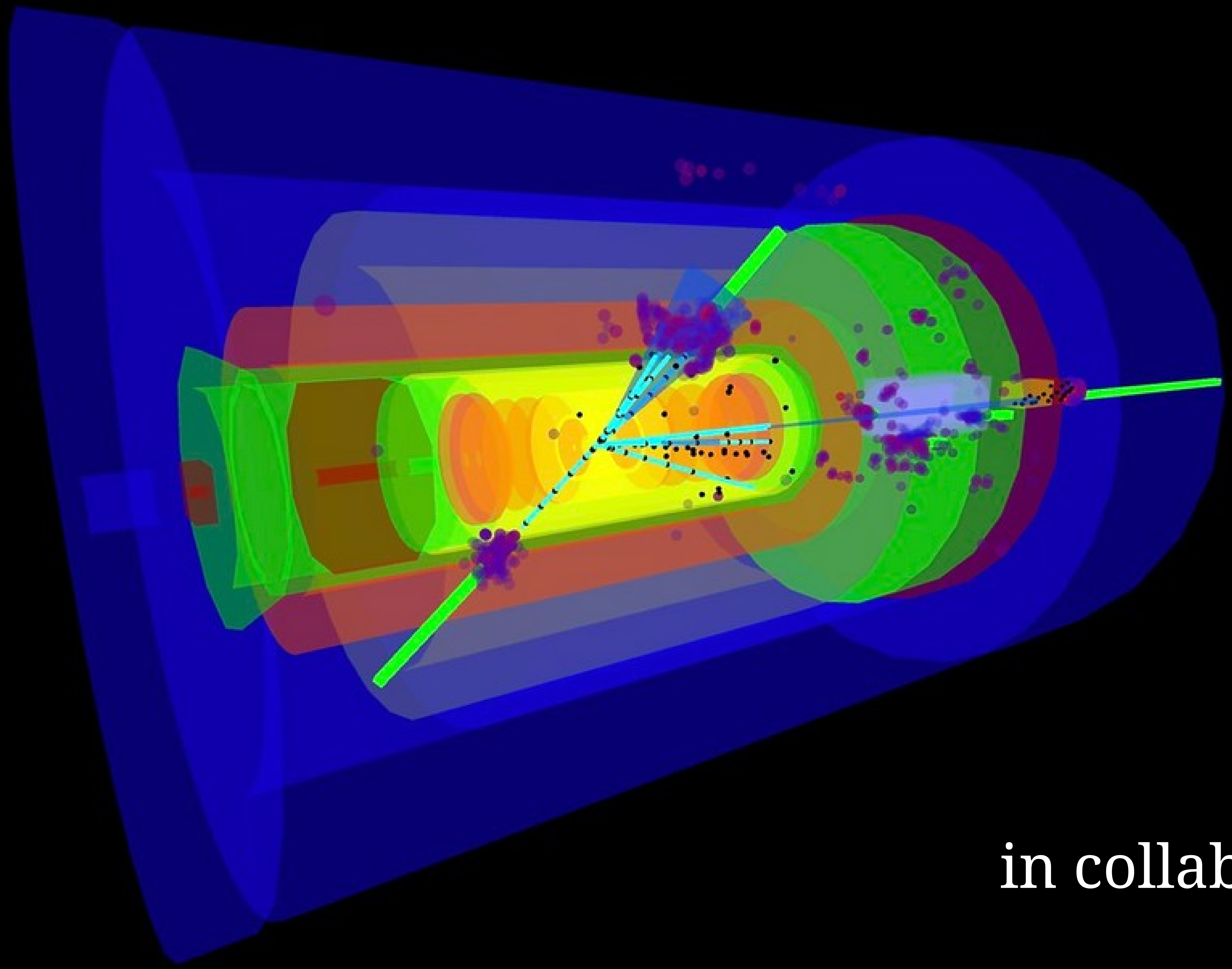
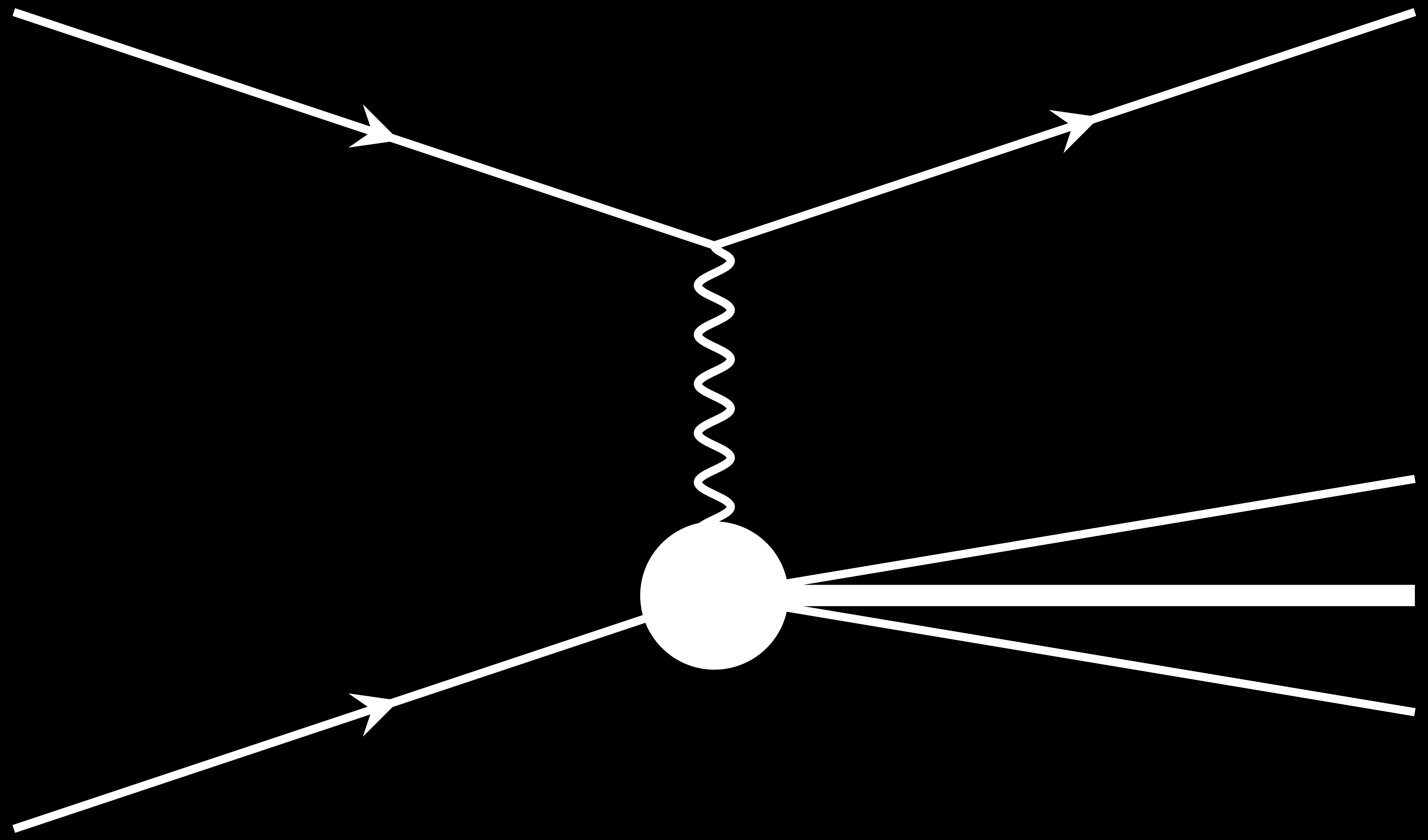
The Charm of Charged Currents





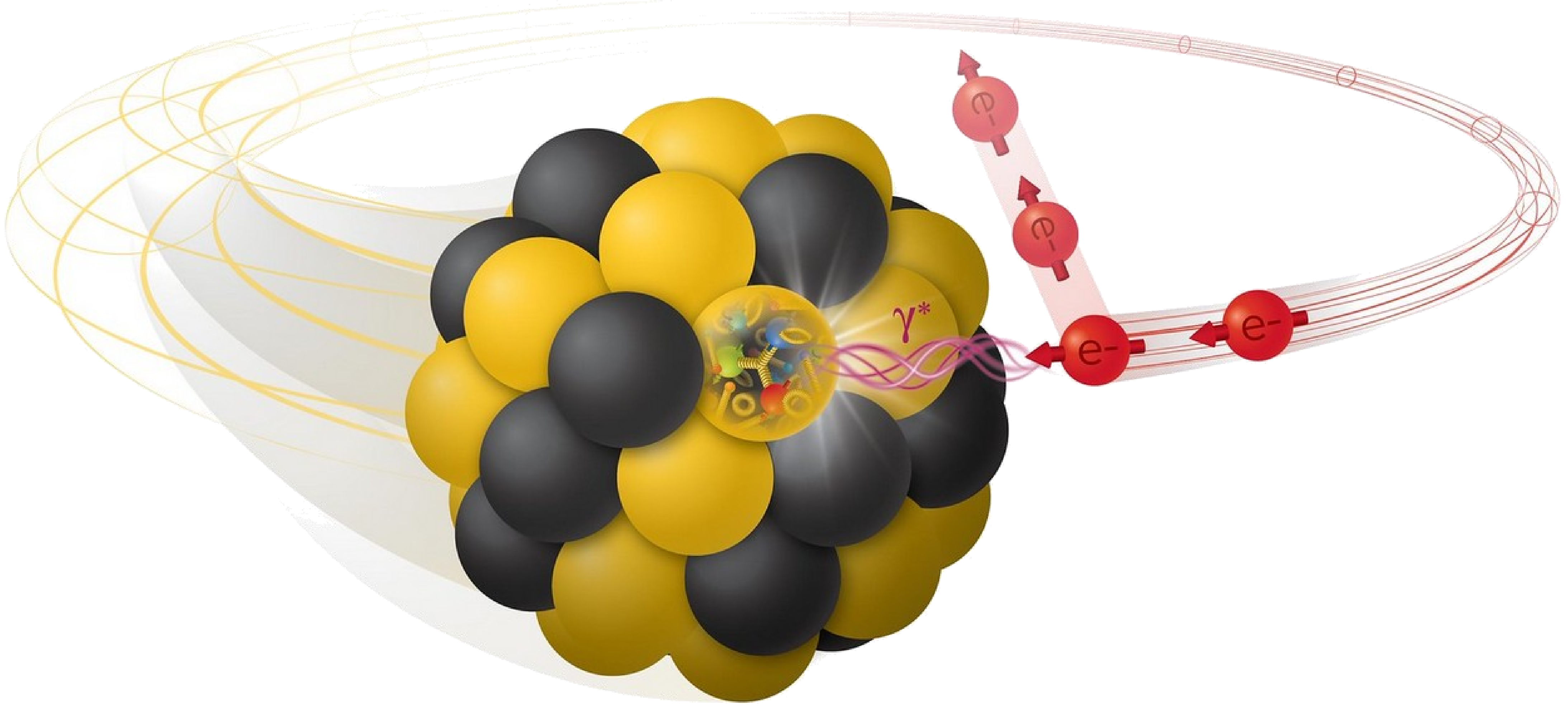
Pushing massive DIS to NNLO

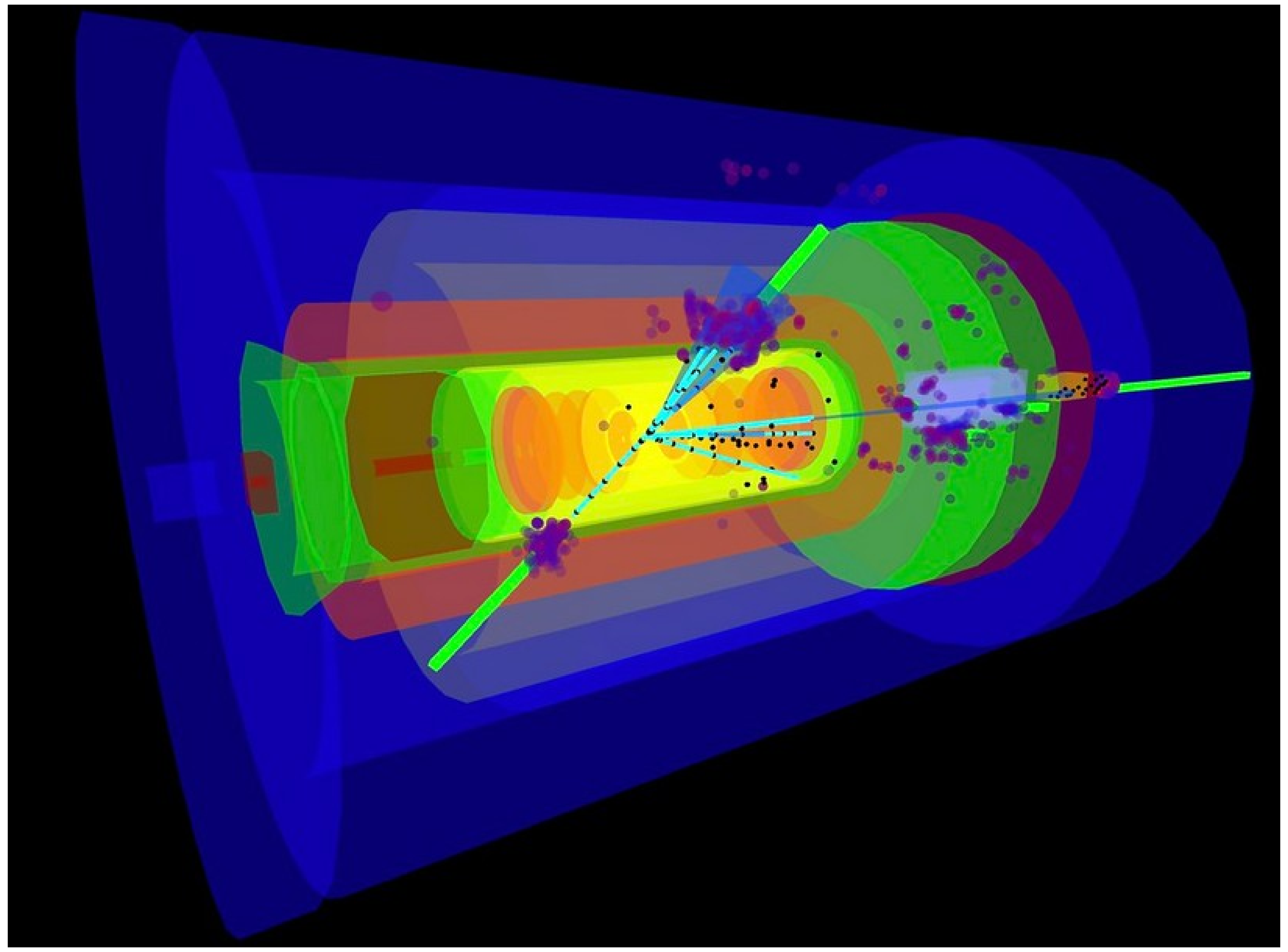
in collaboration with F. Caola and G. Gambuti

Image from
www.bnl.gov/eic/

Icons by flaticon.com
Freepik

The precision microscope of hadron structure



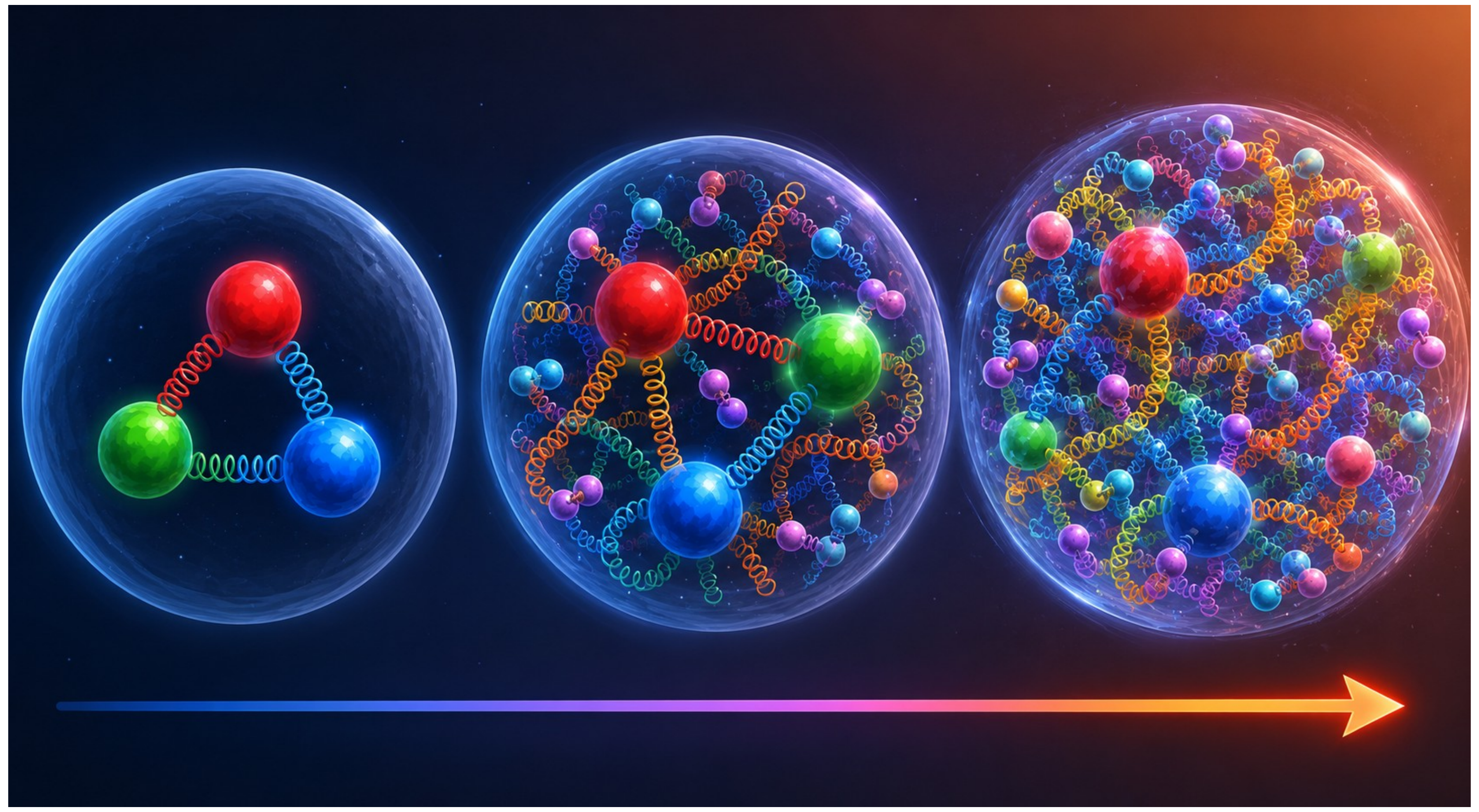


Quark Discovery @ SLAC via Bjorken Scaling → Sea and low-x restraints @ HERA → Future of DIS @ EIC

Images from
h1.desy.de/
www.bnl.gov/eic/
www6.slac.stanford.edu/
https://www.hep.wisc.edu/zeus/

PDF bands are a bottleneck

$$\sigma = \hat{\sigma}_a \otimes f_a$$



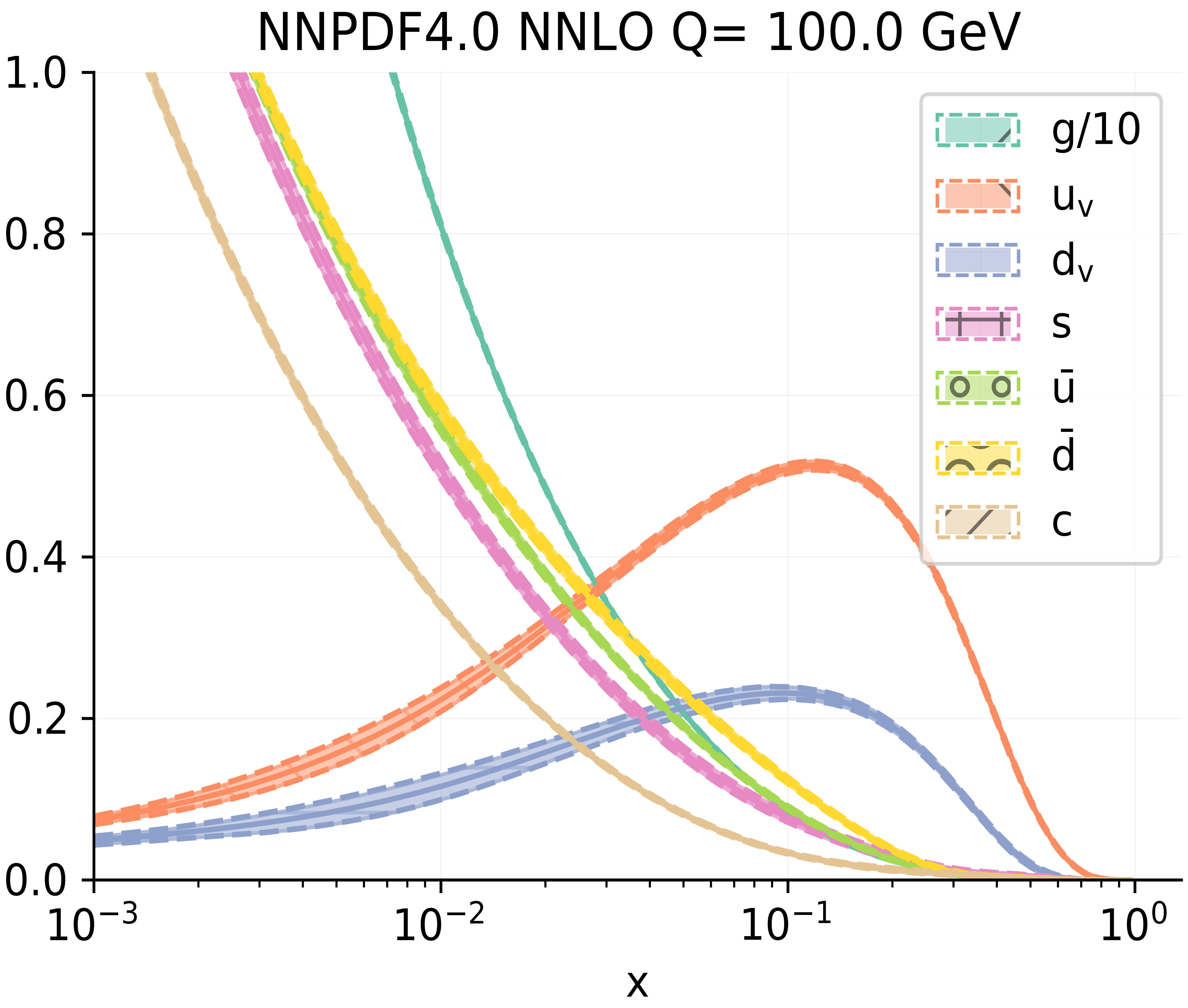
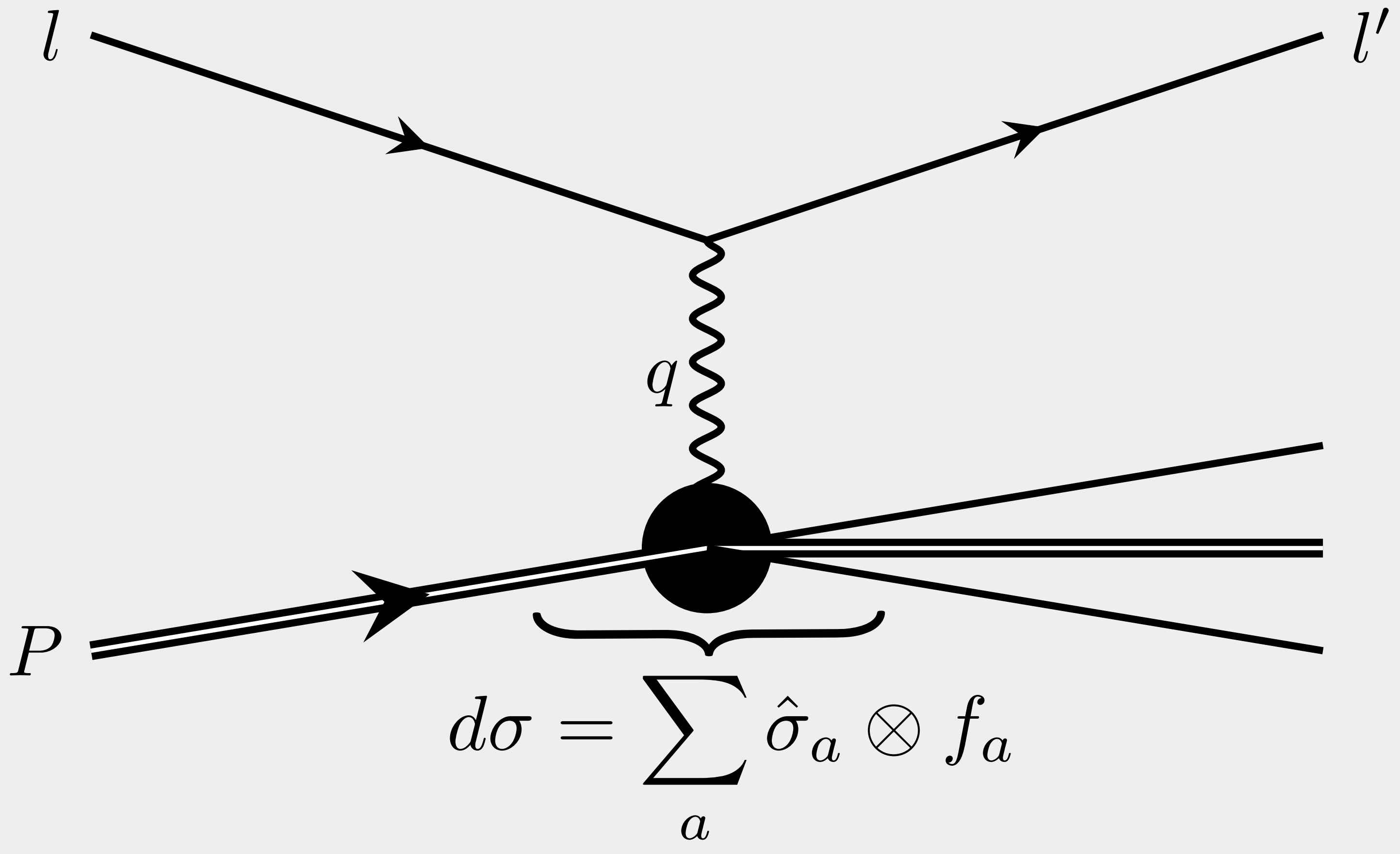


Figure from NNPDF Eur. Phys. J. C (2022) 82:428

Why CC DIS?

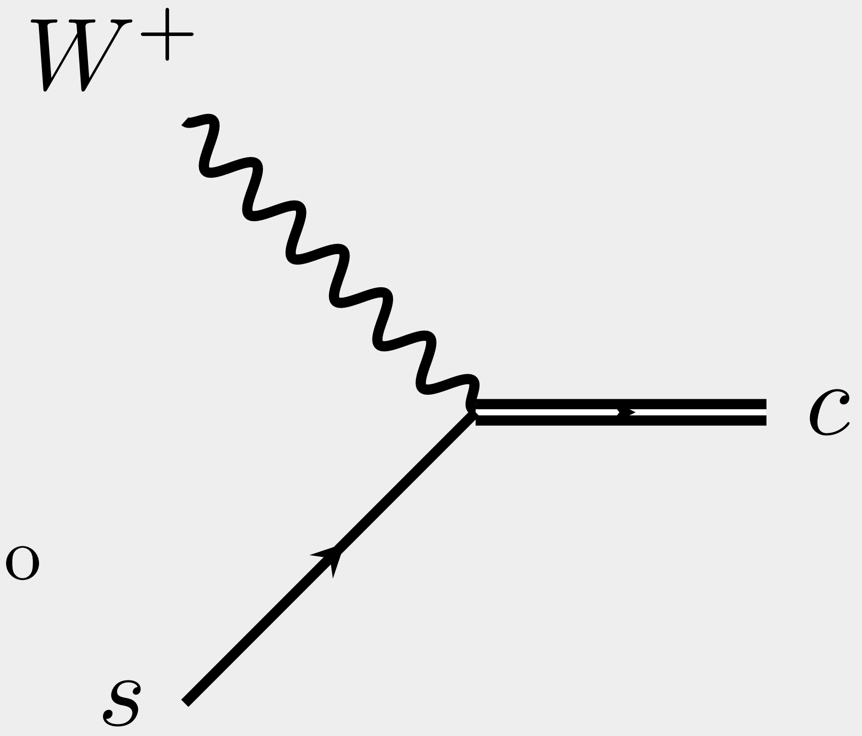


$$d\sigma = \sum_a \hat{\sigma}_a \otimes f_a$$

$$q^2 = -Q^2$$

$$x^{\mathrm{bj}} = \frac{Q^2}{2P \cdot q}$$

CC DIS
gives special access to
flavour structure



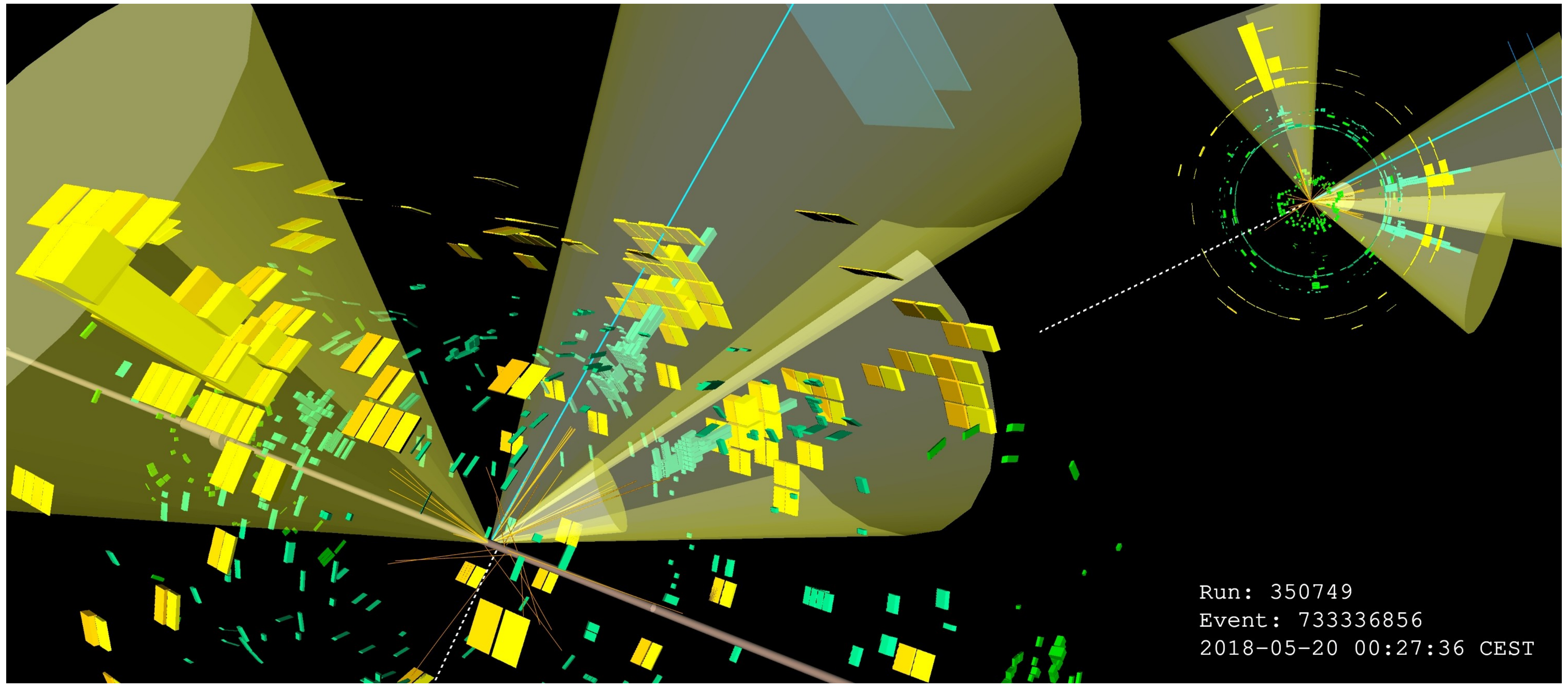
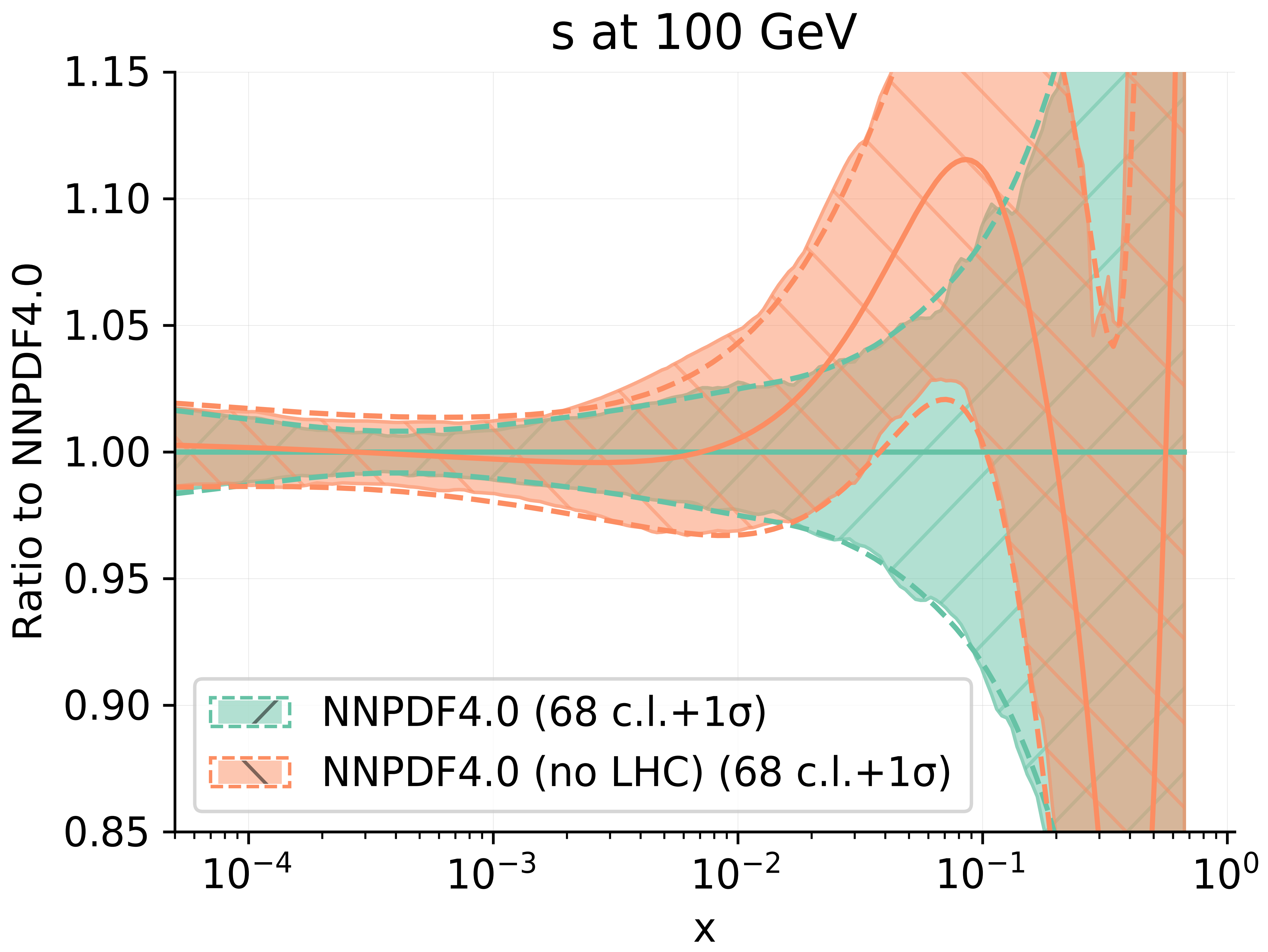


Image from https://atlas.cern/Updates/Briefing/Advanced-Particle-Tagging

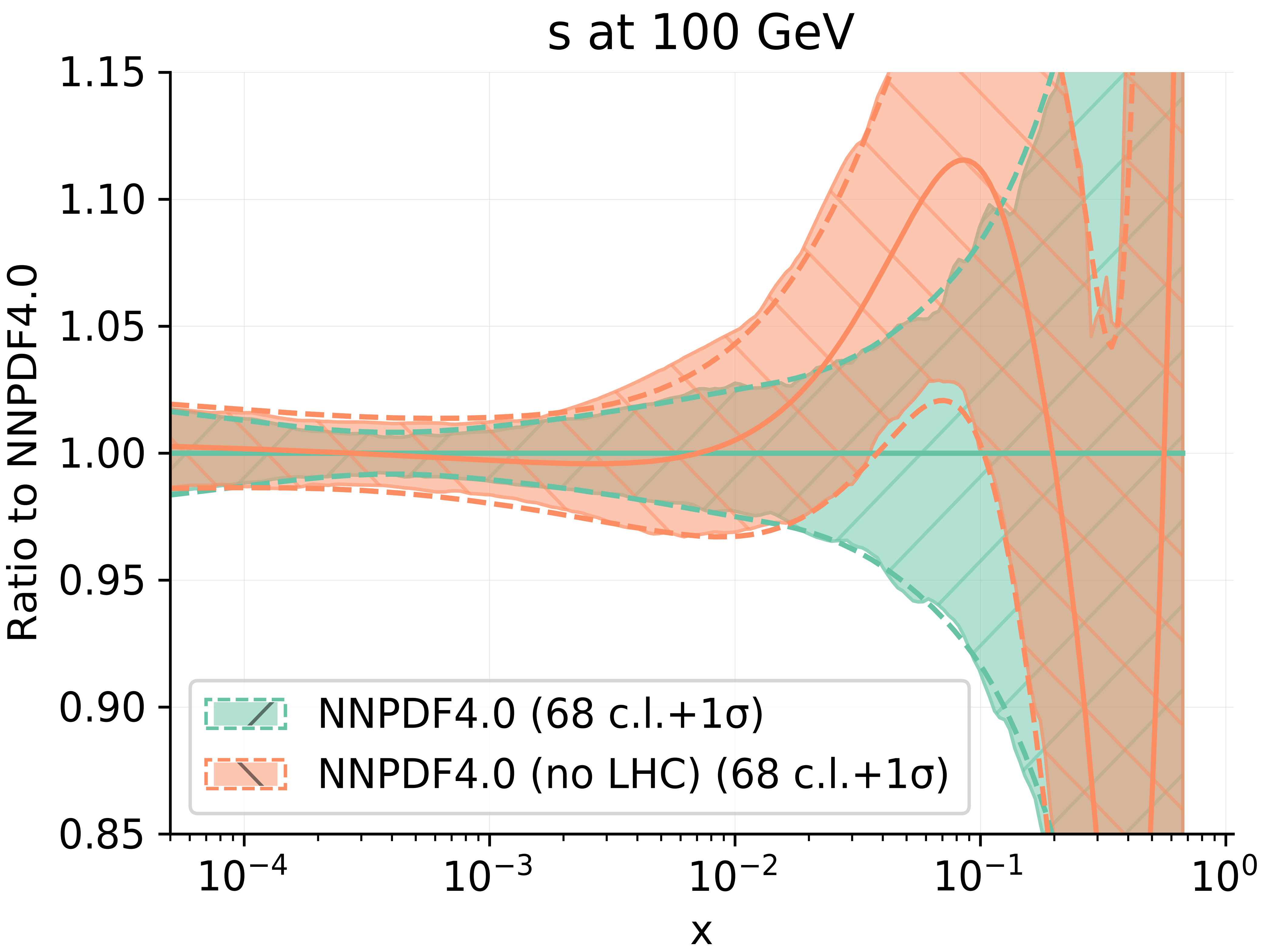
Protons are strange

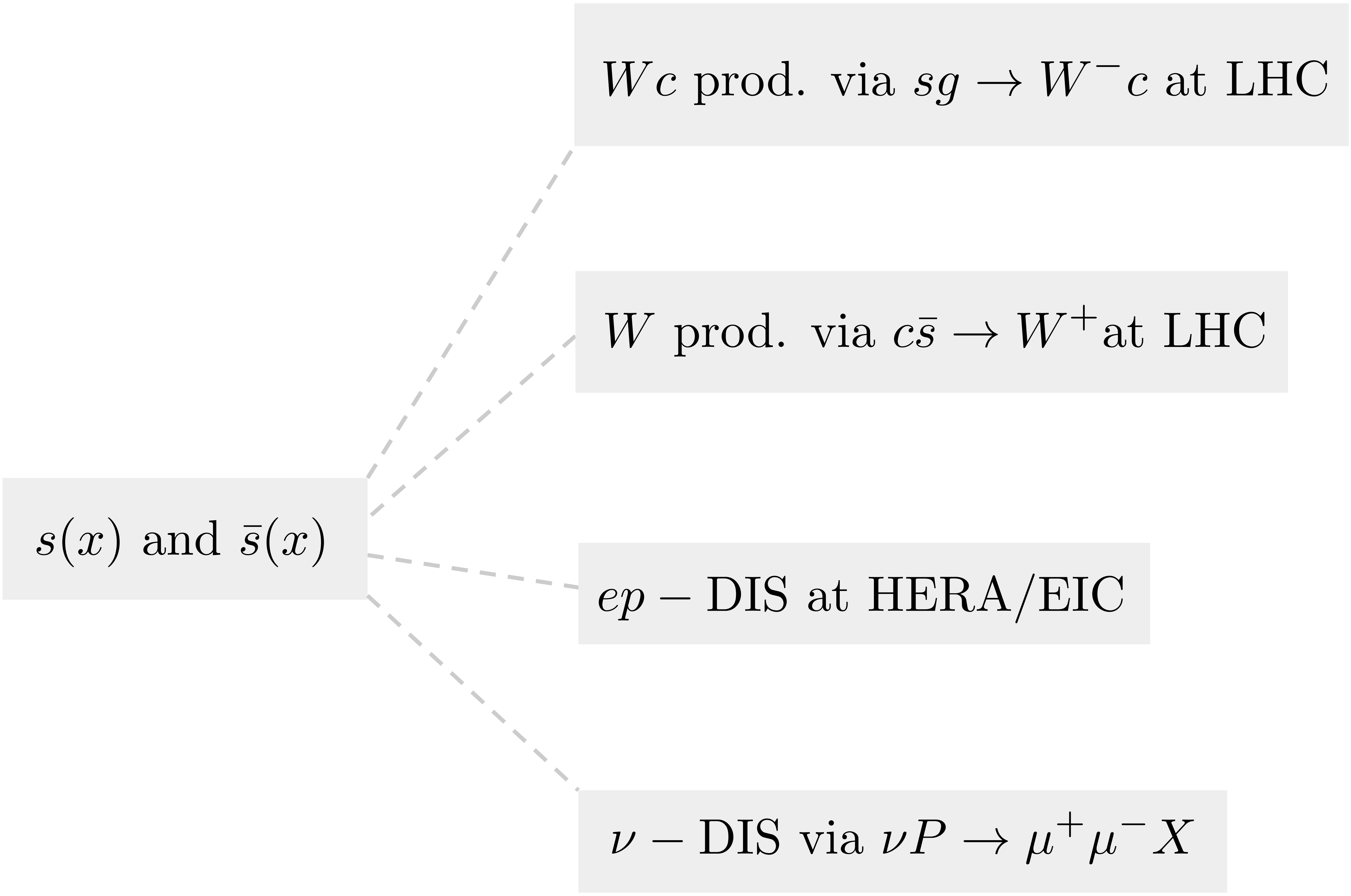


Low x strange fits are mainly constrained by non-LHC data!

Figure from NNPDF Eur. Phys. J. C (2022) 82:428

Protons are strange





$s(x)$ and $\bar{s}(x)$

Wc prod. via $sg \to W^-c$ at LHC

W prod. via $c\bar{s} \to W^+$at LHC

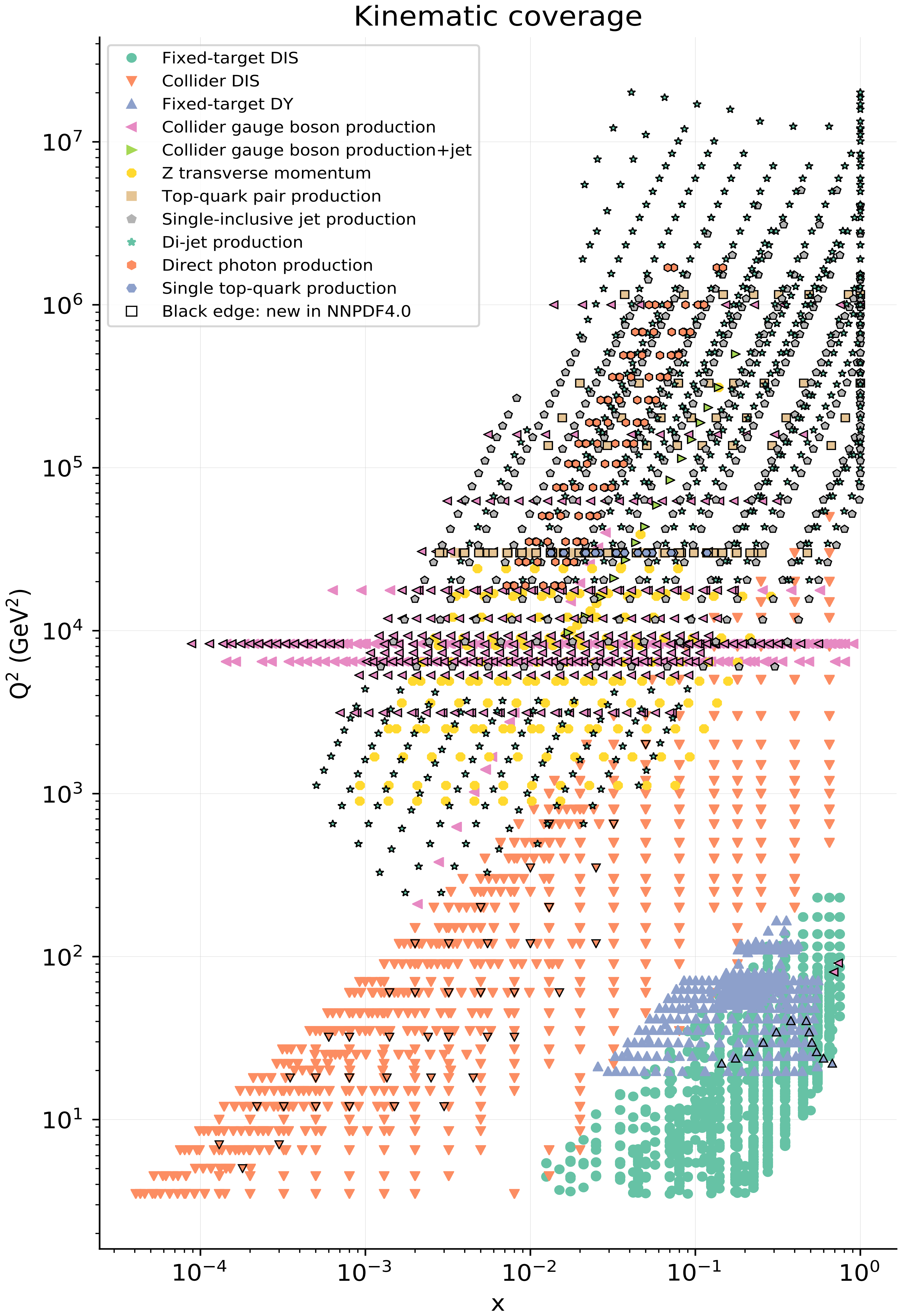
$ep-$DIS at HERA/EIC

$\nu-$DIS via $\nu P \to \mu^+\mu^- X$

Low x strange fits are mainly constrained by non-LHC data!

Figure from NNPDF Eur. Phys. J. C (2022) 82:428

What is special about the charm?



$m_c \sim 1.5$ GeV

DIS data

Figure from NNPDF Eur. Phys. J. C (2022) 82:428

What is special about the charm?

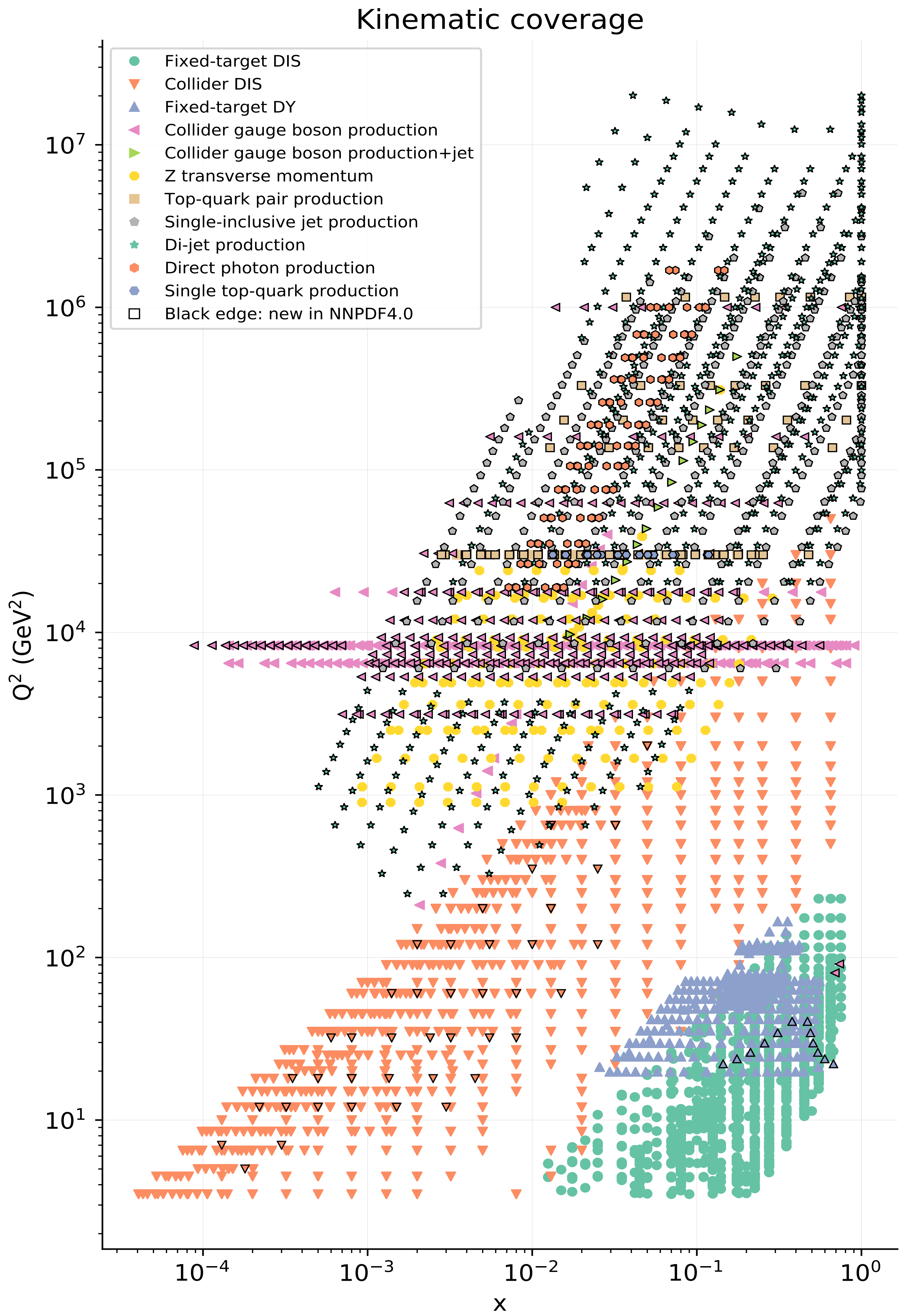


Figure from NNPDF Eur. Phys. J. C (2022) 82:428

DIS data

$m_c \sim 1.5$ GeV

$s(x)$ and $\bar{s}(x)$ are sensitive to $Q^2 \lesssim 100\ \text{GeV}^2$

Charm mass effects $\mathcal{O}\left(\frac{m_c^2}{Q^2}\right)$ are potentially relevant

In modern precision PDF fits need to address m_c

Asymptotic and LO/NLO results

- LO: slow rescaling $x^{\mathrm{bj}} \to \left(1 + \frac{m_c^2}{Q^2}\right) x^{\mathrm{bj}}$
- NLO: Known analytic result [1]
- Exact NNLO is known numerically [2]
- Asymptotic results up to N3LO are available [3] $Q^2 \gg m_c^2$

[1] Gottschalk, Phys. Rev. D 23 (1981) 56
[2] Jun Gao, JHEP 02 (2018) 026 [1710.04258]
[3] Blümlein et al, Nucl. Phys. B 820 (2009) 417

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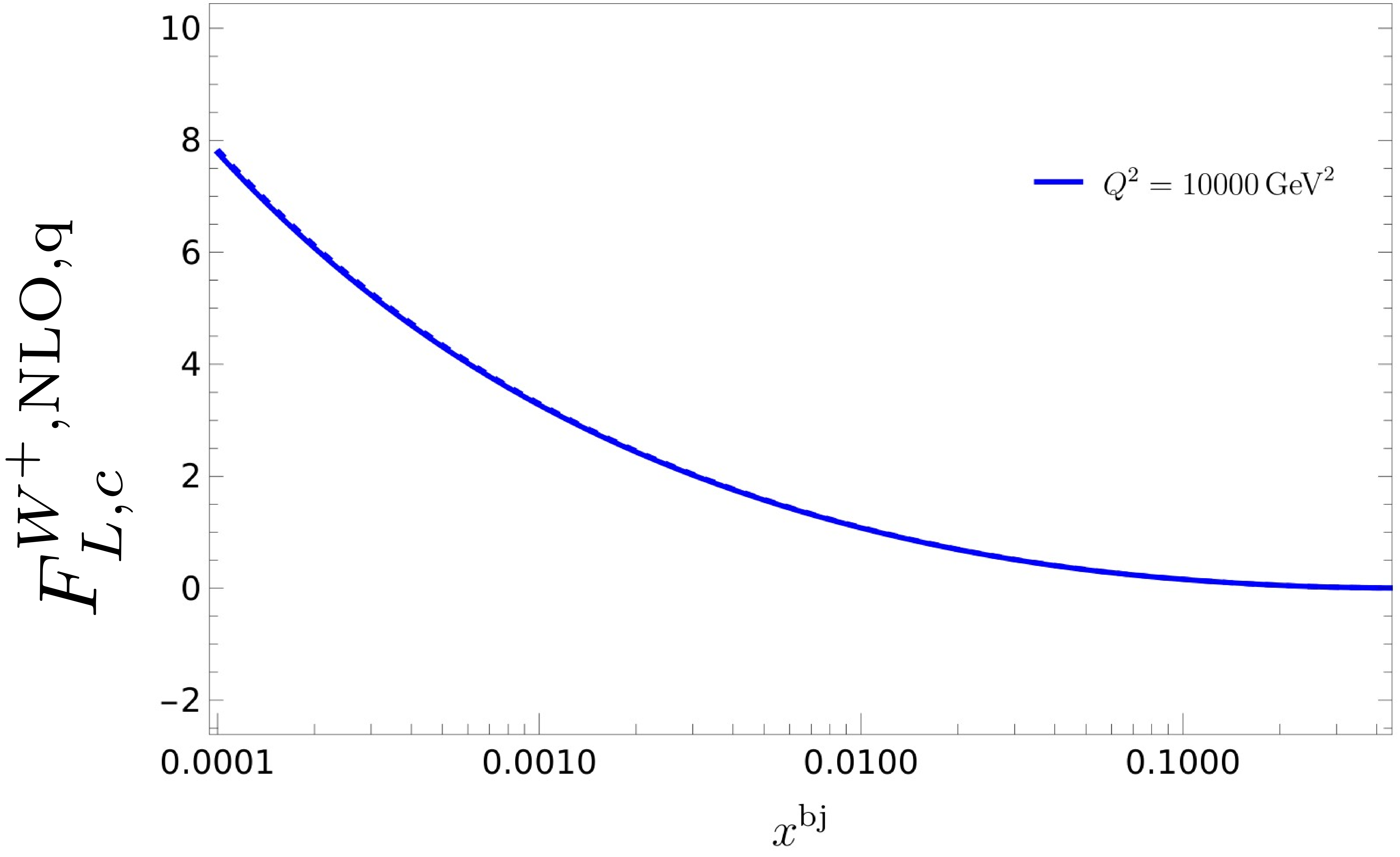

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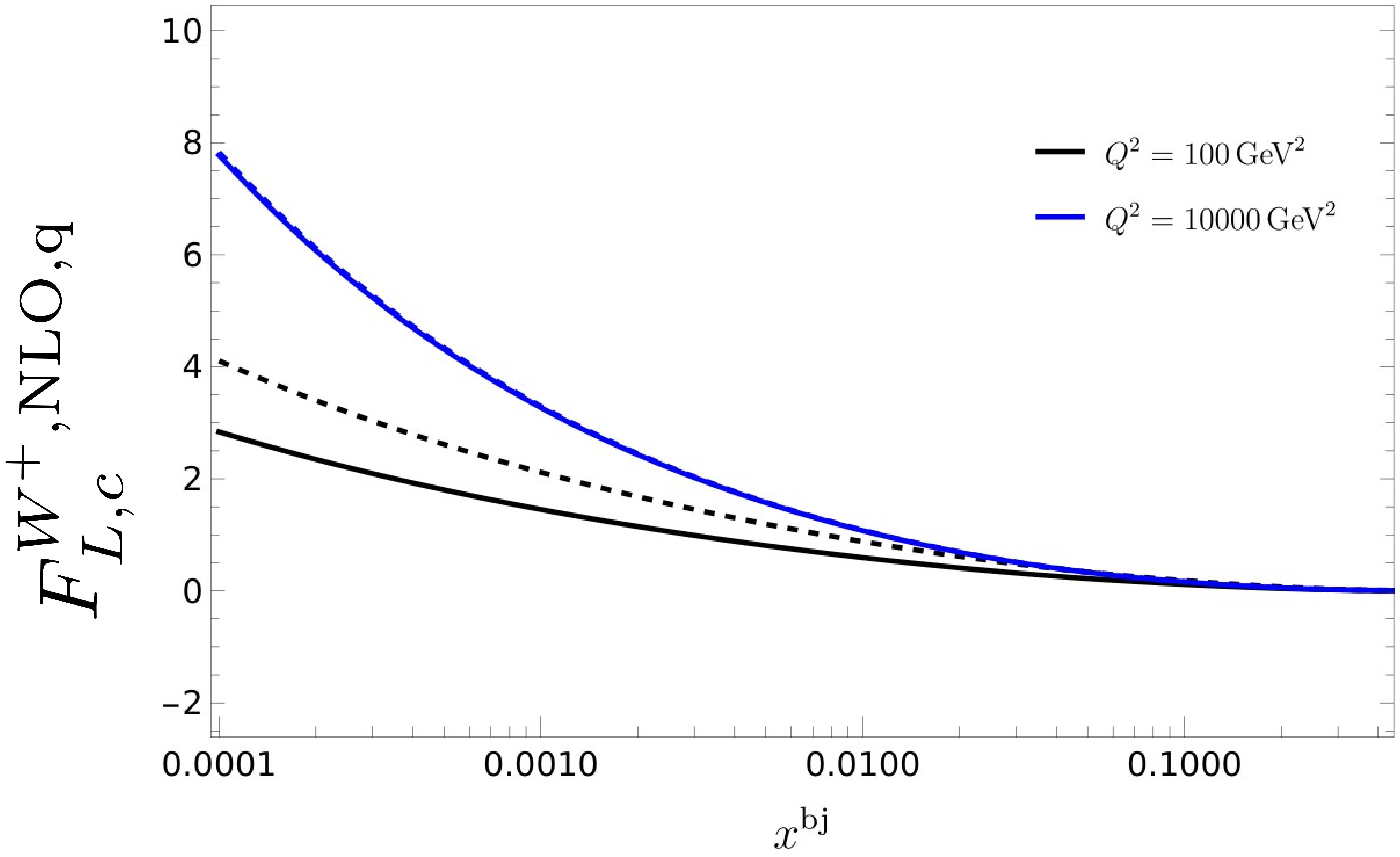

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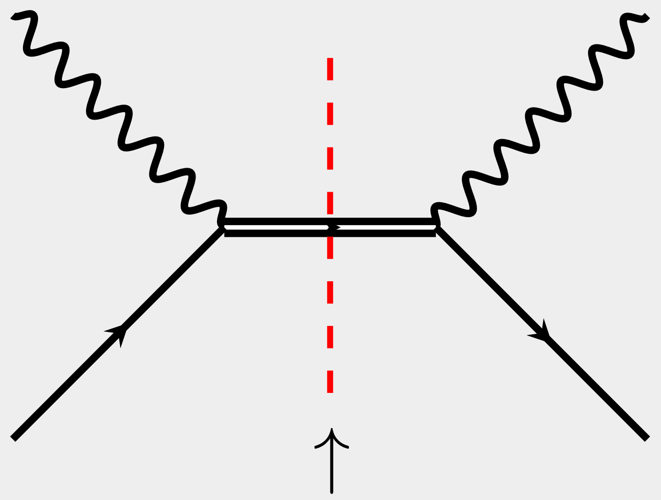
The partonic cross section

Icon by Freepik

The partonic cross section

LO

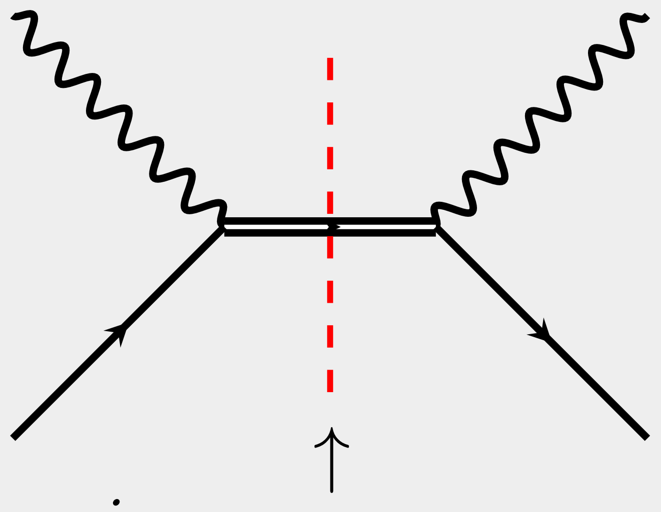


$$\frac{i}{\mathcal{D}} \to (2\pi)\delta_+(\mathcal{D})$$

Icon by Freepik

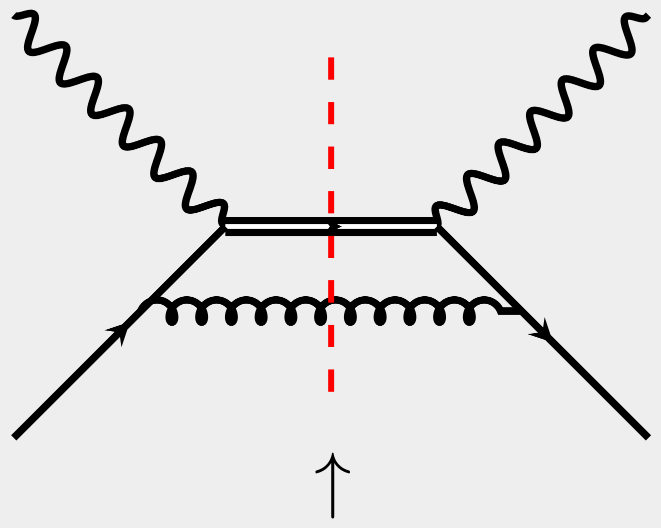
The partonic cross section

LO



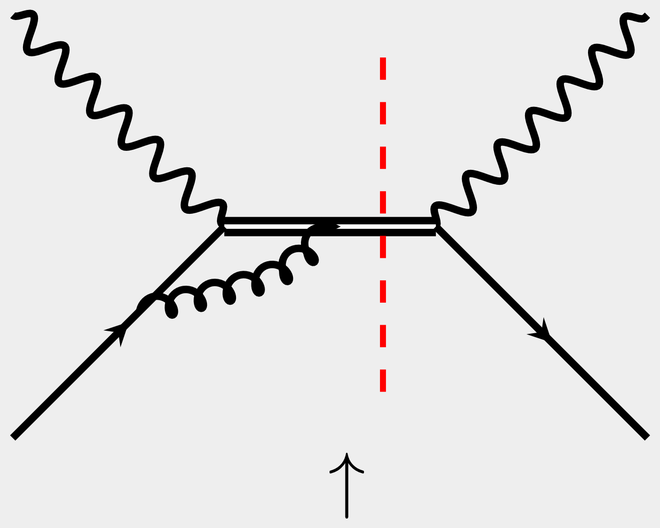
$$\frac{i}{\mathcal{D}} \to (2\pi)\delta_+(\mathcal{D})$$

NLO



Real emissions

+



Virtual emissions

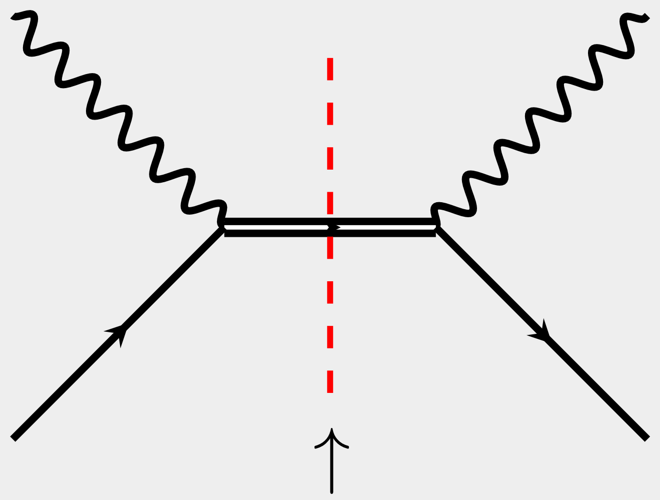
+4 others

→ Relatively easy PS integrals, easy divergence cancellation

Icon by Freepik

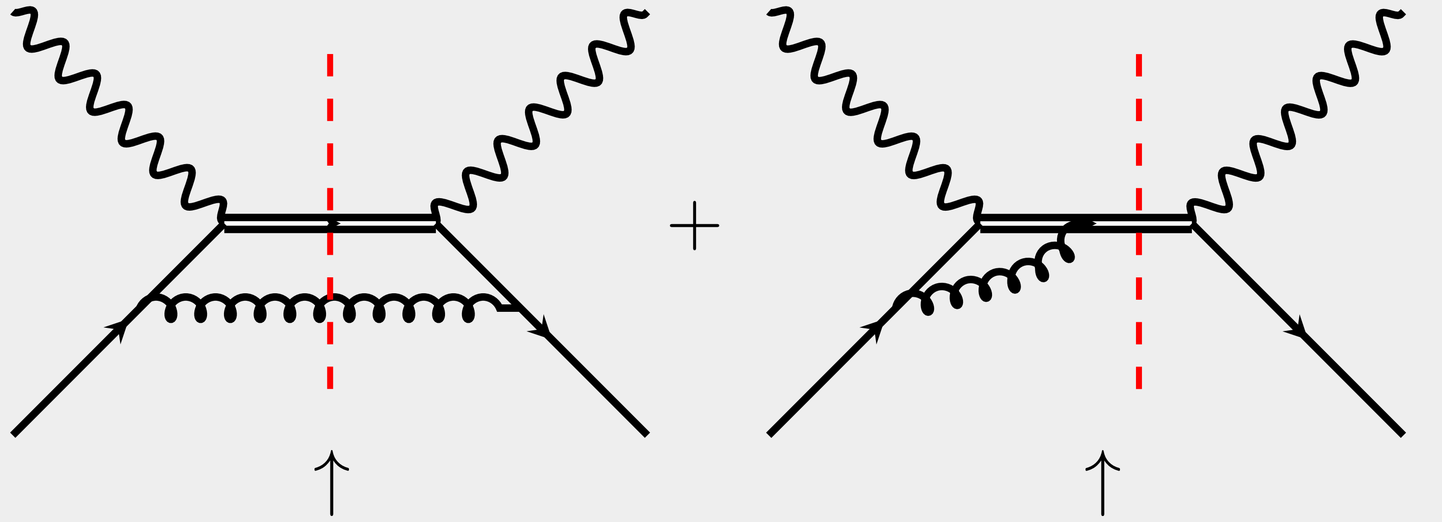
The partonic cross section

LO



$$\frac{i}{\mathcal{D}} \to (2\pi)\delta_+(\mathcal{D})$$

NLO



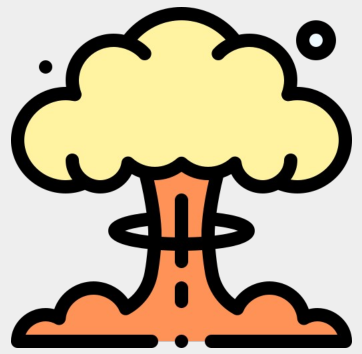
+4 others

Real emissions

Virtual emissions

→ Relatively easy PS integrals, easy divergence cancellation

NNLO



Hard div. cancellation

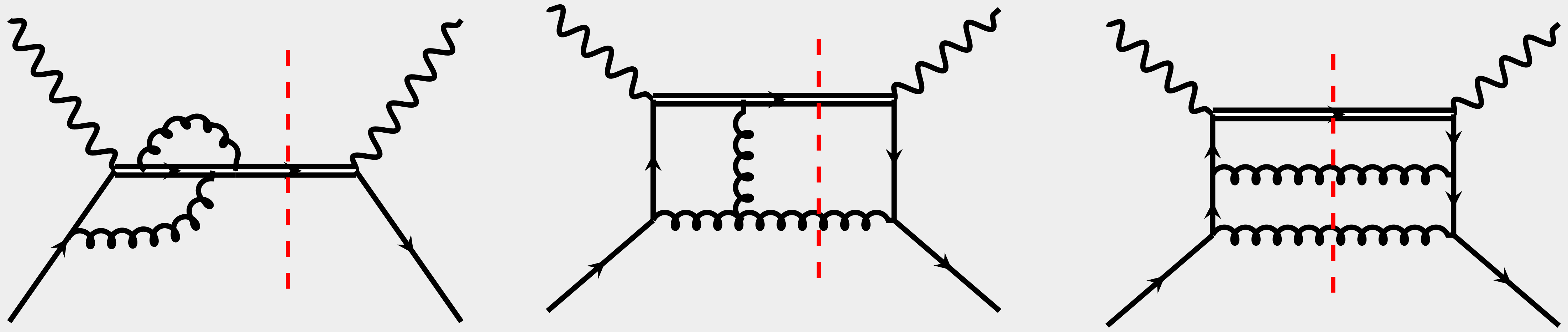
Very hard PS integrals

→ **Need more technology!**

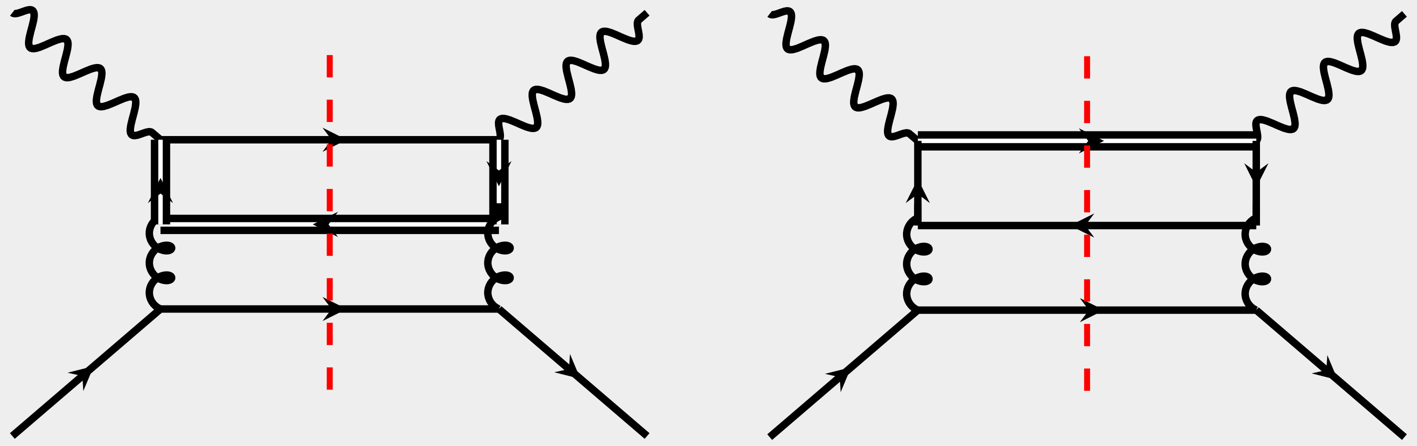
Icon by Freepik

Flavour structure at NNLO

VV1 RV1 RR1

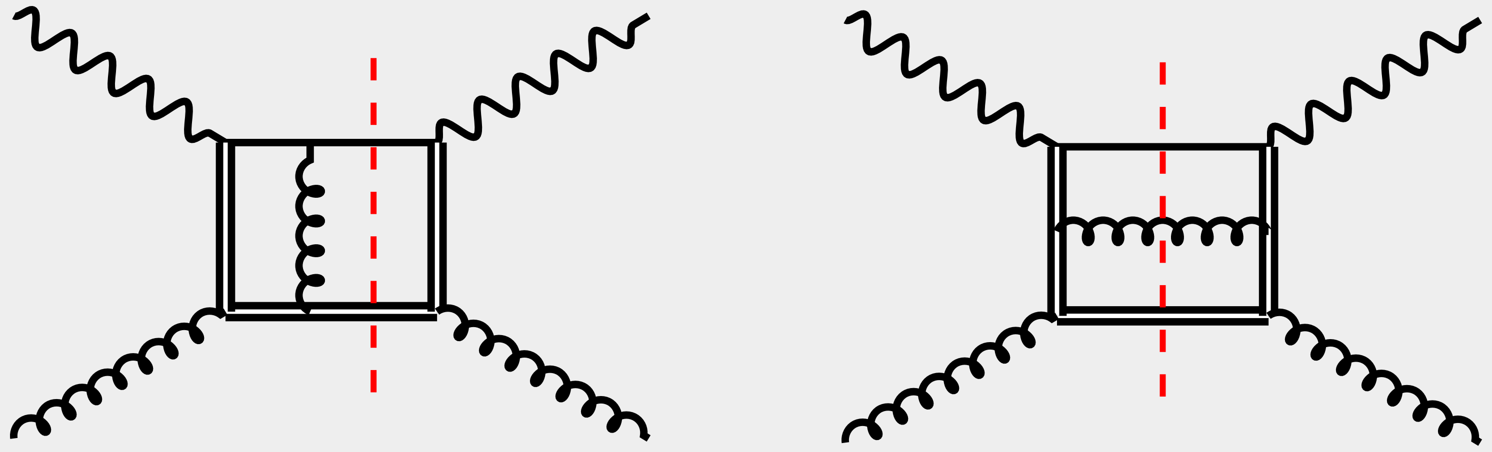


non-singlet



pure-singlet

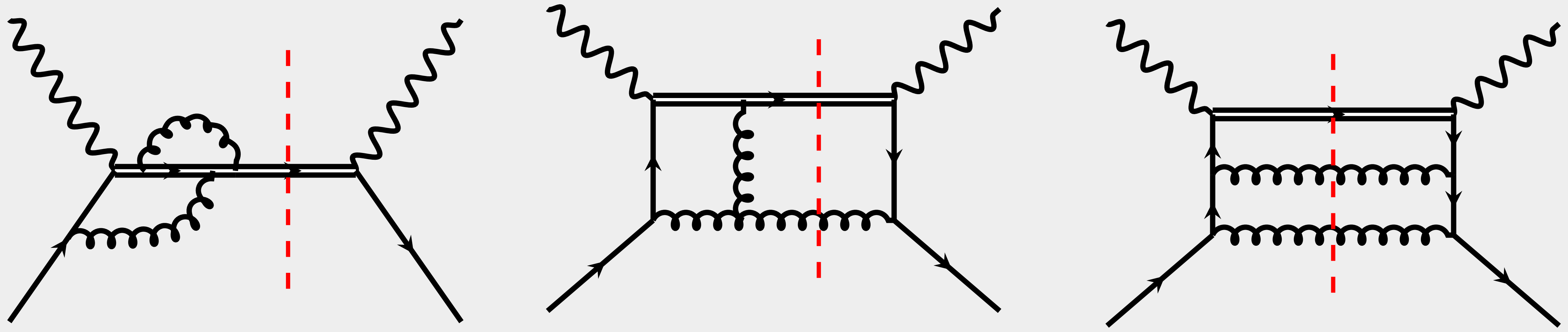
RR1 and RV1



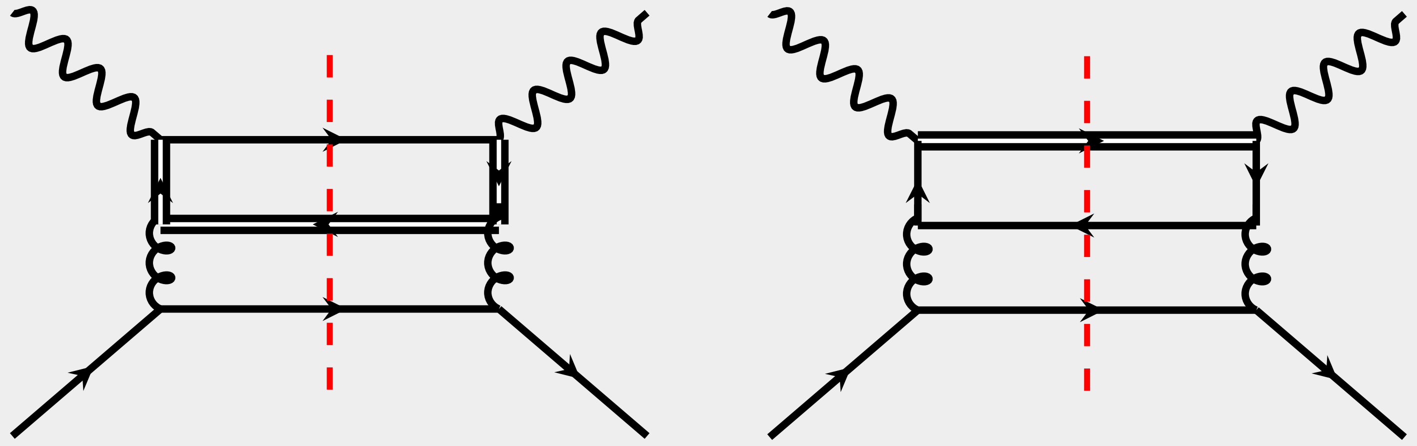
gluon channel

Flavour structure at NNLO

VV1 RV1 RR1

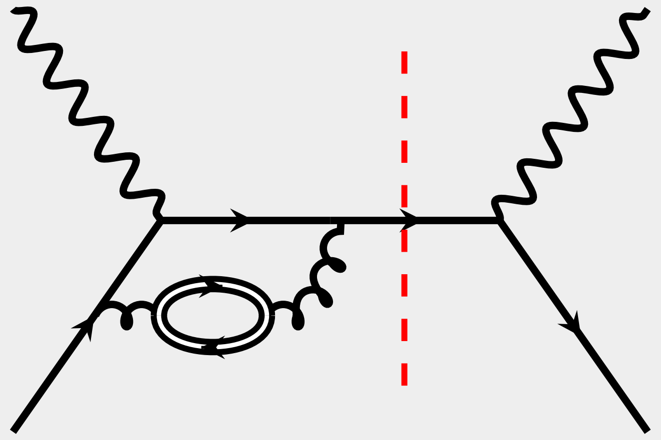


non-singlet

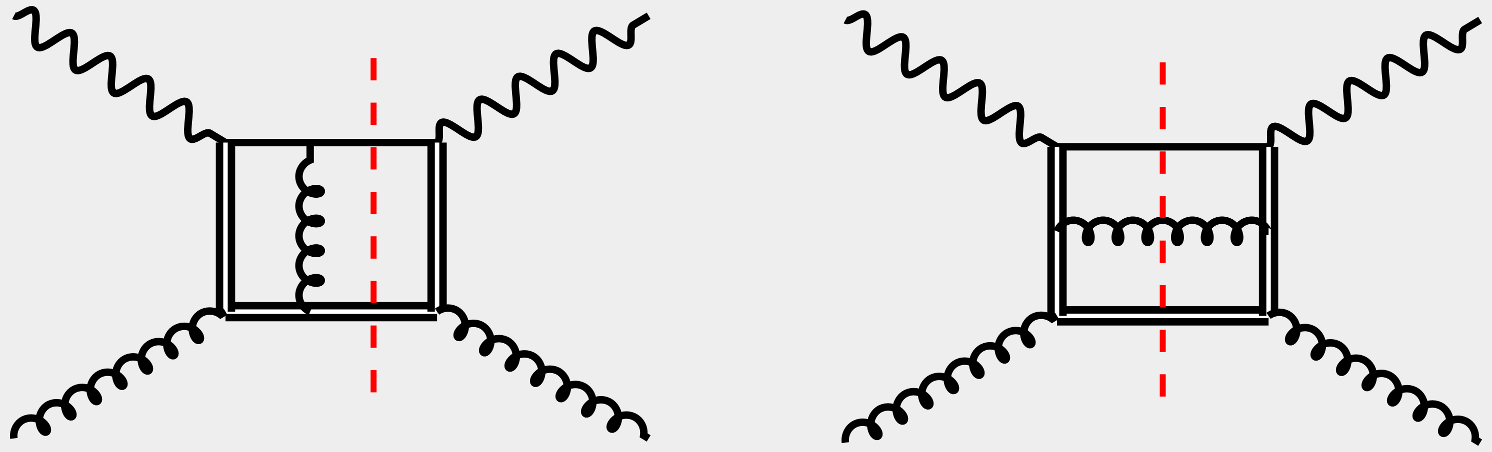


pure-singlet

VV0

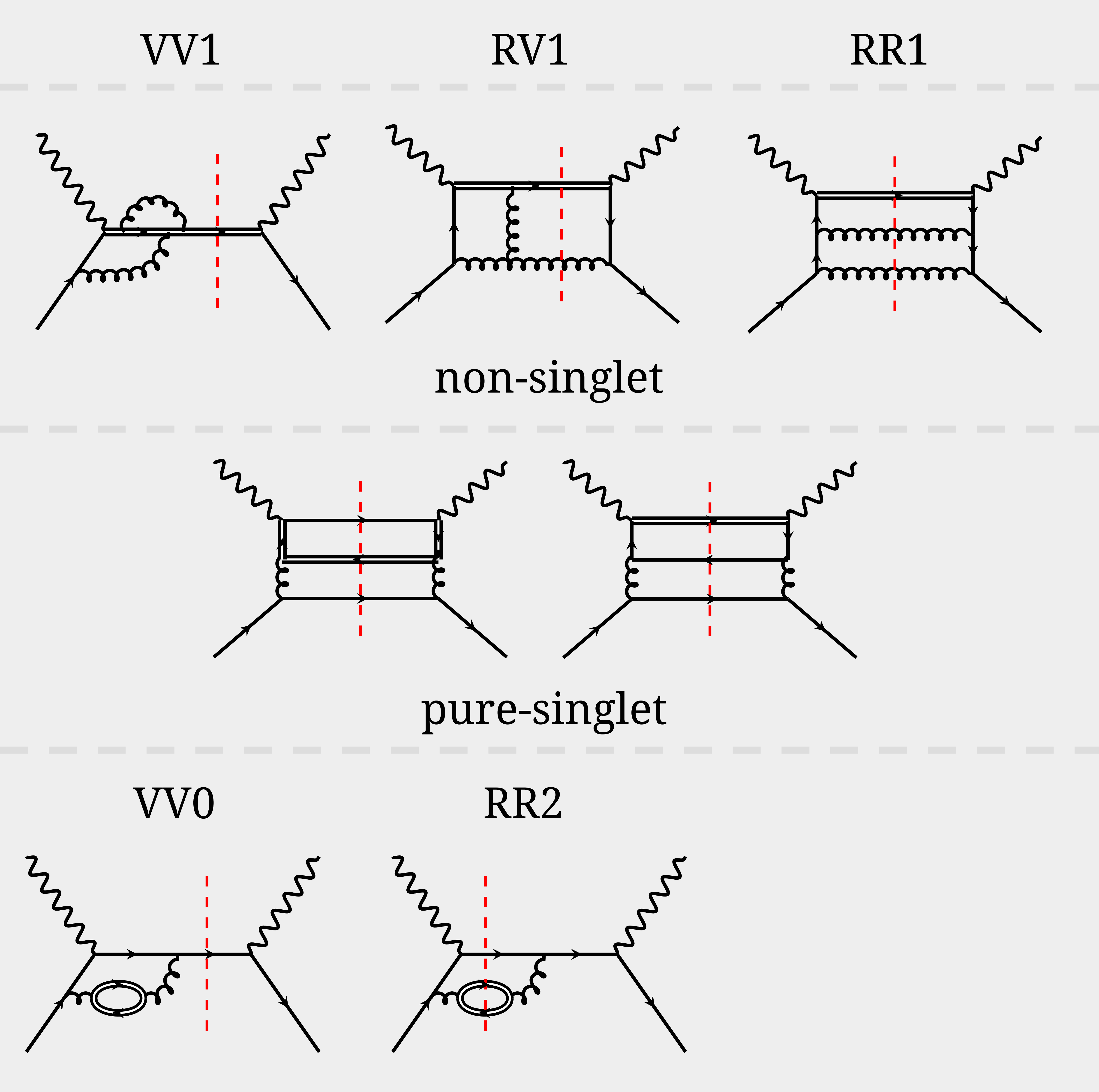


RR1 and RV1



gluon channel

Flavour structure at NNLO

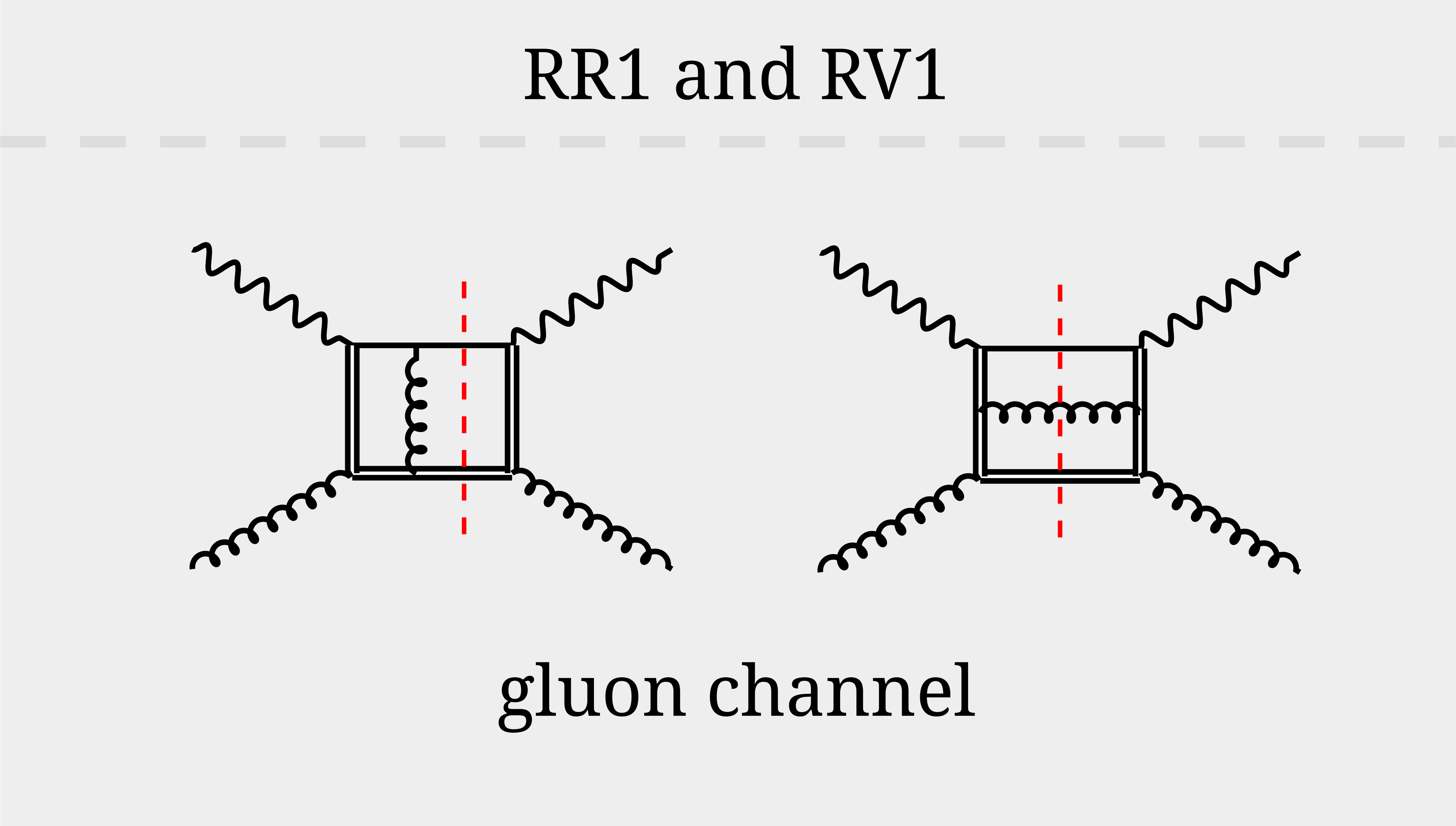


VV1 RV1 RR1

non-singlet

pure-singlet

VV0 RR2

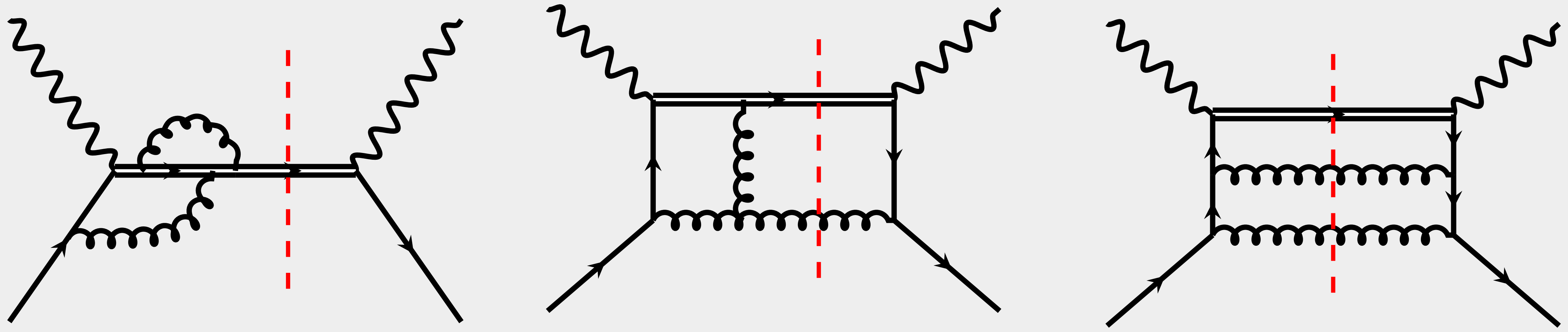


RR1 and RV1

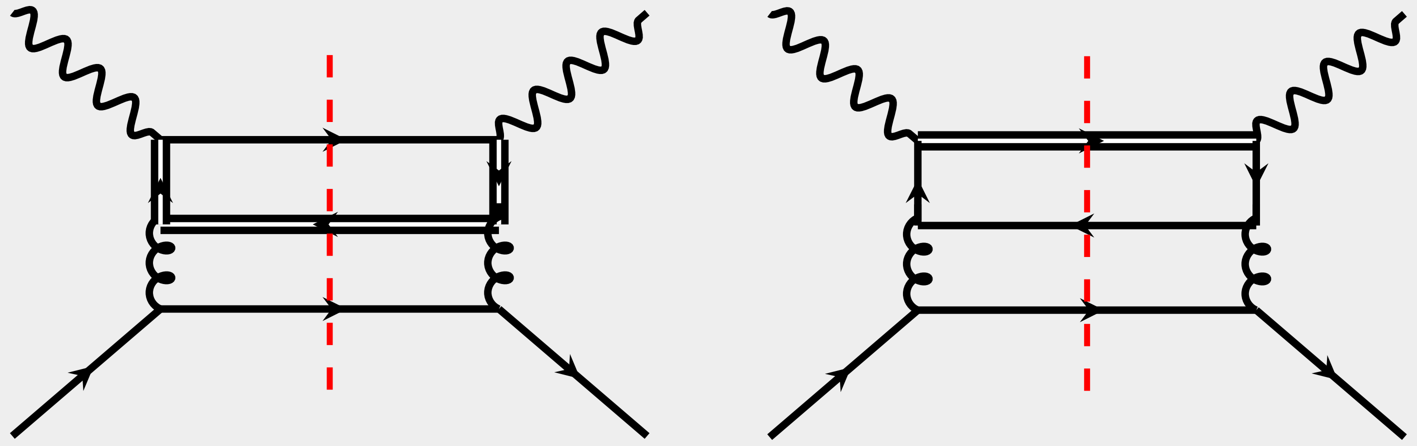
gluon channel

Flavour structure at NNLO

VV1 RV1 RR1

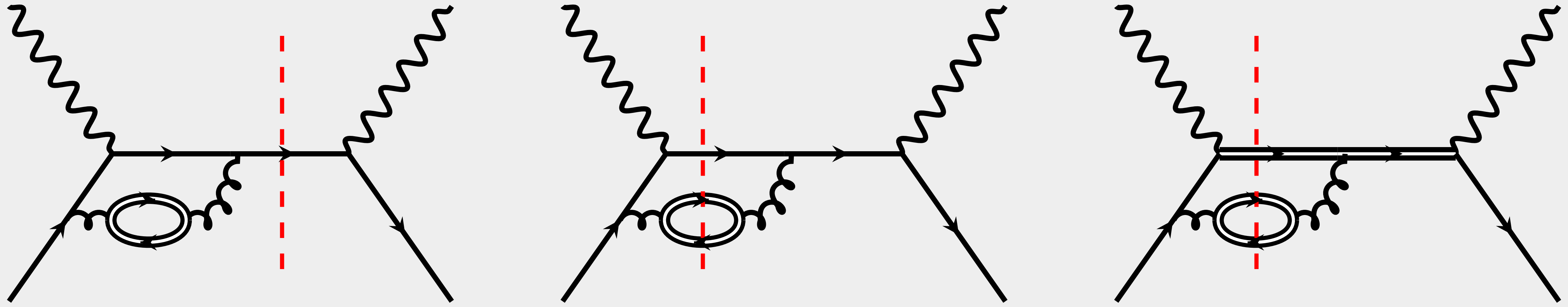


non-singlet

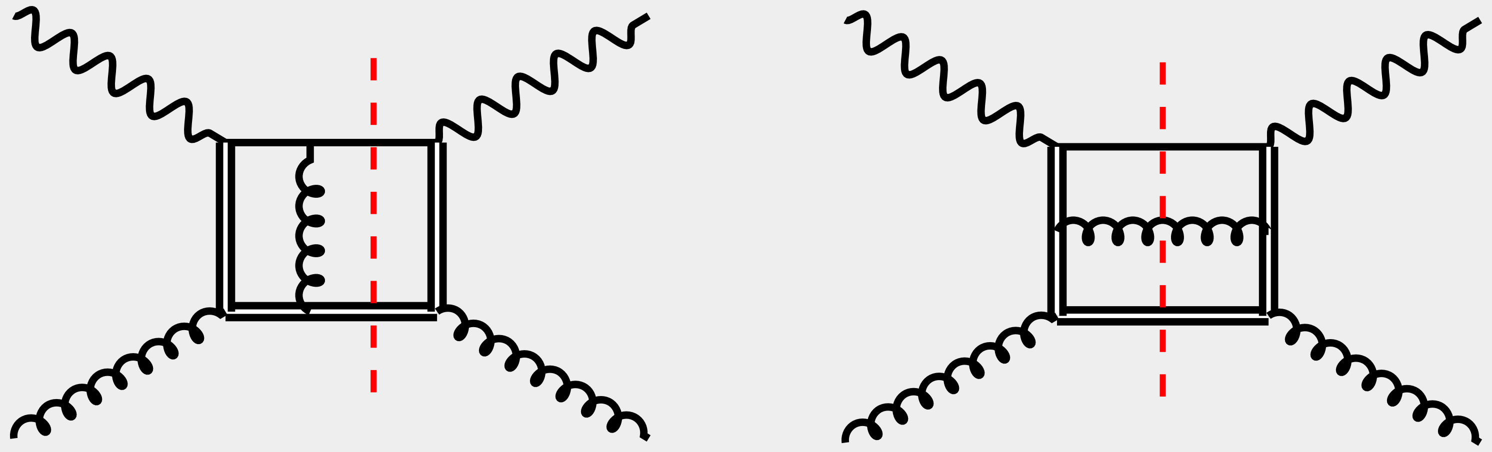


pure-singlet

VV0 RR2 RR3



RR1 and RV1



gluon channel

IBPs and reverse unitarity

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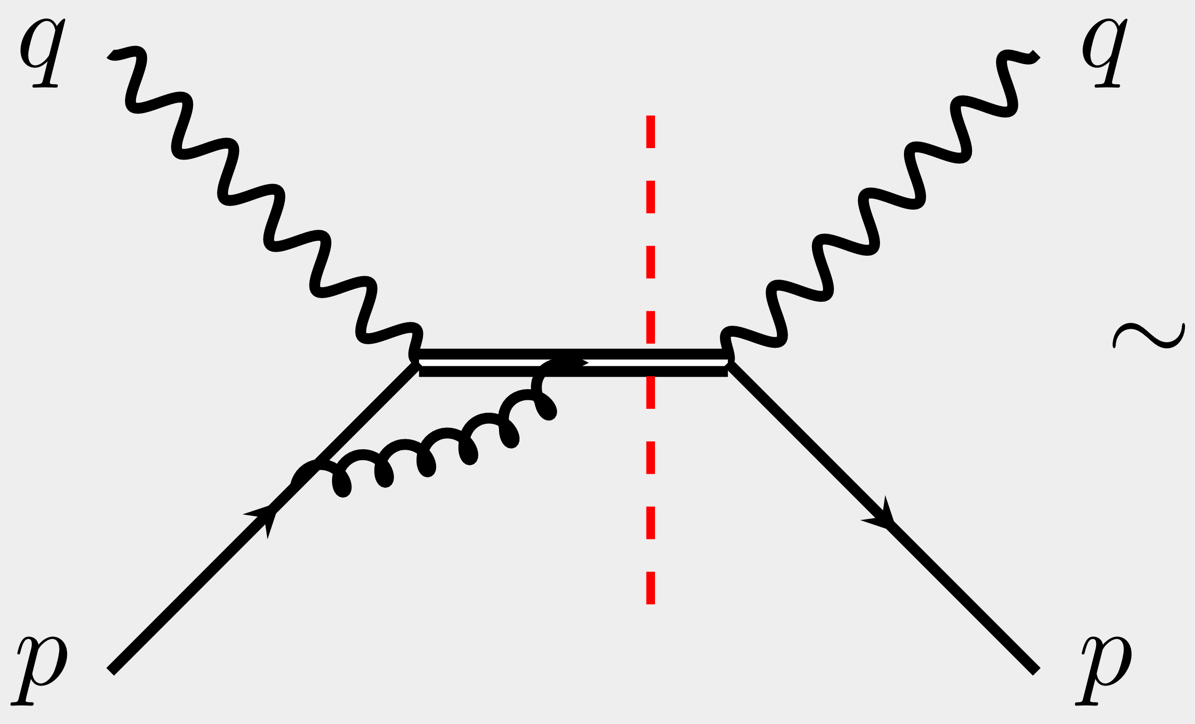
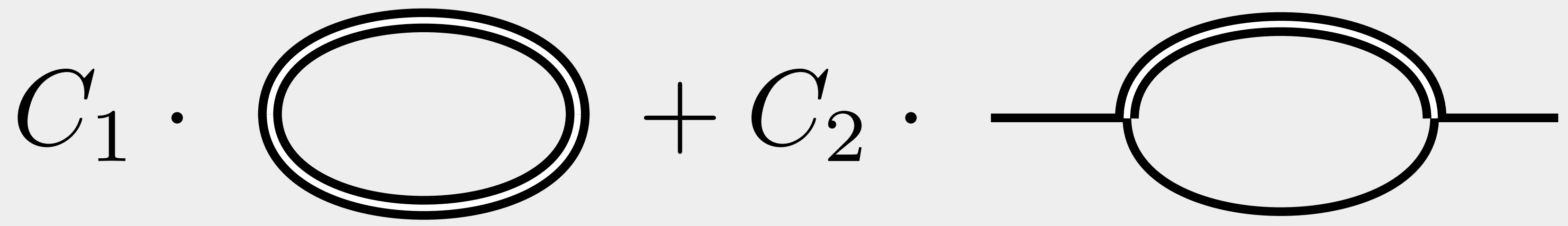


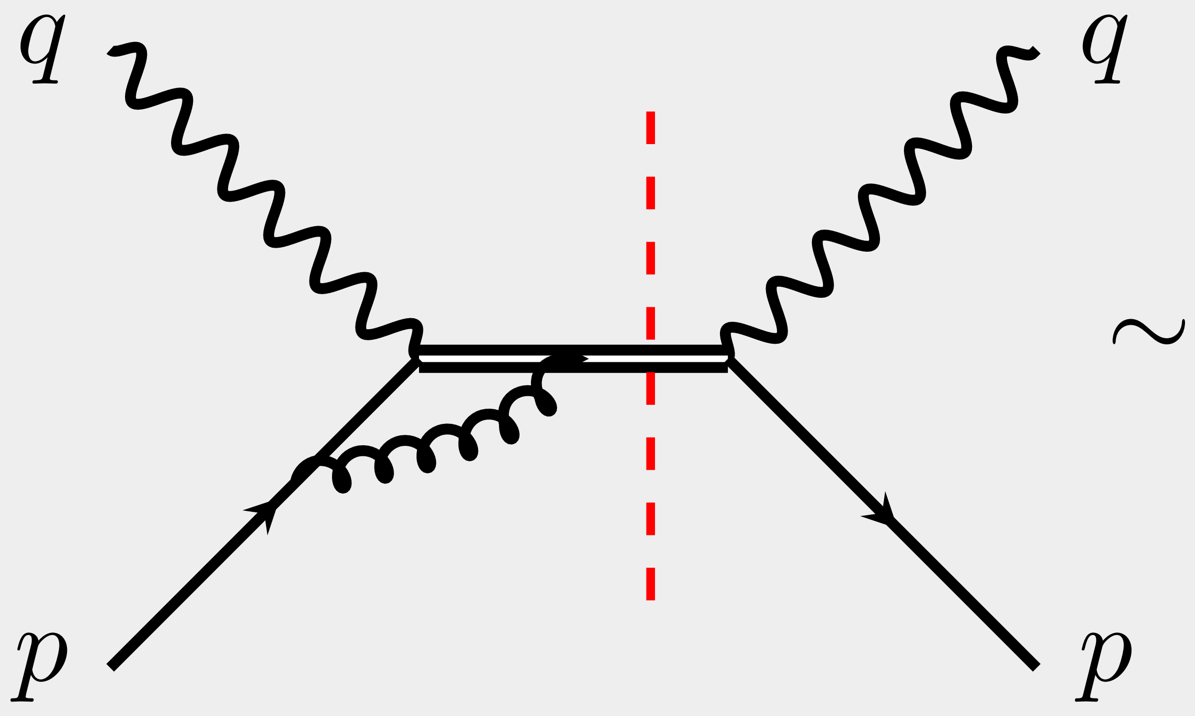
$$\sim \int [dk] \frac{\mathcal{N}_{\mu\nu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3}$$

$$I_{\nu_1 \nu_2 \nu_3} \equiv \int [dk] \frac{1}{\mathcal{D}_1^{\nu_1} \mathcal{D}_2^{\nu_2} \mathcal{D}_3^{\nu_3}}$$

$$0 = \int [dk] \frac{\partial}{\partial k_i^{\mu}} \left[v^{\mu} \frac{1}{\mathcal{D}_1^{\nu_1} \mathcal{D}_2^{\nu_2} \mathcal{D}_3^{\nu_3}} \right]$$ ⬇



IBPs and reverse unitarity



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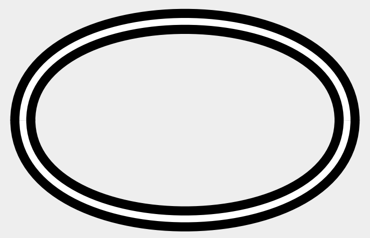
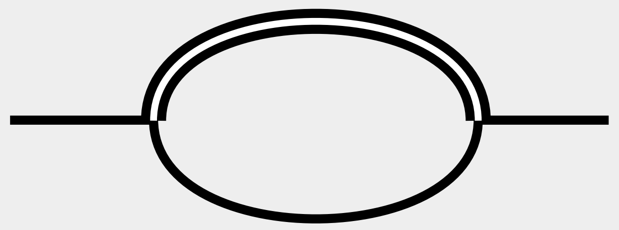
Rev. Unitarity

$C_1 \cdot$



$+ C_2 \cdot$



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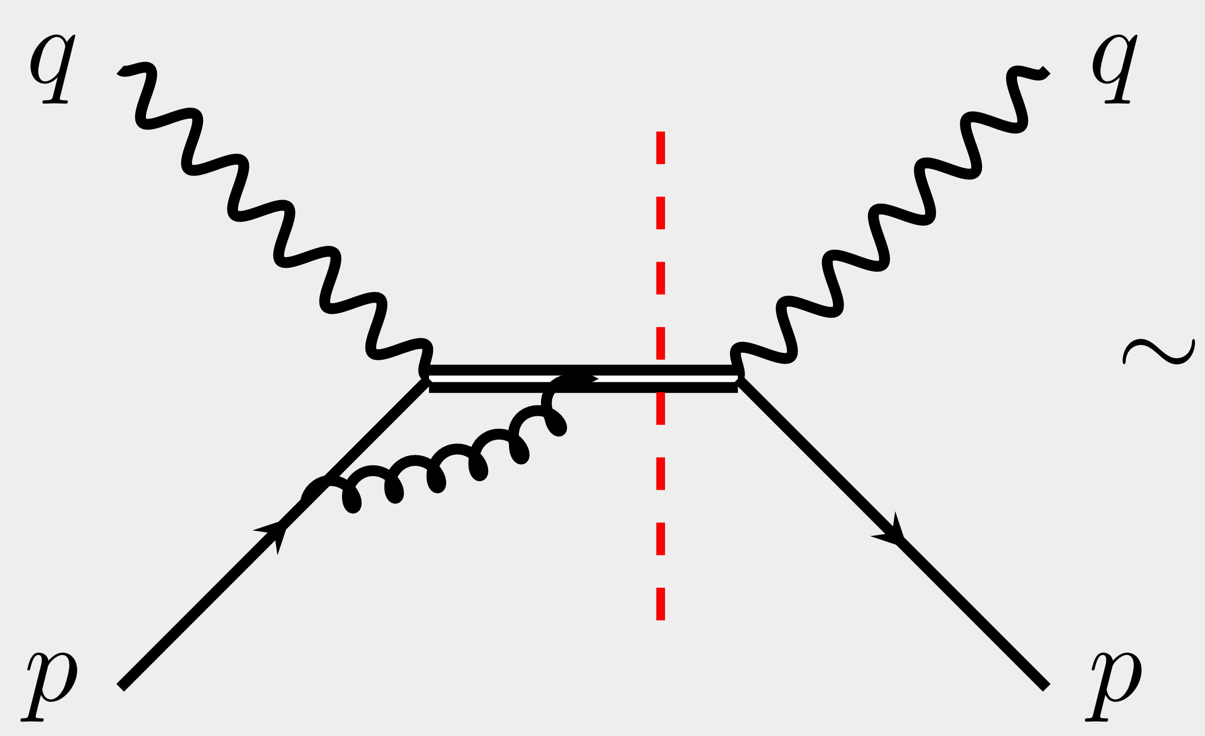

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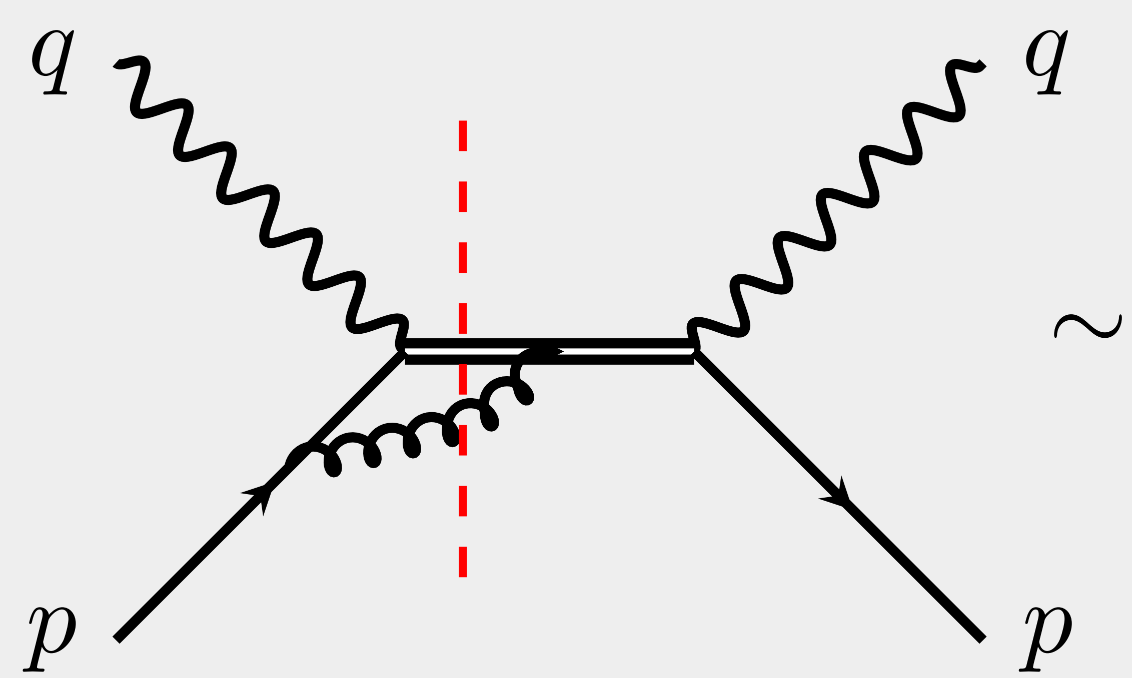


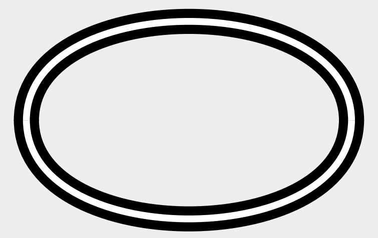
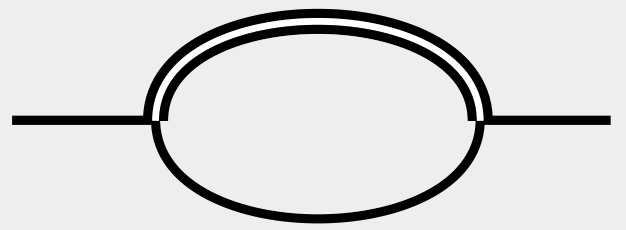
$$\sim \int [dk] \frac{\mathcal{N}_{\mu\nu}}{\mathcal{D}_1\not{\mathcal{D}}_2\not{\mathcal{D}}_3}$$

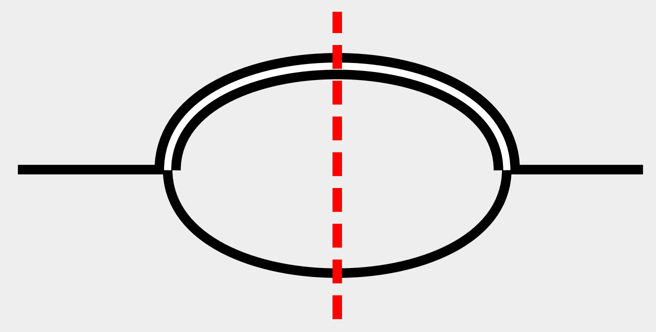
$$\tilde{I}_{\nu_1\nu_2\nu_3} \equiv \int [dk] \frac{1}{\mathcal{D}_1^{\nu_1}\not{\mathcal{D}}_2^{\nu_2}\not{\mathcal{D}}_3^{\nu_3}}$$

$$0 = \int [dk] \frac{\partial}{\partial k_i^{\mu}} \left[v^{\mu} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}} \right]$$

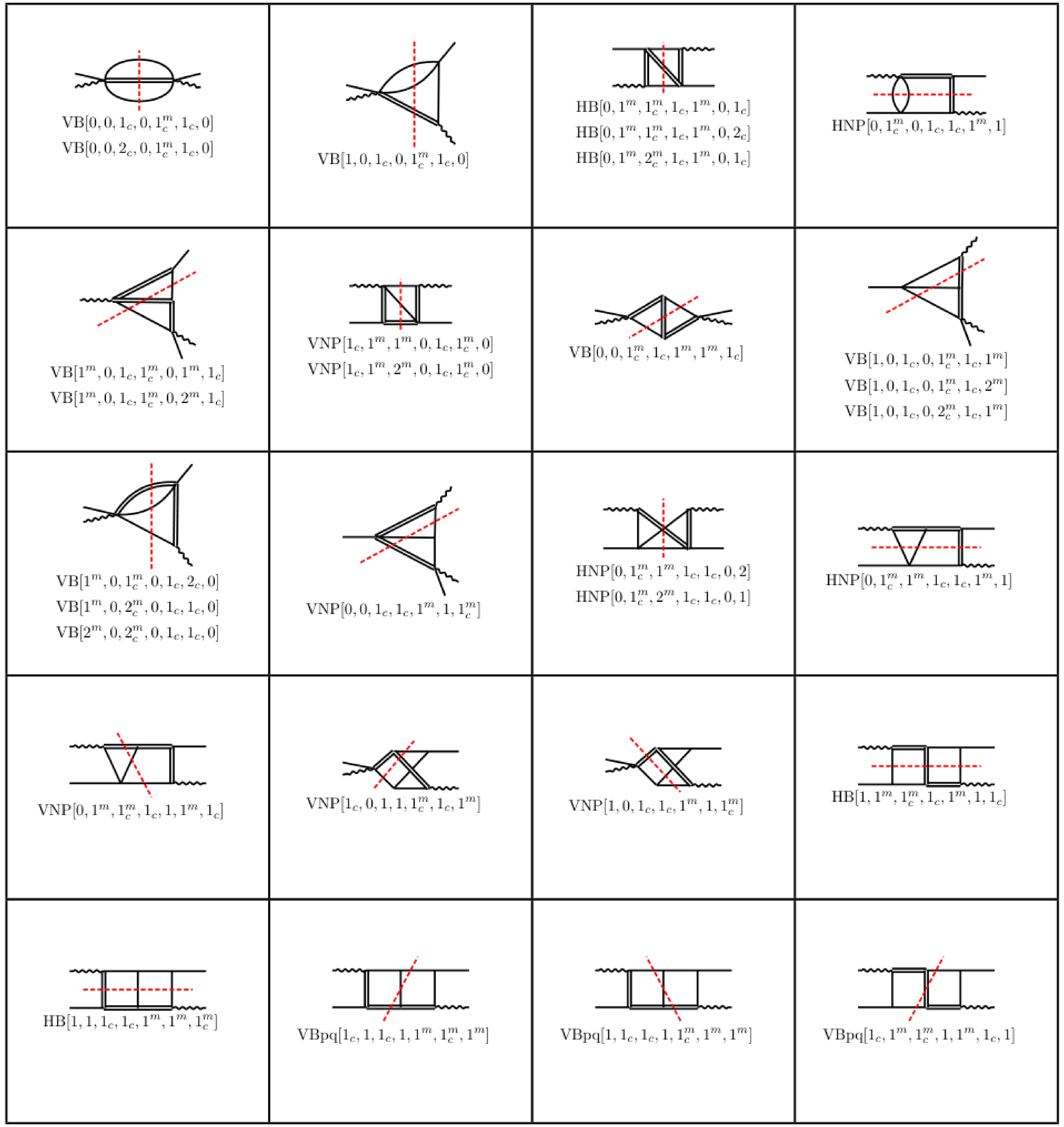
Rev. Unitarity

$$\frac{1}{p_i^2 - m_i^2 + i\epsilon} - \text{h.c.} \sim \delta_+(p_i^2 - m_i^2)$$

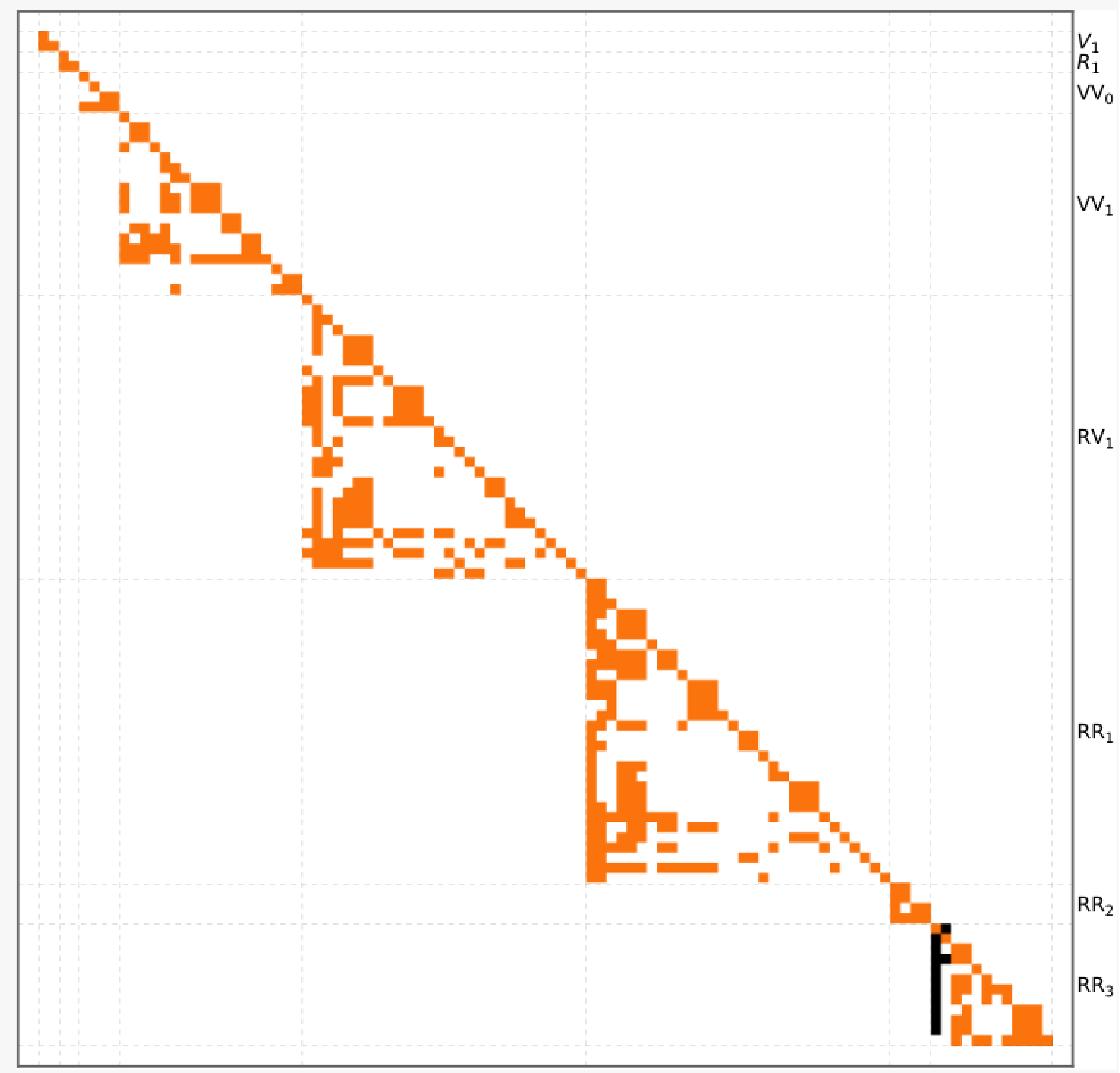
$C_1 \cdot$  $+ C_2 \cdot$ 

$\tilde{C} \cdot$ 

Finding canonical basis



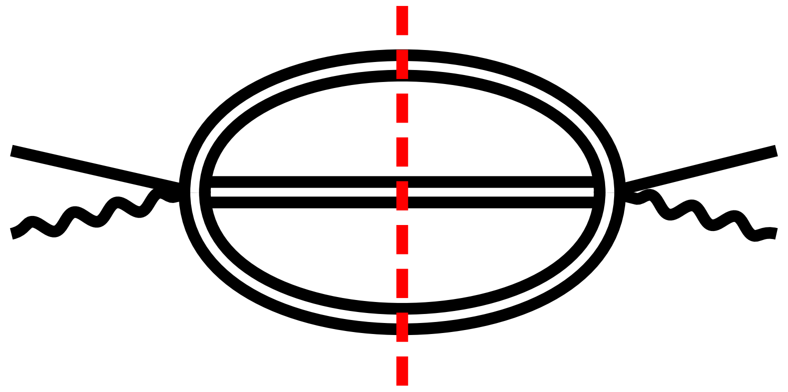
$$dg = \epsilon\cdot$$



$$\cdot g$$

- Look at leading singularities $\oint \frac{dz}{z-w}$
- Choose a "good" dot basis and dimension d to start
- First focus on hom. system
- Fix subsectors

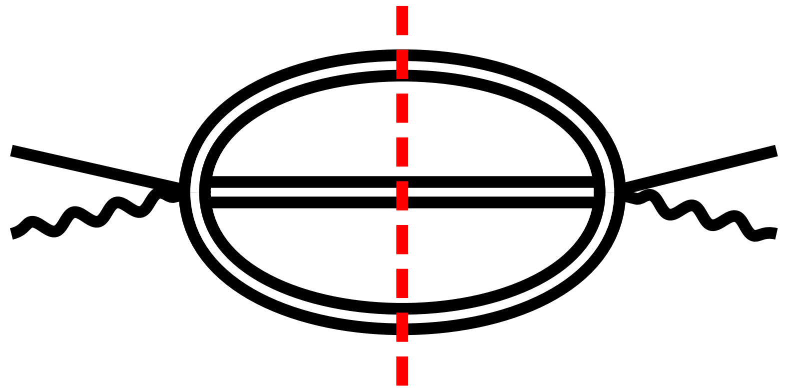
Canonicalisation of 3-mass-sunrise



- 2x2 system in $w \equiv \frac{m_c^2}{(p+q)^2}$
- Start with basis $(f, \partial_w f)$

Tancredi et al, [2503.20655] and [2604.25270v1]

Canonicalisation of 3-mass-sunrise



- 2x2 system in $w \equiv \frac{m_c^2}{(p+q)^2}$
- Start with basis $(f, \partial_w f)$

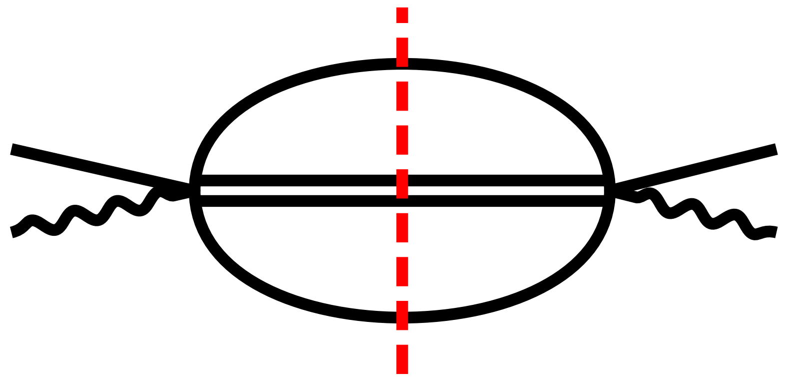
$$\partial_w \boldsymbol{f} = \left(\begin{pmatrix} 0 & 1 \\ \alpha(w) & \beta(w) \end{pmatrix} + \mathcal{O}(\epsilon) \right) \cdot \boldsymbol{f}$$

Solved in terms of Wronskian $\mathbf{W}(w) = \begin{pmatrix} w_0(w) & w_1(w) \\ w_0'(w) & w_1'(w) \end{pmatrix}$

$$w_i''(w) = \alpha(w) w_i(w) + \beta(w) w_i'(w)$$

Tancredi et al, [2503.20655] and [2604.25270v1]

Canonicalisation of 3-mass-sunrise



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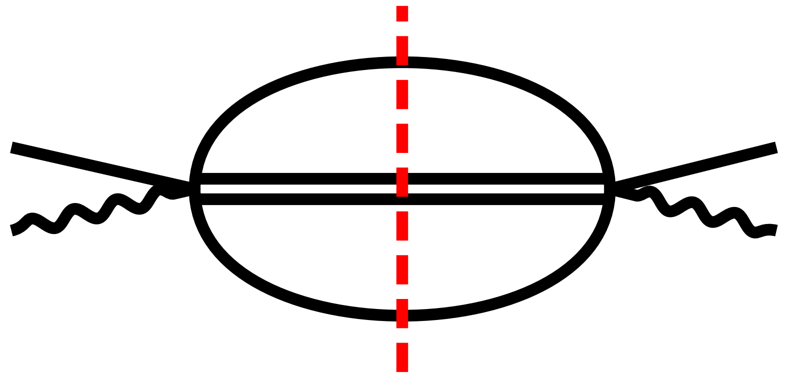
Solved in terms of Wronskian $\mathbf{W}(w) = \begin{pmatrix} w_0(w) & w_1(w) \\ w_0'(w) & w_1'(w) \end{pmatrix}$

$$w_i''(w) = \alpha(w) w_i(w) + \beta(w) w_i'(w)$$

Tancredi et al, [2503.20655] and [2604.25270v1]

Canonicalisation of 3-mass-sunrise



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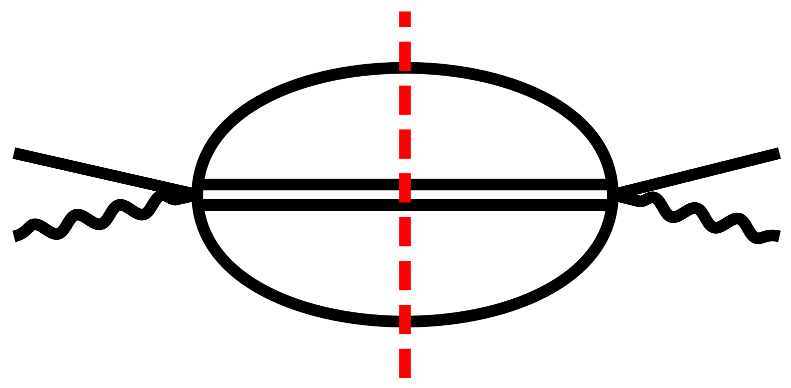

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Tancredi et al, [2503.20655] and [2604.25270v1]

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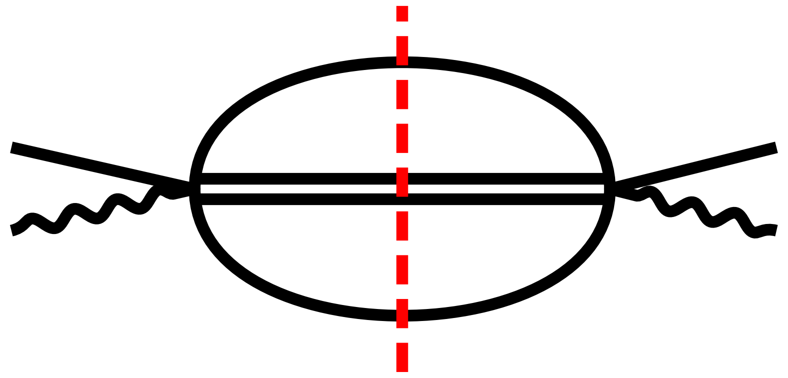

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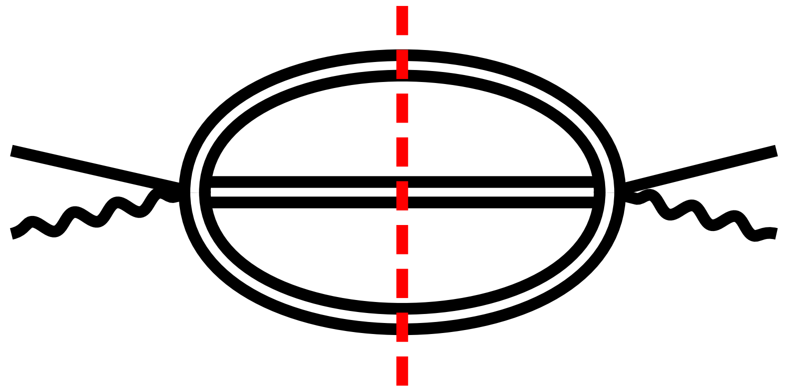
$$\mathbf{W}(w) = \underbrace{\mathbf{W}^{\mathrm{ss}}}_{\text{rational}} \cdot \underbrace{\mathbf{W}^{\mathrm{u}}}_{\sim \begin{pmatrix} 1 & \log \\ 0 & 1 \end{pmatrix}}$$

$$\boldsymbol{g} \equiv \begin{pmatrix} 1 & 0 \\ k(w) & 1 \end{pmatrix} \cdot \mathrm{Diag}(\epsilon, 1) \cdot (\mathbf{W}^{\mathrm{ss}})^{-1} \cdot \boldsymbol{f}$$

is canonical

Tancredi et al, [2503.20655] and [2604.25270v1]

Canonicalisation of 3-mass-sunrise



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Frobenius exp. at singular points (MUM) is the best we can do

$$\mathbf{W}(w) = \underset{\substack{\uparrow \\ \text{locally rational}}}{\mathbf{W}^{\text{ss}}} \cdot \underset{\substack{\uparrow \\ \text{locally}}}{\mathbf{W}^{\text{u}}} \sim \begin{pmatrix} 1 & \log \\ 0 & 1 \end{pmatrix}$$

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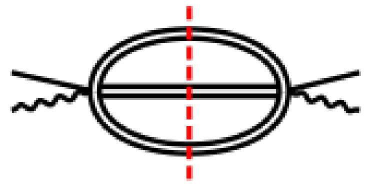
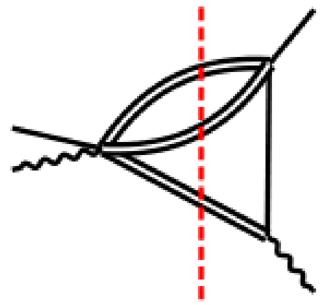
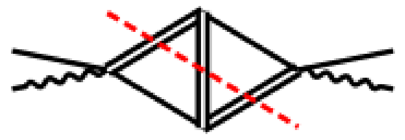
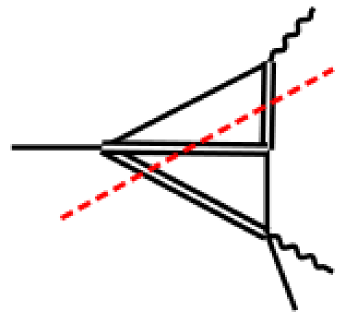
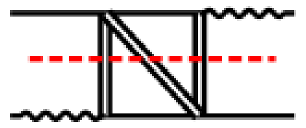
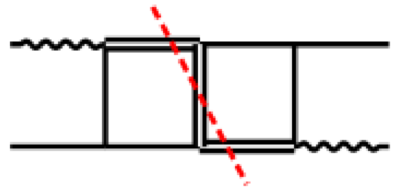
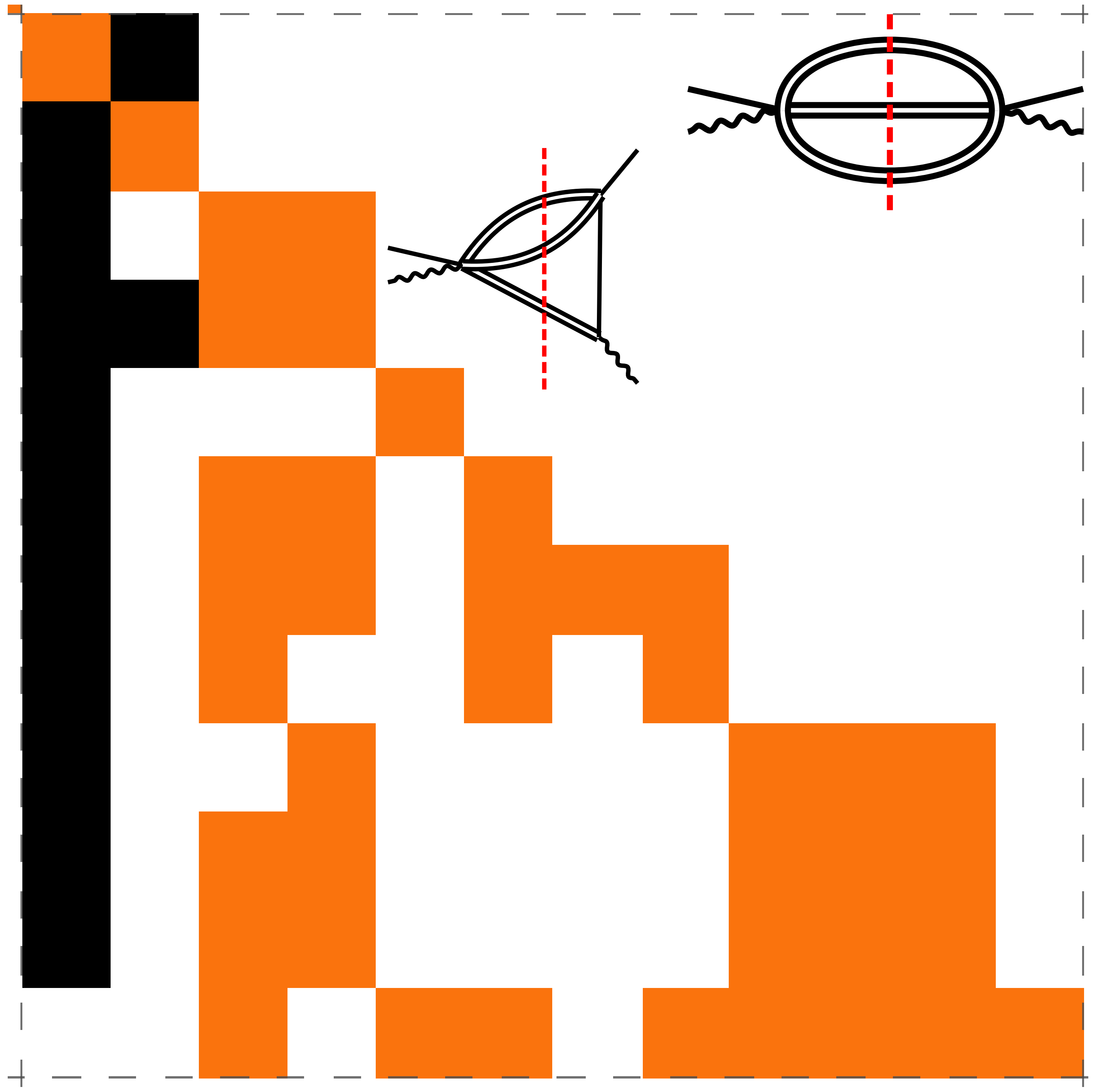
is canonical

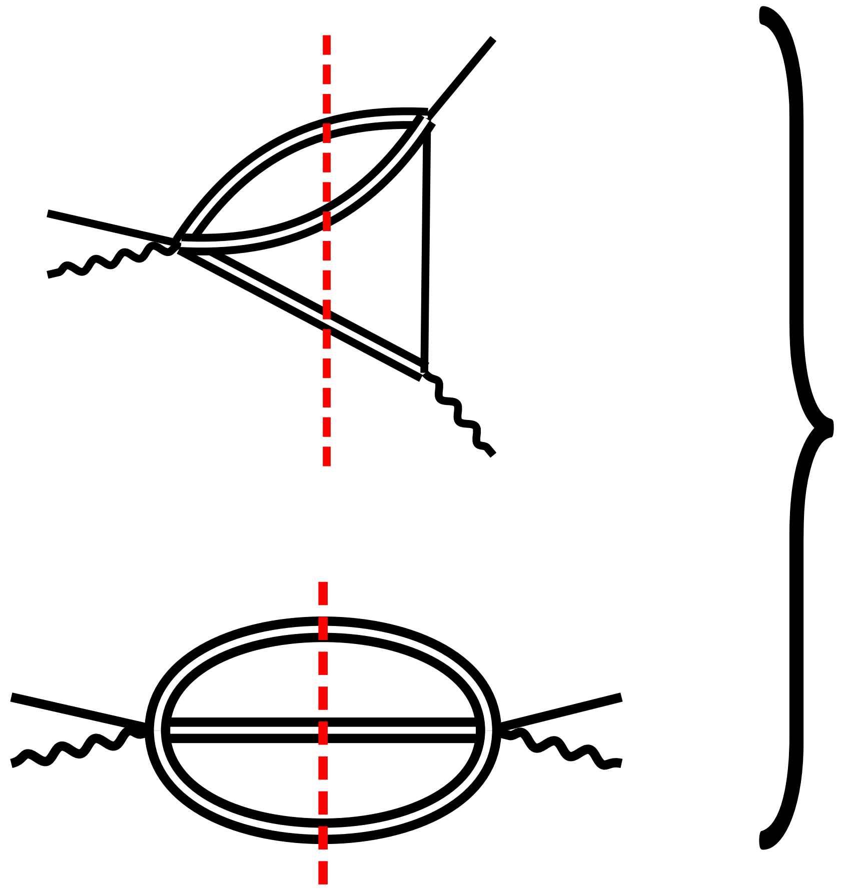
Leaves us with one implicit function $w_0(w)$ (or Frobenius expansion) for hom. sector

Tancredi et al, [2503.20655] and [2604.25270v1]

Canonicalisation of 3-mass-sunrise

 $\mathrm{VB}[0,0,1_c^m,0,1_c^m,1_c^m,0]$ $\mathrm{VB}[0,0,2_c^m,0,1_c^m,1_c^m,0]$	 $\mathrm{VB}[1,0,1_c^m,0,1_c^m,1_c^m,0]$ $\mathrm{VB}[1,0,2_c^m,0,1_c^m,1_c^m,0]$	 $\mathrm{VB}[0,0,1_c^m,1,1_c^m,1_c^m,1]$
 $\mathrm{VB}[1,0,1_c^m,0,1_c^m,1_c^m,1]$ $\mathrm{VB}[1,0,1_c^m,0,2_c^m,1_c^m,1]$ $\mathrm{VB}[1,0,2_c^m,0,1_c^m,1_c^m,1]$	 $\mathrm{VBpq}[0,1_c^m,1_c^m,1,1_c^m,0,1]$ $\mathrm{VBpq}[0,1_c^m,2_c^m,1,1_c^m,0,1]$ $\mathrm{VBpq}[0,2_c^m,1_c^m,1,1_c^m,0,1]$	 $\mathrm{VBpq}[1,1_c^m,1_c^m,1,1_c^m,1,1]$



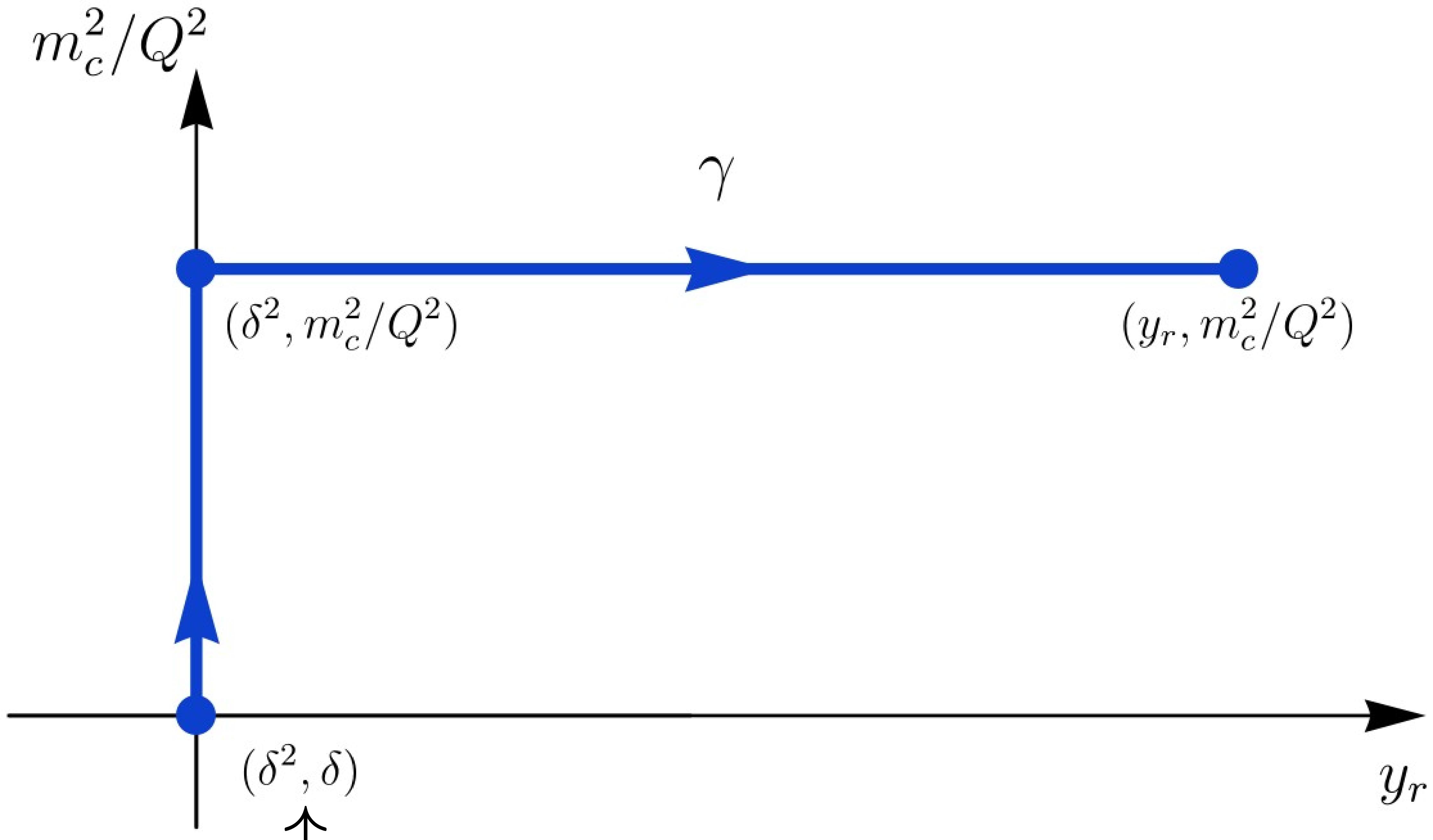


We need another implicit function to make the coupling to one supersector canonical
Define two-variable function via its differential (in terms of w_0)

Boundaries

$$\boxed{d\boldsymbol{f} = \epsilon \cdot \sum_i \underset{\substack{\uparrow \\ \text{45 ind. kernels}}}{dW_i} \mathbf{A}_i \cdot \boldsymbol{f}}$$

$$\boldsymbol{f} = \mathcal{P} \exp\left(\epsilon \int_\gamma d\mathbf{A} \right) \cdot \boldsymbol{f}_0$$



↑
Base point is threshold + massless

$$y_r = 1 - \left(1 + \frac{m_c^2}{Q^2}\right) x$$

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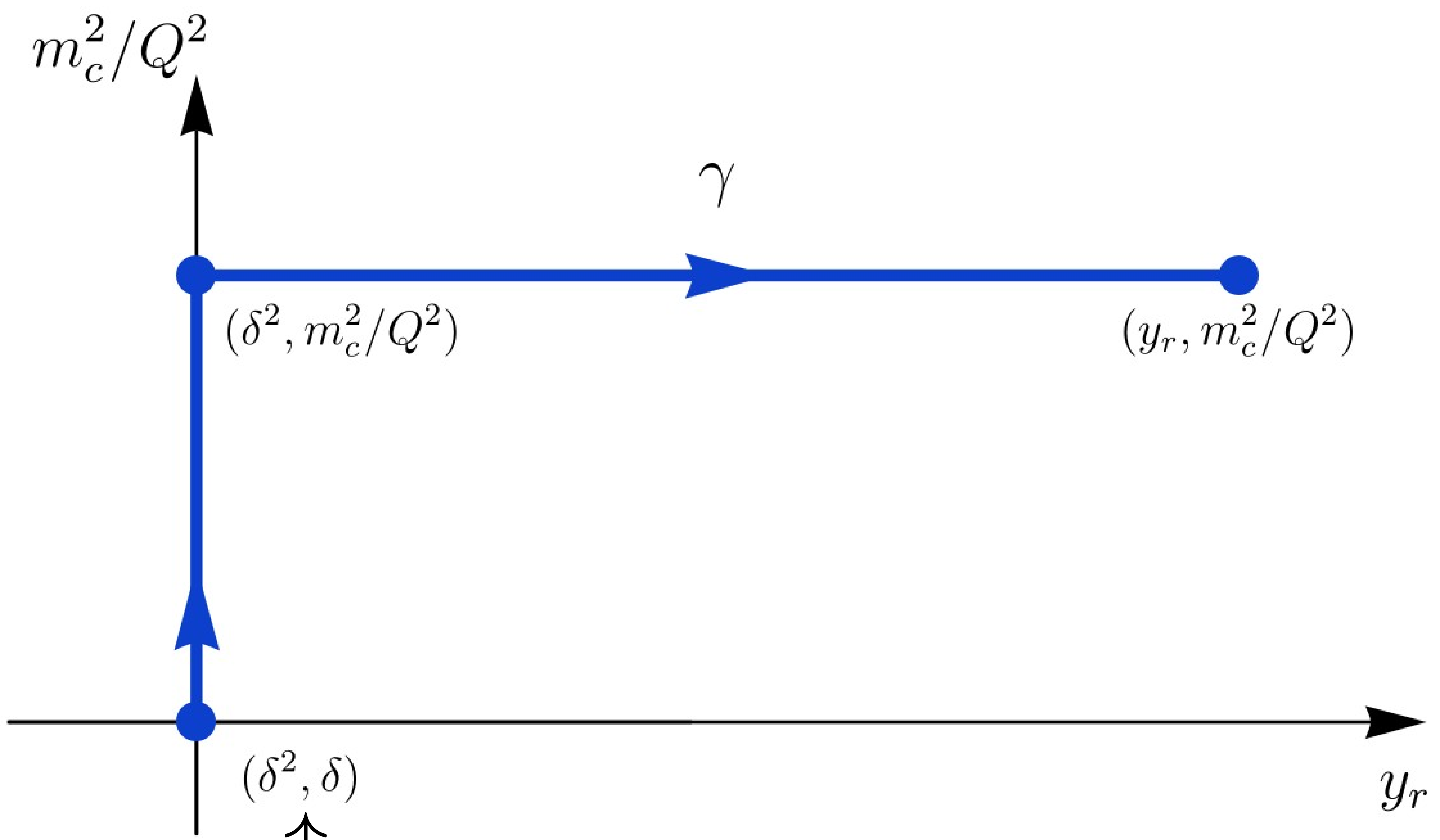

↑
Base point is threshold + massless

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🟥 Need to fix boundary

↳ Approximate close to base point

$$\boldsymbol{f}\left(y_r, \frac{m_c^2}{Q^2}\right) = \sum_{k,m=0}^{\infty} \sum_{l \leq k} \epsilon^k \, y_r^m \, \ln^l(y_r) \, \mathbf{g}_{k,m,l}\left(\frac{m_c^2}{Q^2}\right)$$

`AMFlow` ⇩ `PSLQ`

weight 1: π, $\ln(2)$,
weight 2: π^2, $\ln^2(2)$,
weight 3: π^3, $\pi^2 \ln(2)$, $\ln^3(2)$, ζ_3,
weight 4: π^4, $\pi^2 \ln^2(2)$, $\ln^4(2)$, $\mathrm{Li}_4(1/2)$, $\pi\, \zeta_3$, $\ln(2)\, \zeta_3$.

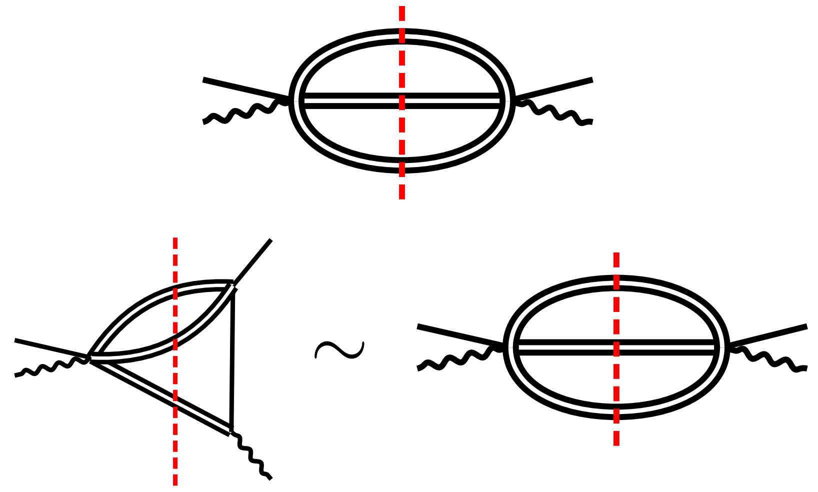
↳ Everything but RR_3 in terms of GPLs

What about elliptic kernels?

Now expand right away:

$$w_0(w) = \sum_{m=1} c_m^{(w_0)}\,(w-1/9)^m$$

$$g_{\mathrm{sc}}(w,z) = \sum_{m=0} c_m^{(\mathrm{sc})}(z)\,(w-1/9)^m$$



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`AMFlow` ⇩ `PSLQ`

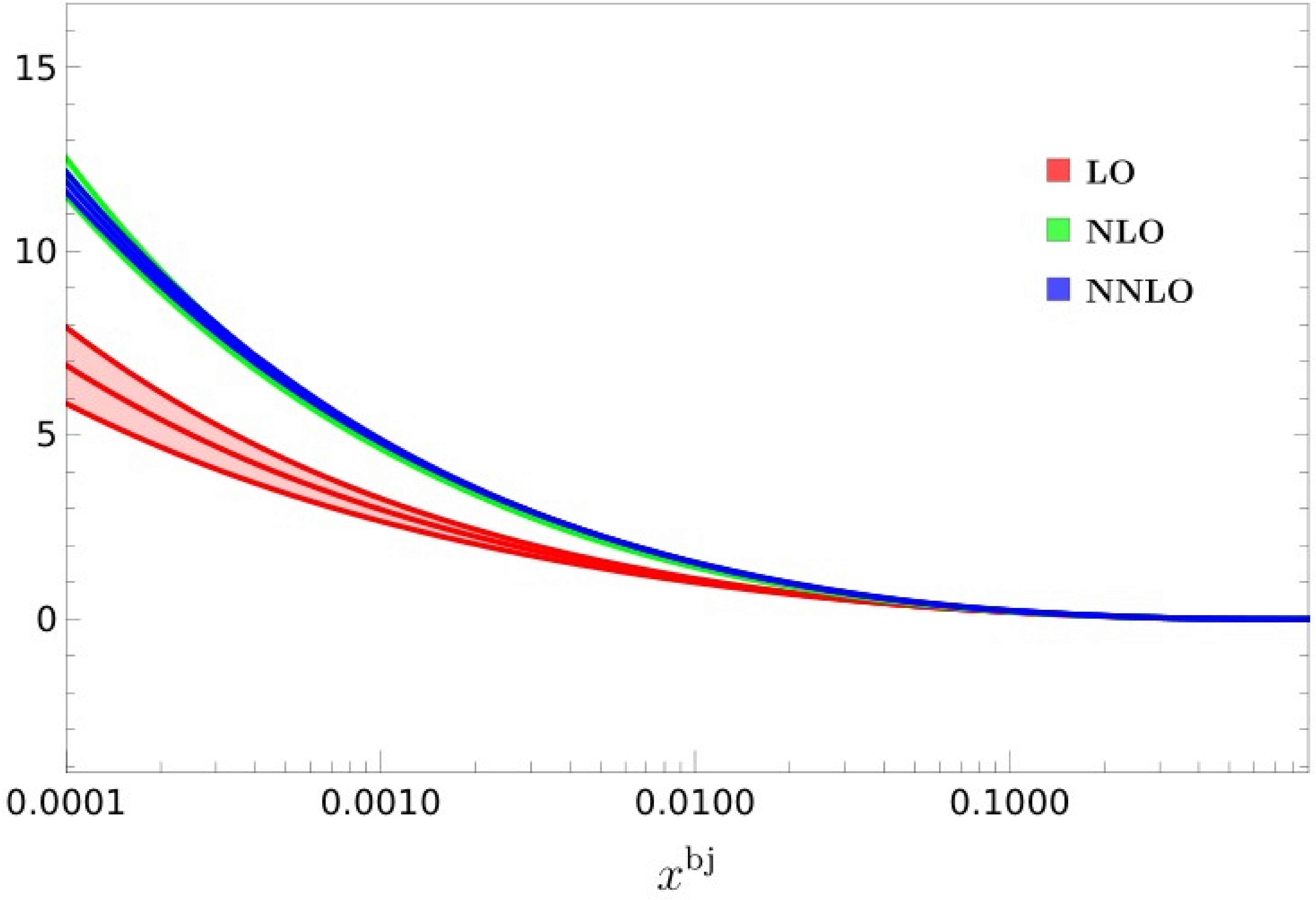
➡ Everything cancels, apart from $\sqrt{3}\pi$
We don't understand why

- To evaluate not at threshold, need expansion there
- Generally possible along the same lines, but: Constrained by the MUM points $w=0$ and $w=1/9$
- Expansions in the massless and the threshold region sufficed for our purposes

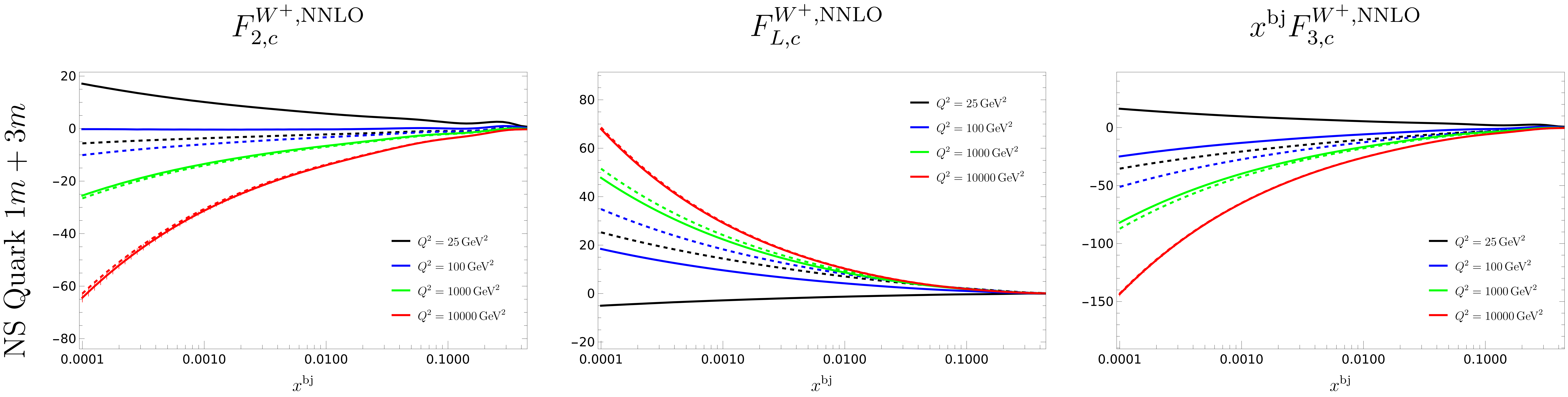
Diagrams to matrix element

The result

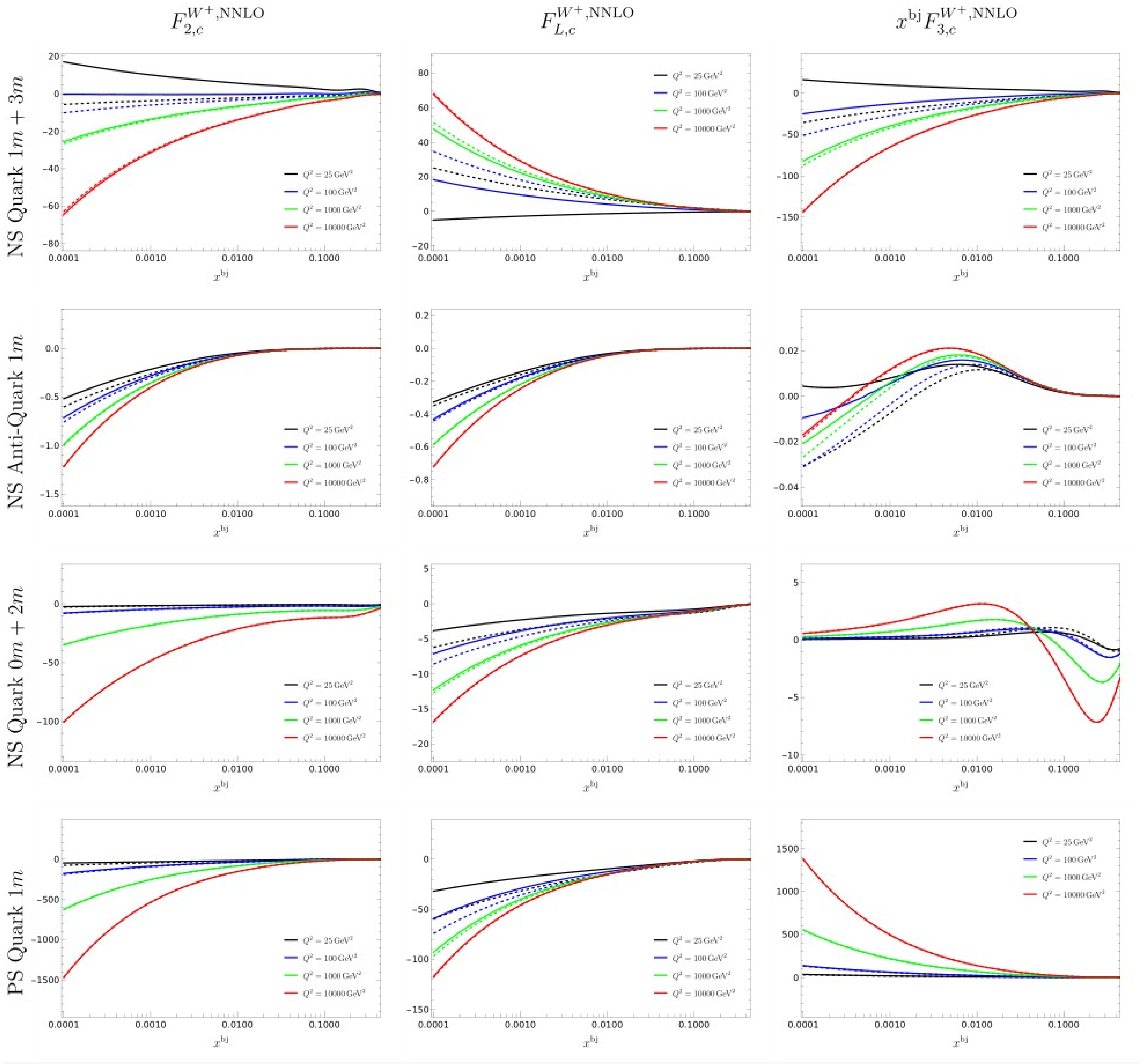
$$F_{2,c}^{W^+}$$



The result


$F_{2,c}^{W^+,\mathrm{NNLO}}$
$F_{L,c}^{W^+,\mathrm{NNLO}}$
$x^{\mathrm{bj}} F_{3,c}^{W^+,\mathrm{NNLO}}$
NS Quark $1m + 3m$
20
0
−20
−40
−60
−80
80
60
40
20
0
−20
0
−50
−100
−150
0.0001
0.0010
0.0100
0.1000
x^{bj}
$Q^2 = 25\,\mathrm{GeV}^2$
$Q^2 = 100\,\mathrm{GeV}^2$
$Q^2 = 1000\,\mathrm{GeV}^2$
$Q^2 = 10000\,\mathrm{GeV}^2$

The result


$F_{2,c}^{W^+,\text{NNLO}}$
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$x^{\text{bj}} F_{3,c}^{W^+,\text{NNLO}}$
NS Quark $1m + 3m$
NS Anti-Quark $1m$
NS Quark $0m + 2m$
PS Quark $1m$
$Q^2 = 25\,\text{GeV}^2$
$Q^2 = 100\,\text{GeV}^2$
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x^{bj}
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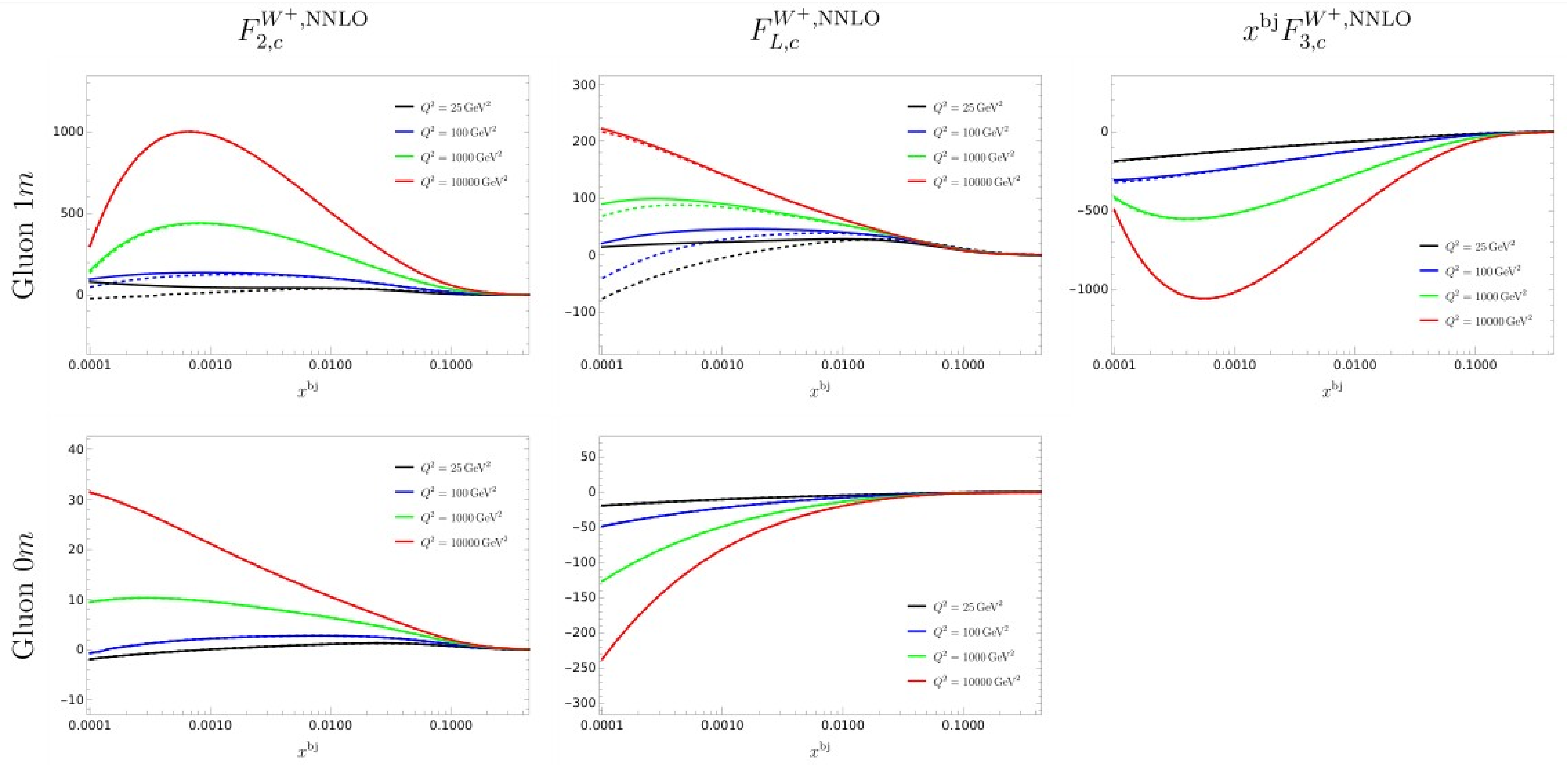
0.1000


$F_{2,c}^{W^+,\text{NNLO}}$
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$x^{\text{bj}} F_{3,c}^{W^+,\text{NNLO}}$
Gluon $1m$
Gluon $0m$
$Q^2 = 25\,\text{GeV}^2$
$Q^2 = 100\,\text{GeV}^2$
$Q^2 = 1000\,\text{GeV}^2$
$Q^2 = 10000\,\text{GeV}^2$
x^{bj}
0.0001
0.0010
0.0100
0.1000

Outlook

Next steps

Pheno

- ? Actual impact on strange fits
- ? ν – DIS

Theory

- ? Soft resummation
- ? Calculate NLP in $\mathcal{O}(m_c^2/Q^2)$
- ? Understand small x corrections

The Charm of Charged Currents

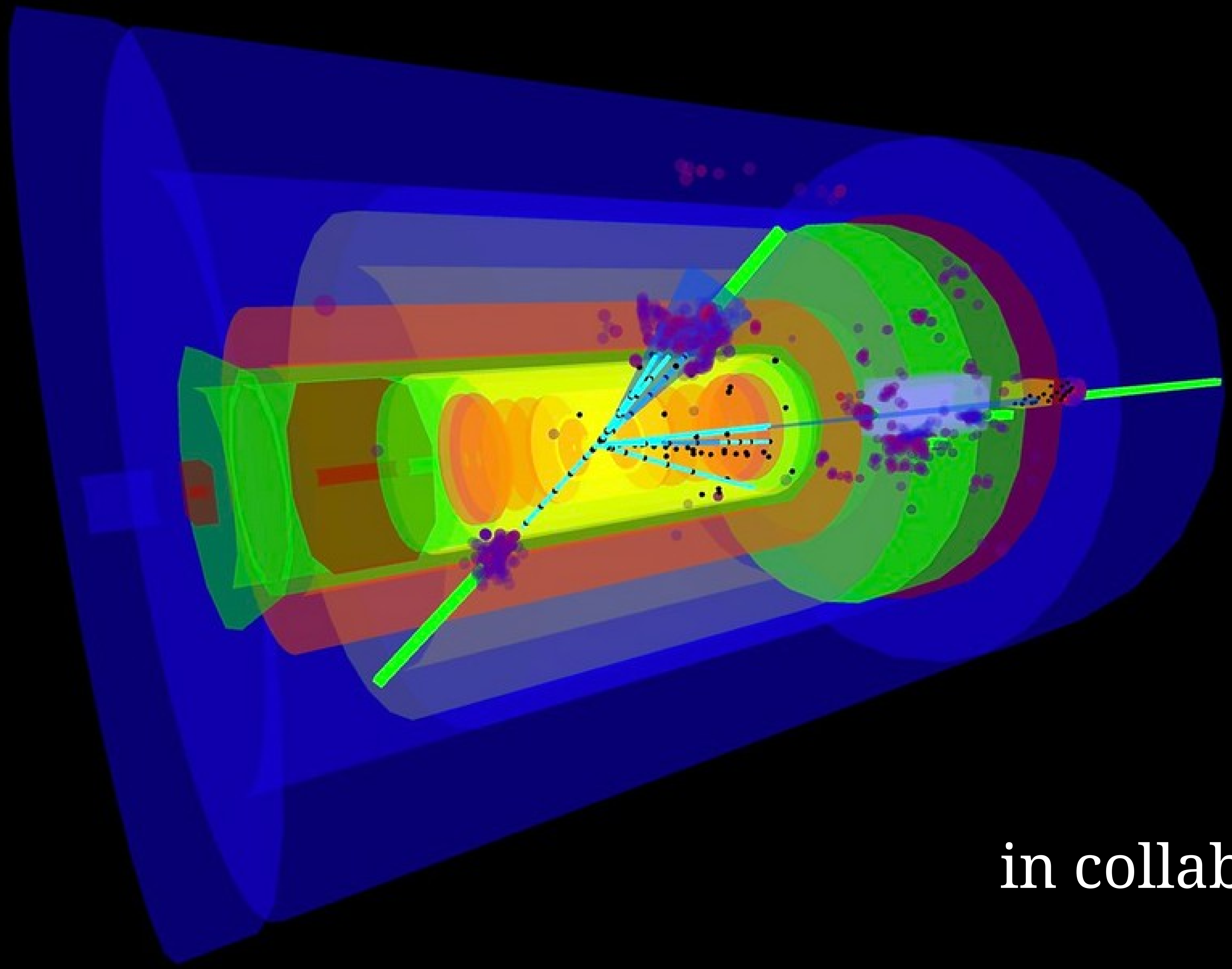
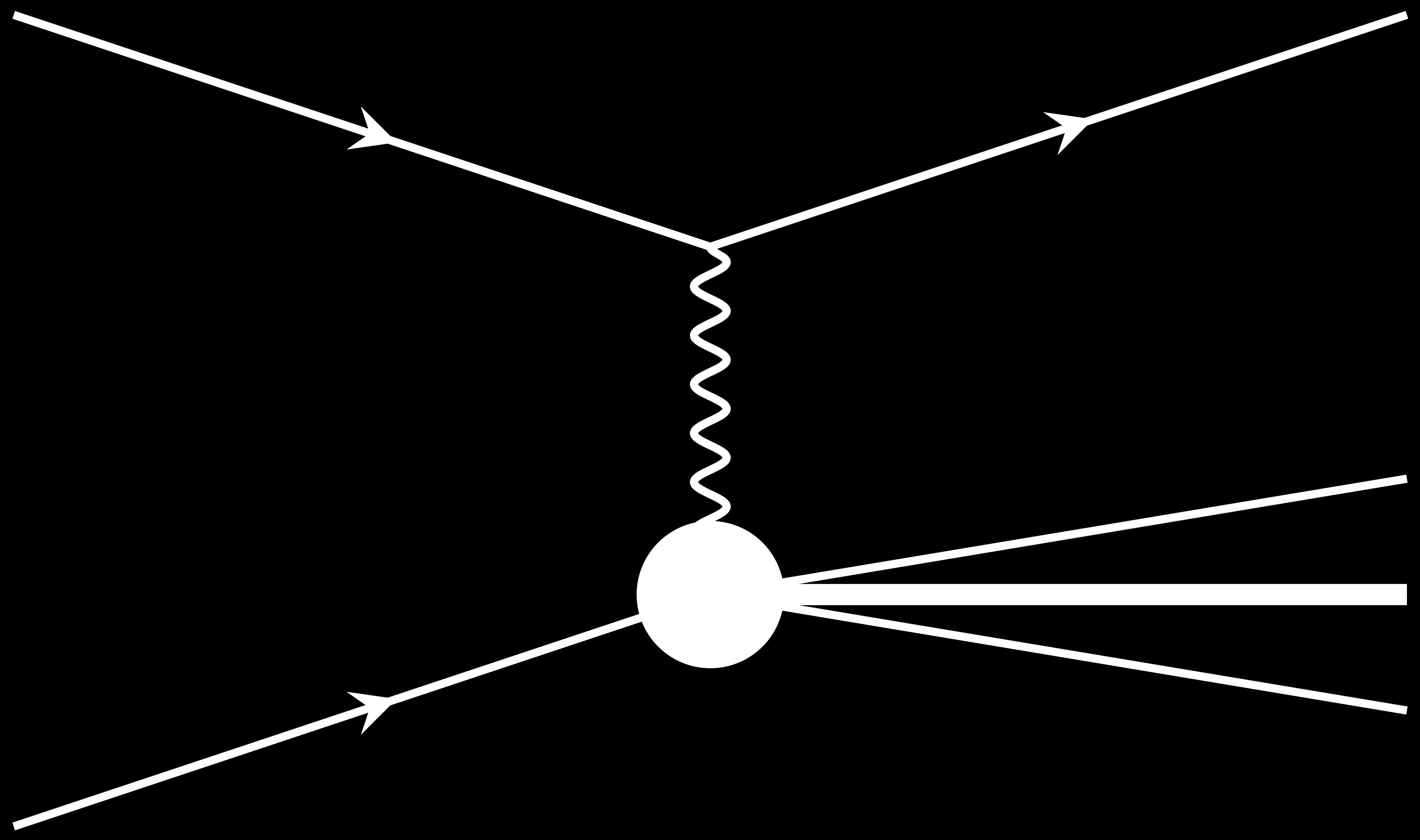


Image from
www.bnl.gov/eic/



Pushing massive DIS to NNLO

in collaboration with F. Caola and G. Gambuti

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