# Higher order corrections and PDF determinations

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**with**

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# Outline

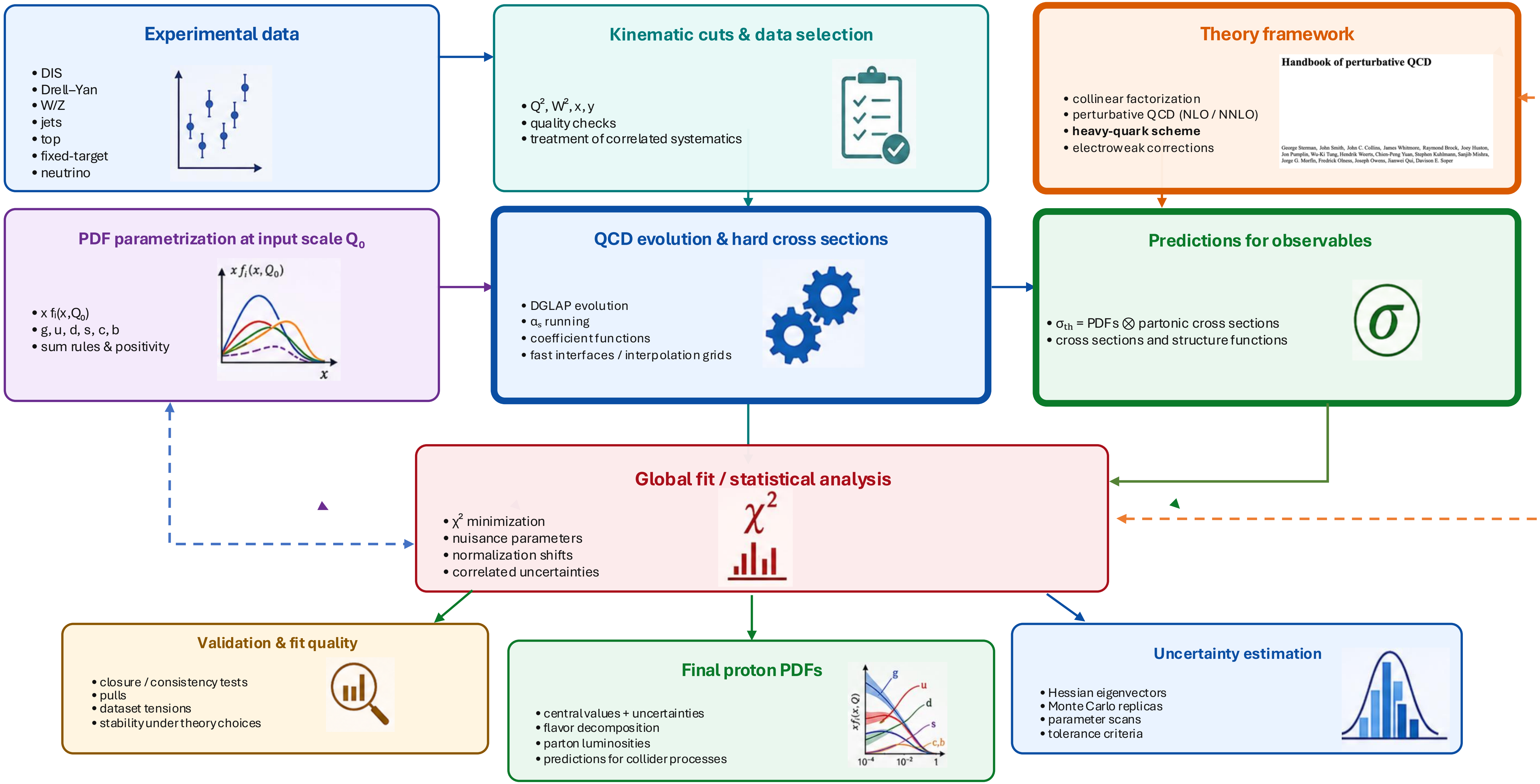
Part I

- Importance of higher orders in PDF fits
- PDFs at approximate or partial N3LO (aN3LO or pN3LO)
- DIS at $\mathcal{O}(\alpha_s^3)$: current approximations for the massive SFs

Part II

- PDF evolution at $\mathcal{O}(\alpha_s^4)$
- Impact from exact 4-loop Non-Singlet DGLAP kernels

# Ingredients of a Global QCD Analysis of Collinear PDFs



# Higher-order calculations in global QCD analyses

**QCD evolution & hard cross sections**

- DGLAP evolution
- $\alpha_s$ running
- coefficient functions
- fast interfaces / interpolation grids

- Progress in the 4-loop DGLAP NS-kernel calc. (Gehrmann, et al. 2604.09534)
- Operator Matrix Elements at 3-loop accomplished (Ablinger, et al. 2510.02175).
- **Singlet sector at 4-loop: work is in progress**
- DGLAP evolution codes publicly available with different approximations

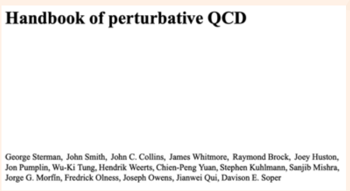
**Predictions for observables**

- $\sigma_{th}$ = PDFs $\otimes$ partonic cross sections
- cross sections and structure functions


- N3LO partonic cross section fully known only for selected processes, e.g., Duhr, Dulat, Mistlberger: pp → H + X, 1904.09990, pp → γ*/Z → ℓ+ℓ− + X 2001.07717, 2007.13313; Baglio et al.: pp → V H + X, V = W±, Z, 2209.06138, Chen, et al. DY dilepton rapidity 2107.09085.
- DIS Wilson Coeff. at $\mathcal{O}(\alpha_s^3)$ in the $Q^2 \gg m^2$ limit recently accomplished (Ablinger, et al. 2509.16124)
- **Massive DIS Wilson Coeff. at $\mathcal{O}(\alpha_s^3)$ in full kinematics still beyond reach**.
- DIS CC (Martin Link's talk on Wed.)

**Theory framework**

- collinear factorization
- perturbative QCD (NLO / NNLO)
- **heavy-quark scheme**
- electroweak corrections


**Handbook of perturbative QCD**

George Sterman, John Smith, John C. Collins, James Whitmore, Raymond Brock, Joey Huston, Jon Pumplin, Wu-Ki Tung, Hendrik Weerts, Chien-Peng Yuan, Stephen Kuhlmann, Sanjib Mishra, Jorge G. Morfin, Fredrick Olness, Joseph Owens, Jianwei Qui, Davison E. Soper

Different HF treatments are used by different PDF groups for the DIS process

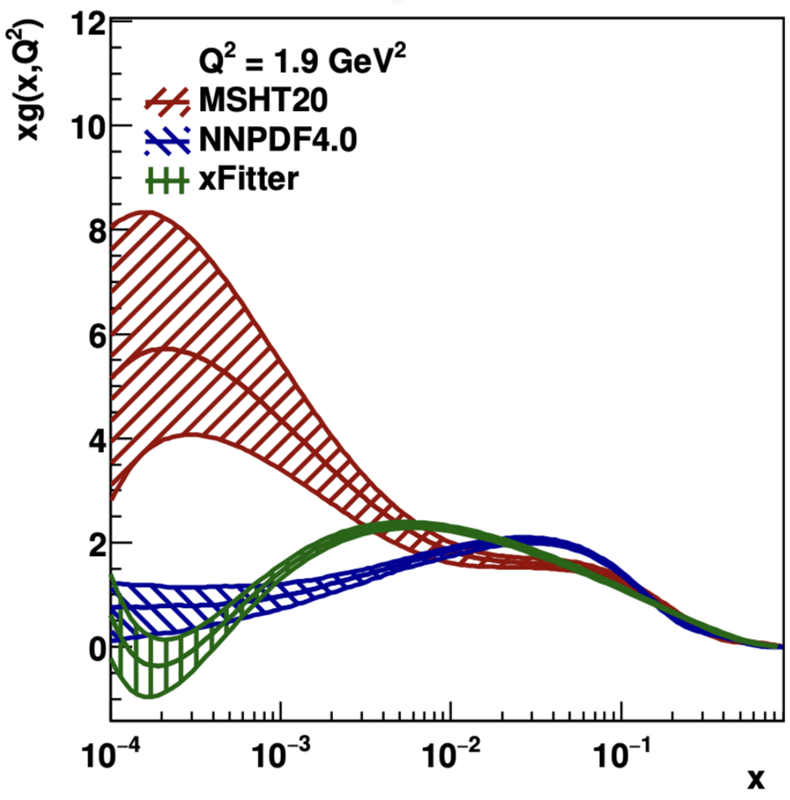
- CTEQ → ACOT-$\chi$
- MSHT → TR'
- NNPDF → FONLL
- ABMP → FFNS

# PDF determinations @aN3LO: where we stand

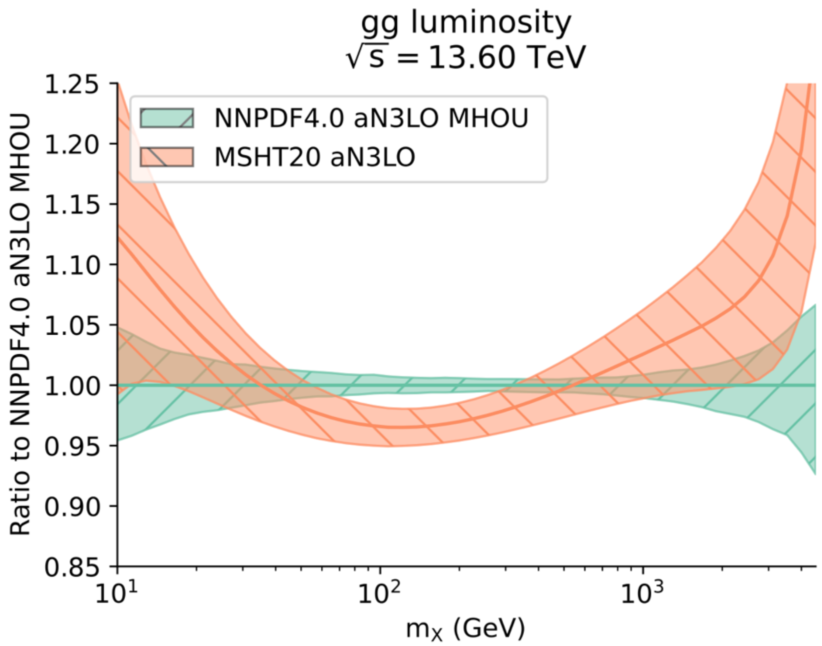
| | **MSHT** [EPJ C83 (2023) 185] | **NNPDF** [EPJ C84 (2024) 659] |
|---|---|---|
| splitting functions | linear combinations of as many interpolating functions, satisfying the known small-x, large-x, and large-nf limits, as the number of known Mellin moments; coefficients fixed by known Mellin moments<br><br>singlet: 4; non-singlet: 8 | linear combinations of as many interpolating functions, satisfying the known small-x, large-x, and large-nf limits, as the number of known Mellin moments; coefficients fixed by known Mellin moments<br><br>singlet: 5, 6 for $P_{ps}^{(3)}$; non-singlet: 8 |
| IHOUs | nuisance parameters (1 per splitting function, for a total of 5) fitted to the data | estimated by varying the basis of interpolation functions and by constructing a corresponding theory covariance matrix |
| transition elements | approximations constructed similarly to splitting functions (5 nuisance parameters in total) | exact inclusion of all transition elements but $A^{gg,H}$ parametrised similarly to splitting functions |
| DIS heavy quarks | massive coefficient functions determined from parametrisations combining known limits and damping functions | |
| hadronic K-factors | parametrised as linear combinations of NNLO and NLO K-factors (2 nuisance parameters per process for a total of 10) | replaced by MHOUs |
| QED effects | $O(\alpha)$, $O(\alpha\alpha s)$, $O(\alpha\hat{}2)$<br>MSHT20qed_an3lo with $\gamma$-PDF | $O(\alpha), O(\alpha\alpha s), O(\alpha\hat{}2)$<br>NNPDF4.0 aN3LO QCD+QED + γ-PDF +MHOU |

E. Nocera's talk at the LHC EW WG General Meeting, April 1, 2026

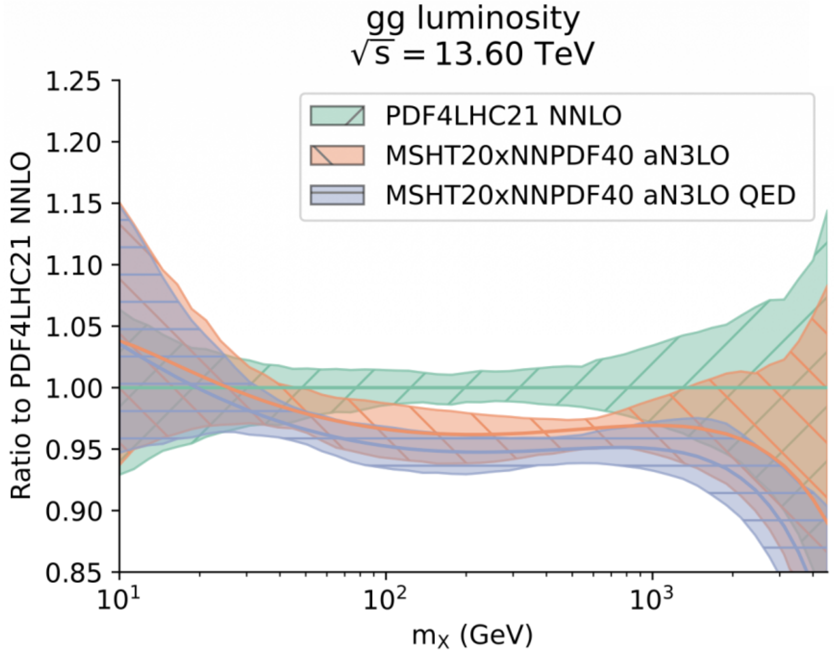
# PDF determinations @aN3LO: where we stand




Recent aN3LO PDF results from xFitter
(from F. Giuli's talk at DIS2026)




From NNPDF4.0 N3LO 2402.18635

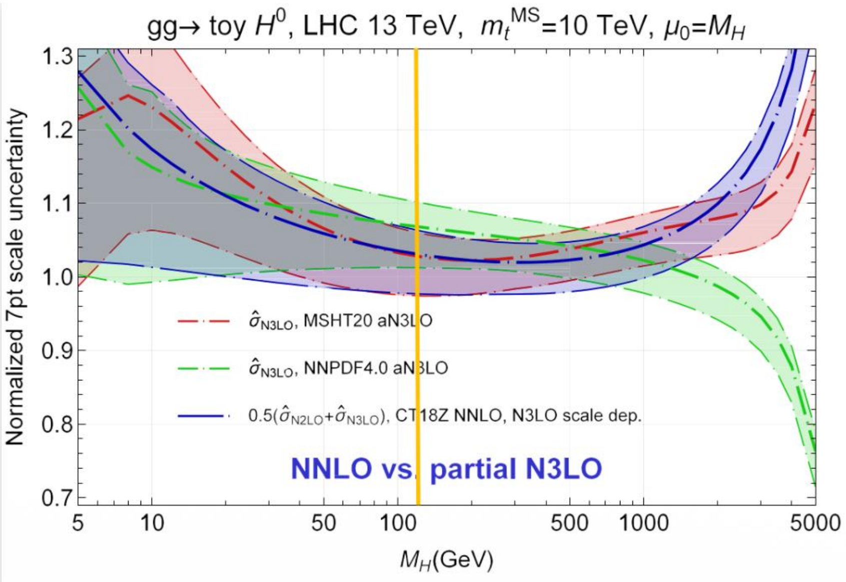



MSHT and NNPDF4.0 statistical combination performed as in PDF4LHC21 [JPG 52 (2025) 065002]

# PDF determinations @aN3LO: where we stand

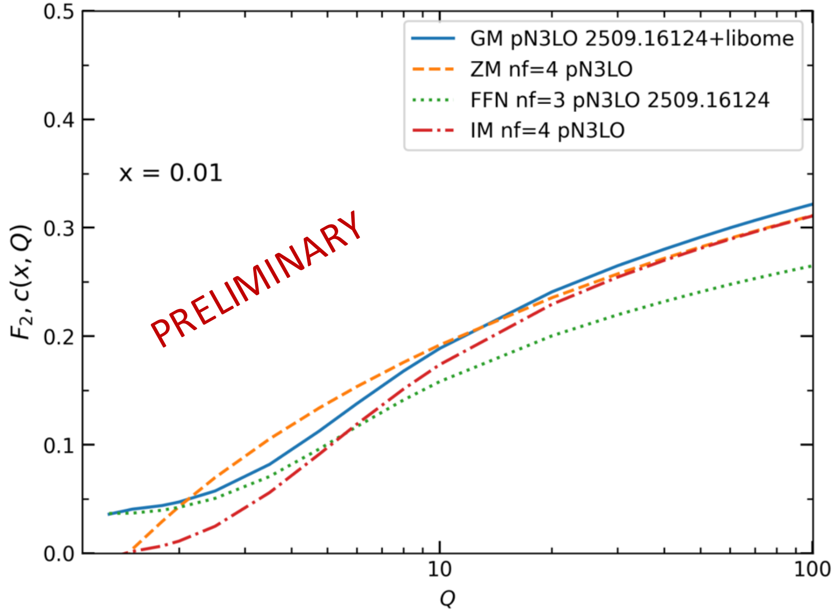
Proposal for a CT18/CT25 **NNLO+ prescription** (T. Hobbs' talk)

- Use CTZ18 NNLO or CT18NNLO error sets
- Central predictions: average of predictions with $\sigma^{(NNLO)}$ and $\sigma^{(N3LO)}$
- Scale uncertainty: compute using $\sigma^{(N3LO)}$




**Toward CTEQ PDF analyses at $N^3LO$**

- ACOT-$\chi$ GMVFN scheme for deep-inelastic scattering (DIS) at partial next-to-next-to-next-to-leading order ($pN^3LO$) in QCD.
- Features of current approximations in $F_{2,c}$ (MG at DIS2026)



# Neutral-current DIS with massive charm quarks at N3LO

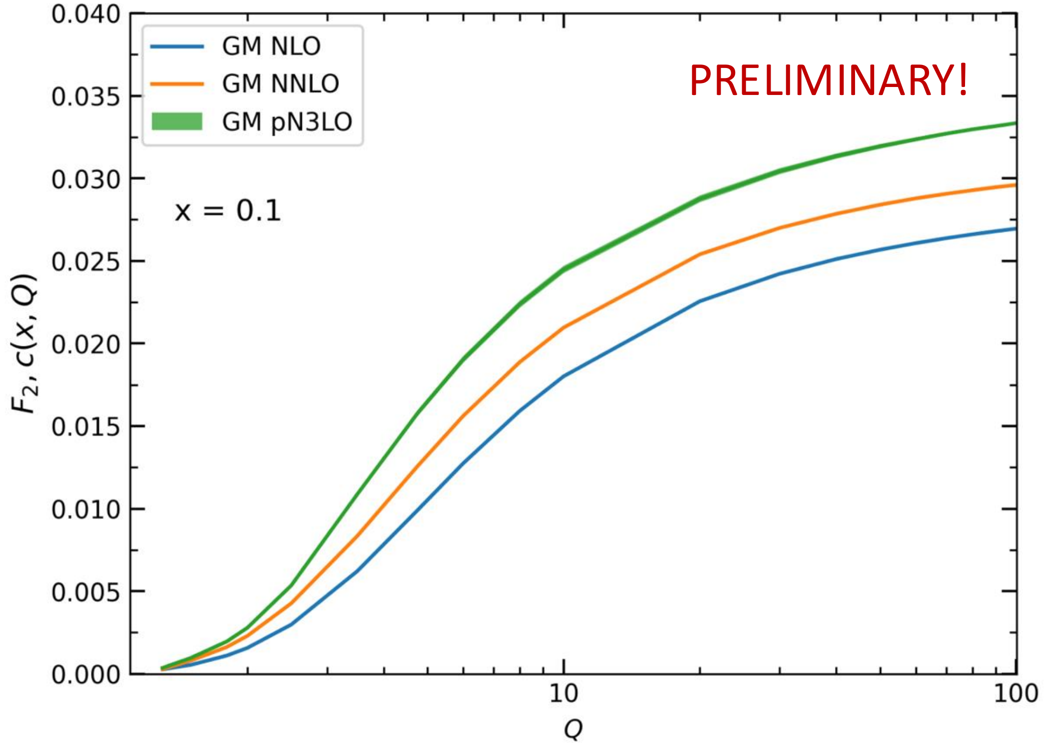
- A significant part (up to a few percent) of the HERA DIS cross section constrains the gluon PDF in the Higgs $x$ region

  - Mass effects at $Q^2 \approx m_c^2$, e.g., phase space suppression, comparable to the rest of (N)NNLO corrections.

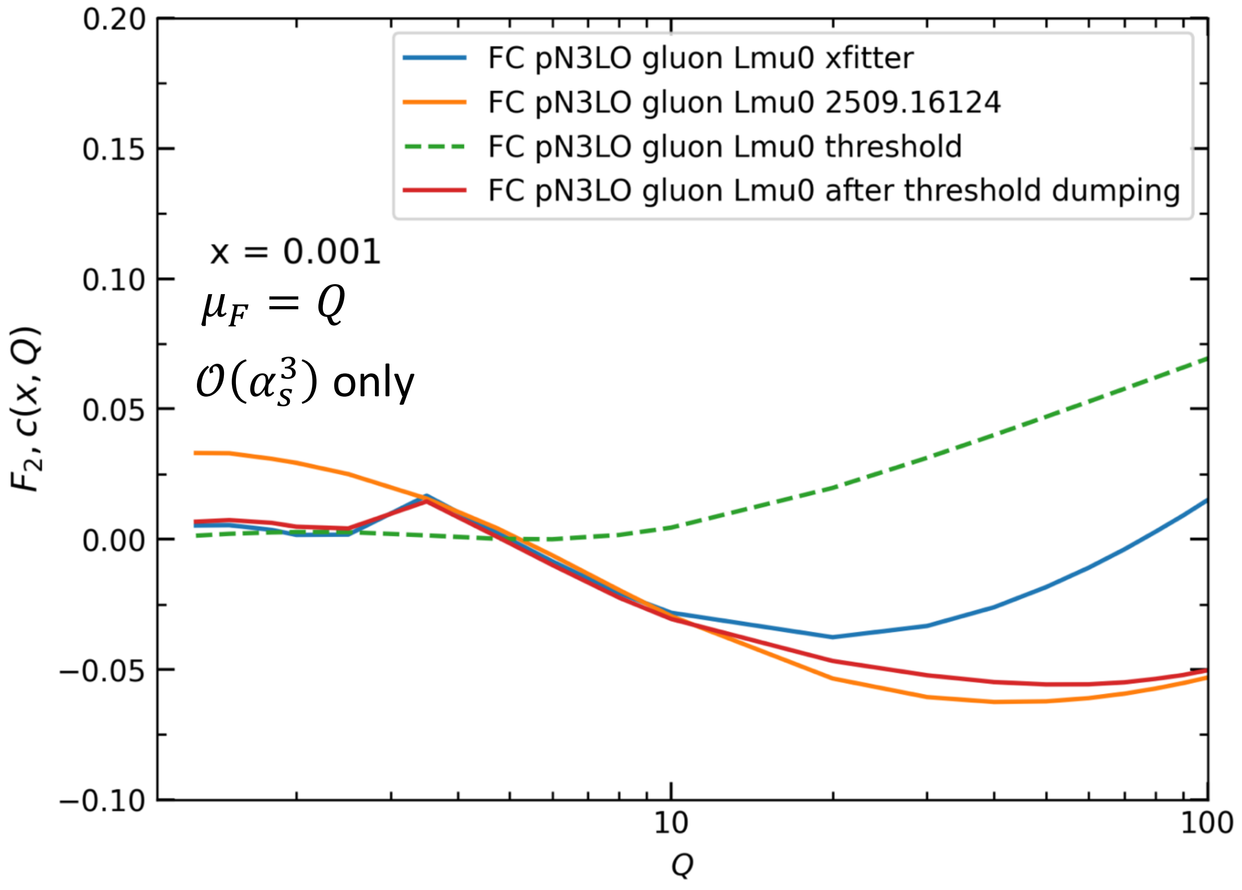
  - A general-mass scheme with varied $N_f$ (GMVFN) is needed to predict the whole $Q$ range

- **NNLO:** fully known, but the existing calculations have numerical inaccuracies at the targeted precision

- **N3LO:** massless coefficient functions (CFs) known exactly; **massive DIS coefficient functions at N3LO are only approximate and depend on matching/dumping functions**



# NC DIS with charm quarks, illustration of ambiguity




Flavor Creation (FC) logarithm-independent (Lmu0) contribution to $F_{2,c}$

The "N3LO" ($\mathcal{O}(\alpha_s^3)$) flavor-creation component is glued together from asymptotic predictions in three limits: $W^2 \to 4\, m_c^2,\, Q^2 \gg m_c^2,\, x \ll 1$

The input asymptotic patches and their interpolations differ among the available codes

Eliminating these differences in the fitted region is a prerequisite for the N3LO accuracy

2509.16124 is Ablinger, Bluemlein, et al.

# DIS is the backbone of all global PDF analyses

The DIS structure Functions

$$F(x,Q)=\sum_{i=1}^{N_f^{fs}} e_i^2 \sum_{a=0}^{N_f} \int_x^1 \frac{d\xi}{\xi} C_{i,a}\left(\frac{x}{\xi},\frac{Q}{\mu},\frac{m_h}{\mu},\alpha_s(\mu)\right)\Phi_{a/p}(\xi,\mu)$$
$$\equiv \sum_{i=1}^{N_f^{fs}} e_i^2 \sum_{a=0}^{N_f} \left[C_{i,a}\otimes\Phi_{a/p}\right](x,Q),$$

To determine $C_{i,a}$ we calculate auxiliary structure functions for scattering on an initial-state parton $b$

$$F(e+b\to e+X)\equiv \sum_{i=1}^{N_f^{fs}} e_i^2 F_{i,b}.$$

$C_{i,a}$ are infrared-safe parts of $F_{i,b}$. They enter convolutions together with parton-level PDFs $\Phi_{a/b}(\xi,\mu)$ for splittings $b \to a$

$$F_{i,b}(\widehat{x},Q)=\sum_{a=0}^{N_f}\left[C_{i,a}\otimes\Phi_{a/b}\right](\widehat{x},Q)$$

# The ACOT recipe (CTEQ)

Expanding perturbatively all the ingredients

$$F_{i,b}(x) = F_{i,b}^{(0)}(x) + a_s\, F_{i,b}^{(1)}(x) + a_s^2\, F_{i,b}^{(2)}(x) + a_s^3\, F_{i,b}^{(3)}(x) + \dots,$$
$$C_{i,a}(\widehat{x}) = C_{i,a}^{(0)}(\widehat{x}) + a_s\, C_{i,a}^{(1)}(\widehat{x}) + a_s^2 C_{i,a}^{(2)}(\widehat{x}) + a_s^3 C_{i,a}^{(3)}(\widehat{x}) + \dots,$$
$$\Phi_{a/b}(\xi) = \delta_{ab}\delta(1-\xi) + a_s\, A_{ab}^{(1)}(\xi) + a_s^2 A_{ab}^{(2)}(\xi) + a_s^3 A_{ab}^{(3)}(\xi) + \dots.$$

and substituting back in the equation,
we can collect equal powers of $\alpha_s$

$$C_{i,b}^{(0)}(\widehat{x}) = F_{i,b}^{(0)}(\widehat{x}),$$
$$C_{i,b}^{(1)}(\widehat{x}) = F_{i,b}^{(1)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(1)}\right](\widehat{x}),$$
$$C_{i,b}^{(2)}(\widehat{x}) = F_{i,b}^{(2)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(2)}\right](\widehat{x}) - \left[C_{i,a}^{(1)} \otimes A_{ab}^{(1)}\right](\widehat{x})$$
$$C_{i,b}^{(3)}(\widehat{x}) = F_{i,b}^{(3)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(3)}\right](\widehat{x}) - \left[C_{i,a}^{(1)} \otimes A_{ab}^{(2)}\right](\widehat{x}) - \left[C_{i,a}^{(2)} \otimes A_{ab}^{(1)}\right](\widehat{x})$$

Procedure rooted in Collins' factorization theorem:

- Collins PRD58 1998
- MG, Lai, Nadolsky, Yuan, PRD86 2012

# Operator Matrix Elements (OMEs)

The functions $A_{i,j}^{(n)}$ (OMEs) can take two different forms depending on the masses of partons i and j.

$$A_{ij}^{(n)}(\xi,\mu^2) = \sum_{l=1}^{n} \left(\frac{1}{\epsilon}\right)^l P_{ij}^{(n,l)}(\xi) + \sum_{l=0}^{n} \ln^l\left(\frac{\mu^2}{\mu_{\mathrm{IR}}^2}\right) P_{ij}^{\prime(n,l)}(\xi),$$

massless $i, j$

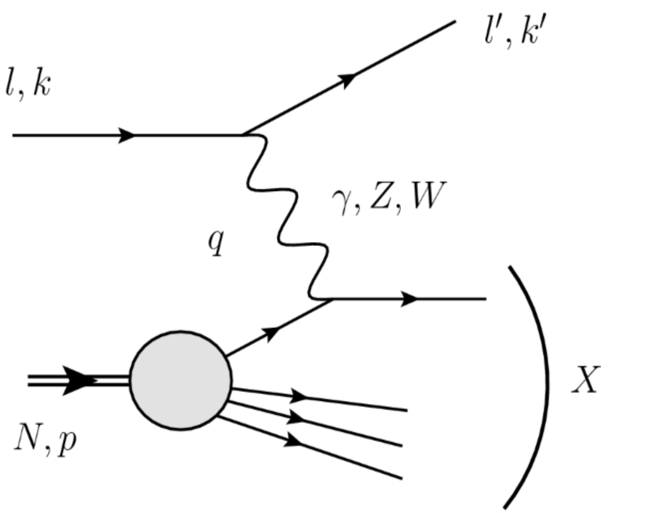
$$A_{Qj}^{(n)}\left(\xi, \frac{\mu^2}{m_Q^2}\right) = \sum_{l=0}^{n} \ln^l\left(\frac{\mu^2}{m_Q^2}\right) a_{Qj}^{(n,l)}(\xi)$$

For massive partons
(at 3-loop -> Blümlein et al.)

Infrared (IR) safe SFs are obtained by subtracting the parton PDFs:

$$\widehat{F}_{i,b}^{(k)}\left(\widehat{x}, \frac{Q^2}{\mu^2}\right) = F_{i,b}^{(k)}\left(\widehat{x}, \frac{Q^2}{\mu_{IR}^2}, \frac{1}{\epsilon}\right) - \sum_{p=0}^{k}\left[C_{i,a}^{(p)} \otimes A_{ab}^{(k-p)}\right]\left(\widehat{x}, \frac{\mu^2}{\mu_{IR}^2}, \frac{1}{\epsilon}\right)$$

# DIS Inclusive Structure Functions




$$F = \sum_{l=1}^{N_l} F_l + F_h$$

Virtual photon coupling to light quark

$$F_l(x,Q) = e_l^2 \sum_a \left[C_{l,a} \otimes \Phi_{a/p}\right](x,Q)$$

Virtual photon coupling to heavy quark

$$F_h(x,Q) = e_h^2 \sum_a \left[C_{h,a} \otimes \Phi_{a/p}\right](x,Q)$$

Here we are concerned with $F_h$ at $\mathcal{O}(\alpha_s^3)$ which is relevant for $F_{2,c/b}$. The ACOT recipe at $\mathcal{O}(\alpha_s^3)$ gives

$$\begin{aligned} a_s^3 F_h^{(3)} =& a_s^3 \left( \widehat{F}_{h,g}^{(3)} - c_{h,h}^{(0)} \otimes A_{hg}^{(3)} - c_{h,h}^{(1)} \otimes A_{hg}^{(2)} - c_{h,h}^{(2)} \otimes A_{hg}^{(1)} \right) \otimes g \\ &+ a_s^3 \left( \widehat{F}_{h,l}^{(3)} - c_{h,h}^{(0)} \otimes A_{hl}^{(3)} - c_{h,h}^{\mathrm{NS}(1)} \otimes A_{hl}^{(2)} \right) \otimes \Sigma + a_s^2 c_{h,h}^{(2)} \otimes f_h \end{aligned}$$

$$c_{hh}^{(2)} = c_{hh}^{\mathrm{NS}(2)} + c_{hh}^{\mathrm{PS}(2)}$$

# Why is full N3LO theory is important?

- Precise and accurate theory predictions are sensitive to heavy-flavor dynamics and rely on delicate cancellation patterns that must in principle be realized with ``Swiss-clock'' precision

# Cancellation patterns and current p$N^3LO$ approx

- ★ $\hat{F}^{(3)}_{h,g}$ represents IR-safe **Flavor Creation (FC)** contributions at $\mathcal{O}(\alpha_s^3)$ which incorporate the full mass dependence
  **No complete results available in the full kinematics**.

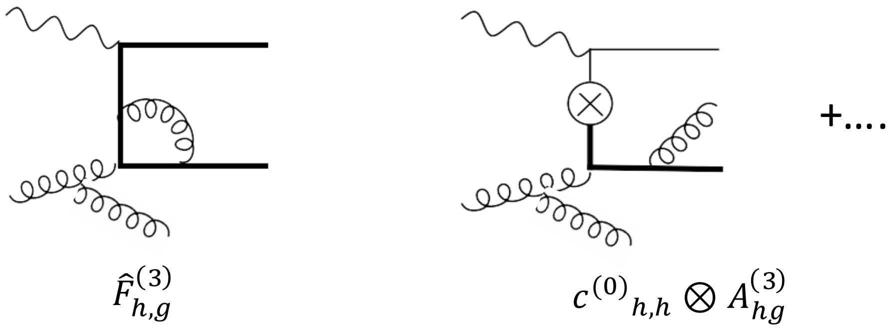
- ★ $- c^{(0)}_{h,h} \otimes A^{(3)}_{hg} - c^{(1)}_{h,h} \otimes A^{(2)}_{hg} - c^{(2)}_{h,h} \otimes A^{(1)}_{hg}$

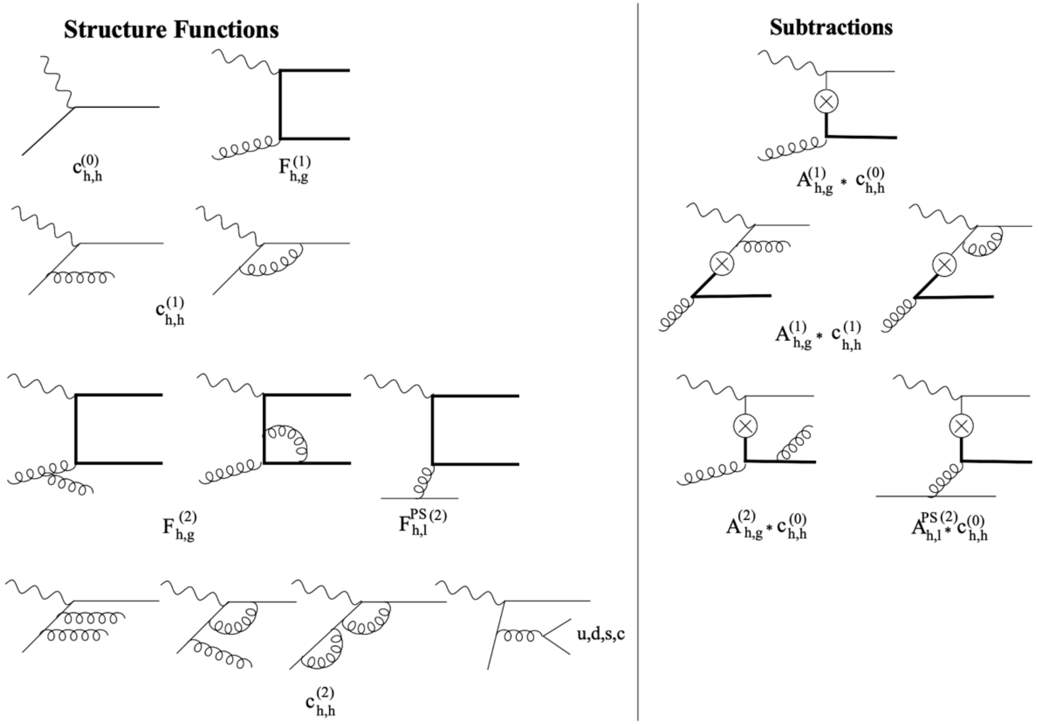
  subtraction terms (Sub) to cancel the singular behavior of the FC terms in the high-scale $(Q^2 \gg m^2)$ limit

- ★ $c^{(2)}_{h,h}$ represents the **Flavor Excitation (FE)** terms

ACOT scheme: delicate cancellations between FC, subtraction terms and FE contributions at each perturbative order

➡ For the FC terms at $\mathcal{O}(\alpha_s^3)$ approximated expressions have been obtained that match three different kinematic regions



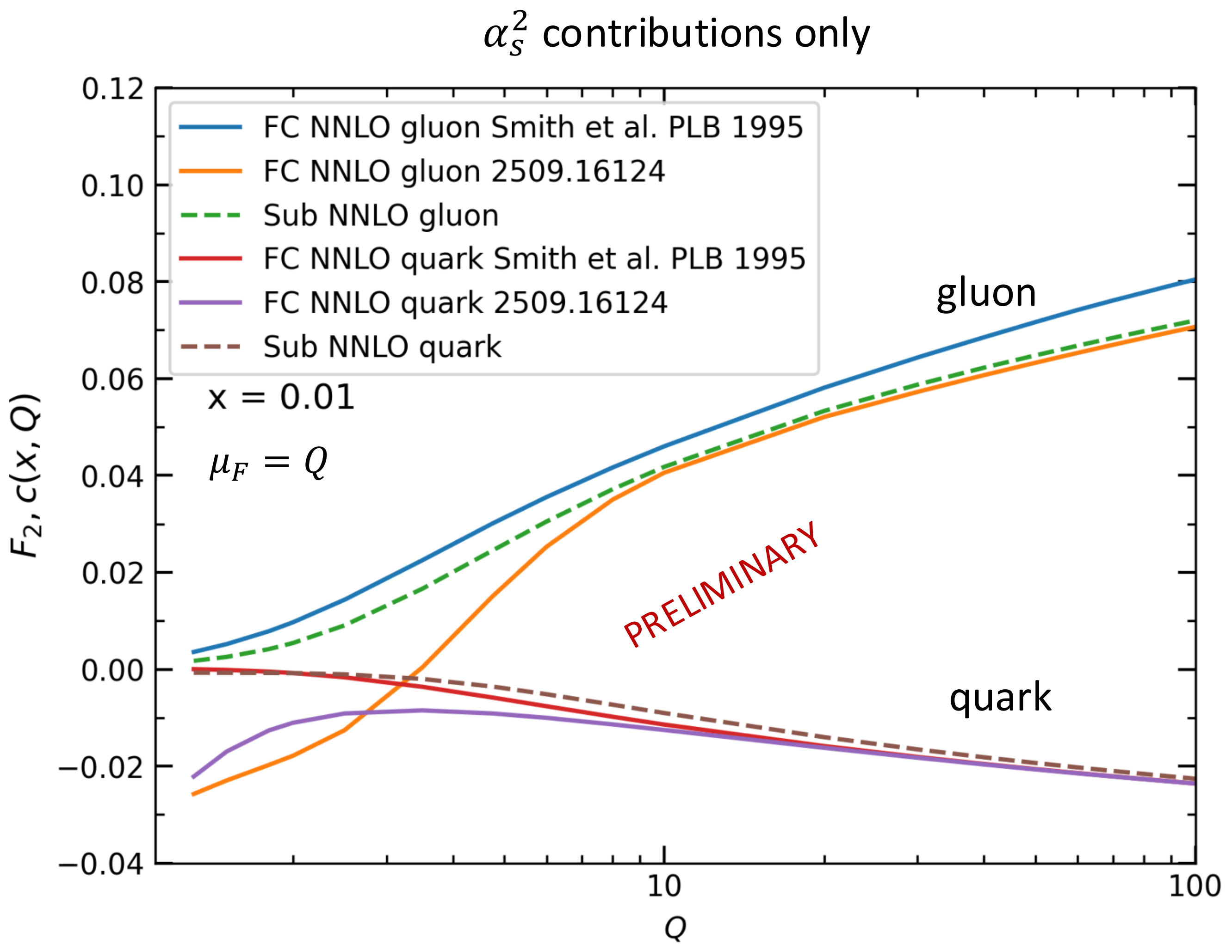

S-ACOT-$\chi$ NNLO: M.G., Lai, Nadolsky, Yuan PRD 2012

# Cancellation patterns @ $\mathcal{O}(\alpha_s^2)$

- NNLO ($\alpha_s^2$) contributions to $F_{2,c}$

$$C^{(2)}_{h,g} = \widehat{F}^{(2)}_{h,g} - A^{(2)}_{hg} - c^{(1)}_{h,h} \otimes A^{(1)}_{hg};$$

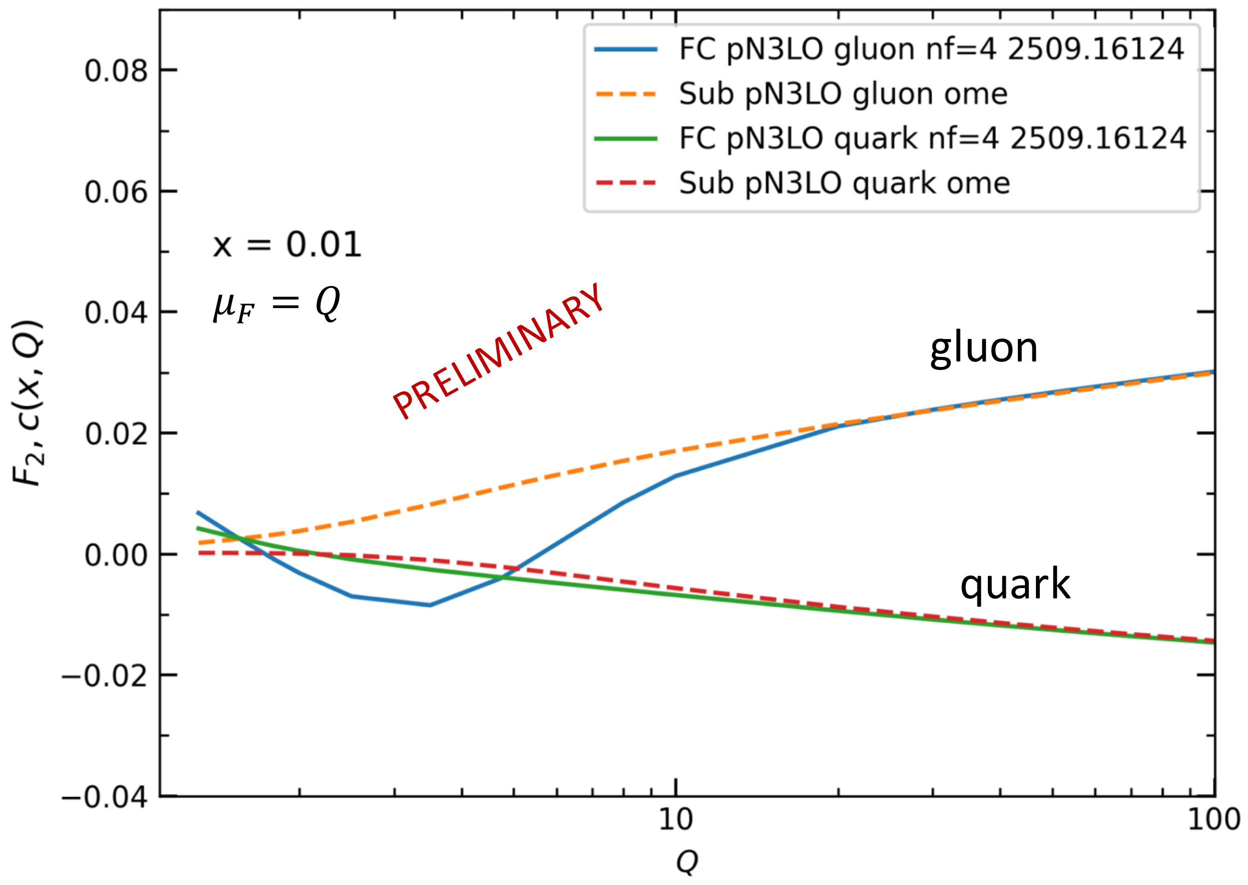
$$C^{(2)}_{h,l} = \widehat{F}^{PS,(2)}_{h,l} - A^{PS,(2)}_{hl}$$



# Cancellation patterns @ $\mathcal{O}(\alpha_s^3)$

- $\mathrm{p}N^3LO$ $(\alpha_s^3)$ contributions to $F_{2,c}$

$\alpha_s^3$ contributions only




$$a_s^3 F_h^{(3)} = a_s^3 \left( \widehat{F}_{h,g}^{(3)} - c_{h,h}^{(0)} \otimes A_{hg}^{(3)} - c_{h,h}^{(1)} \otimes A_{hg}^{(2)} - c_{h,h}^{(2)} \otimes A_{hg}^{(1)} \right) \otimes g$$
$$+ a_s^3 \left( \widehat{F}_{h,l}^{(3)} - c_{h,h}^{(0)} \otimes A_{hl}^{(3)} - c_{h,h}^{\mathrm{NS}(1)} \otimes A_{hl}^{(2)} \right) \otimes \Sigma + a_s^2 c_{h,h}^{(2)} \otimes f_h$$

**OMEs**: https://gitlab.com/libome/libome (Behring, et al. 2510.02175)

**FC**: (`WILS3HQF2`) Ablinger, et al. 2509.16124 ($Q^2 \gg m_c^2$ limit)

# p$N^3LO$ DIS Massive Wilson Coefficients

The current approximations for $\hat{F}^{(3)}_{h,g}$, rely on the matching of different kinematic regions (Kawamura et al. 1205.5727, Alekhin et al. 1701.05838, Laurenti's Thesis 2401.12139)

$$c_{k,i}(\eta,\xi,\mu^2) = \sum_{j=0}^{\infty}(4\pi\alpha_s)^j c_{k,i}^{(j)}(\eta,\xi,\mu^2) = \sum_{j=0}^{\infty}(4\pi\alpha_s)^j \sum_{l=0}^{j} c_{k,i}^{(j,\ell)}(\eta,\xi)\ln^{\ell}\frac{\mu^2}{m^2}$$

Wilson Coeff.

$$\eta = \frac{s}{4m^2} - 1 \quad , \qquad \xi = \frac{Q^2}{m^2} \ , \quad \beta = \sqrt{1-4m^2/s} \quad , \qquad \rho = 4m^2/s$$

## Approx obtained by matching kinematic regions with dumping functions

$$c^{(2)}_{2,g} \simeq c^{(2)\,\mathrm{thr}}_{2,g} + (1-f(\xi))\,\beta^k c^{(2)\,\mathrm{asm}}_{2,g} + f(\xi)\,\beta^3\left(-c^{(2)\mathrm{LLx}}_{2,g}\frac{\ln\eta}{\ln x} + c^{(2)\mathrm{NLL}}_{2,g}\frac{\eta^\gamma}{C+\eta^\gamma}\right)$$

needed at $\mathcal{O}(\alpha_s^3)$

best known piece from OMEs computation (Behring, et al. 2510.02175)

Dumping function

$$f(\xi) = \frac{1}{1+e^{2(\xi-4)}}$$

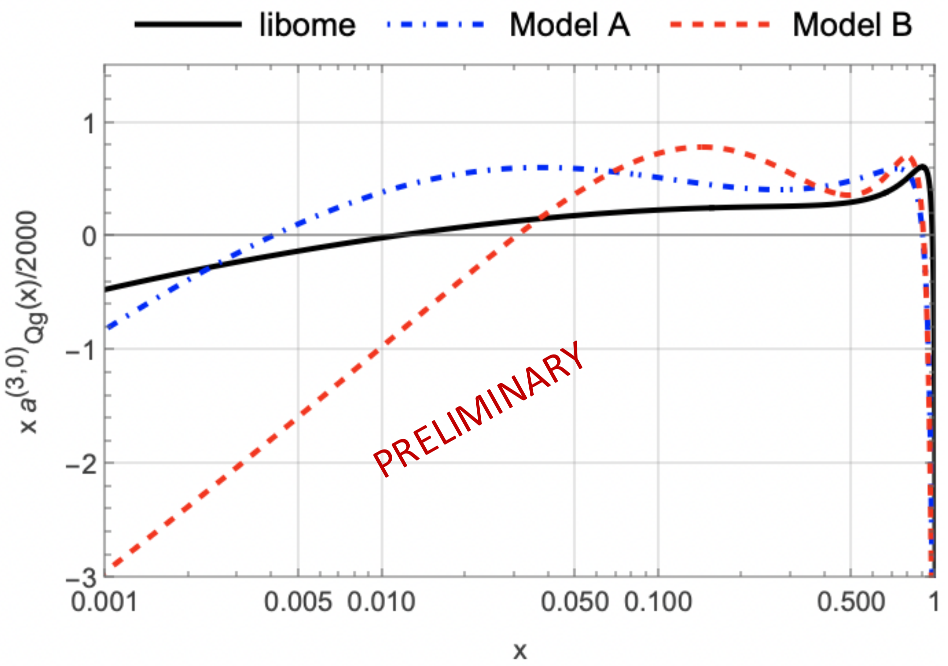
- $s \cong 4\,m^2$ Threshold
- $s \gg 4\,m^2$ High-energy regime $s = Q^2(1/x - 1) \to \infty \Longrightarrow x \to 0$ (small-x limit)
- $Q^2 \gg m^2$ High-scale (asymptotic) limit

# Asymptotic contributions at $N^3LO$

In the $Q^2 \gg m^2$ limit:

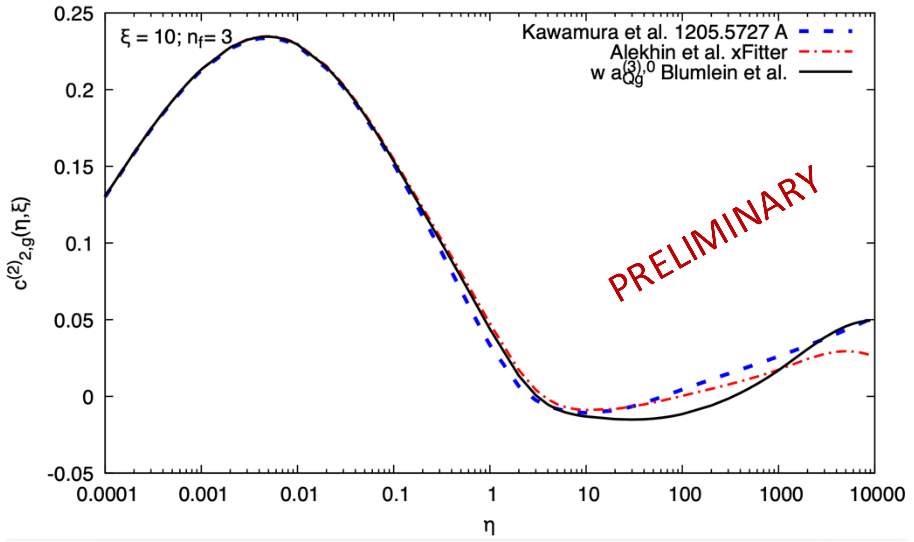
$$H_{2,g}^{(3)}(x,Q^2,m^2) = \text{const}\,[\ldots] + L_Q[\ldots] + L_Q^2[\ldots,\text{nf}+1] + L_Q^3[\ldots,\text{nf}+1] + L_\mu[\ldots] + L_\mu^2[\ldots] + L_Q L_\mu\,[\ldots] + L_Q^2 L_\mu\,[\ldots] + L_\mu^2 L_Q\,[\ldots] + a_{Qg}^{(3)0}(x) + c_{2,g}^{(3)}(x, n_f+1)/(n_f+1)$$

State-of-the-art $a_{Qg}^{(3,0)}(x)$ from libome* differs from the previous approximations (A & B) in Kawamura et al. 1205.5727



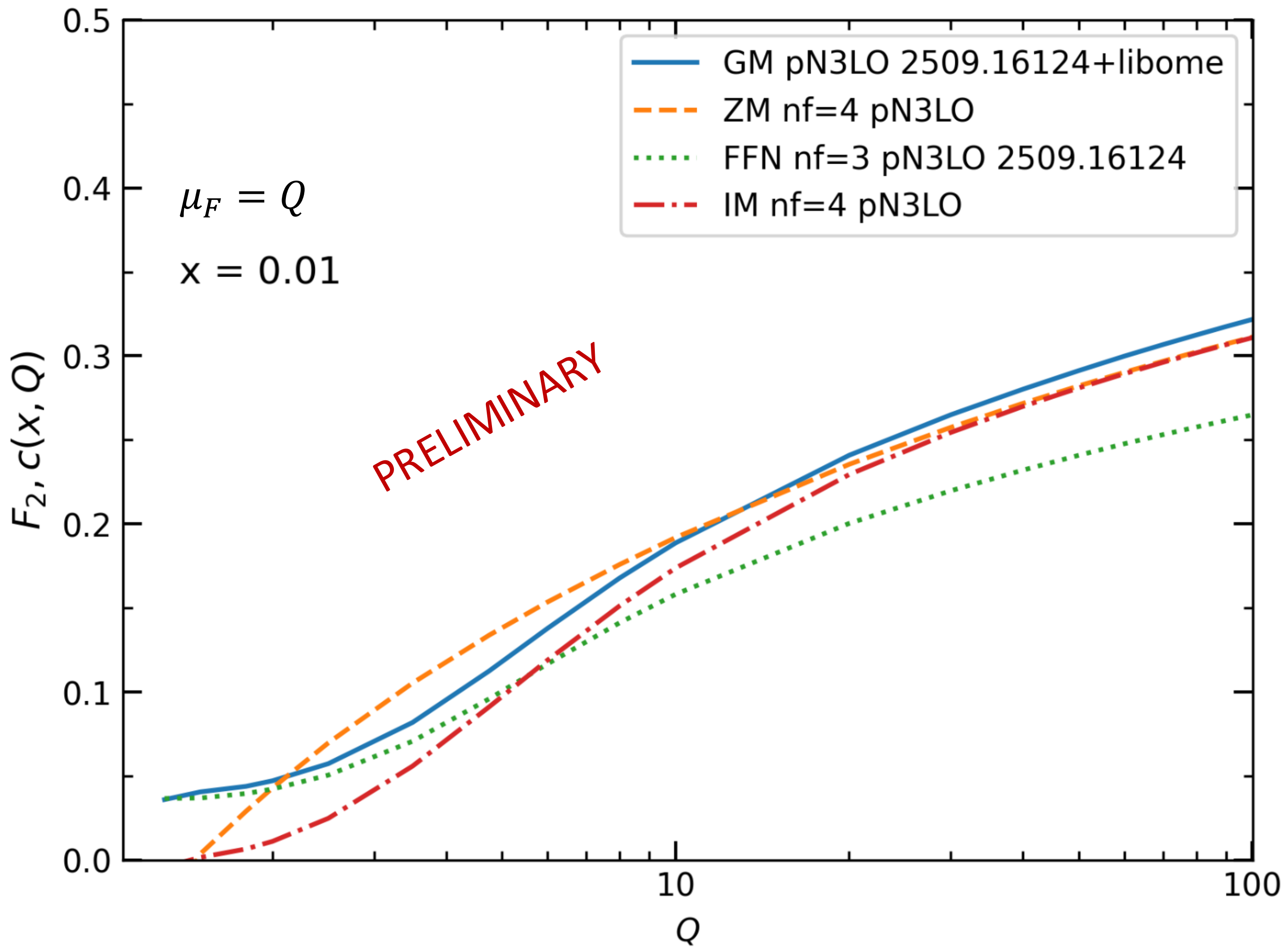

Complete results for the $Q^2 \gg m^2$ limit are provided by Ablinger, et al. in 2509.16124 (`WILS3HQF2`) for $Q^2/m^2 > 10$

$$H_{2,g} = a_s\left[A_{Qg}^{(1)} + \tilde{C}_{2,g}\right] + a_s^2\left[A_{Qg}^{(2)} + A_{Qg}^{(1)}C_{2,q}^{\text{NS},(1)} + A_{gg,Q}^{(1)}\tilde{C}_{2,g}^{(1)} + \tilde{C}_{2,g}^{(2)}\right] + a_s^3\left[A_{Qg}^{(3)} + A_{Qg}^{(2)}C_{2,q}^{\text{NS},(1)} + A_{gg,Q}^{(2)}\tilde{C}_{2,g}^{(1)} + A_{Qg}^{(1)}\left(C_{2,q}^{\text{NS},(2)} + \tilde{C}_{2,q}^{\text{PS},(2)}\right) + A_{gg,Q}^{(1)}\tilde{C}_{2,g}^{(2)} + \tilde{C}_{2,g}^{(3)}\right]$$




* https://gitlab.com/libome/libome -> OMEs repository (Behring, et al. 2510.02175)

# Current CTEQ p$N^3LO$ result for $F_{2,c}$




These predictions are obtained with CT25 PDFs at $Q_0 = 1.3$ GeV and evolved at N3LO with HOPPET-v2

IM = Intermediate Mass scheme: Implementation of $SU(N_f)$ flavor structure at N3LO in the CTEQ fitting code by B. Wang and K. Xie.

- https://scholar.smu.edu/hum_sci_physics_etds/7 (2019). Keping Xie PhD Thesis
- Applications of QCD factorization in multi-scale hadronic scattering, SMU-HEP-15-14 (2015). Bowen Wang, Ph.D. Thesis

# GMVFN schemes simplified: Subtraction & Residual PDFs

$$SUB = c_{hh}^{(0)} \otimes \Big[ \underbrace{\Big(a_s A_{hg}^{(1)} + a_s^2 A_{hg}^{(2)} + a_s^3 A_{hg}^{(3)}\Big) \otimes g + \Big(a_s^2 A_{hl}^{(2),\mathrm{PS}} + a_s^3 A_{hl}^{(3),\mathrm{PS}}\Big) \otimes \Sigma + \cdots}_{\tilde{f}_{\mathrm{h}}^{\mathrm{N3LO}}} \Big]$$

$$FE = c_{hh}^{(0)} \otimes \Big(a_s f_h^{(1)} + a_s^2 f_h^{(2)} + a_s^3 f_h^{(3)} + \cdots\Big) = c_{hh}^{(0)} \otimes f_h + \cdots$$

← This is the *h*-PDF from N3LO DGLAP

$$\tilde{f}_{\mathrm{h}}^{\mathrm{N3LO}} = \tilde{f}_h^{(1)} + \tilde{f}_h^{(2)} + \tilde{f}_h^{(3)}; \qquad \delta f_h^{\mathrm{N3LO}} = f_h - \tilde{f}_h^{\mathrm{N3LO}}$$

Subtraction PDF

Residual PDF → Can be provided as LHPDF grids

The subtraction terms can be grouped together with the FE contributions, as they share the same Wilson Coeff. functions which are convoluted with the corresponding HQ PDFs

$$F_{2,h} \approx \mathrm{FC} - \mathrm{SUB} + \mathrm{FE} = \mathrm{FC} + c_{hh}^{(0)} \otimes \delta f_h^{\mathrm{N3LO}} + \cdots$$

Z+b production: MG, Nadolsky, Reina, Wackeroth, Xie, PRD 2024
Higgs production: Biello, Sankar, Xie, MG, Gauld, Nadolsky, Wieseman, Zanderighi (In preparation)

# Part II

# $N^3LO$ DGLAP evolution

# DGLAP evolution at $\mathrm{N}^3\mathrm{LO}$ in QCD

Recent progress in the analytical computation of the DGLAP splitting function at N3LO in QCD and what included in Our code

- Falcioni, Herzog, Moch, Pelloni, Vogt, PLB (2025) 2410.08089: Pgg @aN3LO
- Falcioni, Herzog, Moch, Pelloni, Vogt, PLB (2024) 2404.09701: Pgq @aN3LO
- Falcioni, Herzog, Moch, Vogt, PLB (2023) 2307.04158: Pqg @aN3LO
- Falcioni, Herzog, Moch, Vogt, PLB (2023)  2302.07593:  Pqq @aN3LO
- Moch, Ruijl, Ueda, Vermaseren, Vogt, JHEP 10 (2017) 1707.08315: First iteration on approximate $P_{ij}^{N3LO}$
- Kniehl, Moch, Velizhanin, Vogt, PRL (2025) 2505.09381 $P_{NS}^{\pm}$ @N3LO → **nf-part exact**

Since last Fall 2025:
- Fast access to OMEs via precise local overlapping series expansions with the LIBOME library
  Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, Schneider, Schönwald (2025) arXiv:2510.02175 [hep-ph]
- Non-zero (but small) contribution to heavy-quark asymmetry OMEs $A_{Qq}^{PS,s,(3)}$
  - Behring, Blümlein, De Freitas, von Manteuffel, Schneider, Schönwald (2025) arXiv:2512.13508 [hep-ph]
- **Gehrmann, von Manteuffel, Sotnikov, Yang, 2604.09534 → exact expressions for $P_{NS}^{\pm}$, $P_{NS}^{s}$ @N3LO**

# DGLAP evolution codes

After the first iteration on the aN3LO splitting functions computation, global PDF fits @aN3LO performed:

1. MSHT20 aN3LO PDFs, J. McGowan, T. Cridge, L. A. Harland-Lang and R. S. Thorne, EPJC (2023) 2207.04739
2. NNPDF4.0 aN3LO PDFs, The NNPDF Collaboration EPJC (2024) 2402.18635
3. xFitter -> F. Giuli's Talk at DIS2026

**DGLAP evolution codes** (LO, NLO, NNLO) **& methods**

- PEGASUS, (Mellin Space; U-matrix), Vogt, CPC (2005) hep-ph/0408244
- CANDIA, (x-space, recursion relations), Cafarella, Corianò, M.G., CPC (2008) 0803.0462 [hep-ph]
- HOPPET, (x-space, brute force, Runge-Kutta), Salam, Rojo, CPC (2009) 0804.3755 [hep-ph]
- QCDNum, (x-space fast evol. + spline interpolation) M. Botje, CPC (2011), 1005.1481 [hep-ph]

**More recently (@ aN3LO):**

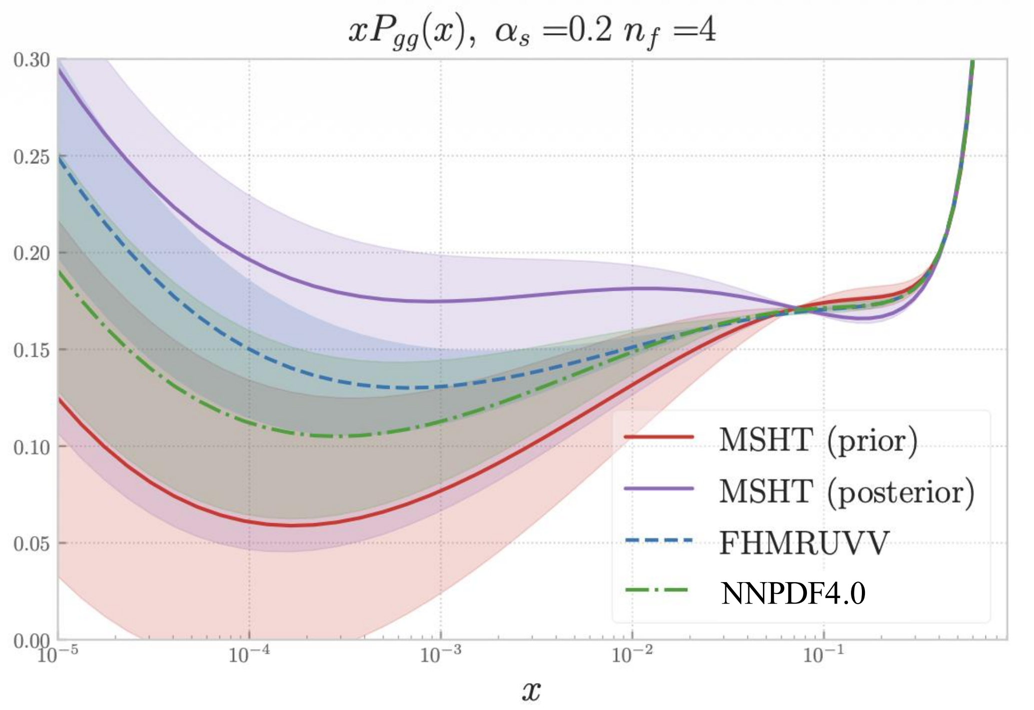
- HOPPETv2, (x-space, brute force, Runge-Kutta) Karlberg, Nason, Salam, Zanderighi, Dreyer, 2510.09310 [hep-ph]
- EKO, (Mellin space, interpolation), Candido, Hekhorn, Magni, EPJC (2022) [2202.02338]
- Apfel & Apfel++, (x-space, interpolation, Runge-Kutta), Bertone, Carrazza, Rojo, CPC (2014) [1310.1394].
- CANDIA-v2, Hampson, M.G., PRD (2026) [2512.22667]

**N3LO Benchmarking**

A Benchmarking of QCD Evolution at aN3LO: Cooper-Sarkar et. al. 2406.16188 [hep-ph]

# 4-loop DGLAP Splitting Functions

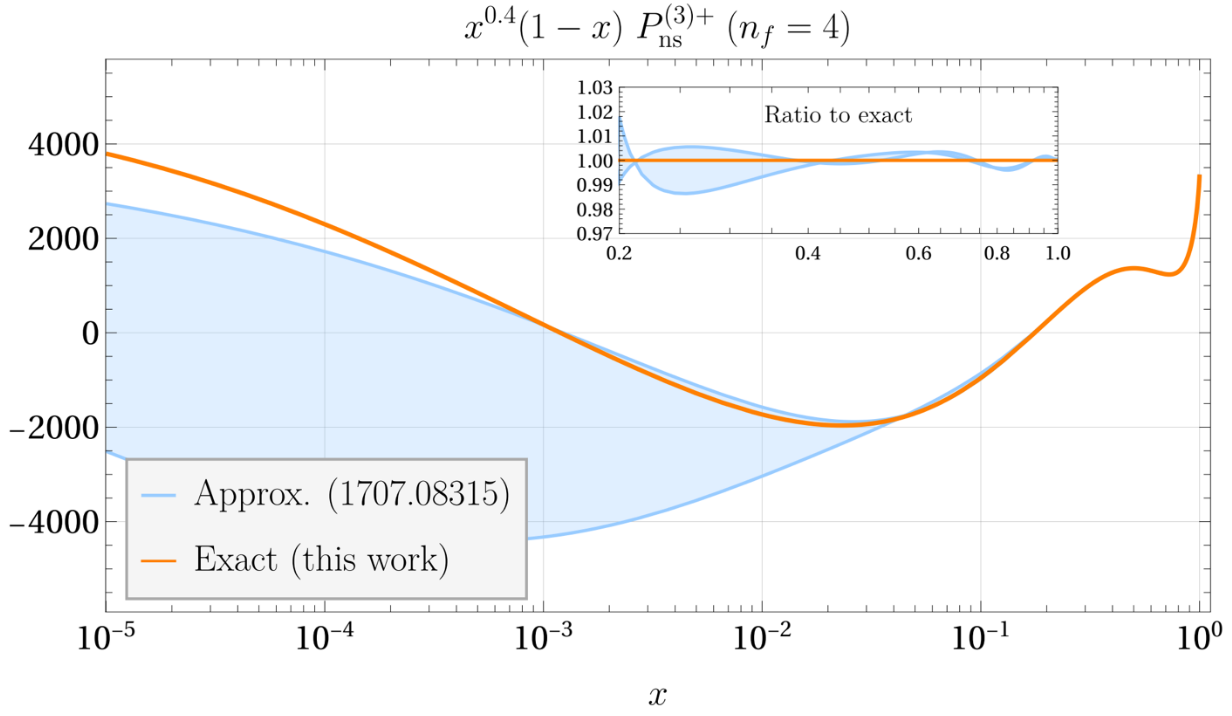
**Singlet Sector**: uses approximations -> aN3LO




From NNPDF4.0 at aN3LO 2402.18635

FHMRUVV = Falcioni, Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt, PLB2022,PLB2023,PLB2024,PLB2025

**Non-Singlet Sector**: known exactly




Gehrmann, von Manteuffel, Sotnikov, Yang, 2604.09534 exact expressions @N3LO!
(See also Sotnikov's talk at DIS2026)

**Impact of exact NS splitting functions recently analyzed with the CANDIA-v2 evolution code**

# `Candia's` algorithm (now in C++)

- ❑ PDF evolution; insight into analytical structure of DGLAP solutions.

- ❑ Recursion relations from an ansatz in x-space where:

- ➢ Exact expansions (equivalent to exact DGLAP solutions as series) in the NS sector
- ➢ Logarithmic expansions for both the NS and Singlet sectors

U-matrix method (Mellin space) ⟺ logarithmic expansions in x-space

(*Cafarella, Corianò Guzzi, NPB* 2006; *Guzzi*, hep-ph/0612355; *Cafarella, Corianò, Guzzi*, CPC 2008).

It extends a method outlined by Rossi (PRD 1984) at LO, and by Da Luz Vieira and Storrow at NLO (ZPC 1991), in their studies of photon PDFs.

# Candia's method in a nutshell

- Non singlet (scalar equations)

$$\frac{\partial q_{\mathrm{NS}}(x,\alpha_s)}{\partial \alpha_s} = \frac{P_{\mathrm{NS}}^{\pm,v}}{\beta(\alpha_s)} \otimes q_{\mathrm{NS}}(x,\alpha_s)$$

Solutions from two types of ansatz:

1. Logarithmic expansion ($\kappa^{\mathrm{th}}$ order)

$$f_{N^{\kappa'}LO}(x,\alpha_s)\big|_{O(\alpha_s^{\kappa'})} = \sum_{n=0}^{\infty}\left(A_n^0(x)+\alpha_s A_n^1(x)+\alpha_s^2 A_n^2(x)+\ldots+\alpha_s^{\kappa'} A_n^{\kappa'}(x)\right)\left[\ln\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)\right]^n$$

2. Exact expansion

$$f(x,Q^2) = \left(\sum_{n=0}^{\infty}\frac{a(x)^n}{n!}L^n\right)_{\otimes}\left(\sum_{m=0}^{\infty}\frac{b(x)^m}{m!}M^m\right)_{\otimes}\left(\sum_{\ell=0}^{\infty}\frac{c(x)^\ell}{\ell!}Q^\ell\right)_{\otimes}\left(\sum_{k=0}^{\infty}\frac{d(x)^k}{k!}P^k\right)_{\otimes}\ldots f(x,Q_0^2)$$

Generalizable to all orders in QCD. $L^n, M^m, Q^l$, ... are scalar functions of $\alpha_s$ and $\alpha_0$, but do not depend on $x$.

- Singlet (matrix equation)

$$\frac{\partial}{\partial \alpha_s}\begin{pmatrix} q^{(+)} \\ g \end{pmatrix} = \frac{1}{\beta(\alpha_s)}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q^{(+)} \\ g \end{pmatrix}$$

Solutions from logarithmic expansions:

$$\mathbf{f}(x,Q^2) = \sum_{n=0}^{\infty}\left\{\left[\sum_{i=0}^{\kappa}\left(\alpha_s(Q^2)\right)^i \frac{\mathbf{S}_n^i(x)}{n!}\right]\ln^n\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right\}$$

*Cafarella, Corianò, Guzzi, NPB* (2006) for NNLO
*C. H., Guzzi* PRD 2026, for $N^3LO$

# Exact power-series expansion for the NS sector at N3LO and new N3LO recursions for the Singlet

The key idea is to observe that in the NS case, the solution has the following structure

$$f(x,\alpha_s) = \left(\frac{\alpha_s}{\alpha_0}\right)^{-\frac{2}{\beta_0}P^{(0)}} \otimes e^{\mathcal{F}(\alpha_s,\alpha_0,P^{(0)},P^{(1)},P^{(2)},P^{(3)}\dots,\beta_0,\dots,\beta_3)} \otimes f(x,\alpha_0)$$

We devise an ansatz in x-space which

$$f(x,Q^2) = \left(\sum_{n=0}^{\infty}\frac{a(x)^n}{n!}\mathcal{L}^n\right)_{\otimes}\left(\sum_{m=0}^{\infty}\frac{b(x)^m}{m!}\mathcal{M}^m\right)_{\otimes}\left(\sum_{\ell=0}^{\infty}\frac{c(x)^\ell}{\ell!}\mathcal{Q}^\ell\right)_{\otimes}\left(\sum_{k=0}^{\infty}\frac{d(x)^k}{k!}\mathcal{P}^k\right)_{\otimes} f(x,Q_0^2)$$

which generates 4 (very cumbersome) recursion relations to determine the coefficients $a(x)$, $b(x)$, .... or $D^s_{t,m,n}(x)$ in

$$f^{\mathrm{N^3LO}}(x,Q^2) = \sum_{s=0}^{\infty}\sum_{t=0}^{s}\sum_{m=0}^{t}\sum_{n=0}^{m}\frac{D^s_{t,m,n}(x)}{n!(m-n)!(t-m)!(s-t)!}\mathcal{L}^n\mathcal{M}^{m-n}\mathcal{Q}^{t-m}\mathcal{P}^{s-t}$$

where we simply rearranged the series and defined

$$D^s_{t,m,n}(x) = a_n(x)\otimes b_{m-n}(x)\otimes c_{t-m}(x)\otimes d_{s-t}(x)\otimes f(x,Q_0^2).$$

# CANDIA−V2 Results & Comparisons

## Candia-v2 N3LO results for different flavors

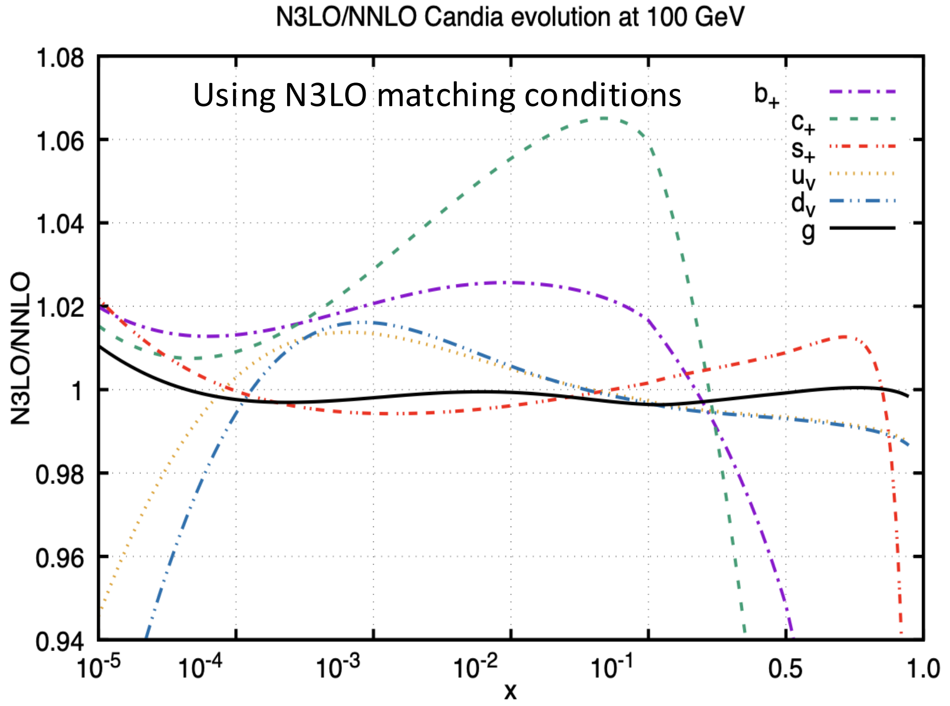
| $n_f = 3\ldots5,\ \mu_f^2 = 10^4$ GeV$^2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $xuv$ | $xdv$ | $xL_-$ | $xL_+$ | $xs_+$ | $xc_+$ | $xb_+$ | $xg$ |
| $\mu_r^2 = 1.0\mu_f^2$ | | | | | | | | |
| 1e−5 | 3.017e−3 | 1.750e−3 | 1.281e−4 | 3.546e+1 | 1.706e+1 | 1.612e+1 | 1.315e+1 | 2.224e+2 |
| 1e−4 | 1.406e−2 | 8.228e−3 | 4.883e−4 | 1.561e+1 | 7.272e+0 | 6.785e+0 | 5.329e+0 | 8.859e+1 |
| 1e−3 | 6.082e−2 | 3.507e−2 | 1.763e−3 | 6.382e+0 | 2.779e+0 | 2.520e+0 | 1.851e+0 | 3.034e+1 |
| 1e−2 | 2.336e−1 | 1.307e−1 | 5.822e−3 | 2.267e+0 | 8.542e−1 | 7.045e−1 | 4.623e−1 | 7.786e+0 |
| 1e−1 | 5.485e−1 | 2.695e−1 | 9.996e−3 | 3.845e−1 | 1.125e−1 | 6.830e−2 | 3.790e−2 | 8.496e−1 |
| 3e−1 | 3.444e−1 | 1.276e−1 | 2.946e−3 | 3.457e−2 | 8.887e−3 | 3.966e−3 | 2.085e−3 | 7.870e−2 |
| 5e−1 | 1.179e−1 | 3.060e−2 | 3.653e−4 | 2.321e−3 | 5.681e−4 | 2.019e−4 | 1.138e−4 | 7.634e−3 |
| 7e−1 | 1.933e−2 | 2.965e−3 | 1.285e−5 | 5.242e−5 | 1.266e−5 | 3.402e−6 | 2.496e−6 | 3.709e−4 |
| 9e−1 | 3.315e−4 | 1.674e−5 | 8.104e−9 | 2.520e−8 | 6.644e−9 | 7.614e−10 | 1.432e−9 | 1.172e−6 |

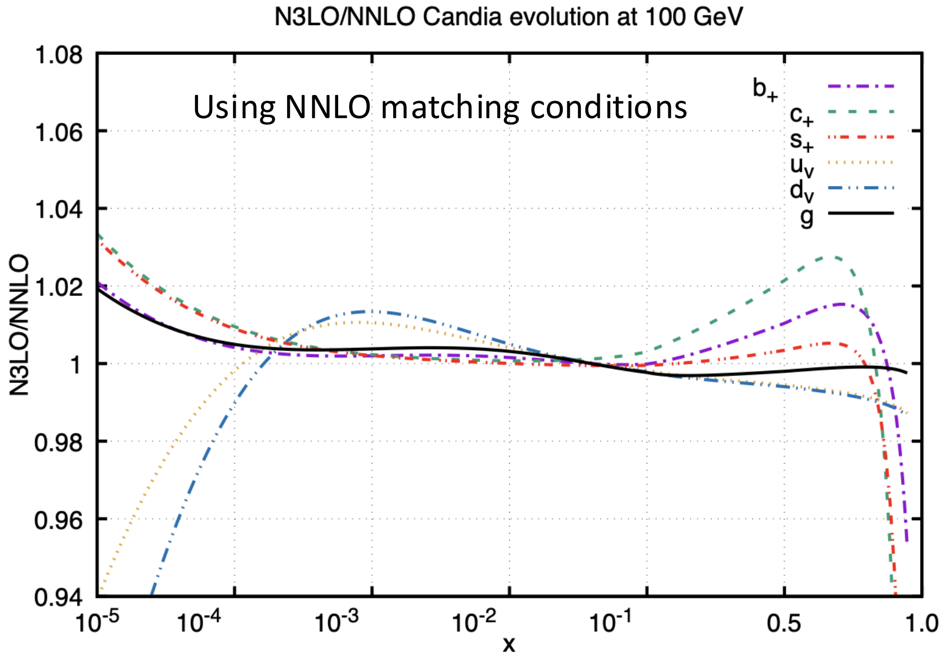
## %Error difference: Candia-v2 vs Hoppet-v2

| $n_f = 3\ldots5,\ \mu_f^2 = 10^4$ GeV$^2$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $g$ | $xu$ | $xd$ | $xs$ | $xc$ | $xb$ | $x\bar{u}$ | $x\bar{d}$ | $x\bar{s}$ | $x\bar{c}$ | $x\bar{b}$ |
| $\mu_r^2 = 1.0\mu_f^2$ **per mille level agreement** | | | | | | | | | | | |
| 1e−5 | 0.01% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 1e−4 | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 1e−3 | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 1e−2 | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 1e−1 | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 3e−1 | 0.00% | 0.00% | 0.00% | 0.00% | 0.01% | 0.00% | 0.00% | 0.00% | 0.00% | 0.01% | 0.00% |
| 5e−1 | 0.00% | 0.00% | 0.00% | 0.00% | 0.02% | 0.01% | 0.01% | 0.00% | 0.00% | 0.02% | 0.01% |
| 7e−1 | 0.00% | 0.00% | 0.00% | 0.00% | 0.06% | 0.03% | 0.04% | 0.00% | 0.00% | 0.06% | 0.03% |
| 9e−1 | 0.01% | 0.00% | 0.00% | 0.01% | 0.00% | 0.02% | 0.32% | 0.01% | 0.01% | 0.00% | 0.02% |

With Les Houches PDFs initial conditions 2406.16188 [hep-ph].

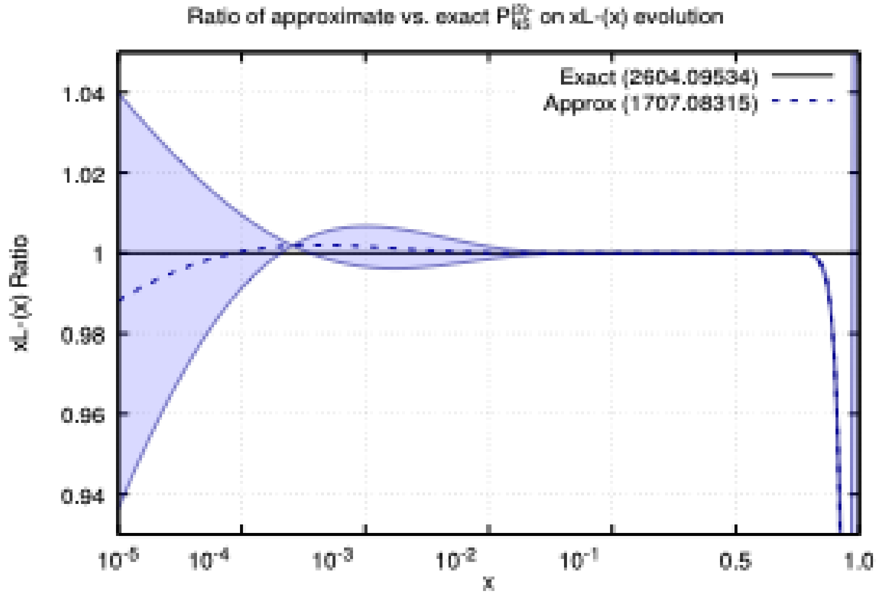
Casey Hampson, undergraduate student at KSU has done most of the work on `Candia-v2` .

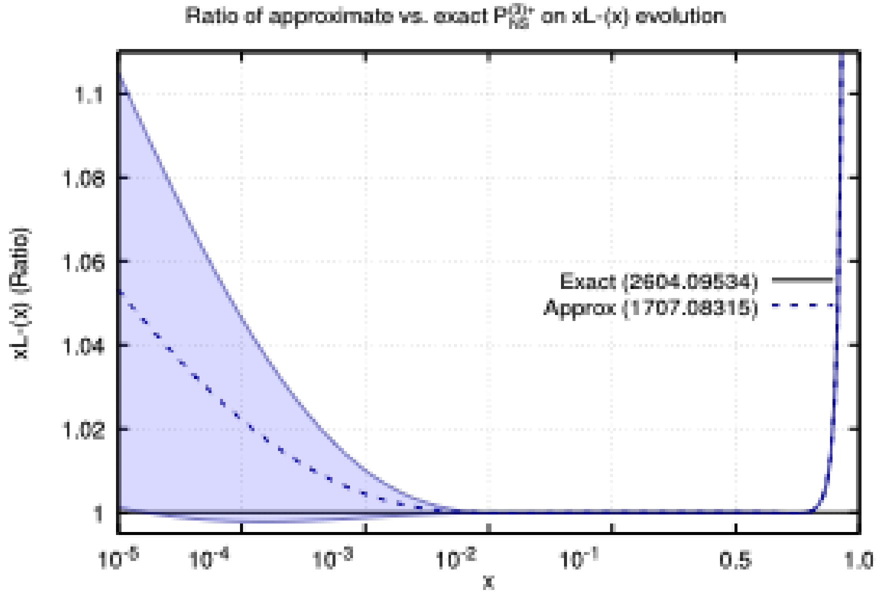


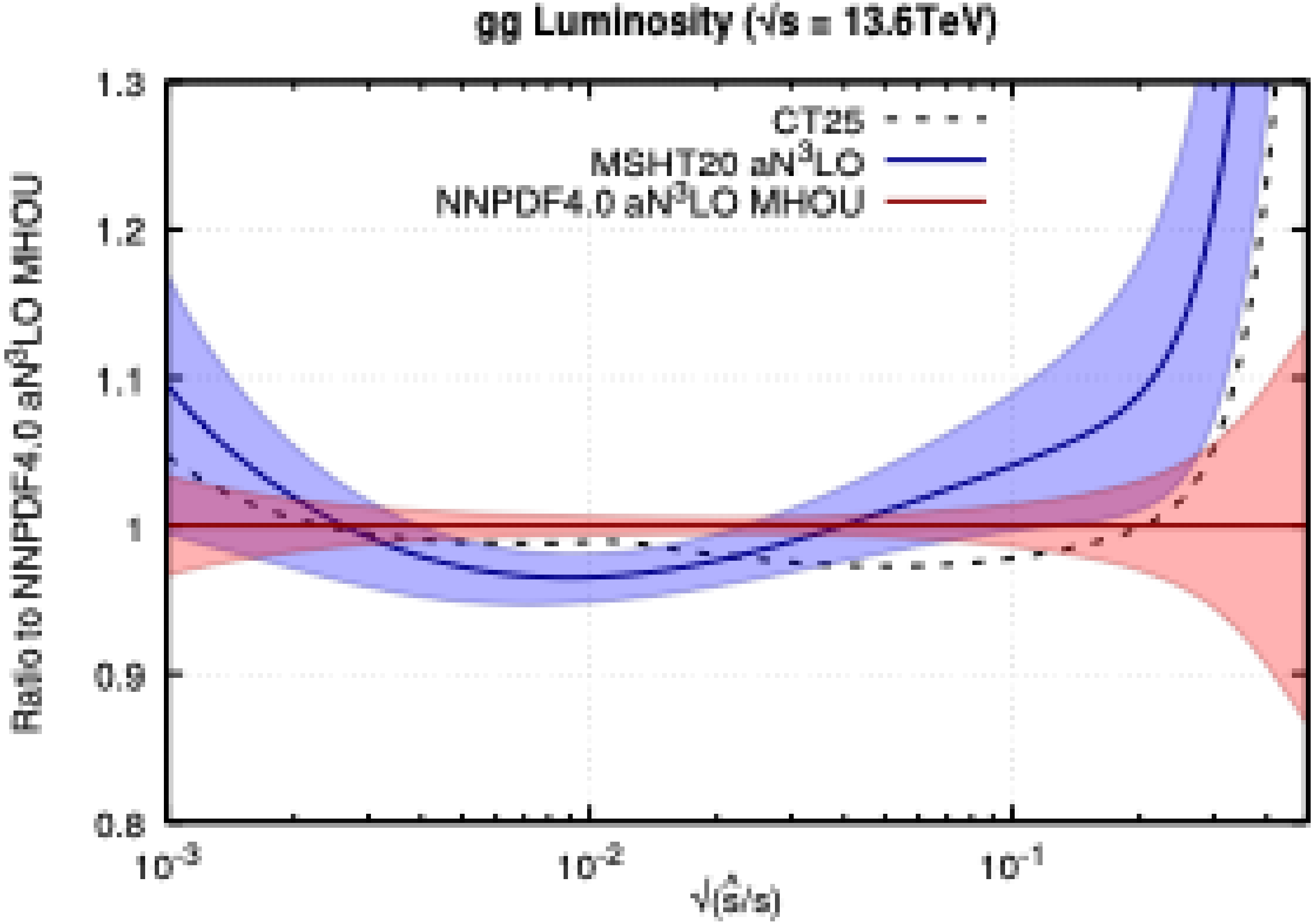
# Impact from new $\boldsymbol{P_{NS}^{(3)\pm}}$ from Gehrmann et al. 2604.09534

| $n_f = 3\ldots5,\ \mu_f^2 = 10^4\ \mathrm{GeV}^2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $xuv$ | $xdv$ | $xL_-$ | $xL_+$ | $xs_+$ | $xc_+$ | $xb_+$ | $xg$ |
| $\mu_r^2 = 1.0\mu_f^2$ | | | | | | | | |
| 1e−5 | 9.22e-2% | 1.72e-2% | 3.92e+0% | 2.16e-4% | 1.26e-4% | 2.62e-4% | 1.18e-4% | 0.00% |
| 1e−4 | 9.42e-2% | 1.66e-1% | 2.23e+0% | 2.11e-4% | 2.98e-4% | 4.52e-4% | 1.41e-4% | 0.00% |
| 1e−3 | 5.30e-3% | 2.59e-2% | 6.10e-1% | 1.84e-4% | 3.79e-4% | 4.54e-4% | 1.12e-4% | 0.00% |
| 1e−2 | 3.86e-4% | 1.23e-4% | 1.47e-3% | 1.13e-4% | 1.56e-4% | 2.30e-4% | 7.42e-5% | 0.00% |
| 1e−1 | 2.89e-5% | 1.67e-5% | 7.13e-4% | 6.61e-5% | 1.96e-5% | 3.80e-5% | 6.86e-6% | 0.00% |
| 3e−1 | 1.04e-5% | 1.15e-5% | 5.88e-5% | 4.40e-5% | 1.81e-4% | 4.19e-4% | 1.17e-4% | 0.00% |
| 5e−1 | 5.28e-6% | 4.21e-6% | 6.21e-4% | 6.91e-5% | 5.51e-4% | 1.54e-3% | 2.71e-4% | 0.00% |
| 7e−1 | 2.64e-5% | 1.50e-5% | 4.62e-3% | 6.53e-3% | 7.46e-3% | 2.77e-2% | 9.43e-3% | 0.00% |
| 9e−1 | 1.39e-4% | 1.69e-4% | 1.12e-1% | 9.64e-1% | 1.69e+0% | 1.57e+1% | 1.82e+0% | 0.00% |



# Gluon-Gluon Luminosity




CT25 is obtained using CT25NNLO PDFs at $Q_0 = 1.3$ GeV evolved at N3LO with CANDIA-v2

# Summary & Conclusions

- Milestone calculations/results obtained in the past few months:

✓4-loop DGLAP NS-kernel completed (Gehrmann, et al. 2604.09534)

✓Operator Matrix Elements at 3-loop accomplished (Ablinger, et al. 2510.02175).

- **4-loop Singlet sector: in progress**
- Several evolution codes publicly available with different approximations
- DIS SFs: eliminating the differences in the fitted region is a prerequisite for the N3LO accuracy
- **4-loop DGLAP and massive Wilson coefficients in full kinematics necessary to go from aN3LO to N3LO global PDF analyses**.
- We hope to see such important future milestones in the next editions of LoopFest!

# BACKUP

# Candia's method: main idea

LO DGLAP

LO Ansatz: $$f(x,Q^2)^{LO} = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!}\left[\ln\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)\right]^n$$ $\Rightarrow$ $$\frac{\partial f(x,Q^2)}{\partial \alpha_s} = -\frac{\left(\frac{\alpha_s}{2\pi}\right)P^{(0)}(x)}{\frac{\beta_0}{4\pi}\alpha_s^2} \otimes f(x,Q^2)$$

Recursion relation in the x-space

$$A_{n+1} = -\frac{2}{\beta_0}P^{(0)} \otimes A_n \quad \Rightarrow \quad A_n(x) = \left(-\frac{2}{\beta_0}P^{(0)}\right)^n f(x,\alpha_s(Q_0^2)), \qquad A_0 = f(x,\alpha_s(Q_0^2))$$

Substituting $A_n$ back in the ansatz gives $$f(x,\alpha_s) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\left[-\frac{2P^{(0)}(x)}{\beta_0}\right]^n \log^n\left(\frac{\alpha_s}{\alpha_0}\right)\right\}_{\otimes} f(x,\alpha_0)$$

Which generates the LO solution once it is resummed

$$f(x,\alpha_s) = \left(\frac{\alpha_s}{\alpha_0}\right)^{-\frac{2P^{(0)}(x)}{\beta_0}} \otimes f(x,\alpha_0) = \exp\left\{-\frac{2P^{(0)}(x)}{\beta_0}\log\left(\frac{\alpha_s}{\alpha_0}\right)\right\}_{\otimes} f(x,\alpha_0)$$

This is generalized to all orders: $$\mathbf{f}(x,Q^2) = \sum_{n=0}^{\infty}\left\{\left[\sum_{i=0}^{\kappa}\left(\alpha_s(Q^2)\right)^i \frac{\mathbf{S}_n^i(x)}{n!}\right]\ln^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right\}$$ → multiple recursion relations

# Exact expansions for the NS sector at N3LO and new N3LO recursions for the Singlet

$$\mathcal{L} = \ln\left(\frac{a_s}{a_0}\right),$$
$$\mathcal{M} = \ln\left(\frac{a_s^2 + b a_s + c}{a_0^2 + b a_0 + c}\right),$$
$$\mathcal{Q} = \ln\left(\frac{a_s - r_1}{a_0 - r_1}\right),$$
$$\mathcal{P} = \frac{1}{\sqrt{-b^2+c}}\tan^{-1}\left(\frac{(a_s - a_0)\sqrt{-b^2+c}}{a_s b + a_0(2a_s + b) + 2c}\right)$$

$L, M, Q$, are $P$ are hard-coded scalar functions that depends on the perturbative order and are directly read off from the analytical Mellin-space solutions (no numeric calc. needed).
$D^0_{0,0,0} = f_0$ specifies the initial conditions.

This algorithm is exact up to N3LO. It's the $P^{(3)}$ splitting function that is approximate and leads to approximate N3LO evolved PDFs.

**Recursion relations at N3LO**

$$D^s_{t,m,n} = -\frac{2}{\beta_0}[P^{(0)} \otimes D^{s-1}_{t-1,m-1,n-1}],$$

$$\gamma \equiv r_1^2 + \bar{b} r_1 + \bar{c}\,,$$

$$\beta_3 \gamma D^s_{t,m,n} = \frac{1}{2}\left[16\pi^2\beta_1 + 4\pi r_1 \beta_2 - (c + b r_1)\beta_3\right] D^s_{t,m,n+1}$$
$$+ 32\pi^2[P^{(1)} \otimes D^{s-1}_{t-1,m-1,n}] + 16\pi r_1[P^{(2)} \otimes D^{s-1}_{t-1,m-1,n}] - 8(c + b r_1)[P^{(3)} \otimes D^{s-1}_{t-1,m-1,n}]$$

$$\beta_3 \gamma D^s_{t,m,n} = (-16\pi^2\beta_1 - 4\pi r_1\beta_2 - r_1^2\beta_3) D^s_{t,m+1,n+1}$$
$$- 64\pi^2[P^{(1)} \otimes D^{s-1}_{t-1,m,n}] - 32\pi r_1[P^{(2)} \otimes D^{s-1}_{t-1,m,n}] - 16 r_1^2[P^{(3)} \otimes D^{s-1}_{t-1,m,n}],$$

$$\beta_3 \gamma D^s_{t,m,n} = -2b\beta_3\gamma D^s_{t+1,m+1,n+1} + \left(32\pi^2(b + r_1)\beta_1 - 8\pi c\beta_2 - 2c r_1\beta_3\right) D^s_{t+1,m+1,n+1}$$
$$+ 128\pi^2(b + r_1)[P^{(1)} \otimes D^{s-1}_{t,m,n}] - 64\pi c[P^{(2)} \otimes D^{s-1}_{t,m,n}] - 32 c r_1[P^{(3)} \otimes D^{s-1}_{t,m,n}],$$

# Exact expansions for the NS sector at N3LO and new N3LO recursions for the Singlet

At LO, there is nothing to truncate in the DGLAP equation, so the logarithmic expansion ansatz valid in the singlet and NS sectors is given by an exact solution with the following recursion relation:

$$\mathbf{S}^0_{n+1} = -\frac{2}{\beta_0}\left[\hat{\mathbf{P}}^{(0)} \otimes \mathbf{S}^0_n\right](x).$$

- At higher orders there is the inclusion of the truncation index $\kappa$, giving, at N3LO, the following for $i \geq 3$:

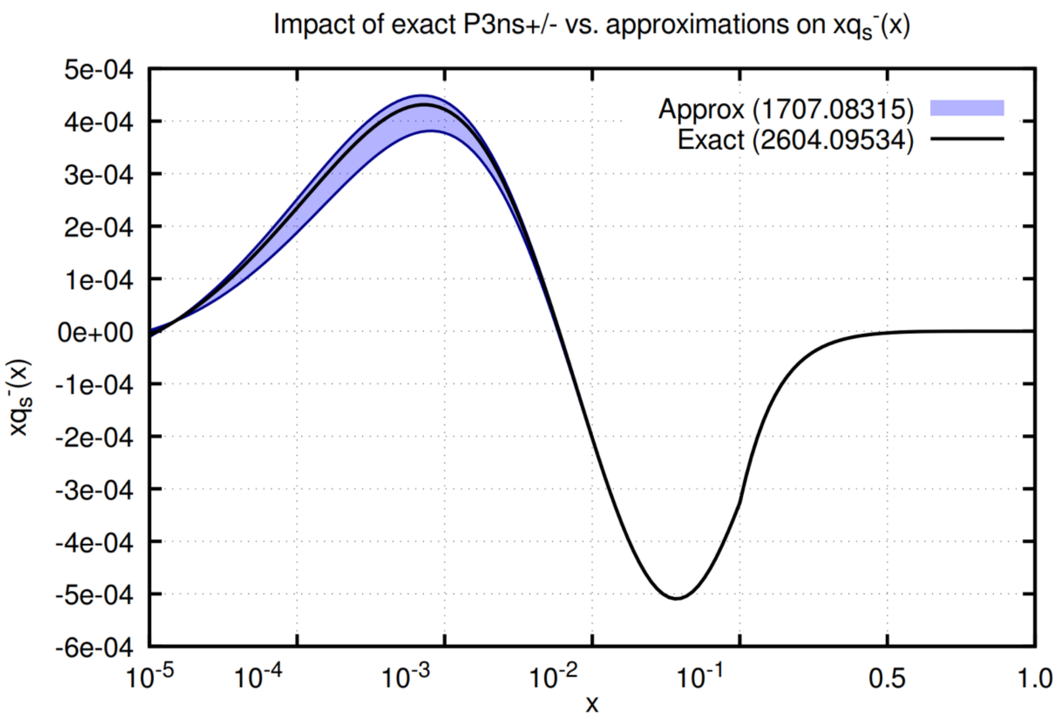
$$\begin{aligned}
\mathbf{S}^i_{n+1}(x) = &-\frac{\beta_1}{4\pi\beta_0}\mathbf{S}^{i-1}_{n+1}(x) - \frac{\beta_2}{16\pi^2\beta_0}\mathbf{S}^{i-2}_{n+1}(x) - \frac{\beta_3}{64\pi^3\beta_0}\mathbf{S}^{i-3}_{n+1}(x) \\
&- i\mathbf{S}^i_n(x) - (i-1)\frac{\beta_1}{4\pi\beta_0}\mathbf{S}^{i-1}_n(x) - (i-2)\frac{\beta_2}{16\pi^2\beta_0}\mathbf{S}^{i-2}_n(x) - (i-3)\frac{\beta_3}{64\pi^3\beta_0}\mathbf{S}^{i-3}_n(x) \\
&- \frac{2}{\beta_0}[\mathbf{P}^{(0)} \otimes \mathbf{S}^i_n](x) - \frac{1}{\pi\beta_0}[\mathbf{P}^{(1)} \otimes \mathbf{S}^{i-1}_n](x) - \frac{1}{2\pi^2\beta_0}[\mathbf{P}^{(2)} \otimes \mathbf{S}^{i-2}_n](x) \\
&- \frac{1}{4\pi^3\beta_0}[\mathbf{P}^{(3)} \otimes \mathbf{S}^{i-3}_n](x).
\end{aligned}$$

- The $i < 3$ pieces are given by the respective lower order contributions that follow a similar structure as above
- These recursion relations generate truncated solutions at arbitrary order κ that are equivalent to

those obtained using the U-matrix method (*Cafarella, Corianò, Guzzi, NPB* 2006; *Guzzi*, hep-ph/0612355; *Cafarella, Corianò, Guzzi*, CPC 2008)

# Impact from new $P_{NS}^{(3)\pm}$ from Gehrmann et al. 2604.09534

| $n_f = 3\ldots5,\ \mu_f^2 = 10^4\ \mathrm{GeV}^2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $xuv$ | $xdv$ | $xL_-$ | $xL_+$ | $xs_+$ | $xc_+$ | $xb_+$ | $xg$ |
| $\mu_r^2 = 1.0\mu_f^2$ | | | | | | | | |
| 1e−5 | 9.22e-2% | 1.72e-2% | 3.92e+0% | 2.16e-4% | 1.26e-4% | 2.62e-4% | 1.18e-4% | 0.00% |
| 1e−4 | 9.42e-2% | 1.66e-1% | 2.23e+0% | 2.11e-4% | 2.98e-4% | 4.52e-4% | 1.41e-4% | 0.00% |
| 1e−3 | 5.30e-3% | 2.59e-2% | 6.10e-1% | 1.84e-4% | 3.79e-4% | 4.54e-4% | 1.12e-4% | 0.00% |
| 1e−2 | 3.86e-4% | 1.23e-4% | 1.47e-3% | 1.13e-4% | 1.56e-4% | 2.30e-4% | 7.42e-5% | 0.00% |
| 1e−1 | 2.89e-5% | 1.67e-5% | 7.13e-4% | 6.61e-5% | 1.96e-5% | 3.80e-5% | 6.86e-6% | 0.00% |
| 3e−1 | 1.04e-5% | 1.15e-5% | 5.88e-5% | 4.40e-5% | 1.81e-4% | 4.19e-4% | 1.17e-4% | 0.00% |
| 5e−1 | 5.28e-6% | 4.21e-6% | 6.21e-4% | 6.91e-5% | 5.51e-4% | 1.54e-3% | 2.71e-4% | 0.00% |
| 7e−1 | 2.64e-5% | 1.50e-5% | 4.62e-3% | 6.53e-3% | 7.46e-3% | 2.77e-2% | 9.43e-3% | 0.00% |
| 9e−1 | 1.39e-4% | 1.69e-4% | 1.12e-1% | 9.64e-1% | 1.69e+0% | 1.57e+1% | 1.82e+0% | 0.00% |



# Computational Improvements in CANDIA-V2 vs. CANDIA(-V1)

- **Multi-threading**:
  - The independent nature of the non-singlet PDFs allows them to be evolved concurrently
  - Thread creation/management very straightforward with `std::thread`
  - The singlet sector cannot do this: but the expensive convolutions/loops can be parallelized and even vectorized with C++ execution policies (requires Intel TBB library, available on all systems)

- **Reduced memory footprint**: recursive nature requires information from only the previous (outermost) iteration CANDIA-V1 allocated memory for *all*, rather than just two

- **Smarter grid setup and convolution routine**:
  - Grid is split up into three segments [1e-5,0.1], [0.1, 0.9], [0.9, 1.0]
  - Each segment is mapped to [0,1] logarithmically, linearly, and quadratically, respectively, with the points in the quadratic interval mapped more densely to 1.0, then a standard Gauss-Legendre quadrature routine is used
  - Reduces the number of total grid points from >1500 to <200 for equivalent, if not better, accuracy

- **More readability** in many routines, cleaner API for end users, platform-independent build system Cmake (though LHAPDF/GSL dependencies still require UNIX-like systems)