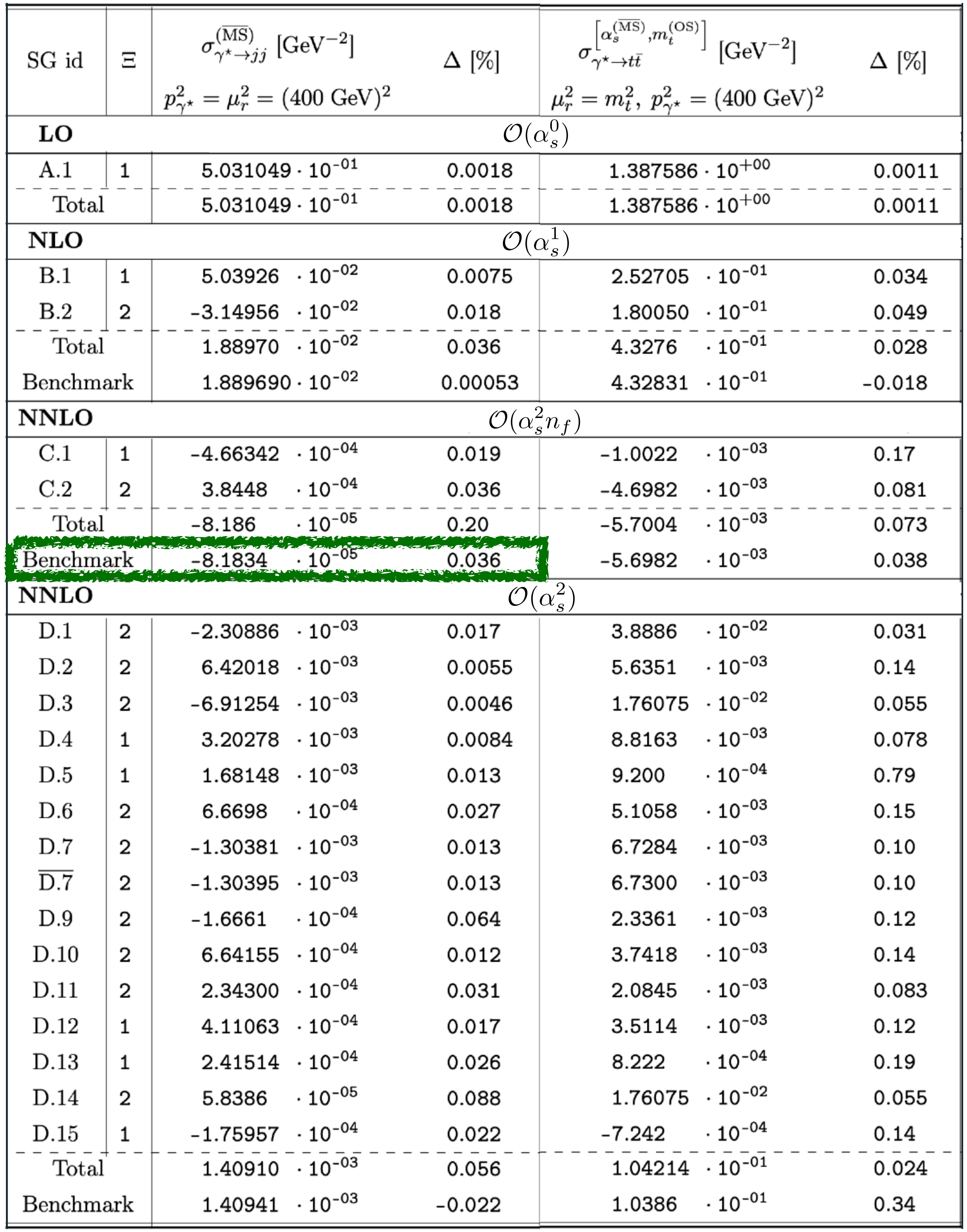

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$$\gamma^\star \to jj$$

$$\boxed{K_{jj}} = -C_F\left[11 - 8\zeta_3\right]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

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$$\gamma^\star\to t\bar{t}$$

$$\boxed{K_{t\bar{t}}}=\quad \delta^{(2)}=-\frac{(3-v^2)\,(1+v^2)}{6}\times$$
$$\Big\{\mathrm{Li}_3(p)-2\,\mathrm{Li}_3(1-p)-3\,\mathrm{Li}_3(p^2)-4\,\mathrm{Li}_3\Big(\frac{p}{1+p}\Big)-5\,\mathrm{Li}_3(1-p^2)+\frac{11}{2}\zeta(3)$$
$$+\mathrm{Li}_2(p)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)+2\,\mathrm{Li}_2(p^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)+2\,\zeta(2)\Big[\ln p-\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]$$
$$-\frac{1}{6}\,\ln\Big(\frac{1+v}{2}\Big)\Big[36\,\ln 2\,\ln p-44\,\ln^2 p+49\,\ln p\,\ln\Big(\frac{1-v^2}{4}\Big)+\ln^2\Big(\frac{1-v^2}{4}\Big)\Big]$$
$$-\frac{1}{2}\,\ln p\,\ln v\Big[36\,\ln 2+21\,\ln p+16\,\ln v-22\,\ln(1-v^2)\Big]\Big\}$$
$$+\frac{1}{24}\Big\{(15-6\,v^2-v^4)\Big(\mathrm{Li}_2(p)+\mathrm{Li}_2(p^2)\Big)+3\,(7-22\,v^2+7\,v^4)\,\mathrm{Li}_2(p)$$
$$-(1-v)\,(51-45\,v-27\,v^2+5\,v^3)\,\zeta(2)$$
$$+\frac{(1+v)\,(-9+33\,v-9\,v^2-15\,v^3+4\,v^4)}{v}\,\ln^2 p$$
$$+\Big[(33+22\,v^2-7\,v^4)\,\ln 2-10\,(3-v^2)\,(1+v^2)\,\ln v$$
$$-(15-22\,v^2+3\,v^4)\,\ln\Big(\frac{1-v^2}{4\,v^2}\Big)\Big]\,\ln p$$
$$+2\,v\,(3-v^2)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)\Big[\ln v-3\,\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]$$
$$+\frac{237-96\,v+62\,v^2+32\,v^3-59\,v^4}{4}\,\ln p-16\,v\,(3-v^2)\,\ln\Big(\frac{1+v}{4}\Big)$$
$$-2\,v\,(39-17\,v^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)-\frac{v\,(75-29\,v^2)}{2}\Big\}\,.\,\ldots \qquad \text{(B.3)}$$

[Chetykrin, Kuehn, Steinhauser, arxiv : 9606230]