

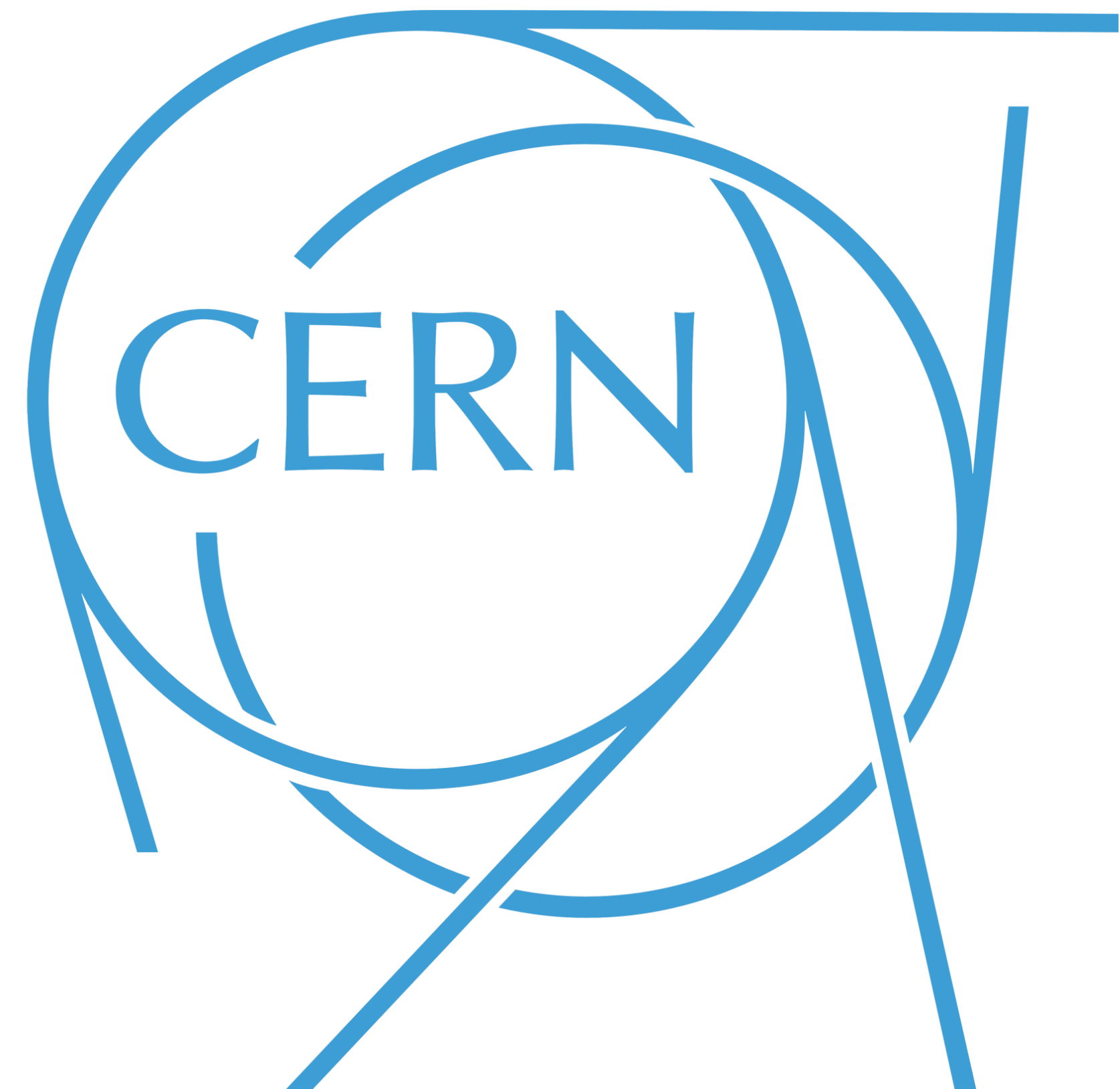
# Two-Loop Spacelike Splitting Amplitudes in Full-Color QCD

**Federico Buccioni**

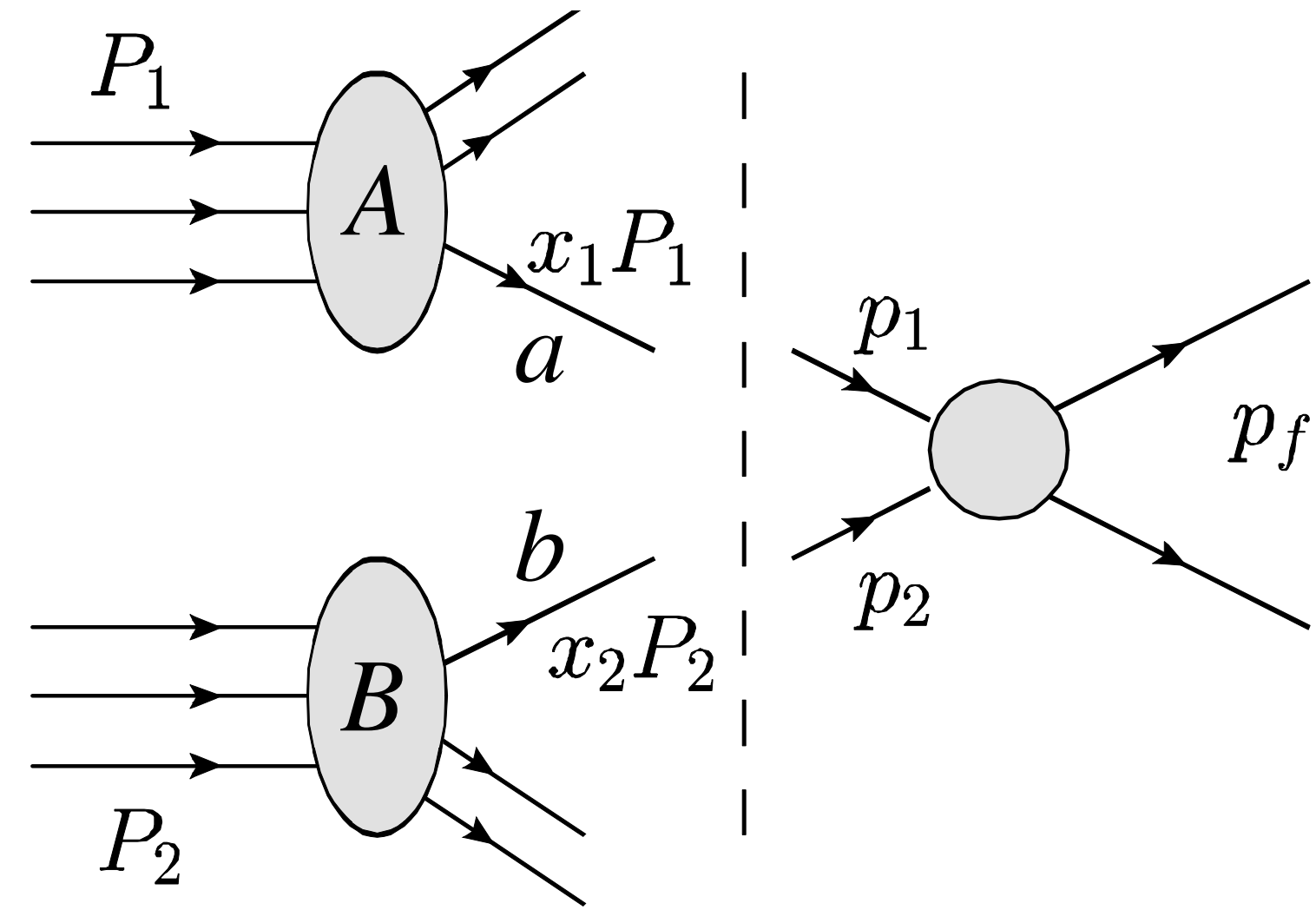
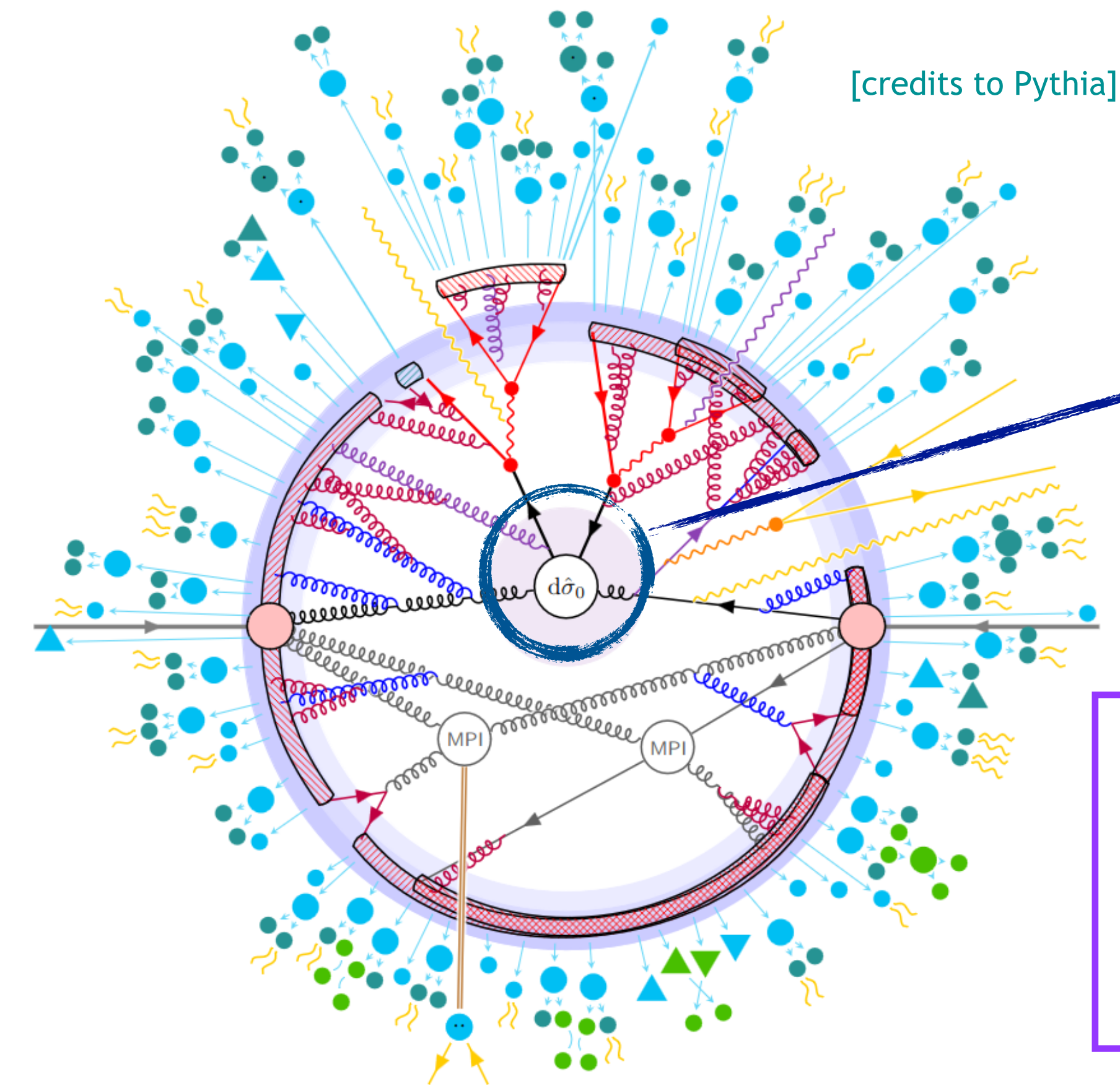
*LoopFest XXIV*

**BNL, 27/05/2026**

based on [FB, H. Fang and K. Yan [2603.27123](#)]



# Hadronic cross sections and collinear factorisation



The master formula

$$\frac{d\sigma_{AB \rightarrow f}}{dO} = \sum_{ab} \int_0^1 dx_1 dx_2 f_{a,A}(x_1, \mu) f_{b,B}(x_2, \mu) \frac{d\hat{\sigma}_{ab \rightarrow f}(x_1, x_2, \mu)}{dO} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^p}{Q^p}\right)$$

Fully-differential hadronic cross section

fundamental property (assumption):

universality of collinear factorisation: separation of long- and short-distance dynamics independent of scattering process

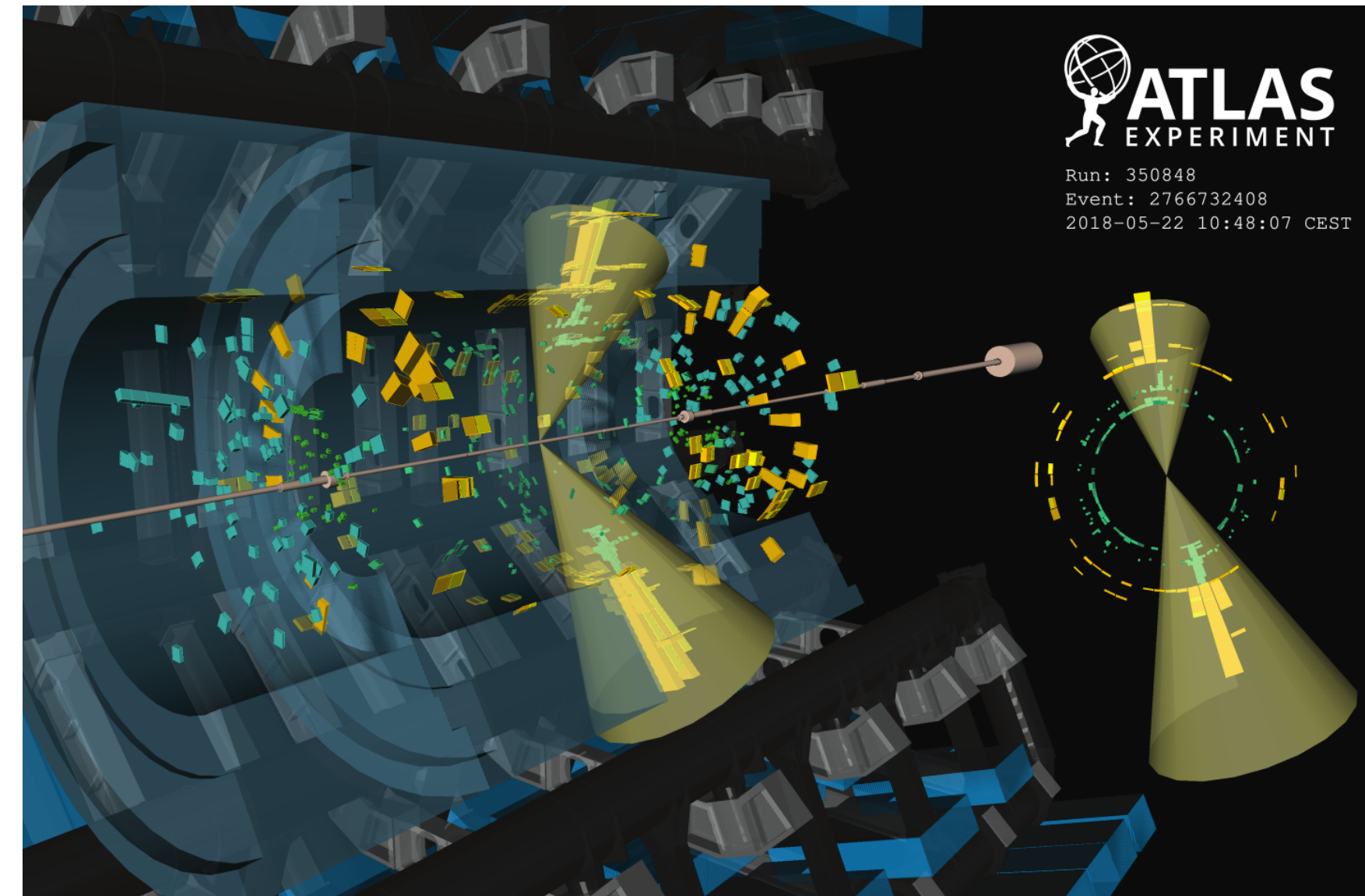
# Fully differential jet cross sections

Fully differential  $m$ -jets cross section in pQCD

need a definition of jets (jet algorithm +  $p_{\perp}$  cut, rapidity intervals, etc);

$\Lambda \rightarrow$  “fiducial volume”

$$\int_{\Lambda} \equiv \int F(\vec{p}) \rightarrow \text{eg} \rightarrow F(\vec{p}) := \prod_i \theta(p_{i,\perp} > 50 \text{ GeV}) \times \theta(2.5 - y_i) \times \theta(y_i + 2.5)$$



this allows the investigation of **any (multi-)jet observable** (provided it is *IR safe*) at fixed order in pQCD

$$d\hat{\sigma}_m^{\text{LO}} = \int_{\Lambda} d\Phi_m |\mathcal{M}_{2 \rightarrow m}^{(0l)}|^2$$

$$d\hat{\sigma}_m^{\delta\text{NLO}} = \int_{\Lambda} d\Phi_m |\mathcal{M}_{2 \rightarrow m}^{(1l)}|^2 + \int_{\Lambda} d\Phi_{m+1} |\mathcal{M}_{2 \rightarrow m+1}^{(0l)}|^2$$

...

$$d\hat{\sigma}_m^{\delta\text{N}^3\text{LO}} = \int_{\Lambda} d\Phi_m |\mathcal{M}_{2 \rightarrow m}^{(3l)}|^2 + \int_{\Lambda} d\Phi_{m+1} |\mathcal{M}_{2 \rightarrow m+1}^{(2l)}|^2 + \int_{\Lambda} d\Phi_{m+2} |\mathcal{M}_{2 \rightarrow m+2}^{(1l)}|^2 + \int_{\Lambda} d\Phi_{m+3} |\mathcal{M}_{2 \rightarrow m+3}^{(0l)}|^2$$

after complete sum of V+R: only **collinear singularities** left



**PDF renormalisation**  
(universal?)

# Hadronic cross sections and collinear factorisation

a **proof** (hadronic XS) exists only for inclusive colour-singlet production [Collins, Soper, Sterman '85, '88]

many have questioned/investigated the **validity/limitations** in more involved cases: **differential jet cross sections**

some (illustrative) examples:

- “*factorisation is violated in production of high-transverse-momentum particles in hh collisions*” [Collins and Qiu [0705.2141](#)]
- “*Glauber Gluons and Multiple Parton interactions*” [Gaunt [1405.2080](#)]
- “*Drell-Yan with jet vetoes: breaking of generalized factorization*” [Zeng [1507.01652](#)]
- “*Coloumb gluons will generally destroy coherence*” [Forshaw and Holguin [2109.03665](#)]

and many more...

THIS TALK



Amplitude level: **breakdown of collinear factorisation in spacelike collinear splittings**: indication of **PDF factorisation breaking?**

- “*Space-like (vs time-like) collinear limits in QCD: Is factorization violated?*” [Catani, de Florian, Rodrigo [1112.4405](#)]
- shortly followed by “*On the Breaking of Collinear Factorization in QCD*” [Forshaw, Seymour, Siodmok [1206.6363](#)]
- more recently “*Catani’s generalization of collinear factorization breaking*” [Cieri, Dhani, Rodrigo [2402.14749](#)]

# Is factorisation violated?

The strict factorization breaking effects uncovered in the simple example of  $2 \rightarrow 3$  parton scattering have more general implications in the context of perturbative QCD computations of jet and hadron production in hadron-hadron collisions. Starting from the  $N^3\text{LO}$  in perturbation theory, these effects severely complicate the mechanism of cancellation of IR divergences that leads to the factorization theorem of mass (collinear) singularities [29]. These complications challenge the universal (process-independent) validity of mass-singularity factorization, and they are related to issues that arise in the context of factorization of transverse-momentum dependent distributions [30, 31, 32]. The perturbative resummation of large logarithmic terms produced by collinear parton evolution is also affected by the violation of strict collinear factorization: parton evolution gets tangled with the colour and kinematical structure of the hard-scattering subprocess, and this leads to the appearance of ‘entangled logarithms’. An example of entangled logarithms is represented by the class of ‘super-leading’ non-global logarithms discovered [33] in the  $N^4\text{LO}$  computation of the dijet cross section with a large rapidity gap between the two jets. Indeed, the physical mechanism that produces those super-leading logarithms [34] is directly related to the mechanism that generates the violation of strict collinear factorization.

THIS TALK

Super Leading Logs  
(beyond this talk)

# Plan of the talk

- (breaking of strict) Collinear factorisation
- Spacelike splitting from full-colour QCD amplitudes
- (two-loop) Generalised splitting amplitudes
- Cross-section level results
- Summary and outlook

## Disclaimer:

this is a talk in fixed-order pQCD

I won't discuss all-orders effects (neither resummation)

I will work in full QCD (not in an EFT)

# Beyond this talk

(in the context of QCD observables)

Critical developments and results from many “orthogonal” directions (to a fairly large degree)

- An EFT for forward scattering and factorization violation (Glauber SCET) [[Rothstein and Stewart 1601.04695](#)]
- 2-loop spacelike splitting amplitude in  $\mathcal{N} = 4$  sYM [[Henn, Ma, Xu, Yan, Zhang, Zhu 2406.14604](#)]
- Factorisation restoration through Glauber gluons [[Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 2408.10308](#)]
- Low-energy theory of jet processes and PDF factorization [[Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 2509.07082](#)]

Connections to Multi-Regge Kinematics (MRK)

Reggeon dynamics vs Glauber dynamics

- Double spacelike collinear from MRK (in  $\mathcal{N} = 4$  sYM) [[Duhr, Venkata, Zhang 2507.05355](#)]

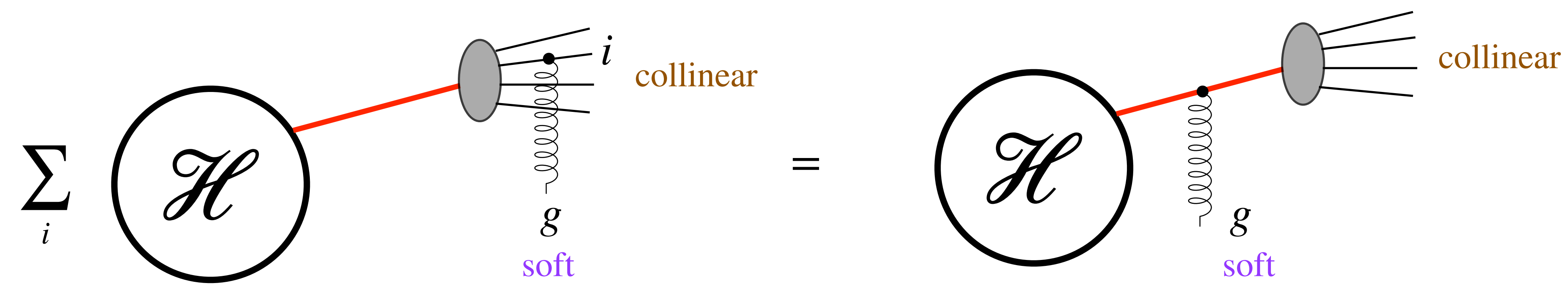
**(breaking of strict)**

# **Collinear factorisation**

# Colour coherence and strict collinear factorisation

- Colour coherence:

a soft gluon ( $g$ ) cannot resolve the internal structure of a bunch of collinear partons ( $i$ ): only sensitive to the net colour charge



- Strict collinear factorisation:

a collinear factorisation formula depends only on the momenta and quantum numbers (flavour, colour, spin) of collinear partons

↓  
*strict collinear factorisation*

↓  
**violated when**

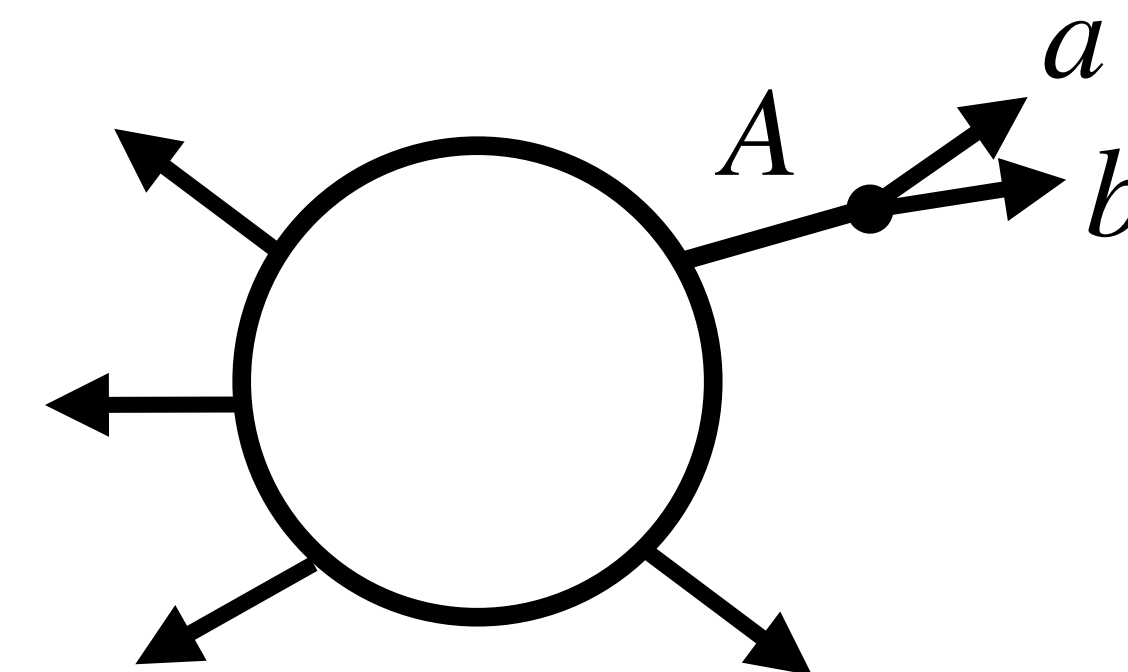
↓  
there is a **non-trivial dependence on** quantum numbers of non-collinear (**spectator**) **partons**

# Amplitude level collinear factorisation

Consider an  $(n + 1)$ -point scattering amplitude in QCD  $\mathcal{A}_{n+1}$

convention: all particles with momenta  $p_i$  are **outgoing**  $\rightarrow p_i^0 > 0$  final state,  $p_i^0 < 0$  initial state

pair of particles  $(a, b)$  with helicities  $(\lambda_a, \lambda_b)$  **collinear**  $\rightarrow A$  with helicity  $\lambda_A$



$$\mathcal{A}_{n+1}(a^{\lambda_a}, b^{\lambda_b}, \dots) \simeq \sum_{\lambda_A = \pm} \mathbf{Sp}_{\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}_n(A^{\lambda_A}, \dots)$$

operator in colour space

in pQCD, both  $\mathbf{Sp}$  and  $\mathcal{A}$  admit an expansion in  $\alpha_s$

$$\mathbf{Sp}_{\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) = \sum_{\ell=0} \left( \frac{\alpha_s}{4\pi} \right)^\ell \mathbf{Sp}_{\lambda_A}^{(\ell)}(a^{\lambda_a}, b^{\lambda_b})$$

At two-loop order:

$$\mathcal{A}_{n+1}^{(2)}(a^{\lambda_a}, b^{\lambda_b}, \dots) \simeq \sum_{\lambda_A = \pm} \left[ \mathbf{Sp}_{\lambda_A}^{(0)} \times \mathcal{A}_n^{(2)} + \mathbf{Sp}_{\lambda_A}^{(1)} \times \mathcal{A}_n^{(1)} + \mathbf{Sp}_{\lambda_A}^{(2)} \times \mathcal{A}_n^{(0)} \right]$$

this talk

- spacelike
- all partonic config.
- all helicity config.

# Timelike vs spacelike splittings

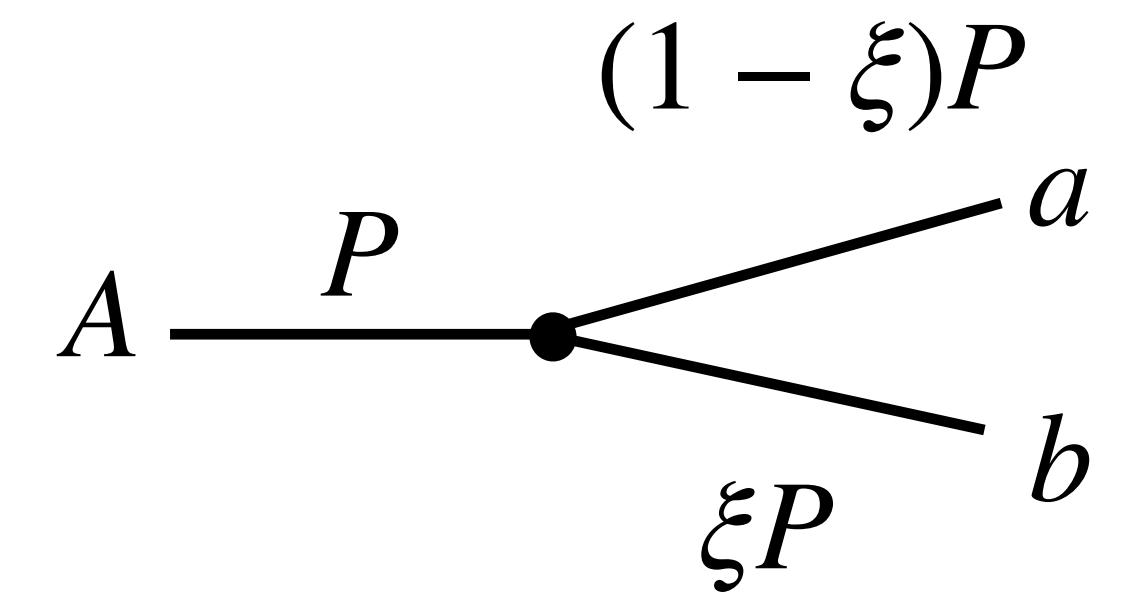
Timelike splitting:

$(p_a^0 p_b^0 > 0, P^0 > 0)$ , i.e. either both  $(a, b)$  in the final state or both in the initial state

$$A(P) \rightarrow a((1 - \xi)P) + b(\xi P), \quad \text{with} \quad 0 < \xi < 1$$

state of the art: three-loop [\[Guan, Herzog, Ma, Mistlberger, Suresh 2408.03019\]](#)

no violation of strict collinear factorisation



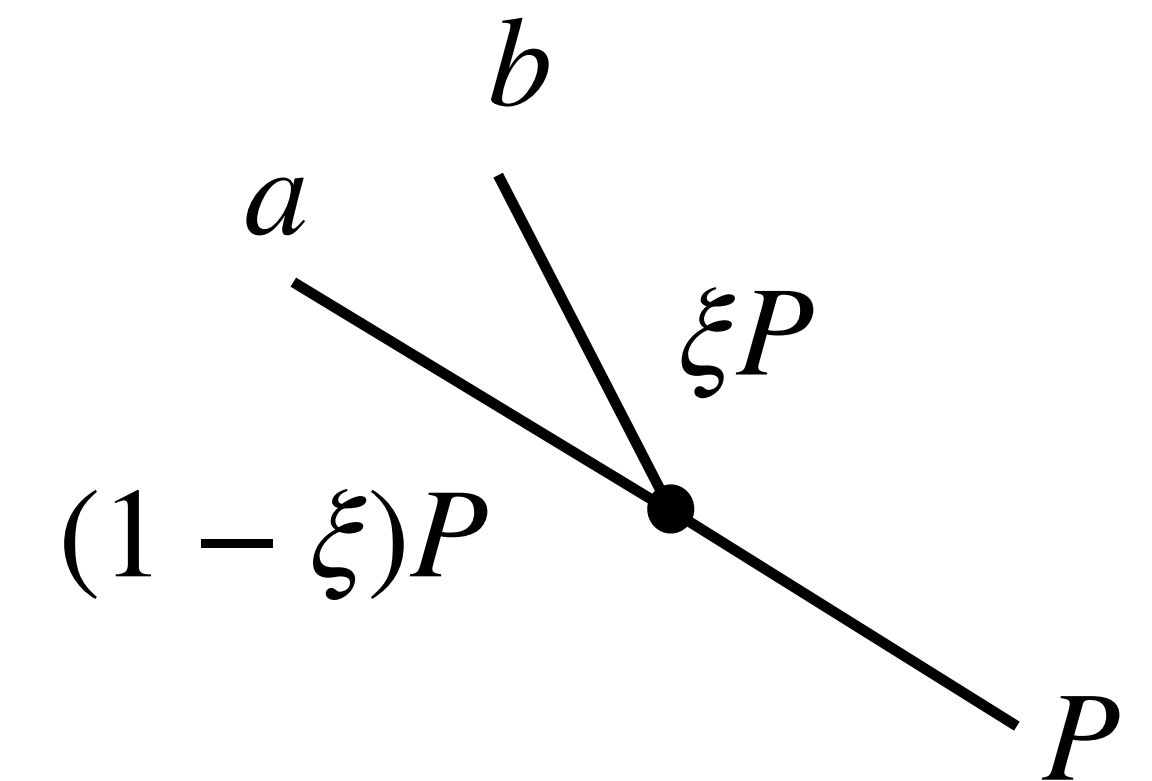
Spacelike splitting:

$(p_a^0 p_b^0 < 0, P^0 < 0)$ , i.e.  $a$  in the initial state and  $b$  in the final state (or viceversa)

$$a((1 - \xi)P) \rightarrow A(P) + b(\xi P), \quad \text{with} \quad \xi < 0$$

loop corrections induce a non-trivial dependence on quantum numbers of spectators

**violation of strict collinear factorisation**



# Collinear factorisation at one loop

Let's see this explicitly at 1-loop. Simplified framework: IR structure (**poles**) of scattering amplitudes

[Catani, De Florian, Rodrigo [1112.4405](#)]

[Forshaw, Seymour, Siodmok [1206.6363](#)]

Consider  $(n + 1)$ -scattering process, with  $p_{1,2}$  in the initial state i.e.  $p_{1,2}^0 < 0$

IR factorisation

$$|\mathcal{A}_{n+1}^{(1)}\rangle = \mathbf{I}_{n+1}^{(1)} |\mathcal{A}_{n+1}^{(0)}\rangle + |\mathcal{A}_{n+1}^{(1,\text{fin})}\rangle$$

coll factorisation

$$|\mathcal{A}_{n+1}^{(1)}\rangle = \mathbf{I}_{n+1}^{(1)} \cdot \mathbf{Sp}^{(0)} |\mathcal{A}_n^{(0)}\rangle + |\mathcal{A}_{n+1}^{(1,\text{fin})}\rangle$$

collinear factorisation

$$|\mathcal{A}_{n+1}^{(1)}\rangle = \mathbf{Sp}^{(0)} |\mathcal{A}_n^{(1)}\rangle + \mathbf{Sp}^{(1)} |\mathcal{A}_n^{(0)}\rangle$$

IR factorisation

$$|\mathcal{A}_{n+1}^{(1)}\rangle = \mathbf{Sp}^{(0)} \cdot \mathbf{I}_n^{(1)} |\mathcal{A}_n^{(0)}\rangle + \mathbf{Sp}^{(1)} |\mathcal{A}_n^{(0)}\rangle + \text{fin}$$

define  $\mathbf{I}_C^{(1)}$  via:  $\mathbf{Sp}^{(1)} = \mathbf{I}_C^{(1)} \cdot \mathbf{Sp}^{(0)}$

at poles level one can establish:

$$\mathbf{I}_C^{(1)} \cdot \mathbf{Sp}^{(0)} = \mathbf{I}_{n+1}^{(1)} \cdot \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)} \cdot \mathbf{I}_n^{(1)}$$

# Collinear factorisation at one loop (TL)

The infrared operator can be written as

$$\mathbf{I}_{n+1}^{(1)} = \frac{\alpha_s}{4\pi} \left\{ - \sum_{i=1}^{n+1} \left( \frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} + i\pi \frac{C_i}{\epsilon} \right) + \frac{1}{\epsilon} \sum_{(i,j)}^{n+1} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left( \frac{\mu^2}{|s_{ij}|} \right) + \frac{2i\pi}{\epsilon} \mathbf{T}_{12}^2 \right\} \quad \mathbf{T}_{12} = \mathbf{T}_1 + \mathbf{T}_2$$

Coloumb/Glauber phase

an equivalent structure holds for the underlying  $n$ -point scattering process

**Timelike splitting**, e.g.  $p_n \parallel p_{n+1}: p_{\tilde{n}} \rightarrow p_n(z_n p_{\tilde{n}}) + p_{n+1}(z_{n+1} p_{\tilde{n}})$ , with  $z_{n,n+1} > 0$ . Moreover:  $\mathbf{T}_{\tilde{n}} = \mathbf{T}_n + \mathbf{T}_{n+1}$

$$\mathbf{I}_C^{(1)} \cdot \mathbf{Sp}^{(0)} = \mathbf{I}_{n+1}^{(1)} \cdot \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)} \cdot \mathbf{I}_n^{(1)} \quad \text{terms that do not involve } \tilde{n}, n \text{ or } n+1 \text{ commute, so vanish in the difference}$$

$$\mathbf{I}_C^{(1)} = \frac{\alpha_s}{4\pi} \left\{ (C_{\tilde{n}} - C_n - C_{n+1}) \left( \frac{1}{\epsilon^2} + \frac{i\pi}{\epsilon} \right) + \frac{\gamma_{\tilde{n}} - \gamma_n - \gamma_{n+1}}{\epsilon} + \frac{2}{\epsilon} C_n \ln z_n + \frac{2}{\epsilon} C_{n+1} \ln z_{n+1} + \frac{2}{\epsilon} \mathbf{T}_n \cdot \mathbf{T}_{n+1} \ln \left( \frac{\mu^2 z_n z_{n+1}}{|s_{n+1,n}|} \right) \right\}$$

**strict collinear factorisation**: depends only on collinear partons  
consequence of **colour coherence** (+ colour conservation)

# Collinear factorisation at one loop (SL)

Space like splitting, e.g.  $p_1 \parallel p_{n+1}: p_{\tilde{n}} \rightarrow p_1(z_1 p_{\tilde{n}}) + p_{n+1}(z_{n+1} p_{\tilde{n}})$ , with  $p_{\tilde{n}}^0 < 0$ ,  $z_1 > 0$ ,  $z_{n+1} < 0$

$$\mathbf{I}_{n+1}^{(1)} = \frac{\alpha_s}{4\pi} \left\{ - \sum_{i=1}^{n+1} \left( \frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} + i\pi \frac{C_i}{\epsilon} \right) + \frac{1}{\epsilon} \sum_{(i,j)}^{n+1} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left( \frac{\mu^2}{|s_{ij}|} \right) + \frac{2i\pi}{\epsilon} \mathbf{T}_{12}^2 \right\}$$

does not trivially vanish anymore in the difference

recall:  $\mathbf{I}_C^{(1)} \cdot \mathbf{Sp}^{(0)} = \mathbf{I}_{n+1}^{(1)} \cdot \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)} \cdot \mathbf{I}_n^{(1)}$

in SL kinematics we get:

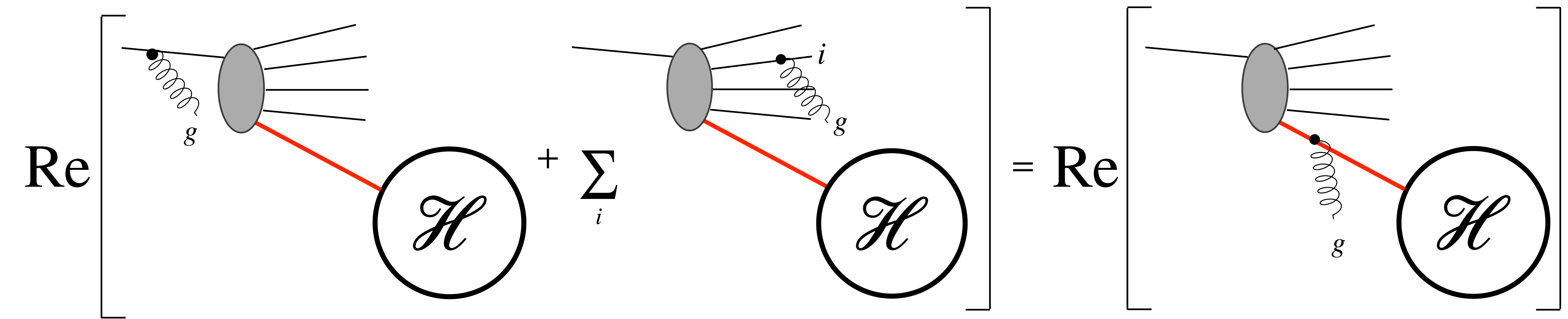
$$\mathbf{I}_C^{(1)} = \frac{\alpha_s}{4\pi} \left\{ \frac{C_{\tilde{n}} - C_1 - C_{n+1}}{\epsilon^2} + \frac{2i\pi}{\epsilon} (C_1 - C_{n+1} - C_{\tilde{n}}) + \frac{\gamma_{\tilde{n}} - \gamma_1 - \gamma_{n+1}}{\epsilon} + \frac{2}{\epsilon} C_1 \ln z_1 + \frac{2}{\epsilon} C_{n+1} \ln |z_{n+1}| \right. \\ \left. + \frac{2}{\epsilon} \mathbf{T}_1 \cdot \mathbf{T}_{n+1} \ln \left( \frac{\mu^2 z_1 |z_{n+1}|}{s_{1,n+1}} \right) + \tilde{\Delta}_C^{(1)} \right\}$$

$$\tilde{\Delta}_C^{(1)} = -4 \frac{i\pi}{\epsilon} \mathbf{T}_2 \cdot \mathbf{T}_{n+1}$$

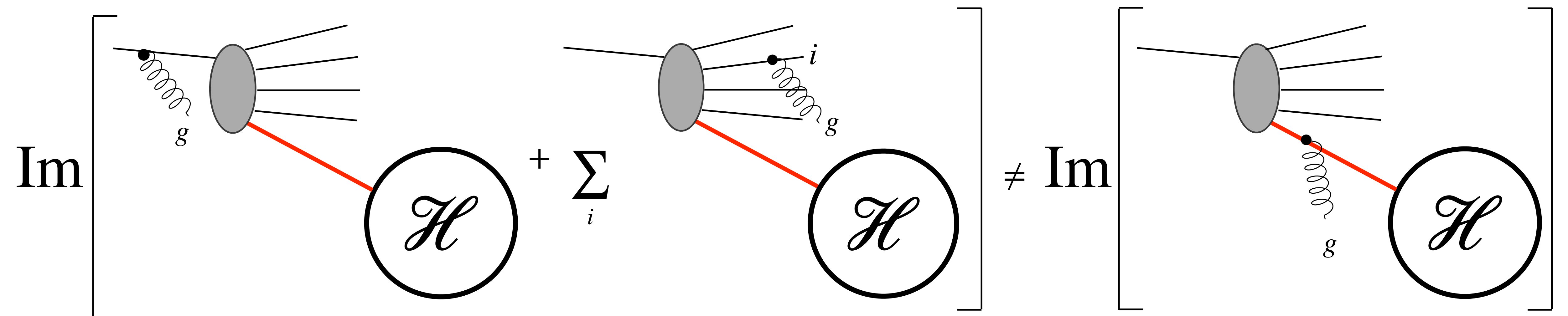
depends on colour charge of spectator 2

violates strict collinear factorisation

# Collinear factorisation at one loop (SL)



violation of colour coherence  $\leftrightarrow$  violation of strict collinear factorisation



# A glimpse at two-loop amplitudes

source of strict CFV at one-loop: **dipole operator**

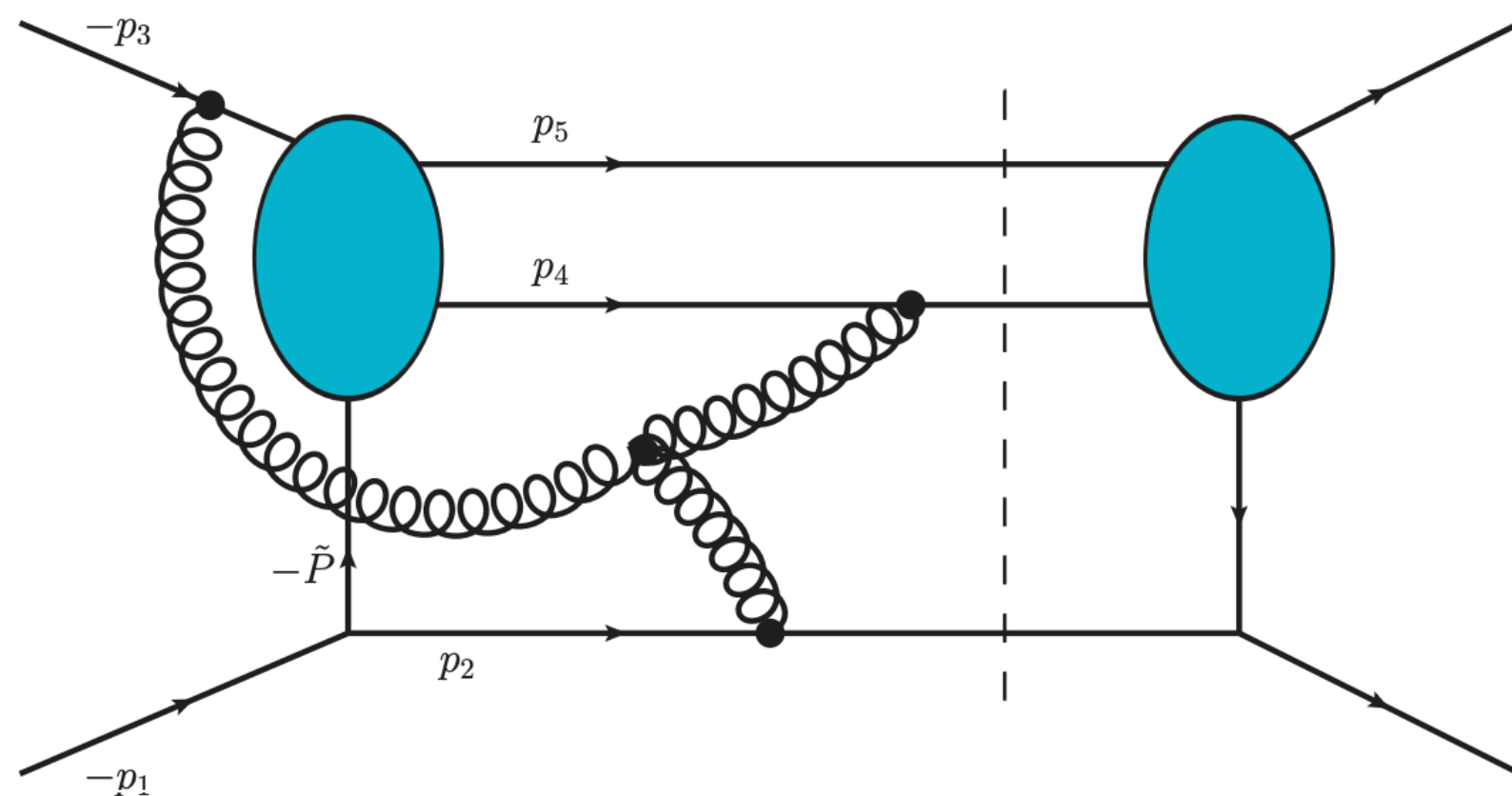
responsible: **soft (Glauber) gluon exchange** between **incoming spectator parton** + **collinear outgoing parton**

this will repeat at two-loops: **two dipole insertions**

responsible: **double (soft) Glauber gluon exchange** between **incoming spectator parton** + **collinear outgoing parton**

These contributions were argued to cancel at XS level [\[Catani, De Florian, Rodrigo 1112.4405\]](#)

Genuine new colour operator at 2-loop for  $n \geq 5$  colour partons: violates strict collinear factorisation



[\[Catani, De Florian, Rodrigo 1112.4405\]](#)

**Tripole operator** ([\[Catani, De Florian, Rodrigo 1112.4405\]](#), [\[Dixon, Herrmann, Yan, Zhu 1912.09370\]](#))

- pure non-abelian origin
- absorptive contribution
- relevant in the 2-loop single-collinear (RVV) of dijet cross-sections

are there more CFV contributions? **YES** (this talk, new)

# Spacelike splitting from full-colour QCD amplitudes

# Five-parton scattering in QCD at two loops

Start from 2-loop 5-pt amplitudes in full-colour QCD [Agarwal, FB, Devoto, Gambuti, von Manteuffel, Tancredi 2311.09870]

$$\mathcal{A}_5^{(L)} = \sum_{k=-2L} \sum_m \epsilon^k R_m^{(k)}(s_{ij}; \text{tr}_5, \epsilon_5) f_m^{(w_k)}(\{W\}) \quad L = 1, 2$$

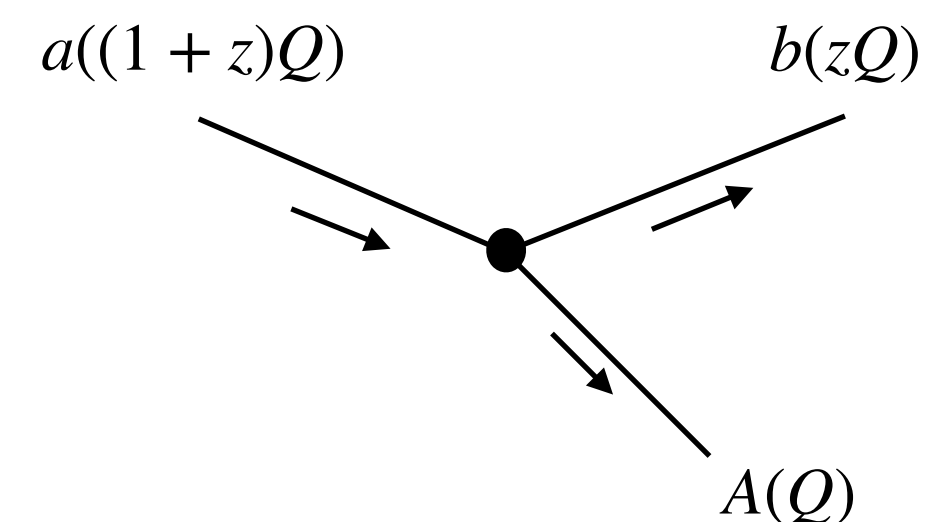
- $R_m^{(k)}$  rational functions
- $f_m^{(w_k)}$  massless pentagon functions (iterated integrals) [Gehrmann et al 1807.09812, Abreu et al 1812.08941, Chicherin et al 1812.11160, Chicherin et al 2009.07803]

Parametrisation of the kinematics in the  $p_2 \parallel p_3$  collinear limit:  $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \rightarrow \{s, x, z, y, \delta\}$

$$s_{12} = s(1+z), \quad s_{23} = -sz\delta^2, \quad s_{34} = s(1-x)z, \quad s_{45} = s, \quad s_{51} = -s \left( 1 - x - \frac{4y\sqrt{x(1-x)(1+z)}}{1+y^2} \delta \right)$$

- $z$  energy fraction carried by 3 in the splitting. Here:  $z > 0$

moreover:  $z \rightarrow 0$  is the soft limit and  $z \rightarrow \infty$  is the high-energy limit (*small-x*)



# Expansion in the collinear limit

## Strategy:

1. expand 1- and 2-loop scattering amplitudes in  $\delta \rightarrow 0$
2. match result to generalised 1- and 2-loop splitting amplitudes (colour operators) acting on four point amplitudes
3. extract 2-loop corrections to splitting amplitude in spacelike kinematics

## Implementation:

- A. leading singularity  $1/\delta$  captured by tree-level (factored out): loop corrections can only induce  $\ln^\ell(\delta)$  corrections
- B. expand rational functions: straightforward but painful
- C. expand pentagon functions: **two alternative methods**
  - C1. systematically take **discontinuity of pentagon functions** (work at symbol level) and move from SL to TL kinematics
  - C2. solve **differential equations** wrt  $s_{ij}$  satisfied by pentagon functions as a **power series in  $\delta$**

agreement between the two methods

in rational functions spurious  $1/\delta^3$  poles  $\rightarrow$  need to expand PF up to  $\delta^3$

# Expansion via differential equations

At leading power (LP) in  $\delta \sim 0$  the pentagon alphabet reduces to 16 letters

$\{\delta\}, \{s\}, \{1 \pm y, i \pm y\}, \{x, 1 - x, z, 1 + z, 1 + z - x,$

$x + z, 1 - x - xz, 1 + z - xz, 1 + xz, x - z + xz\}$

pentagon functions defined as iterated integrals with boundary point in  $X_0$

step #1:

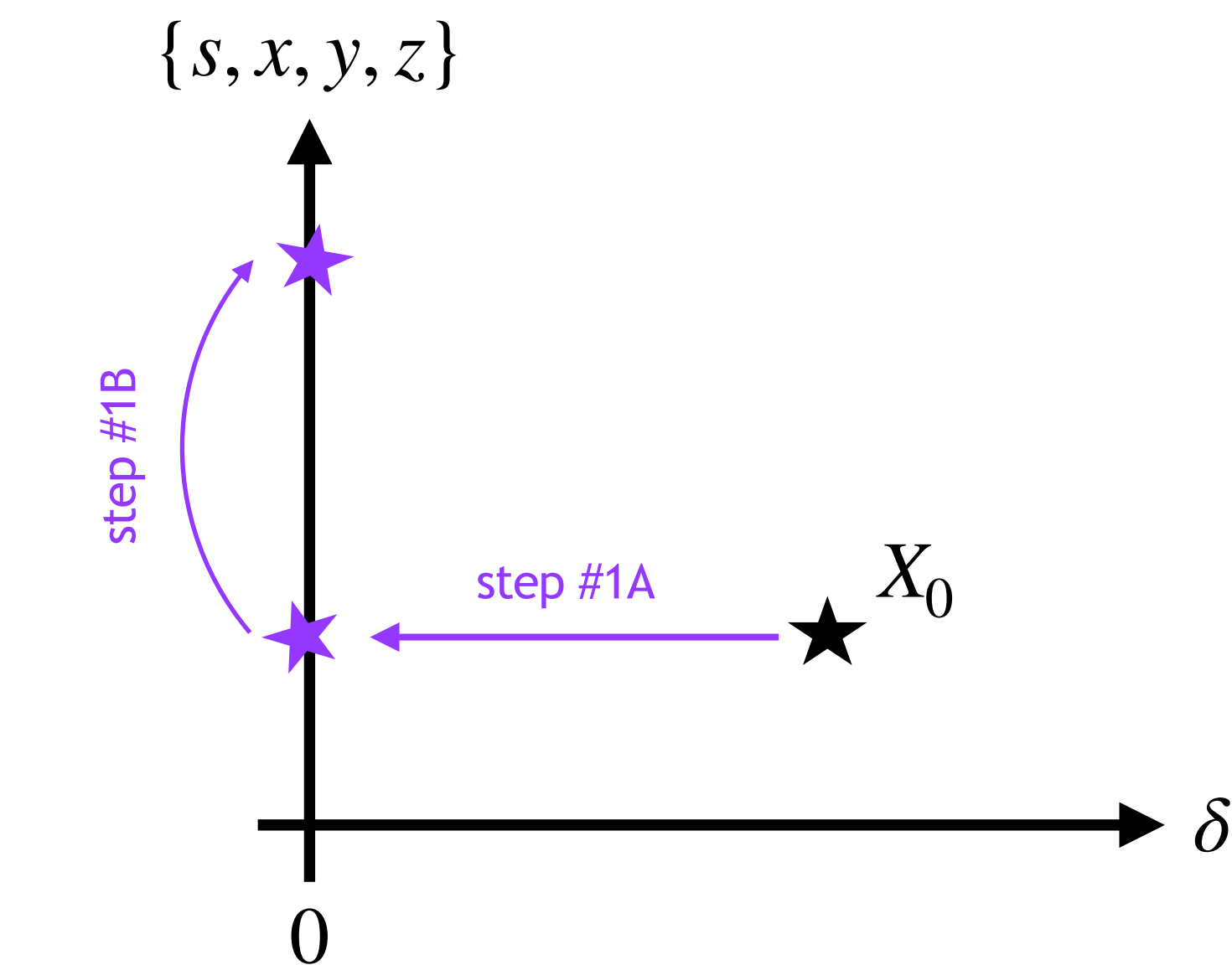
A. at fixed  $\{s, x, y, z\}$  solve 1-dimension deq in  $\delta \rightarrow$  complete tower of logarithms  $\ln^\ell \delta$

B. integrate deqs in  $\{s, x, y, z\}$ : write (compact) ansatz in terms of GPLs and fix coefficients

step #2:

solutions of PF beyond LP: write pentagon function as a generalised series around  $\delta = 0$

$$f_i^{(w)} = \sum_{n=0} \sum_{\ell=0}^w \delta^n \ln^\ell(\delta) g_{i,n\ell}^{(w-\ell)}(\{s\}, \{y\}, \{x, z\})$$



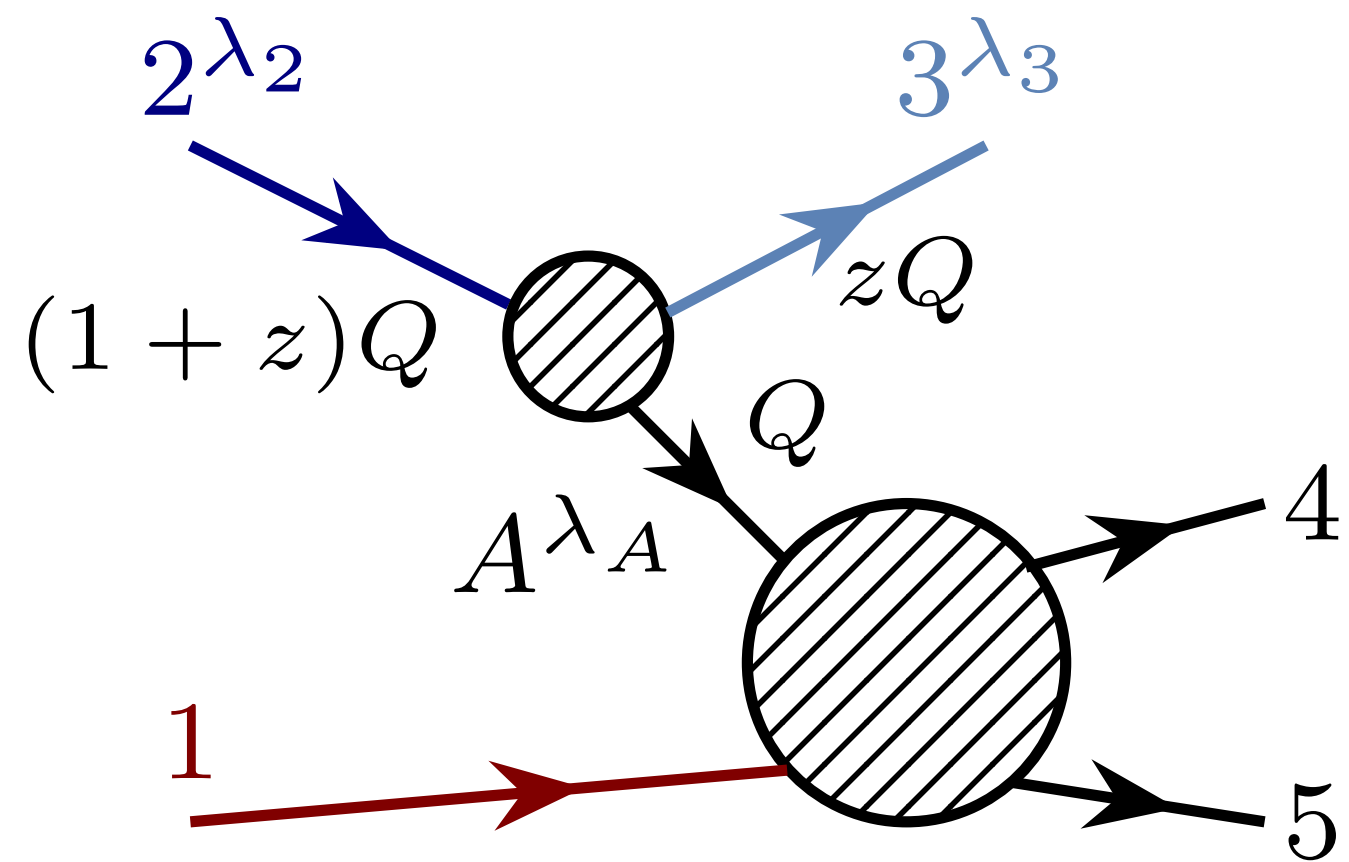
scattering amplitudes expressed in terms of  
**simple single-valued polylogarithms**

+  $\{i\pi, \zeta_2, \zeta_3\}$

**(two-loop)**  
**Generalised splitting amplitudes**

# General form and main ingredients

Consider the spacelike splitting:  $a^{\lambda_a}((1+z)Q) \rightarrow A^{\lambda_A}(Q) + b^{\lambda_b}(zQ), \quad z > 0$



alternatively, upon performing simultaneously:  $Q \rightarrow zQ, \quad z \rightarrow (1-z)/z$

$$a^{\lambda_a}(Q) \rightarrow A^{\lambda_A}(zQ) + b^{\lambda_b}((1-z)Q), \quad 0 < z < 1$$

here, take  $a=2$  and  $b=3$

$$\mathbf{Sp}_{\lambda_A}(2^{\lambda_2}, 3^{\lambda_3}) = i\sqrt{4\pi\alpha_s} \text{Split}_{\lambda_A}(2^{\lambda_2}, 3^{\lambda_3}) \times \left\{ \exp \left[ \mathcal{G}_{\vec{\lambda}}(z; \epsilon) + \Delta_{\vec{\lambda}}(z_I, \bar{z}_I; \epsilon) \right] \mathbf{R}_{\vec{\lambda}}(z) \right\}$$

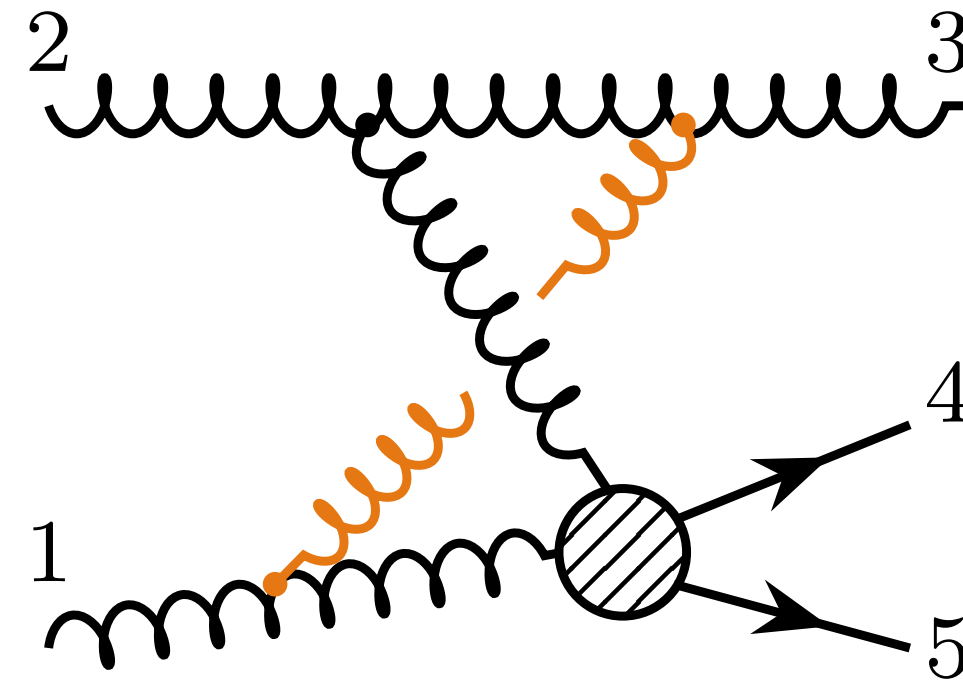
Some definitions:

$$\vec{\lambda} = \{\lambda_A; \lambda_2, \lambda_3\} \quad z_I = \frac{\langle ba \rangle \langle \text{in } I \rangle}{\langle \text{in } b \rangle \langle aI \rangle}, \quad \bar{z}_I = z_I^*, \quad \text{with } a \parallel b$$

$\text{Split}_{\lambda_A}$  captures the leading singular behaviour

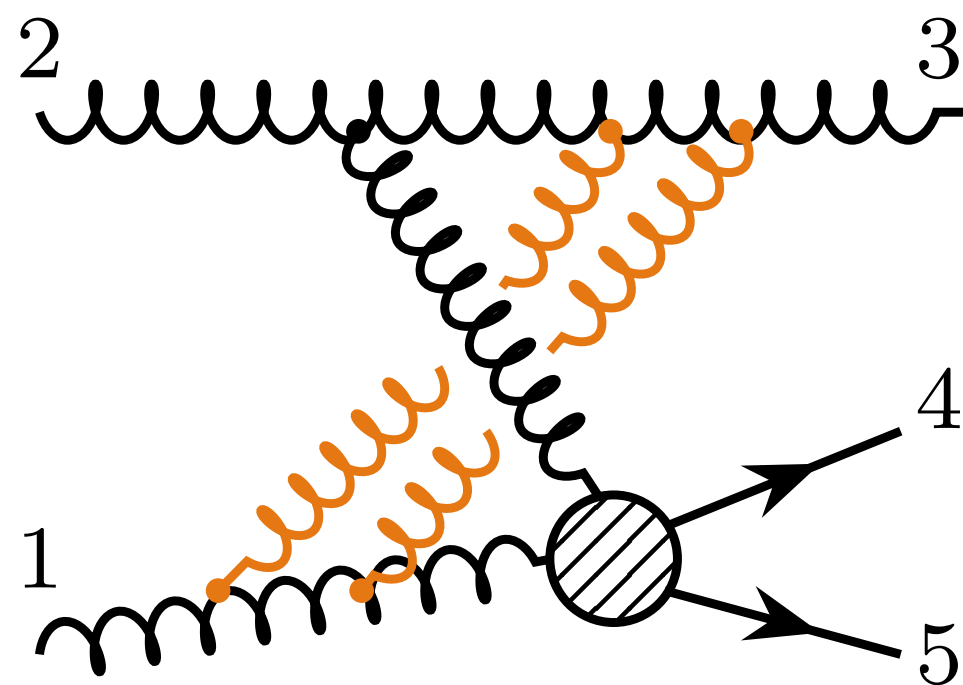
# Dipole and tripole diagrams

strict-collinear factorisation violation at 1-loop

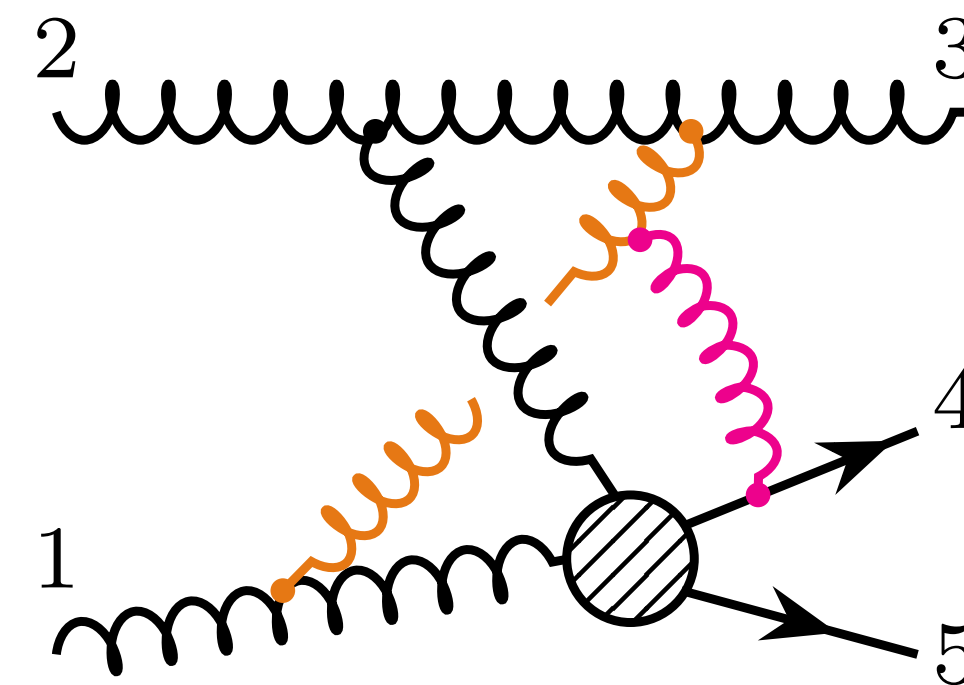


one-loop dipole

strict-collinear factorisation violation at 2-loop



two-loop dipole



two-loop tripole

# Singlet and dipole contributions

perturbative expansion:  $\mathcal{G}_{\vec{\lambda}}(z; \epsilon) = \sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n \mathcal{G}_{\vec{\lambda}}^{(n)}(z; \epsilon)$

$$\bar{\mu} = \frac{\mu^2}{-S_{ab}} \frac{1+z}{z}$$

$$\vec{r}^{(n)} \equiv \bar{\mu}^{-n\epsilon} r^{(n)}(z, \epsilon)$$

$$\mathcal{G}^{(n)}(z; \epsilon) = r^{(n)}(z, \epsilon) + \mathcal{C}_{ab} \mathbf{T}_3 \cdot \mathbf{T}_1 \bar{\mu}^{n\epsilon} \text{disc}[\vec{r}^{(n)}(z, \epsilon)]$$

strictly factorised

singlet

dipole

anti-hermitian (breaks coll. factorisation)

all singlet functions up to two loops in all partonic channels available in the literature [Badger, Glover [hep-ph/0405236](https://arxiv.org/abs/hep-ph/0405236)]

**caveat:** our splitting amplitude is written in exponential form. Beyond one-loop  $\neq$  from Badger, Glover's (@ 2L: subtract 1-loop<sup>2</sup>)

procedure:

- $r^\ell(z; \epsilon)$  from [Badger, Glover, [hep-ph/0405236](https://arxiv.org/abs/hep-ph/0405236)] and in their results take  $w = -z - i0^+$ , where  $w$  fraction of energy in the timelike splitting
- extract discontinuity as  $\text{disc}[f(z)] = 2i\text{Im}[f(z)]$

# Tripole contributions

perturbative expansion:  $\Delta_{\vec{\lambda}}(z_I, \bar{z}_I; \epsilon) = \sum_{n=2} \left(\frac{\alpha_s}{4\pi}\right)^n \Delta_{\vec{\lambda}}^{(n)}(z_I, \bar{z}_I; \epsilon)$   $h_\lambda = \pm 1$

$$\Delta_{\vec{\lambda}}^{(2)} = 2i\pi \bar{\mu}^{2\epsilon} \sum_{I \in \text{out}} \mathcal{T}_{3,1,I} \left[ \left( \frac{2}{\epsilon^2} - 2\zeta_2 \right) (\ln |z_I|^2 + 2i\pi) + 8\zeta_3 + h_{\vec{\lambda}} \frac{2}{3} \left( \ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I} \right],$$

colour tripole operator:  $\mathcal{T}_{i,j,k} \equiv [\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_i \cdot \mathbf{T}_k] = if^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$

properties:

- it appears also in the two-loop corrected single-soft gluon emission [Dixon, Herrmann, Yan, Zhu [1912.09370](#)] (and it's the same)
- it is the same as in  $\mathcal{N} = 4$  sYM [Henn, Ma, Xu, Yan, Zhang, Zhu [2406.14604](#)]
- encodes **non-trivial dependence** on **colour charge** and **kinematics of spectators** and on **spin of collinear partons**
- NB: tripole operator is anti-hermitian (actually anti-symmetric):  $\left(\mathcal{T}_{i,j,k}\right)^\dagger = -\mathcal{T}_{i,j,k}$

# Beyond dipole/tripole

perturbative expansion:  $\mathbf{R}_{ab,\vec{\lambda}}(z) = \sum_{n=0} \left(\frac{\alpha_s}{4\pi}\right)^n \mathbf{R}_{ab,\vec{\lambda}}^{(n)}(z)$        $\mathbf{R}_{ab,\vec{\lambda}}^{(0)}(z) = \mathbf{T}_{ab}^c$ ,  $\mathbf{R}_{ab,\vec{\lambda}}^{(1)}(z) = 0$

small caveat for  $\mathbf{Sp}_{\pm}(g^{\pm}, g^{\pm})$  splitting  
(see [paper](#) for details)

new contributions at two loops (unknown in the literature before); not present in  $\mathcal{N} = 4$  sYM:

$$\mathbf{R}_{ab}^{(2)} = (2\pi i) \mathcal{R}_{1,\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) \mathcal{T}_{b,1,a} \cdot \mathbf{T}_{ab}^c + (2\pi i) \mathcal{R}_{2,\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) \mathbf{C}_{ab}^{c,e} \cdot \mathbf{T}_1^e$$

colour triple operator

non-dipole/tripole

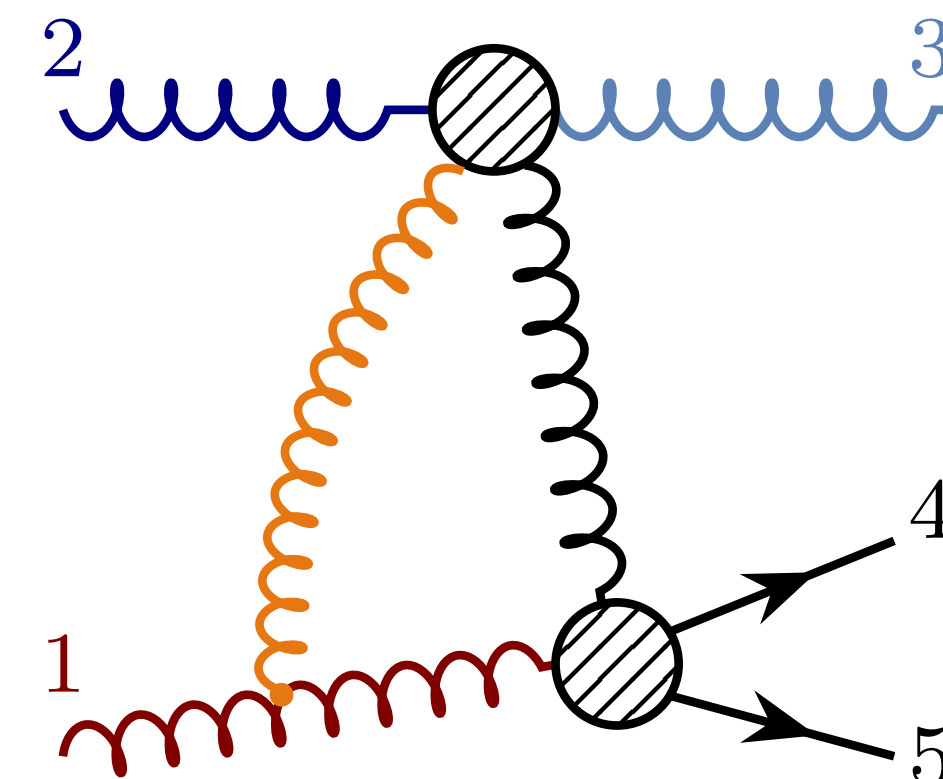
violate strict coll. fact.

- depends on **quantum numbers** of **collinear partons** and the **colour charge** of the **incoming spectator**

- it is flavour dependent

- the operator  $\mathbf{C}_{ab}^{c,e}$  vanishes when the outgoing collinear parton  $b$  is a quark

- the functions  $\mathcal{R}_{1,2}$  are IR finite



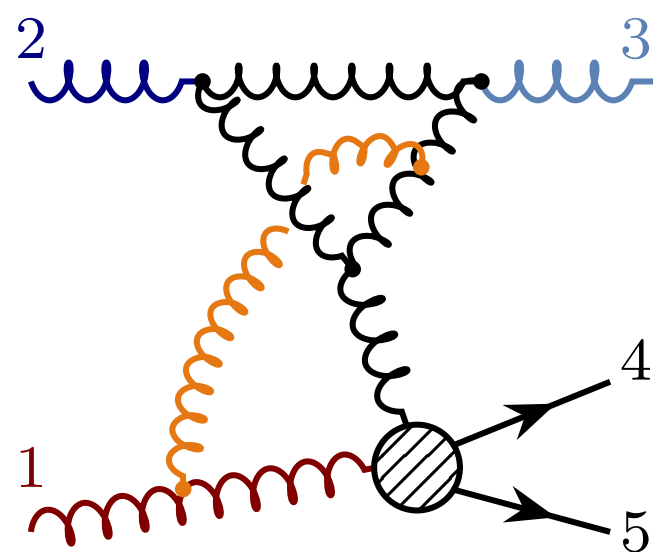
# Beyond dipole/tripole: operators

When the outgoing collinear parton is a gluon we find:

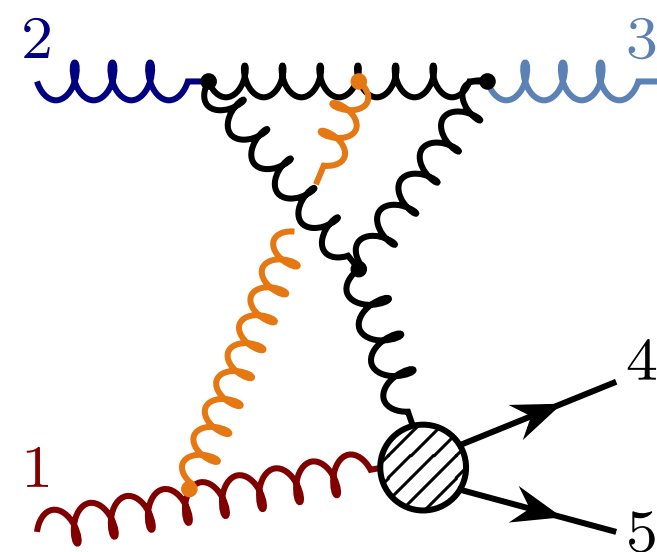
$$\mathbf{C}_{gg}^{c,e} = N_f \left[ \text{Tr} (T^{a_2} T^e T^{a_3} T^c) + \text{Tr} (T^c T^{a_3} T^e T^{a_2}) \right]$$

$$\mathbf{C}_{qg}^{c,e} = (T^d T^e T^{a_3} T^d)_{ci_2} + \frac{1}{N_c^2} (T^d T^{[e} T^{a_3]} T^d)_{ci_2}$$

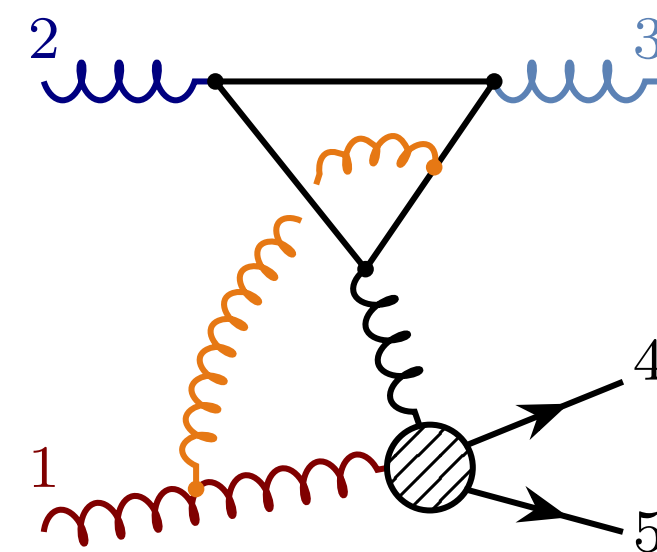
Example:  $g \rightarrow gg$  splitting



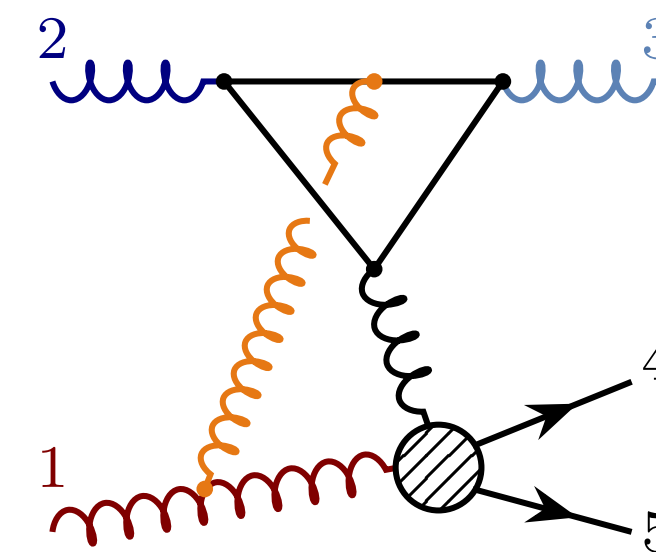
(a1) planar



(a2) non-planar



(b1) planar



(b2) non-planar

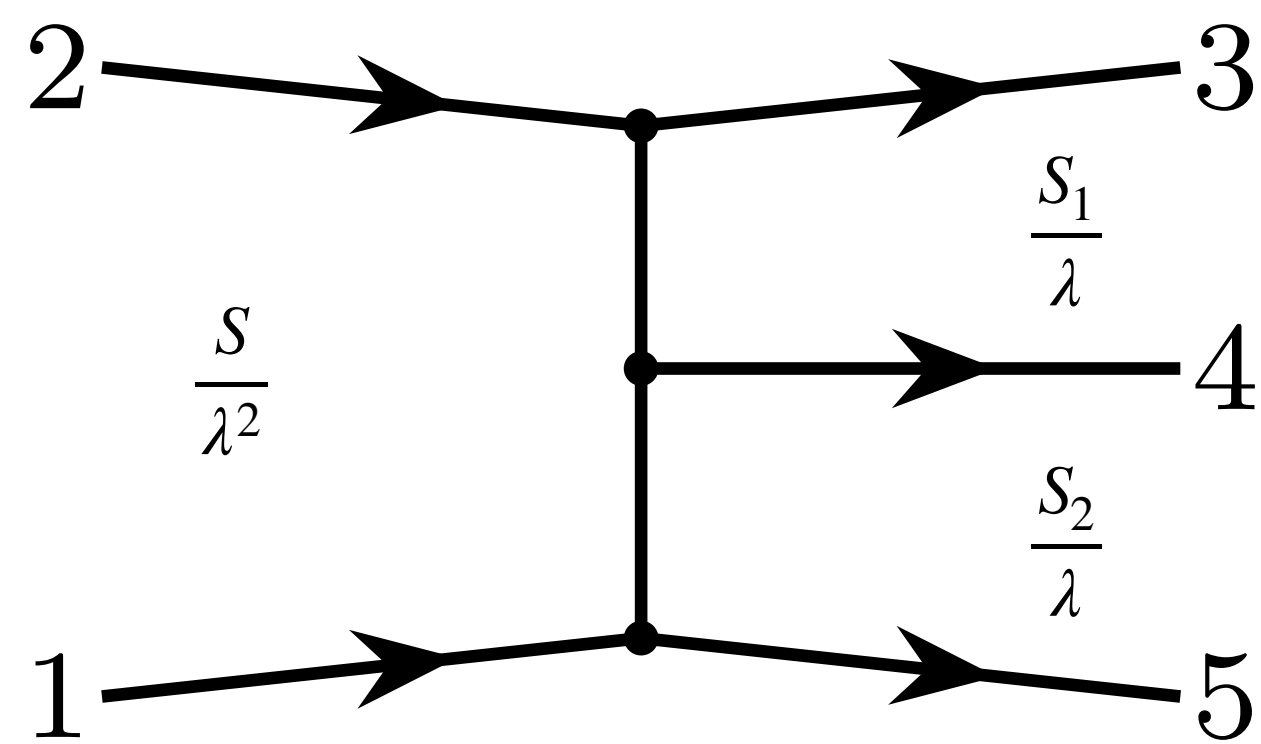
- non-planar: (a2) ~ tripole operator  $\mathcal{T}_{b,1,a}$ ; (b2) contributes to  $\mathbf{C}_{gg}^{c,e}$ ; planar ones can be decomposed onto those + dipole (1,3)
- (a2) and (b2) vanish if the system (4,5) is a colour singlet (Higgs or Z boson) → cannot be extracted from [\[Badger and Glover hep-ph/0405236\]](https://arxiv.org/abs/hep-ph/0405236)
- (b2) also present in an abelian theory (QED): real emission to Compton scattering  $e(p_1)\gamma(p_2) \rightarrow \gamma(p_3)\gamma(p_4)e(p_5)$

# Connection with Multi-Regge Kinematics (MRK)

Interesting to study the space like collinear limit from a high-energy perspective (and viceversa)

**MRK:** a) final-state particles strongly ordered in rapidity and b) their transverse components  $p_{T,i}$  commensurate and  $p_{T,i} \ll \sqrt{s_{12}}$

$$s_{12} = \frac{S}{\lambda^2}, \quad s_{23} = -\frac{S_1 S_2}{S} u \bar{u}, \quad s_{34} = \frac{S_1}{\lambda}, \quad s_{45} = \frac{S_2}{\lambda}, \quad s_{51} = -\frac{S_1 S_2}{S} (1-u)(1-\bar{u}), \quad \lambda \ll 1$$



physically: we are investigating the *small-x limit* of the splitting amplitudes

technically:

1.  $\delta \rightarrow 0$  controls collinear limit,  $\lambda \rightarrow 0$  controls MRK limit
2. rescale invariants accordingly and take limits in opposite orders
3. check vs our previous MRK at 2-loop QCD [FB, Caola, Devoto and Gambuti 2411.14050]

collinear kinematics  $\rightarrow$  MRK

$$\left\{ x \rightarrow 1 - x\lambda, z \rightarrow \frac{1+z}{\lambda} - 1, \delta \rightarrow \lambda\delta\sqrt{\frac{z}{1+z}} \right\}$$

take  $\lambda \rightarrow 0$  limit

MRK  $\rightarrow$  collinear kinematics

$$\left\{ s \rightarrow S(1+z), S_1 \rightarrow xs(1+z), S_2 \rightarrow s, u\bar{u} \rightarrow \frac{4}{x}z\delta^2 \right\}$$

take  $\delta \rightarrow 0$  limit

full agreement

# Connection with Multi-Regge Kinematics (MRK)

observation from kinematics:  $\left| \frac{z_I}{\bar{z}_I} \right| = \left| \frac{u}{\bar{u}} \right|$  describe only the kinematics in the transverse plane

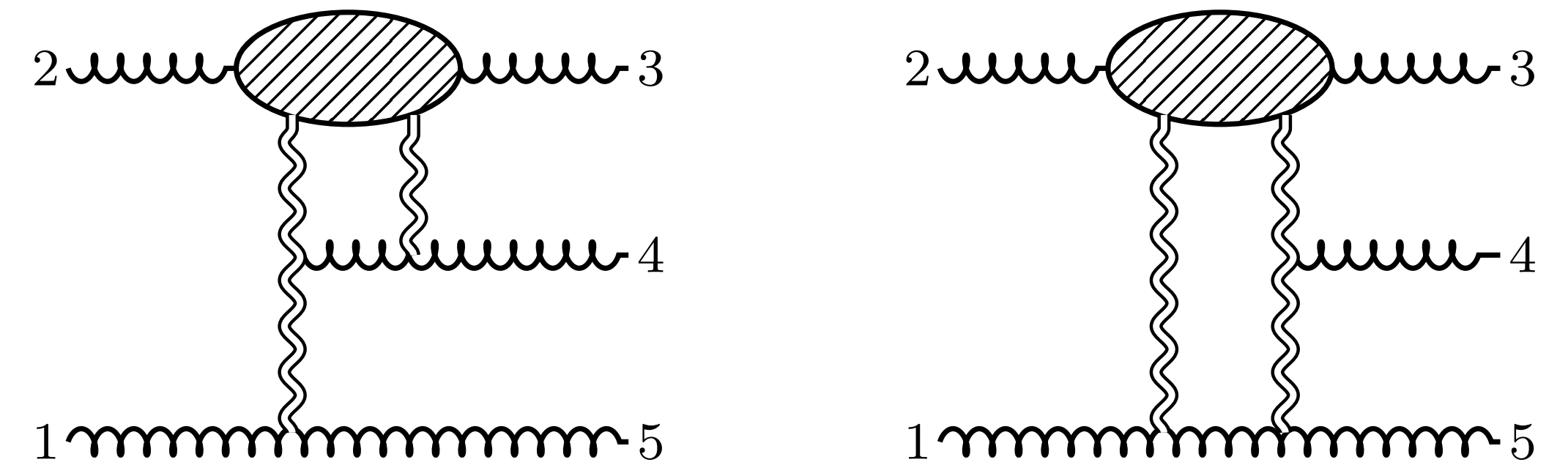
signature amplitudes and  $C_{ab}$  operator:

in MRK signature amplitudes very clear meaning: exchange or odd/even # of reggeons  $\rightarrow$  irreducible representations of  $SU(N)$

Example:  $gg \rightarrow ggg$  scattering

$(r_1, r_2)$	$C_i^{[gg]}$	$(r_1, r_2)$	$C_i^{[gg]}$
$(8_a, 8_a)$	$if^{a_5 b a_1} if^{a_3 c a_2} if^{b c a_4}$	$(10, 8_a)$	$\mathcal{T}_{10+10}^{a_4 b a_1 a_5} if^{a_3 b a_2}$
★ $(8_a, 8_s)$	$\frac{N_c^2}{N_c^2-4} if^{a_5 b a_1} d^{a_3 c a_2} d^{b c a_4}$	$(8_s, 10)$	$\frac{N_c^2}{N_c^2-4} d^{a_5 b a_1} \mathcal{T}_{10-10}^{a_4 b a_2 a_3}$
$(8_s, 8_a)$	$\frac{N_c^2}{N_c^2-4} d^{a_5 b a_1} if^{a_3 c a_2} d^{b c a_4}$	$(10, 8_s)$	$\frac{N_c^2}{N_c^2-4} \mathcal{T}_{10-10}^{a_4 b a_1 a_5} d^{a_3 b a_2}$
★ $(8_s, 8_s)$	$\frac{N_c^2}{N_c^2-4} d^{a_5 b a_1} d^{a_3 c a_2} if^{b c a_4}$	$(27, 27)$ ★	$\mathcal{T}_{27}^{b c a_2 a_3} \mathcal{T}_{27}^{b e a_1 a_5} if^{c a_4 e}$
$(1, 8_a)$	$\frac{N_c^2}{N_c^2-1} \delta^{a_5 a_1} if^{a_3 a_4 a_2}$	$(0, 0)$ ★	$\mathcal{T}_0^{b c a_2 a_3} \mathcal{T}_0^{b e a_1 a_5} if^{c a_4 e}$
★ $(8_a, 1)$	$\frac{N_c^2}{N_c^2-1} if^{a_5 a_4 a_1} \delta^{a_3 a_2}$	$(10, 0)$	$\mathcal{T}_{10+10}^{b c a_1 a_5} \mathcal{T}_0^{b e a_2 a_3} if^{e a_4 c}$
★ $(8_a, 0)$	$if^{a_5 b a_1} \mathcal{T}_0^{a_4 b a_2 a_3}$	$(0, 10)$	$\mathcal{T}_0^{b e a_1 a_5} \mathcal{T}_{10+10}^{b c a_2 a_3} if^{c a_4 e}$
$(0, 8_a)$	$\mathcal{T}_0^{a_4 b a_1 a_5} if^{a_3 b a_2}$	$(10, 27)$	$\mathcal{T}_{10+10}^{b e a_1 a_5} \mathcal{T}_{27}^{b c a_2 a_3} if^{c a_4 e}$
★ $(8_a, 27)$	$if^{a_5 b a_1} \mathcal{T}_{27}^{a_4 b a_2 a_3}$	$(27, 10)$	$\mathcal{T}_{27}^{b e a_1 a_5} \mathcal{T}_{10+10}^{b c a_2 a_3} if^{c a_4 e}$
$(27, 8_a)$	$\mathcal{T}_{27}^{a_4 b a_1 a_5} if^{a_3 b a_2}$	$(10, 10)_1$	$\mathcal{T}_{10+10}^{b e a_1 a_5} \mathcal{T}_{10+10}^{b c a_2 a_3} if^{c a_4 e}$
$(8_a, 10)$	$if^{a_5 b a_1} \mathcal{T}_{10+10}^{a_4 b a_2 a_3}$	$(10, 10)_2$	$\mathcal{T}_{10+10}^{b e a_1 a_5} (\mathcal{T}_{10-10}^{b c a_2 a_3} d^{c a_4 e} - \mathcal{T}_{10+10}^{b c a_2 a_3} if^{c a_4 e} / N_c)$

two possible contributions:



one-loop corrected impact factor with two-reggeon emission

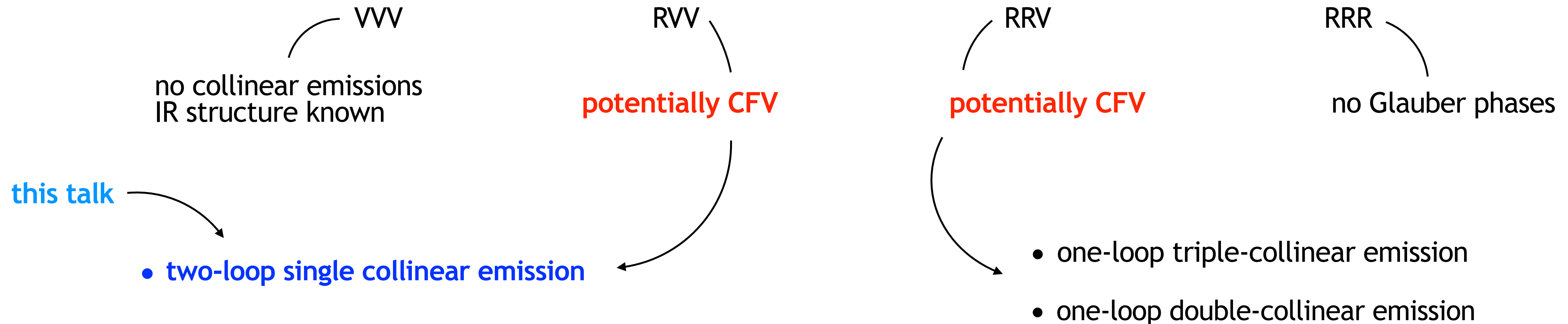
[Fadin, Fiore, Kotsky, Papa [hep-ph/9908264](https://arxiv.org/abs/hep-ph/9908264)]

# Cross-section level results

# Jet cross sections at N3LO QCD

Fiducial  $m$ -jets cross section at N3LO in QCD

$$d\sigma_m^{\delta N^3\text{LO}} = \int_{\Lambda} d\Phi_m |\mathcal{M}_{2 \rightarrow m}^{(3l)}|^2 + \int_{\Lambda} d\Phi_{m+1} |\mathcal{M}_{2 \rightarrow m+1}^{(2l)}|^2 + \int_{\Lambda} d\Phi_{m+2} |\mathcal{M}_{2 \rightarrow m+2}^{(1l)}|^2 + \int_{\Lambda} d\Phi_{m+3} |\mathcal{M}_{2 \rightarrow m+3}^{(0l)}|^2$$



**Q1:** do CFV terms survive at cross-section level? (in the collinear limit, i.e. DGLAP factorisation)

**Q2:** if they cancel, what is the mechanism?

**NB:** for cross-section predictions, we need to **sum over colour**

# Cancellation of CFV contributions. I

Let's start with the IR-finite operator  $\mathbf{R}_{ab}^{(2)}$ : we have a contraction of the type  $\mathbf{R}^{(0),\dagger} \cdot \mathbf{R}^{(2)}$ , with  $\mathbf{R}^{(0),\dagger} = (\mathbf{T}_{ab}^{\bar{c}})^\dagger$

A. tripole contribution  $\mathcal{T}_{b,1,a}$ , easy to check that

$$(\mathbf{T}_{ab}^{\bar{c}})^\dagger \cdot \left( i f^{e_1 e_2 e_3} \mathbf{T}_b^{e_1} \mathbf{T}_1^{e_2} \mathbf{T}_a^{e_3} \right) \cdot \mathbf{T}_{ab}^c = 0$$

B. non-tripole contribution ( $\mathbf{C}_{ab}$  operator):

$$\mathbf{R}^{(0),\dagger} \cdot \mathbf{R}^{(2)} \sim (\mathbf{T}_{ab}^{\bar{c}})^\dagger \cdot \mathbf{C}_{ab}^{c,e} \cdot \mathbf{T}_1^e$$

1.  $g \rightarrow gg$  splitting:

for  $(a, b) = (g, g)$   $\mathbf{T}_{ab}^{\bar{c}}$  is anti-symmetric in  $a_2, a_3$  whereas  $\mathbf{C}_{ab}^{c,e}$  is symmetric  $\rightarrow$  **contraction is zero** upon summing over colour

2.  $q \rightarrow qg$  splitting:

$$\text{explicitly: } (\mathbf{T}_{ab}^{\bar{c}})^\dagger \cdot \mathbf{C}_{ab}^{c,e} = \left( T^d T^e T^{a_3} T^d T^{a_3} \right)_{c\bar{c}} + \frac{1}{N_c^2} \left( T^d T^{[e} T^{a_3]} T^d T^{a_3} \right)_{c\bar{c}} = 0$$

# Cancellation of CFV contributions. II

operator:  $\Delta^{(2)} = c_1 \mathbf{O}_{\mathcal{F}} + ic_2 \mathbf{O}_{\mathcal{F}}$ , with  $\mathbf{O}_{\mathcal{F}} \equiv$  tripole, thus  $\mathbf{O}_{\mathcal{F}}^T = -\mathbf{O}_{\mathcal{F}}$

at cross-section level we are interested in:

$$\sum_{\text{col}} (\langle \mathcal{A}^{(2)} | \mathcal{A}^{(0)} \rangle + \langle \mathcal{A}^{(0)} | \mathcal{A}^{(2)} \rangle) = \sum_{\text{col}} (\langle \mathcal{A}^{(0)} | \Delta^{(2),\dagger} + \Delta^{(2)} | \mathcal{A}^{(0)} \rangle) = 2ic_2 \sum_{\text{col}} \langle \mathcal{A}^{(0)} | \mathbf{O}_{\mathcal{F}} | \mathcal{A}^{(0)} \rangle$$

cancellation of tripole operator based on reality of tree-level amplitudes (or common trivial phase) [Forshaw, Seymour, Siodmok [1206.6363](#)]

$$\langle \mathcal{A}^{(0)} | \mathbf{O}_{\mathcal{F}} | \mathcal{A}^{(0)} \rangle = \text{Tr} \left[ \underbrace{(| \mathcal{A}^{(0)} \rangle \langle | \mathcal{A}^{(0)} |)}_{\text{symmetric}} \underbrace{\mathbf{O}_{\mathcal{F}}}_{\text{anti-symmetric}} \right] = 0$$

this is strictly valid only if  $\mathcal{A}_0$  are real: in general, **even in QCD, tree-level amplitudes are not real!**

starting from  $2 \rightarrow n$  with  $n > 2$ , non-trivial phases: phases cancel pairwise in parity conjugated helicity configurations

for cancellation at XS level (in a parity invariant theory): **colour + spin sum**

**Q1:** breaks down in the **EW theory** (CP violation, phases do not cancel upon spin sum). Factorisation breaking? see also [Baumgart et al [1811.04120](#)]

**Q2:** possible to devise **spin-correlated/sensitive observables**? see eg. very recently [CMS [2603.03689](#)]

# Cancellation of CFV contributions. III

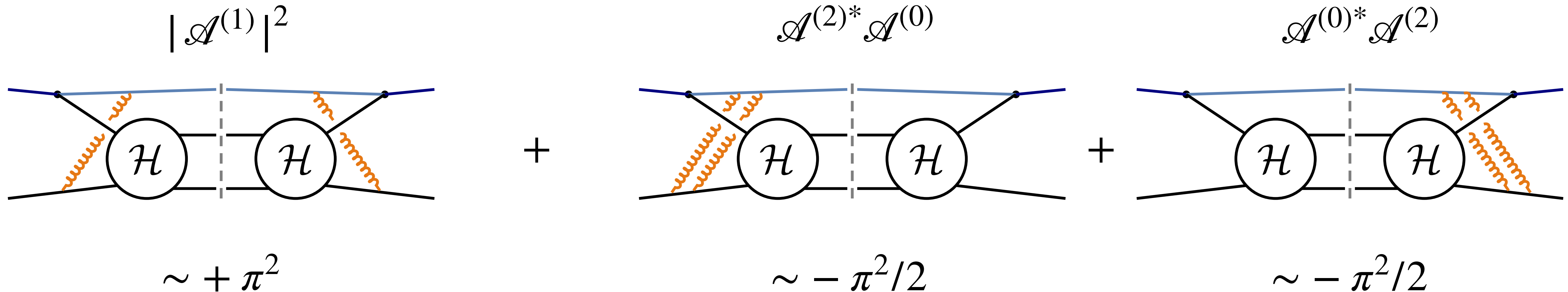
both  $\mathbf{\Delta}^{(2)}$  and  $\mathbf{R}^{(2)}$  cancel upon summing over colour (and spin)

dipole operator (up to this perturbative order) can be written in exponential form

$$\exp \left[ \mathcal{G}_{\vec{\lambda}}(z; \epsilon) \right] \longrightarrow \text{with } \mathcal{G}_{\vec{\lambda}}(z; \epsilon) \sim \text{singlet} + \text{dipole} \longrightarrow [\text{singlet}, \text{dipole}] = 0 \quad + \quad \text{dipole} \sim \text{anti-hermitian}$$

**dipole operator**  $\rightarrow$  **pure phase**: trivially vanishes in a squared matrix element

perturbatively, up to two-loops:



# Summary and outlook

# Summary and outlook

- strict collinear factorisation breaking at amplitude level challenges the factorisation formula for hadronic cross-sections
- potential issuer in pQCD: first at 3rd order for jet cross sections (Glauber exchanges)
- clear presence of **strict collinear factorisation violation in 2-loop 5-point QCD amplitudes** with one collinear emission
- thanks to **advances in scattering amplitude calculation** → **possible to extract quantitative information**

in **this talk**:

- extracted **complete set of two-loop spacelike splitting amplitudes in QCD** (all partonic configurations, all helicities)
- (in pure QCD) after **colour + spin sum**: all CFV terms cancel at squared amplitude level, hence at XS level (single coll emission)

**beyond**:

- role of **parity** and **EW theory**? Collinear factorisation definitely broken at **N4LO QCD**? Even higher orders?
- intimate **connection** between **Glauber** and **Reggeon** dynamics

**Backup**

# CFV, Coloumb/Glauber gluons, SLL

## On the breaking of collinear factorization in QCD

---

Jeffrey R. Forshaw, Michael H. Seymour, Andrzej Siódmok

[Forshaw, Seymour, Siodmok [1206.6363](#)]

ABSTRACT: We investigate the breakdown of collinear factorization for non-inclusive observables in hadron-hadron collisions. For pure QCD processes, factorization is violated at the three-loop level and it has a structure identical to that encountered previously in the case of super-leading logarithms. In particular, it is driven by the non-commutation of Coulomb/Glauber gluon exchanges with other soft exchanges. Beyond QCD, factorization may be violated at the two-loop level provided that the hard subprocess contains matrix element contributions with phase differences between different colour topologies.

will come back to this at the end of the talk

At one-loop level the violation of factorization in the amplitude is anti-Hermitian and cannot contribute to physical cross sections. At two-loop level the sum of all contributions corresponding to two Coulomb gluon exchanges (in the amplitude, the conjugate amplitude and one in each) similarly cancels. However, a non-zero commutator between one Coulomb gluon exchange and one eikonal exchange gives a Hermitian term that may contribute to physical cross sections. We find that this contribution does vanish for pure QCD processes.

with a caveat (final answer?)

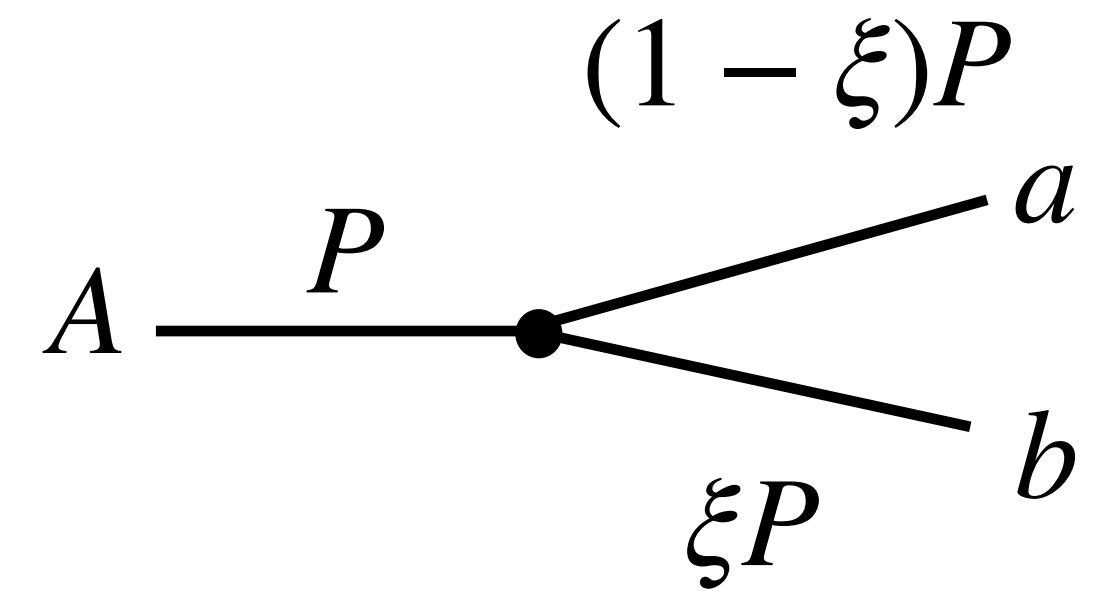
# Spacelike vs timelike kinematics

Consider the splitting:

$$A(P) \rightarrow a((1 - \xi)P) + b(\xi P), \quad \text{with} \quad 0 < \xi < 1$$

**Timelike splitting:**

$(p_a^0 p_b^0 > 0, P^0 > 0)$ , i.e. either both  $(a, b)$  in the final state or both in the initial state



Results available in the literature:

- one-loop (d-dimensions) [Bern, Del Duca, Kilgore, Schmidt [hep-ph/9903516](#)] [Kosower, Uwer [hep-ph/9903515](#)]
- two-loop [Bern, Dixon, Kosher [hep-ph/0404293](#)] [Badger, Glover [hep-ph/0405236](#)] + higher-orders in the dimensional regulator [Duhr, Gehrmann, Jacquier [1411.3587](#)]
- three-loop (d=4) [Guan, Herzog, Ma, Mistlberger, Suresh [2408.03019](#)]

**In the TL case: no violation of strict collinear factorisation**

universal splitting amplitudes, depend only on quantum numbers of collinear partons

# Connection with Multi-Regge Kinematics (MRK)

observation from kinematics:  $\left| \frac{z_I}{\bar{z}_I} \right| = \left| \frac{u}{\bar{u}} \right|$  describe only the kinematics in the transverse plane

$$\Delta_{\vec{\lambda}}^{(2)} = 2i\pi \bar{\mu}^{2\epsilon} \sum_{I \in \text{out}} \mathcal{T}_{3,1,I} \left[ \left( \frac{2}{\epsilon^2} - 2\zeta_2 \right) (\ln |z_I|^2 + 2i\pi) + 8\zeta_3 + h_{\vec{\lambda}} \frac{2}{3} \left( \ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I} \right],$$

MRK in  $\mathcal{N} = 4$  (equivalent to QCD in this matter) [Caron-Huot et al 2003.03120]

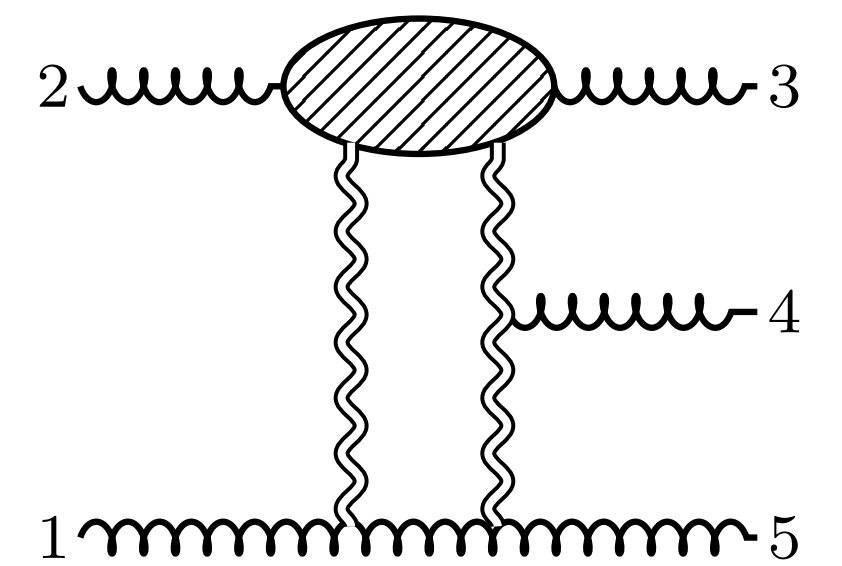
As we have seen in Section 4.4, the  $\mathcal{N} = 4$  super Yang-Mills hard function in the multi-Regge limit involves several functions which are not manifestly real analytic in the complex  $z$  plane. It is worthwhile to highlight that they appear only in the order- $\log^0 x$  components  $h_{a,0}^{(2)}$  with  $a \in \{1, 2, 5, 6, 11, 12, 14, 15\}$ . Among these,  $h_{12,0}^{(2)}$  and  $h_{15,0}^{(2)}$  are particularly simple, as they only involve – of the functions which are not manifestly real analytic – the logarithms (4.51). For instance,

$$h_{12,0}^{(2)} = \frac{i\pi}{6} \left( (g_7^{(1)})^3 - (g_6^{(1)})^3 \right) + \frac{2i\pi^3}{3} (g_7^{(1)} - g_6^{(1)}) + (\text{analytic}). \quad (5.28)$$

$$\begin{aligned} I_{\text{tri}}^{(1)'} &= \int \frac{d^{2-2\epsilon} \ell}{\pi^{1-\epsilon} e^{-\epsilon\gamma}} \frac{\bar{\mathbf{p}}_3 (\mathbf{p}_3 - \ell) (\bar{\mathbf{p}}_5 + \bar{\ell}) \mathbf{p}_5}{|(\mathbf{p}_3 - \ell)^2 (\mathbf{p}_5 + \ell)^2 \ell^2|} \times \frac{1}{4} \left[ \log \left( \frac{\ell}{\ell + \mathbf{p}_5} \right) - \log \left( \frac{\bar{\ell}}{\bar{\ell} + \bar{\mathbf{p}}_5} \right) \right]^2 \\ &= \frac{1}{12} (\log(z) - \log(\bar{z}))^3 + \frac{\pi^2}{3} (\log(z) - \log(\bar{z})) + (\text{analytic}). \end{aligned}$$

$$g_6^{(1)} = \ln(u) - \ln(\bar{u})$$

$$g_7^{(1)} = \ln(1 - u) - \ln(1 - \bar{u})$$



one-loop corrected impact factor with 2-reggeon emission

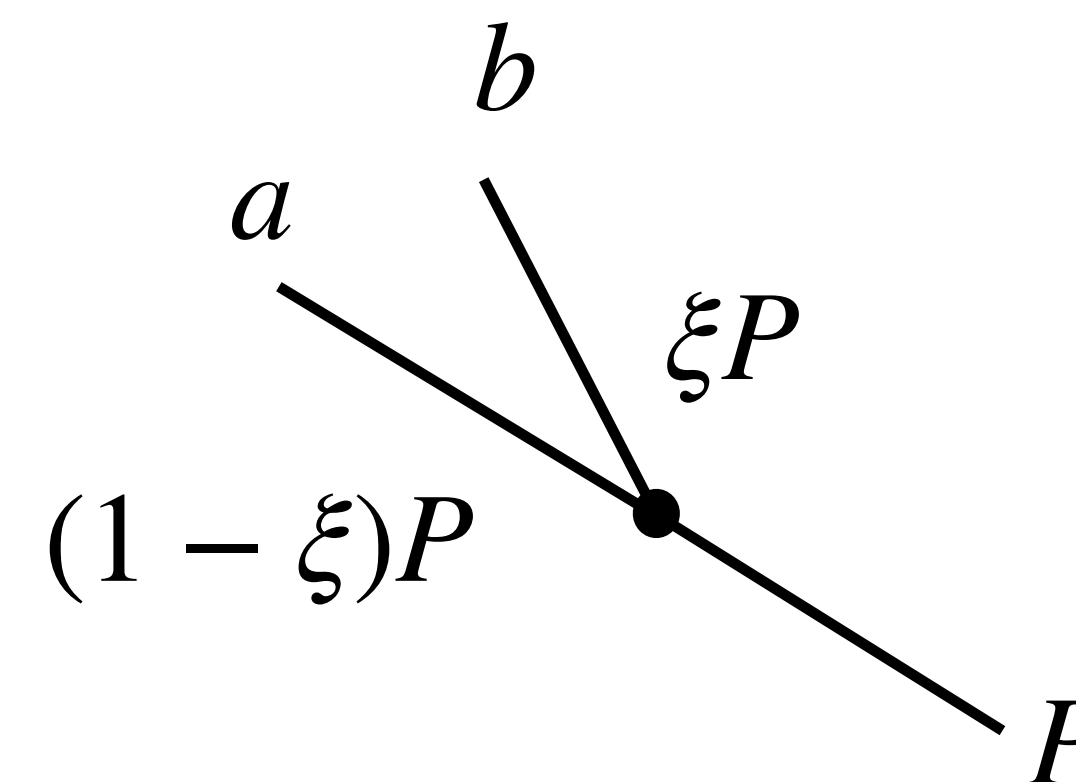
[Fadin, Fiore, Kotsky, Papa hep-ph/9908264]

# Spacelike vs timelike kinematics

Spacelike splitting:

$(p_a^0 p_b^0 < 0, P^0 < 0)$ , i.e.  $a$  in the initial state and  $b$  in the final state (or viceversa)

$$a((1 - \xi)P) \rightarrow A(P) + b(\xi P), \quad \text{with} \quad \xi < 0$$



loop corrections induce a non-trivial dependence on quantum numbers of spectators: strict collinear factorisation violation (CFV)

“mathematical issue” in extracting the SL from TL:

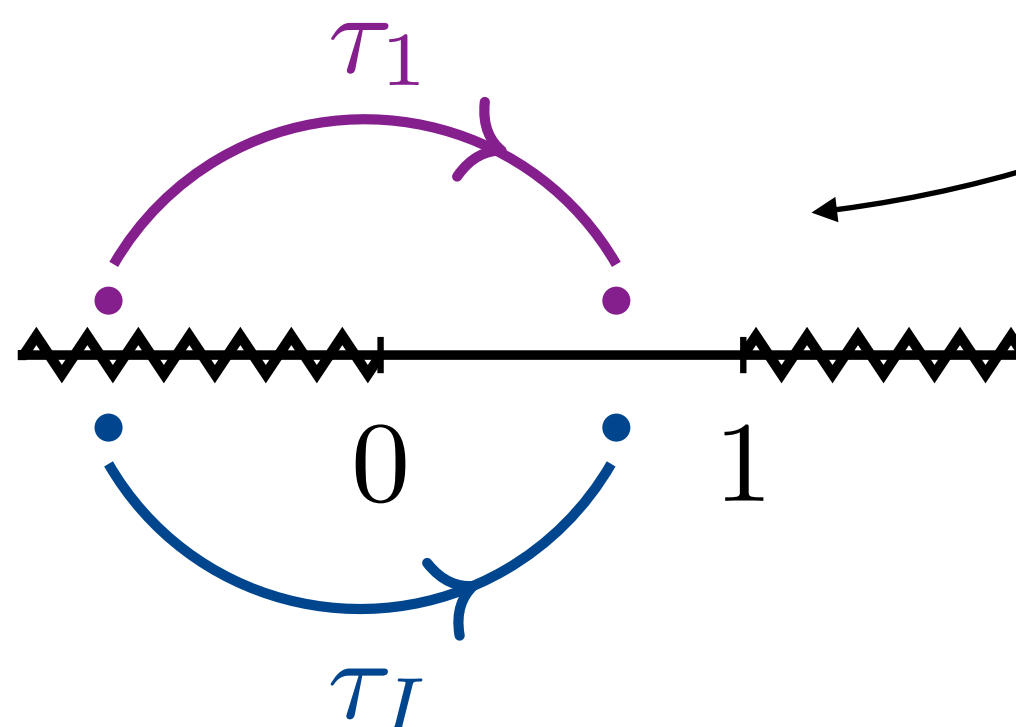
ambiguity in the sign of the infinitesimal imaginary part of energy fraction  $\xi$ : prevents a direct analytic continuation from TL

consider a set of variables  $\tau_I$ , where  $I$  is a spectator parton (all reduce to  $\xi$  in coll. limit)

$$\tau_I = \frac{s_{Ib}}{s_{Ib} + s_{Ia}} \xrightarrow{a \parallel b} |\xi| \exp[-i\pi(\eta_{Ib} - \eta_{IA})]$$

$$\eta_{ij} = 1 \quad i, j \text{ both outgoing}$$

$$\eta_{ij} = 0 \quad \text{otherwise}$$



$\tau_I$  unambiguous in TL splitting

$$\tau_I \rightarrow -|\xi| \pm i0^\pm \quad \text{SL: for incoming/outgoing}$$

# Five-parton scattering in QCD at two loops

Start from 2-loop 5-pt amplitudes in full-colour QCD [Agarwal, FB, Devoto, Gambuti, von Manteuffel, Tancredi [2311.09870](#)]

$$\mathcal{A}_5^{(L)} = \sum_{k=-2L} \sum_m \epsilon^k R_m^{(k)}(s_{ij}; \text{tr}_5, \epsilon_5) f_m^{(w_k)}(\{W\}) \quad L = 1, 2$$

- $R_m^{(k)}$  rational functions
- $f_m^{(w_k)}$  massless pentagon functions (iterated integrals) [Gehrmann et al [1807.09812](#), Abreu et al [1812.08941](#), Chicherin et al [1812.11160](#), Chicherin et al [2009.07803](#)]
- $s_{ij}$  Mandelstam invariants + parity odd invariant  $\text{tr}_5 = 4i\epsilon(p_1, p_2, p_3, p_4)$  and  $\epsilon_5 = i\sqrt{|\det(s_{ij})|}$
- $\{W\}$  pentagon alphabet: 31 letters
- $w_K$ : transcendental weight of the pentagon function  $\rightarrow$  we need up to  $w_K = 4$

# Expansion in the collinear limit

## Strategy:

1. expand 1- and 2-loop scattering amplitudes for all relevant partonic channels in  $\delta \rightarrow 0$
2. match result to generalised 1- and 2-loop splitting amplitudes (colour operators) acting on four point amplitudes
3. extract 2-loop corrections to splitting amplitude in spacelike kinematics

## Implementation:

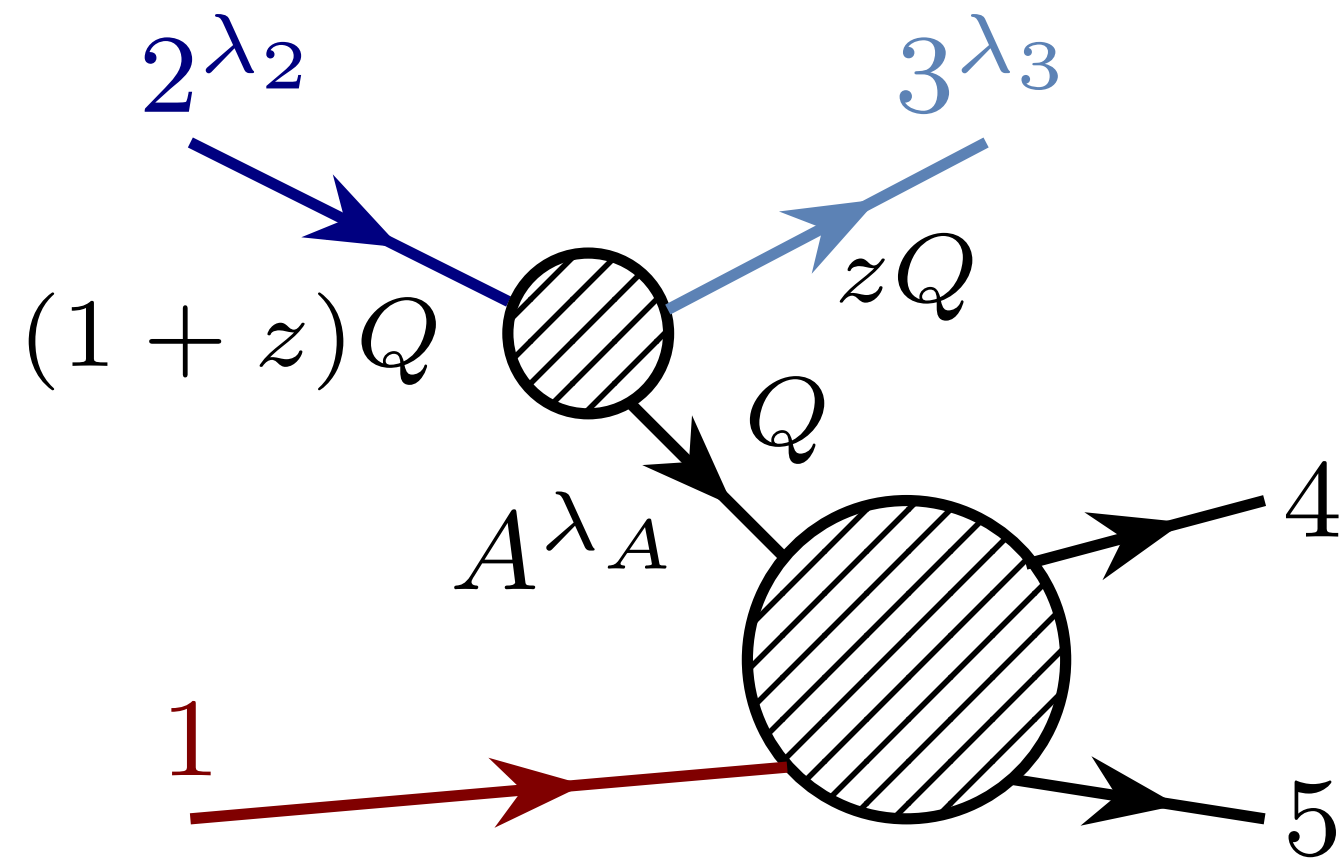
- A. leading singularity  $1/\delta$  captured by tree-level (factored out): loop corrections can only induce  $\ln^\ell(\delta)$  corrections
- B. expand rational functions: straightforward but painful
- C. expand pentagon functions: **two alternative methods**
  - C1. systematically take **discontinuity of pentagon functions** (work at symbol level) and move from SL to TL kinematics
  - C2. solve **differential equations** wrt  $s_{ij}$  satisfied by pentagon functions as a **power series in  $\delta$**

agreement between the two methods

in rational functions spurious  $1/\delta^3$  poles  $\rightarrow$  need to expand PF up to  $\delta^3$

# General form and main ingredients

Consider the spacelike splitting:  $a^{\lambda_a}((1+z)Q) \rightarrow A^{\lambda_A}(Q) + b^{\lambda_b}(zQ), \quad z > 0$



alternatively, upon performing simultaneously:  $Q \rightarrow zQ, \quad z \rightarrow (1-z)/z$

$$a^{\lambda_a}(Q) \rightarrow A^{\lambda_A}(zQ) + b^{\lambda_b}((1-z)Q), \quad 0 < z < 1$$

here, take a=2 and b=3

$$\mathbf{Sp}_{\lambda_A}(2^{\lambda_2}, 3^{\lambda_3}) = i\sqrt{4\pi\alpha_s} \text{Split}_{\lambda_A}(2^{\lambda_2}, 3^{\lambda_3}) \times \left\{ \exp \left[ \mathcal{G}_{\vec{\lambda}}(z; \epsilon) + \Delta_{\vec{\lambda}}(z_I, \bar{z}_I; \epsilon) \right] \mathbf{R}_{\vec{\lambda}}(z) \right\}$$

At tree-level:

$$\mathbf{T}_{qq}^c = T_{i_3 i_2}^c, \quad \mathbf{T}_{qg}^c = T_{c i_2}^{a_3}, \quad \mathbf{T}_{gq}^c = T_{i_3 c}^{a_2}, \quad \mathbf{T}_{gg}^c = i f^{a_2 c a_3}$$

# Connection with Multi-Regge Kinematics (MRK)

observation from kinematics:  $\left| \frac{z_I}{\bar{z}_I} \right| = \left| \frac{u}{\bar{u}} \right|$  describe only the kinematics in the transverse plane

$$\Delta_{\vec{\lambda}}^{(2)} = 2i\pi \bar{\mu}^{2\epsilon} \sum_{I \in \text{out}} \mathcal{T}_{3,1,I} \left[ \left( \frac{2}{\epsilon^2} - 2\zeta_2 \right) (\ln |z_I|^2 + 2i\pi) + 8\zeta_3 + h_{\vec{\lambda}} \frac{2}{3} \left( \ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I} \right],$$

MRK in  $\mathcal{N} = 4$  (equivalent to QCD in this matter) [Caron-Huot et al 2003.03120]

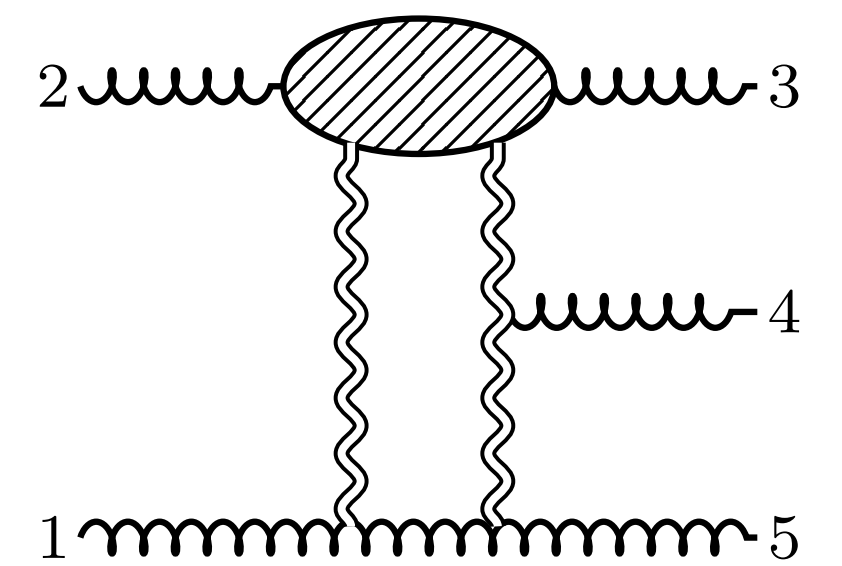
As we have seen in Section 4.4, the  $\mathcal{N} = 4$  super Yang-Mills hard function in the multi-Regge limit involves several functions which are not manifestly real analytic in the complex  $z$  plane. It is worthwhile to highlight that they appear only in the order- $\log^0 x$  components  $h_{a,0}^{(2)}$  with  $a \in \{1, 2, 5, 6, 11, 12, 14, 15\}$ . Among these,  $h_{12,0}^{(2)}$  and  $h_{15,0}^{(2)}$  are particularly simple, as they only involve – of the functions which are not manifestly real analytic – the logarithms (4.51). For instance,

$$h_{12,0}^{(2)} = \frac{i\pi}{6} \left( (g_7^{(1)})^3 - (g_6^{(1)})^3 \right) + \frac{2i\pi^3}{3} (g_7^{(1)} - g_6^{(1)}) + (\text{analytic}). \quad (5.28)$$

$$\begin{aligned} I_{\text{tri}}^{(1)'} &= \int \frac{d^{2-2\epsilon} \ell}{\pi^{1-\epsilon} e^{-\epsilon\gamma}} \frac{\bar{\mathbf{p}}_3 (\mathbf{p}_3 - \ell) (\bar{\mathbf{p}}_5 + \bar{\ell}) \mathbf{p}_5}{|(\mathbf{p}_3 - \ell)^2 (\mathbf{p}_5 + \ell)^2 \ell^2|} \times \frac{1}{4} \left[ \log \left( \frac{\ell}{\ell + \mathbf{p}_5} \right) - \log \left( \frac{\bar{\ell}}{\bar{\ell} + \bar{\mathbf{p}}_5} \right) \right]^2 \\ &= \frac{1}{12} (\log(z) - \log(\bar{z}))^3 + \frac{\pi^2}{3} (\log(z) - \log(\bar{z})) + (\text{analytic}). \end{aligned}$$

$$g_6^{(1)} = \ln(u) - \ln(\bar{u})$$

$$g_7^{(1)} = \ln(1 - u) - \ln(1 - \bar{u})$$



one-loop corrected impact factor with 2-reggeon emission

[Fadin, Fiore, Kotsky, Papa hep-ph/9908264]

# Beyond dipole/tripole: functions

The functions  $\mathcal{R}_{1,2}$  are **remarkably simple**: consider the splitting  $g^\pm \rightarrow g^\pm g^\pm$

$$\mathcal{R}_{1,\pm}(g^\pm, g^\pm) = \frac{4z(11z^2 + 21z + 12)}{3(1+z)^3} (\text{Li}_2(-z) - \zeta_2) + \frac{22}{3} \ln^2(1+z) - \frac{4z(3+2z)}{3(1+z)^2} - \frac{8z \ln(1+z)}{3(1+z)^2}$$

$$\mathcal{R}_{2,\pm}(g^\pm, g^\pm) = \mathcal{R}_{1,\pm}(g^\pm, g^\pm) - \frac{12z}{1+z} \left( \zeta_2 - \frac{(1+z)}{2z} \ln^2(1+z) - \text{Li}_2(-z) \right)$$

- they **vanish in the soft limit**  $z \rightarrow 0$  (couldn't be predicted from 2-loop soft function [\[Dixon, Herrmann, Yan, Zhu 1912.09370\]](#))
- in the **high-energy** limit (*small x*) defined by  $z \rightarrow \infty$  they **reduce to constants**

$$\mathcal{R}_1 = \frac{8}{3} + \frac{44\pi^2}{9}, \quad \mathcal{R}_2 = \frac{8}{3} + \frac{8\pi^2}{9}$$

- interesting observations:

- I. if both  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are non vanishing, then  $\mathcal{R}_1 - \mathcal{R}_2 \sim$  uniform transcendental weight 2  $\in \{\zeta_2, \ln^2(1+z), \text{Li}_2(-z)\}$
- II. if  $\mathcal{R}_2$  vanishes, then  $\mathcal{R}_1$  uniform transcendental weight 2

# Beyond RVV: CFV at XS level in RRV?

Last place with potential CFV terms remaining at XS level: RRV (double-unresolved emissions)

Relevant ingredients available in the literature:

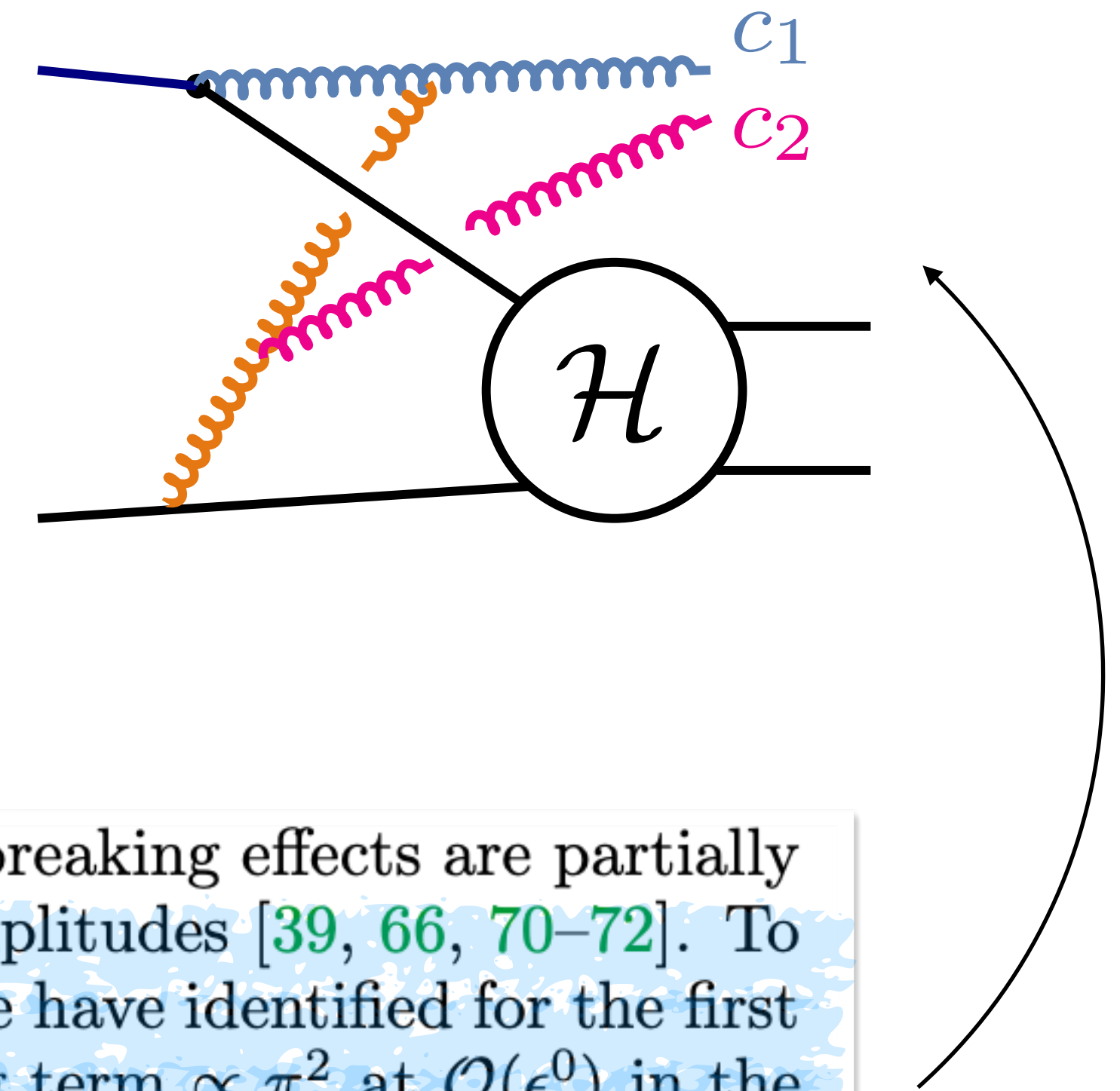
- one-loop double soft current [[Zhu 2009.08919](#); [Czakon, Eschment, Schellenberger 2211.06465](#)]
- one-loop triple collinear splitting [[Czakon, Sapeta 2204.11801](#)]

very recently investigated in the literature [[Cieri, Dhani, Rodrigo 2402.14749](#)]

(spacelike) generalised collinear and soft-collinear factorisation: multi collinear case

does it survive at XS level?

only source of CFV at XS level in dijet at N3LO?



and higher orders, the SCF breaking effects are partially canceled also for squared amplitudes [39, 66, 70–72]. To the best of our knowledge, we have identified for the first time a factorization breaking term  $\propto \pi^2$  at  $\mathcal{O}(\epsilon^0)$  in the SL triple-collinear limit of one-loop double soft-gluon current, which contributes at the level of squared amplitude. Further details will be provided in Ref. [69].