The Fermi Function, Factorization and the Neutron's Lifetime

LoopFest 2026

Peter Vander Griend based on Phys.Lett.B 868 (2025) 139678, Phys.Rev.D 112 (2025) 11, 113006 and 2511.05446

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Introduction

why (neutron) beta decay?

- background: discovery of parity violation and the neutrino
- historical developments:
  - **Fermi interaction** to describe decay
  - **Fermi function** to account for long distance corrections
- status today: precision observable
  - **experiment**[1]: $\tau_n = 877.82(30)\mathrm{s}$ (sub permille precision)
  - **theory**: probe of fundamental parameter of Standard Model $|V_{ud}|$ and, therefore, CKM unitarity

[1] arXiv:2409.05560 (Musedinovic, et.al.)

Fermi function[2]

- radiative corrections: large Z, small electron velocity β corrections treated using Fermi function
- neutron beta decay: Z small and β large
- corrections to Γ_n:

$$\delta\Gamma_n = 1 + 4.6\alpha + 16\alpha^2 + 35\alpha^3$$

α^2 term beyond permille level experimental precision and expansion not controlled beyond order α

What replaces the Fermi function for neutron beta decay and how can we get to permille level uncertainty?

[2] Z.Phys. 88 (1934) 161-177

Method

- EFT: widely separated energy scales $\Longrightarrow$ effective field theories
- factorization: calculate objects in sequence of EFTs
- resummation: QFT analog of Fermi function for neutron beta decay associated with renormalization group resummation
- neutron lifetime: long-distance corrections from product of factorized contributions; combine with short-distance corrections to get $\Gamma_{n\to pe\bar{\nu}}$, τ_n and $|V_{ud}|$

Physical Setup

energy scales and parameters

- nucleon mass $M \sim 1$ GeV
- nucleon mass difference $M_N - M_P = \Delta \gtrsim 1$ MeV
- electron mass $m \lesssim 1$ MeV
- electron energy E: $m \leq |E| \leq \Delta$
- electron velocity $\beta = \sqrt{1 - \frac{m^2}{E^2}}$: $0 < \beta < 0.92$

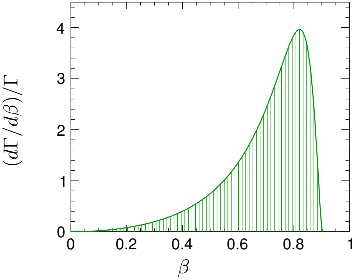
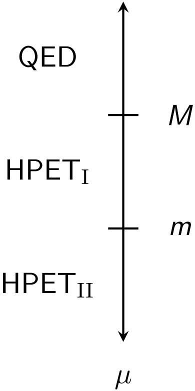


Figure: Tree level neutron decay rate as a function of electron velocity β. Electron velocity not small, i.e., Fermi function (valid at small β) doesn't apply.

Heavy Particle EFTs



- heavy particle effective theory (HPET): considering a system with characteristic energy scales below a particle mass, one can integrate the heavy particle out
- heavy particles described by two-component spinors, no pair creation

Neutron beta decay Lagrangian

- below the nucleon mass scale M, neutron beta decay is described by a four point effective interaction[3]

$$\mathcal{L}_{\rm eff} = -\bar{h}_v^{(p)} \left(\mathcal{C}_V \gamma^\mu + \mathcal{C}_A \gamma^\mu \gamma_5\right) h_v^{(n)} \bar{e} \gamma^\mu (1-\gamma_5) \nu_e + \text{H.c.}$$

- leading order in M^{-1} expansion (with Δ fixed)
- $h_v^{(n,p)}$ two-component spinor which annihilates neutrons, proton with velocity label v
- $\mathcal{C}_{V,A} = -\sqrt{2} G_F V_{ud} g_{V,A}$ matching coefficients encoding hadronic structure

[3] Phys. Rev. Lett. 133 (2024) 2, 021803 (Hill, Plestid)

Factorization

matrix element factorizes

$$\mathcal{M} \sim \mathcal{M}_{\mathrm{UV}}(\mu_{\mathrm{UV}}^2)\mathcal{M}_H(\mu_{\mathrm{UV}}^2, \mu^2)\mathcal{M}_S(\mu^2)$$

- μ_{UV} and μ are factorization scales separating UV, hard and soft regions; $\mathcal{M}$ is $\mu_{(\mathrm{UV})}$ independent
- $\mathcal{M}_{\mathrm{UV}}$ related to $\mathcal{C}_{V,A}$
- for small soft cutoff, $\mathcal{M}_S$ exponentiates, i.e., is known to all orders from a one-loop calculation
- $\mathcal{M}_H$ extractable from calculations in the literature

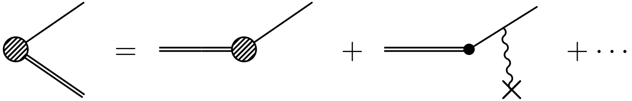
One-loop analysis

- focusing on the kinematics, we rewrite the hard amplitude as

$$\mathcal{M}_H(w) = \mathcal{M}_H(-w) + [\mathcal{M}_H(w) - \mathcal{M}_H(-w)]$$

where $v^\mu = (1,0,0,0)$ in the nucleon rest frame, p is the electron momentum and $w = v \cdot p/m$

- this corresponds to the sum of a spacelike process ($\nu_e \overline{p} \to \overline{n} e^-$) plus all possible insertions of a $Z = +1$ background field[4]



[4] Phys. Rev. D 109 (2024) 11, 113007 (Borah, Hill, Plestid); JHEP 07 (2024) 216 (Plestid)

π enhancements at one loop

- amplitude is function of logs and polylogs of w
- spacelike amplitude $\mathcal{M}_H(-w)$ has kinematic divergence as $w \to \infty$ but is well behaved for $w \sim 1$
- timelike amplitude $\mathcal{M}_H(w)$ contains factors $\sim \pi^2$ even in regime without kinematic enhancements

$$\mathcal{M}_H(w) - \mathcal{M}_H(-w) \supset \frac{\alpha}{2\pi}\left[\frac{\mathrm{i}\pi w}{\sqrt{w^2-1}}\left(\log\left(\frac{-4\boldsymbol{p}^2 - \mathrm{i}0}{\mu^2}\right) - 1\right)\right]$$

- such large π enhancements can be eliminated by setting $\mu^2 = -4\boldsymbol{p}^2 - \mathrm{i}0$

idea: resum large π's by evaluating $\mathcal{M}_H$ at $\mu^2 = -4\boldsymbol{p}^2$

Renormalization analysis

- evolution of $\mathcal{M}_H(\mu^2)$ given by

$$\frac{\mathcal{M}_H(\mu^2)}{\mathcal{M}_H(\hat{\mu}^2)} = \exp\left[\frac{\alpha}{2\pi}\left(-1+\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta}\right)\log\frac{\mu^2}{\hat{\mu}^2}\right]\exp\left[\frac{-i\alpha}{2\beta}\log\frac{\mu^2}{\hat{\mu}^2}\right].$$

- for $\mu^2 = 4\boldsymbol{p}^2$ and $\hat{\mu}^2 = -4\boldsymbol{p}^2 - i0$, $\log\frac{\mu^2}{\hat{\mu}^2} = +i\pi$,first exponential is a phase and hard function ($H(\mu^2) = |M_H(\mu^2)|^2$) receives a numerical enhancement of

$$\frac{H(4\boldsymbol{p}^2)}{H(-4\boldsymbol{p}^2 - i0)} = \exp\left[\frac{\pi\alpha}{\beta}\right]$$

- enhancements associated with Fermi function due to renormalization group resummation; differs from non-relativistic Fermi function at order

$$F_{\rm NR} = \frac{\frac{2\pi\alpha}{\beta}}{1-\exp\left(\frac{-2\pi\alpha}{\beta}\right)} = 1+\frac{\pi\alpha}{\beta}+\frac{1}{3}\frac{\pi^2\alpha^2}{\beta^2}+\ldots$$

Factorization

- for hierarchy $M > m$, hard-soft factorization:

$$|\mathcal{M}|^2 \sim H(\mu^2)S(\mu^2)$$

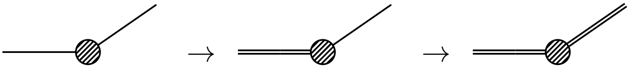
  - hard function: $H(\mu^2)$ encodes physics below the scale M
  - soft function: $S(\mu^2)$ encodes physics below the scale m
- for hierarchy $E > m$, hard-jet-remainder factorization:

$$H(\mu^2) = |F_H(E,\mu)|^2 |F_J(m,\mu)|^2 |F_R(w,m,\mu)|^2$$

  - $F_H(E,\mu)$ encodes physics below M and above m
  - $F_J F_R(w,m,\mu)$ encodes physics of the scale m

Diagramatic factorization

- how to get $H(\mu^2)$ and $S(\mu^2)$?
- consider a sequence of effective field theories



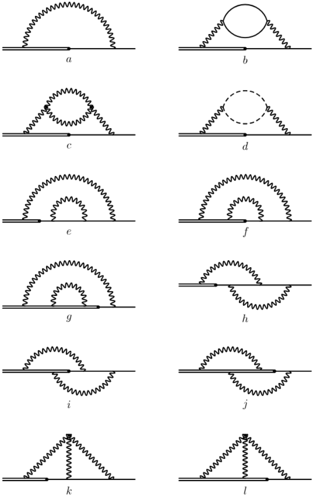
in which the mass M of the nucleon and m of the electron are successively integrated out

HQET to QCD matching[5]

matching coefficient for heavy-light current between QCD and HQET known to two loops

$$C(\mu) = 1 + \frac{\overline{\alpha}}{4\pi}\left(\frac{3}{2}L_M - 4\right) + \cdots$$

where $L_M = \log \frac{M^2}{\mu^2}$



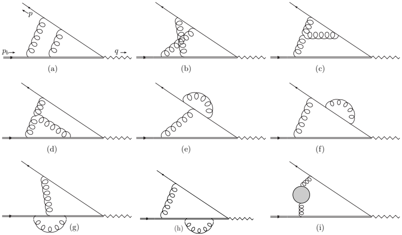
[5] Phys. Rev. 52 (1995) 4082-4098 (Broadhurst, Grozin)

Heavy-light form factors in QCD[6]

heavy-light form factors (for massless light particle) known to two loops

$$\lim_{M\to\infty} F_1 = 1 + \frac{\overline{\alpha}}{4\pi}\left(-2L_E^2 + 2L_E + \frac{3}{2}L_M - 6 - \frac{5\pi^2}{12}\right) + \cdots$$

where $L_E = \log 2E/\mu$



[6] Nucl. Phys. B 811 (2009) 77-97 (Beneke, Huber, Li)

Hard amplitude

- ratio $\lim_{M\to\infty} F_1$ to $C(\mu)$ gives

$$F_H(-E,\mu) = 1 + \frac{\overline{\alpha}}{4\pi}\left[-2L_E^2 + 2L_E - 2 - \frac{5\pi^2}{12}\right] + \cdots$$

  where $\overline{\alpha}$ is the $n_e = 1$ flavor $\overline{\mathrm{MS}}$ coupling given in terms of on-shell α as

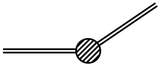
$$\overline{\alpha} = \alpha\left(1 - \frac{4n_e}{3}\frac{\alpha}{4\pi}L_m + \dots\right)$$

  where $L_m = \log\frac{m^2}{\mu^2}$

- $F_H(-E,\mu)$ related to the spacelike amplitude in a theory below the nucleon mass ($M\to\infty$) and above the electron mass ($m\to 0$)

Soft function

- $F_H(-E, \mu)$ encodes physics below M and above m
- F_S related to amplitude in a theory in which m is integrated out, i.e., soft function encodes physics below m

$F_S \sim$ 

- $F_S = 1$ in dim. reg. as all integrals are scaleless; with photon mass regulator

$$F_S = \exp\left[\frac{\alpha}{4\pi}\left(2(L-1)\log\frac{\lambda^2}{\mu^2}\right)\right]$$

where $L = \log 2w$

Resummation

- $F_H(-E, \mu) = F_H(E, -\mu)$ computed for spacelike kinematics; need to resum from $-\mu - i0$ to μ
- large logarithms ($\sim \pi$) spoil naive power counting
- define ($\hat{\mu} = -\mu - i0$)

$$\log \frac{\hat{\mu}^2}{\mu^2} = \int_{\alpha(\mu^2)}^{\alpha(\hat{\mu}^2)} \frac{d\alpha}{\beta(\alpha)} = -2i\pi = -X_*$$

and assign power counting

$$|X_*| \sim \alpha^{-\frac{1}{4}}$$

- factors $\alpha^3 X_*^4$ numerically relevant at order α^2

Resummation

- renormalization group evolution of F_H given by

$$\frac{dF_H(\mu)}{d\mu} = \gamma_{\text{cusp}} \log \frac{-2E}{\mu} + \gamma^{\psi} + \gamma^{h}$$

  where the terms are the massive cusp anomalous dimension and the light and heavy particle anomalous dimensions, respectively
- solve the renormalization group running including π-enhanced term of order α^3

$$\left|\frac{F_H(E,\mu=2E)}{F_H(-E,\mu=2E)}\right|^2 = \left|\frac{F_H(E,\mu=2E)}{F_H(E,-\mu=2E)}\right|^2$$

$$= \exp\left[-X_*^2\frac{\overline{\alpha}}{4\pi} + \frac{32}{9}n_e X_*^2\left(\frac{\overline{\alpha}}{4\pi}\right)^2 - \frac{8}{27}n_e^2 X_*^4\left(\frac{\overline{\alpha}}{4\pi}\right)^3 + \dots\right],$$

Neutron lifetime

- decay rate factorizes and expressed in terms of tree rate as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E} = \left(\frac{\mathrm{d}\Gamma}{\mathrm{d}E}\right)_{\mathrm{tree}} S(\varepsilon_\gamma, \mu^2)\, H(\varepsilon_\gamma, \mu^2)$$

  where ε_γ is a soft-photon energy cutoff which cancels order by order between S and H

- tree level phase space given by

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}E}\right)_{\mathrm{tree}} \propto E\sqrt{E^2 - m^2}(\Delta - E)^2$$

Outer corrections

	$S(\mu^2)H(\mu^2)$			$S(-\mu^2)H(-\mu^2)$		
1	0.3	$\pm$ 3.5	$\pm$ 2.1	34.5	$\pm$ 3.6	$\pm$ 2.2
$1 + H_V^{(1)}$	32.6	$\pm$ 0.1	$\pm$ 2.2	33.2	$\pm$ 0.004	$\pm$ 2.2
$1 + H^{(1)}$	28.8	$\pm$ 0.08	$\pm$ 0.05	29.32	$\pm$ 0.02	$\pm$ 0.01
$1 + H^{(1)} + H_V^{(2)}$	29.04	$\pm$ 0.05	$\pm$ 0.05	29.31	$\pm$ 0.02	$\pm$ 0.01

Table: Long-distance radiative correction to Γ_n in units of 10^{-3}. Columns computed at timelike (left) and spacelike (right) renormalization scale. Central values evaluated at $\mu^2 = m\Delta$, $\Lambda_\gamma = \Delta$, and $\mu_{\mathrm{UV}} = \Delta$ (where Λ_γ parametrizes uncertainty due to imposing cancelation of ε_γ in $H^{(2)}$); errors denote scale variation $\mu = m/2..2\Delta$, and $\Lambda_\gamma = \Delta/2..2\Delta$.

Resummation of π enhancements in H leads to better convergence.

Neutron decay width

- neutron decay width given by

$$\Gamma_n = \frac{G_F^2 |V_{ud}|^2 \Delta^5}{2\pi^3} f_{\rm static} (1 + 3\lambda^2) \Big[1 + \Delta_R(\mu_{\rm UV}) \Big] \times \Big[1 + \delta_{R,\rm static} + \delta_{\rm recoil} + \delta_{\rm rad.rec.} \Big]$$

- $f_{\rm static} = 0.0157528$
- $\lambda = g_A/g_V = -1.27641(56)$[7]
- $\Delta_R = g_V^2 - 1 = 45.37(27) \times 10^{-3}$[8]

Quantity	Value $[10^{-3}]$
Δ_R	45.37 ± 0.27
$\delta_{R,\rm static}$	29.18 ± 0.07
$\delta_{\rm recoil}$	-2.06
$\delta_{\rm rad.rec.}$	-0.08

[7] Phys. Rev. Lett. 122 (2019) 24, 242501 (Märkisch, et.al.)
[8] Phys. Rev. D 108 (2023) 5, 053003 (Cirigliano, Dekens, Mereghetti, Tomalak)

Neutron lifetime

- neutron lifetime given by

$$\tau_n \times |V_{ud}|^2(1+3\lambda^2)\Big[1+\Delta_R(\mu_{\rm UV}=\Delta)\Big]\Big[1+27.04(7)\times 10^{-3}\Big]$$
$$=\frac{2\pi^3\hbar}{G_F^2\Delta^5 f_{\rm static}}=5263.284(17)\,\mathrm{s}$$

- taking the most recent UCNτ average for the neutron lifetime $\tau_n = 877.82(30)\,\mathrm{s}$ gives

$$|V_{ud}| = 0.97393(17)_\tau(35)_\lambda(13)_{\Delta_R}(3)_{\delta_R}$$
$$= 0.97393(41)$$

Conclusions

- first $\mathcal{O}(\alpha^2)$ input to the long-distance radiative corrections to neutron beta decay beyond the Fermi function ansatz
- Fermi function replaced by renormalization group running
- results include leading contributions and uncertainties from power corrections and real radiation at two loop order and all relevant recoil and radiative recoil corrections
- complete analysis of the two-loop virtual corrections in the limit of small m^2/Δ^2
- updated, precise determination of $|V_{ud}|$

Thank you!

Backup Slides

HPET Propagator

- consider a nonrelativistic heavy fermion of momentum $p = Mv + k$ in the limit $M \to \infty$

$$
\begin{aligned}
\frac{i\left(\not{p} + M\right)}{p^2 - M^2 + i\epsilon} &= i\frac{M\left(1 + \not{v}\right) + \not{k}}{2Mv \cdot k + k^2 + i\epsilon} \\
&= \frac{1 + \not{v}}{2}\frac{i}{v \cdot k + i\epsilon} + \mathcal{O}\left(\tfrac{1}{M}\right) \\
&\to \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} \frac{i}{v \cdot k + i\epsilon} = \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} \frac{i}{k_0 + i\epsilon}
\end{aligned}
$$

- what Lagrangian yields such a propagator?

HPET Lagrangian[9]

- HPET Lagrangian:

$$\mathcal{L}_{HPET} = \psi^\dagger i\partial_0 \psi$$

where ψ is a nonrelativistic 2-component spinor which annihilates a heavy particle

- derived rigorously at the Lagrangian level by field redefinitions

$$\mathcal{L}_\psi = \psi^\dagger \Big\{ iD_0 + \frac{c_k}{2M}\mathbf{D}^2 + \frac{c_4}{8M^3}\mathbf{D}^4 + \frac{c_F}{2M}\sigma \cdot g\mathbf{B} + \frac{c_D}{8M^2}\left(\mathbf{D}\cdot g\mathbf{E} - g\mathbf{E}\cdot\mathbf{D}\right) + i\frac{c_S}{8M^2}\sigma\cdot\left(\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times\mathbf{D}\right)\Big\}\psi$$

where $E_i = F_{i0}$ and $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$ are electric and magnetic fields and c are Wilson coefficients

[9] Phys.Lett.B 167 (1986) 437-442 (Caswell, Lepage), Phys.Rev.D 51 (1995) 1125-1171 (Bodwin, Braaten, Lepage)

Full HPET Lagrangian

$$\mathcal{L}_{\text{HPET}} = \mathcal{L}_A + \mathcal{L}_l + \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi\chi}$$

- $\mathcal{L}_A$: gauge fields, $\mathcal{L}_l$: light particles, $\mathcal{L}_\psi$: heavy particle, $\mathcal{L}_\psi^c$: heavy antiparticle, $\mathcal{L}_{\psi\chi}$: fourpoint interactions
- antiparticle described by charge conjugate spinors: $\psi^c = -i\sigma^2\chi^*$ where χ is a 2 spinor which creates a heavy antiparticle
- imaginary part of $\mathcal{L}_{\psi\chi}$ related to decay width of bound states; no pair creation or annihilation after integrating out M

Amplitude structure

- hard amplitude has struture

$$\mathcal{M}_H(w,\mu^2) = \mathcal{A}_H(w,\mu^2) + \frac{\not{v}}{w}\mathcal{B}_H(w,\mu^2)$$

- simple one-loop calculation yields result

$$\mathcal{A}_H(-w) = 1 + \frac{\alpha}{2\pi}\left[\frac{3}{4}\log\frac{\mu_{\rm UV}^2}{m^2} + \log\frac{\mu^2}{m^2}\left[wj(w)-1\right] + wj(w) - wJ(w)\right]$$
$$\mathcal{B}_H(-w) = \frac{\alpha}{2\pi}\left[-wj(w)\right]$$

where

$$wj(w) = \frac{w}{\sqrt{w^2-1}}\log\left(w+\sqrt{w^2-1}\right),$$
$$wJ(w) = \frac{w}{\sqrt{w^2-1}}\left[\mathrm{Li}_2(1-(w-\sqrt{w^2-1})^2) + \log^2\left(w+\sqrt{w^2-1}\right)\right].$$

Large logs

- spacelike amplitude contains kinematically large logarithms

$$\lim_{w\to\infty} A_H(-w) = 1 + \frac{\alpha}{2\pi}\left[\frac{3}{4}\log\frac{\mu_{\rm UV}^2}{m^2} + \log\frac{\mu^2}{m^2}(\log(2w)-1) + \log(2w) - \frac{\pi^2}{6} - \log^2(2w)\right]$$

$$\lim_{w\to\infty} B_H(-w) = \frac{\alpha}{2\pi}\left[-\log(2w)\right]$$

- which are absent in $w \to 1$ limit

$$\lim_{w\to 1} A_H(-w) = 1 + \frac{\alpha}{2\pi}\left[\frac{3}{4}\log\frac{\mu_{\rm UV}^2}{m^2} - 1\right],$$

$$\lim_{w\to 1} B_H(-w) = \frac{\alpha}{2\pi}(-1).$$

π enhancements

- timelike amplitude contains factors $\sim \pi^2$ even in regime without kinematic enhancements

$$\mathcal{A}_H(w) - \mathcal{A}_H(-w) = \frac{\alpha}{2\pi}\left[\frac{\mathrm{i}\pi w}{\sqrt{w^2-1}}\left(\log\left(\frac{-4\boldsymbol{p}^2 - \mathrm{i}0}{\mu^2}\right) - 1\right)\right]$$

$$\mathcal{B}_H(w) - \mathcal{B}_H(-w) = \frac{\alpha}{2\pi}\left[\frac{\mathrm{i}\pi w}{\sqrt{w^2-1}}\right]$$

- such large π enhancements can be eliminated by setting $\mu^2 = -4\boldsymbol{p}^2 - \mathrm{i}0$

idea: resum large π's by evaluating $\mathcal{M}_H$ at $\mu^2 = -4\boldsymbol{p}^2$

Anomalous dimension

- resum from negative to positive μ^2 using anomalous dimension of hard operator

$$M_H(w,\mu^2) = \exp\left[\int_{-\mu^2}^{\mu^2} d\log\mu\,\gamma_H\right] M_H(-w,-\mu^2)$$

- product $\mathcal{M}_H(\mu^2)\mathcal{M}_S(\mu^2)$ independent of μ^2
- scale dependence of soft matrix element given to all orders by cusp anomalous dimension

$$\frac{d\log\mathcal{M}_S(\mu^2)}{d\log\mu^2} = -\frac{\alpha}{2\pi}\left[-1+\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta}-\frac{i\pi}{\beta}\right]$$

Jet and remainder functions[10]

- jet function encodes physics at scale m

$$F_J(m,\mu) = 1 + \frac{\overline{\alpha}}{4\pi}\left(\frac{1}{2}L_m^2 - \frac{1}{2}L_m + 2 + \frac{\pi^2}{12}\right) + \cdots$$

- remainder function accounts for different number of dynamical fermions in two theories, i.e., diagram with fermion loop in photon propagator

$$F_R(-w,m,\mu) = 1 + \left(\frac{\bar{\alpha}}{4\pi}\right)^2 (\log(2w) - 1)\, n_e \left(-\frac{4}{3}L_m^2 - \frac{40}{9}L_m - \frac{112}{27}\right)$$

- product $F_R F_J$ related to matching from a theory with $n_e = 1$ electrons of mass m to a theory with $n_e = 0$
- product $F_H F_R F_J$ represents the matching coefficient onto theory in which m is integrated out, i.e., F_S

[10] Phys. Rev. D 95 (2017) 1, 013001 (Hill); see also references in arXiv:2501.17916

Hard function resummation

- decompose hard function as

$$H(\mu^2) = \left|\frac{F_H(\mu)}{F_H(-\mu)}\right|^2 |F_H(-\mu)|^2 \, |F_R(\mu)|^2 \, |F_J(\mu)|^2$$

- eliminate large-π enhancements by evaluating spacelike μ

$$H(-\mu^2) = \exp\left[-\frac{\pi\alpha}{\beta}\right] \left|\frac{F_H(\mu)}{F_H(-\mu)}\right|^2 |F_H(-\mu)|^2 \, |F_R(\mu)|^2 \, |F_J(\mu)|^2$$

- full amplitude independent of μ

$$\Gamma \sim S(-\mu^2) H(-\mu^2)$$

Hard function decomposition

- decomposition of hard function

$$H(\mu^2) = 1 + \frac{\alpha}{2\pi} H^{(1)} + \left(\frac{\alpha}{2\pi}\right)^2 H^{(2)} + \cdots$$
$$= \exp\left[\frac{\pi\alpha}{\beta}\right] H(-\mu^2)$$
$$= \left(1 + \frac{\pi\alpha}{\beta} + \frac{\pi^2\alpha^2}{2\beta^2}\right)\left[1 + \frac{\alpha}{2\pi}\hat{H}^{(1)} + \left(\frac{\alpha}{2\pi}\right)^2 \hat{H}^{(2)}\right] + \cdots$$

- real and virtual parts of $H^{(1)}$ known with full m dependence
- virtual part of $H^{(2)}$ known in limit $m \to 0$ from hard-remainder-jet factorization

Full one-loop results

- full, one-loop hard function

$$H_V^{(1)} = 3\log\frac{\mu_{\rm UV}}{m} + \log\frac{\mu}{m}\left(\frac{2}{\beta}\log\frac{1+\beta}{1-\beta} - 4\right) + \beta\log\frac{1+\beta}{1-\beta} + \frac{2\pi^2}{\beta} - \frac{2}{\beta}\mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{2\beta}\log^2\left(\frac{1+\beta}{1-\beta}\right)$$

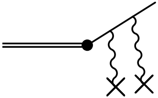
$$H_R^{(1)} = \log\frac{\varepsilon_\gamma}{\Delta - E}\left(4 - \frac{2}{\beta}\log\frac{1+\beta}{1-\beta}\right) + \frac{1}{\beta}\log\frac{1+\beta}{1-\beta}\left[\frac{(\Delta - E)^2}{12E^2} + \frac{2(\Delta - E)}{3E} - 3\right] - \frac{4(\Delta - E)}{3E} + 6$$

- soft function

$$\log S(\varepsilon_\gamma) = \frac{\alpha}{2\pi}\left[\log\frac{2\varepsilon_\gamma}{\mu}\left(\frac{2}{\beta}\log\frac{1+\beta}{1-\beta} - 4\right) - \frac{1}{2\beta}\log^2\left(\frac{1+\beta}{1-\beta}\right) - \frac{2}{\beta}\mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{\beta}\log\frac{1+\beta}{1-\beta} + 2\right]$$

Power corrections

- to estimate effect of small m expansion, we calculate the leading two-loop contribution



$$= -\frac{2\pi^4}{3}\frac{m^2}{E^2 - m^2}\left(\frac{\alpha}{2\pi}\right)^2$$

- which shifts the central value of the outer corrections $29.31 \to 29.18$; we assign an uncertainty of half this shift
- outer corrections given by

$$\delta_{R,\mathrm{static}} = (29.18 \pm 0.07 \pm 0.01 \pm 0.02) \times 10^{-3}$$