THREE LOOP $\mathcal{O}(\alpha\alpha_s^2)$ CORRECTIONS TO THE TOP QUARK MASS

Daniele Gaggero, Ciaran Williams

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University at Buffalo
Department of Physics
College of Arts and Sciences

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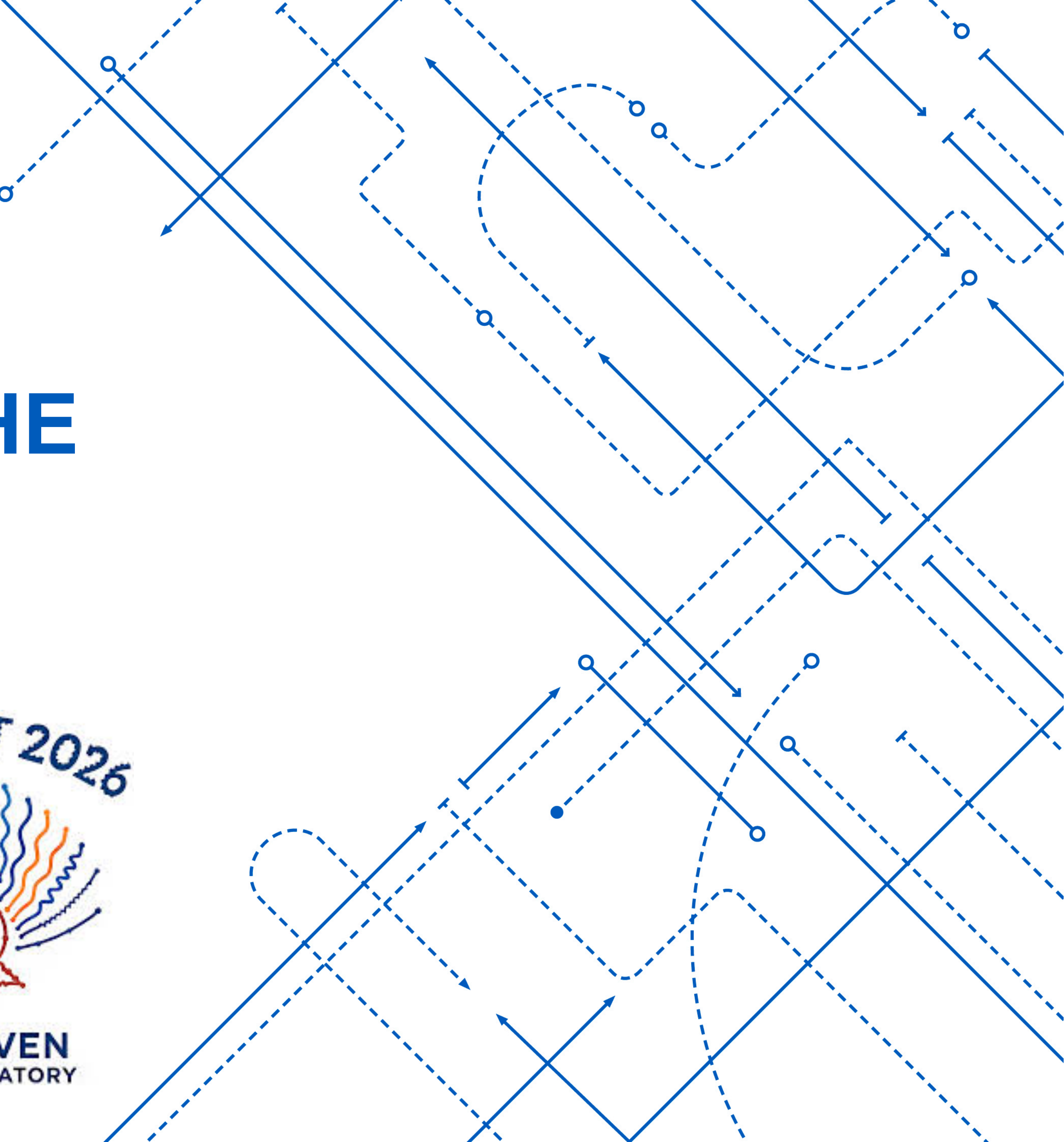


MOTIVATION

Top Quark and SM Parameters

A portrait of the Top Quark

From the PDG (https://pdg.lbl.gov)

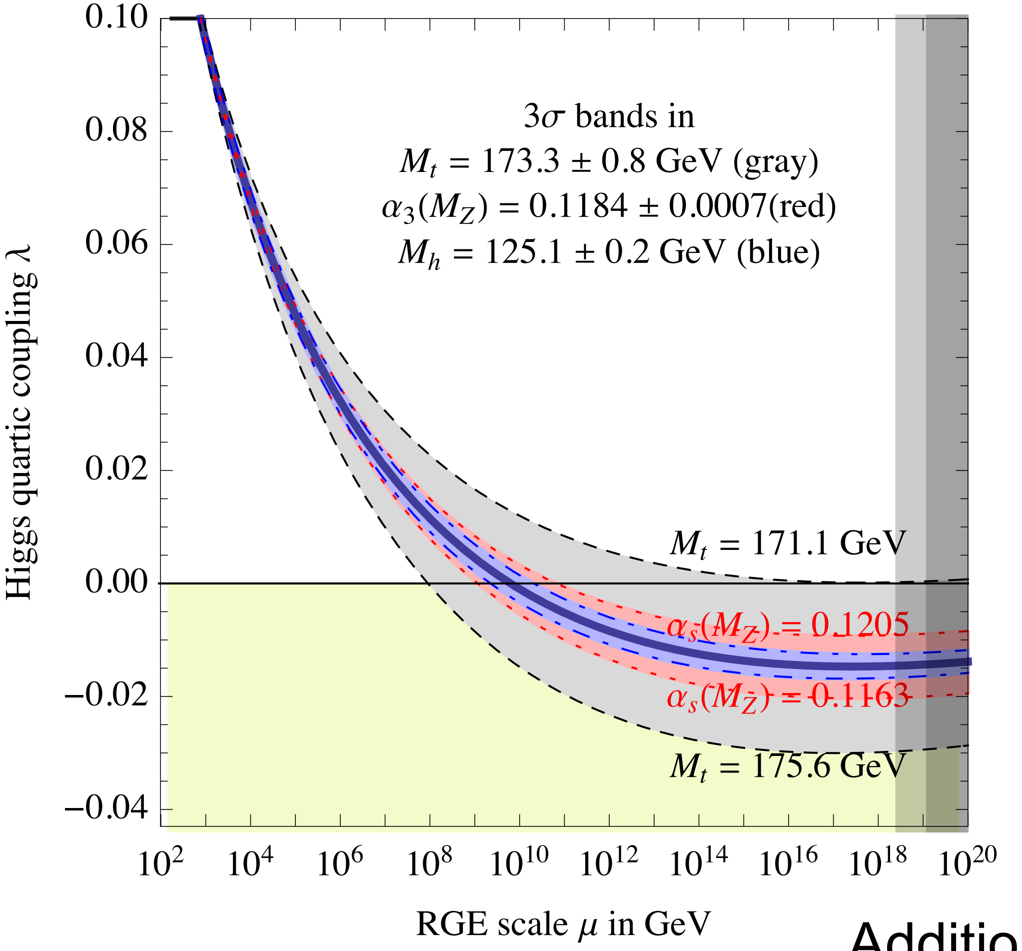
t

$$I(J^P) = 0(\tfrac{1}{2}^+)$$

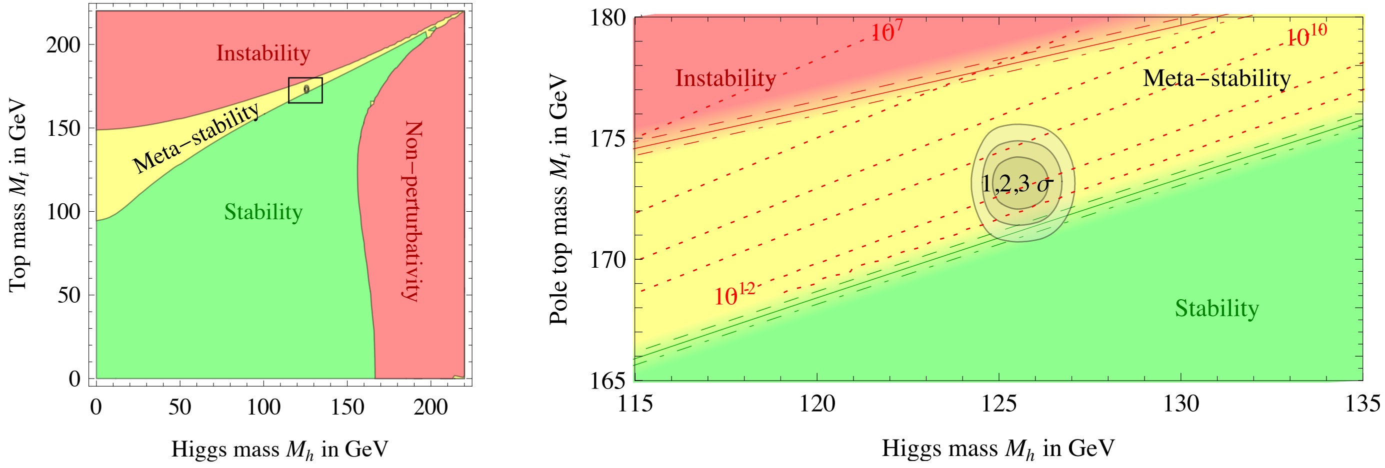
Charge = $\frac{2}{3}\ e$ Top = +1

Mass (direct measurements) $m = 172.56 \pm 0.31$ GeV [a,b] (S = 1.6)
Mass (from cross-section measurements) $m = 162.5^{+2.1}_{-1.5}$ GeV [a]
Mass (Pole from cross-section measurements) $m = 172.4 \pm 0.7$ GeV

Phenomenological Motivation



Buttazzo et al 1307.3536



In the SM, not all parameters are independent and there is a relationship between the masses of the top, W and Higgs. The precision of the theoretical predictions is critical to compare with the observed values.

Additionally the stability of the EW vacuum is very close to the stable/meta-stable boundary. The precise definition of the top pole mass plays a crucial role here.

RELATIONSHIPS BETWEEN BARE AND POLE MASS

Overview and three-loop relationships

QFT basics : Pole mass in perturbation theory

It is well known that the two point correlation function of a fermion can be expanded as a geometric series in terms of the 1PI Self-Energies

$$\int d^4x\, e^{ip\cdot x} \langle\Omega|T\psi(x)\overline{\psi}(0)|\Omega\rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})}$$

This is the master equation for this talk, since it allows us to define a (perturbative) relationship between the **bare** mass m_0 and the **pole** mass m.

$$\boxed{(\not{p} - m_0 - \Sigma(\not{p}))\big|_{\not{p}=m} = 0}$$

Relating M_t to $m_{0,t}$ perturbatively

Kniehl, Pichlum, Steinhauser, 04

We begin expanding out the possible Lorentz structures which can enter into the 1PI 2-point correlation function

$$\Sigma_f(\not{q}) = \sum_l \left[\not{q}\,\Sigma_V^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right) + \not{q}\gamma_5\Sigma_A^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right) + m_{0,f}\Sigma_S^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right)\right],$$

Each coefficient can be separately isolated via a projection operator, i.e. $P_V\Sigma_f = \Sigma_V(q^2)$

We expand the solution around the bare mass:

$$\Sigma_f(\not{q}) = \Sigma_f(m_{0,f}) + (\not{q} - m_{0,f})\left.\frac{d\Sigma_f(\not{q})}{d\not{q}}\right|_{\not{q}=m_{0,f}}$$

Through two-loops $\Sigma_f^{(l)} = \Sigma_V^{(l)} + \Sigma_S^{(l)}$

$$M_f = m_{0,f}\left[1 - \Sigma_f^{(1)}(1) - \Sigma_f^{(2)}(1) - \Sigma_f^{(1)}(1)\left(\Sigma_V^{(1)}(1) - 2\Sigma_f^{(1)\prime}(1)\right)\right].$$

The relationship through 3-loops

We will need to solve this equation through 3-loops either relating the bare mass to the poles mass

$$
\begin{aligned}
m_{0,f} =& M_f \Big\{ 1 + \Sigma_f^{(1)}(1) + \Sigma_f^{(2)}(1) + \Sigma_f^{(3)}(1) + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right] \\
& + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(2)}(1) + 2\Sigma_f^{\prime(2)}(1) \right] + \Sigma_f^{(2)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right] + \\
& + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right]^2 + \Sigma_f^{(1)}(1)^2 \left[\Sigma_f^{\prime(1)}(1) + 2\Sigma_S^{\prime(1)}(1) + 2\Sigma_f^{\prime\prime(1)}(1) \right] \Big\}
\end{aligned}
$$

Or its inverse

$$
\begin{aligned}
M_f =& m_{0,f} \Big\{ 1 - \Sigma_f^{(1)}(1) - \Sigma_f^{(2)}(1) - \Sigma_f^{(3)}(1) + \Sigma_f^{(1)}(1) \left[\Sigma_V^{(1)}(1) - 2\Sigma_f^{\prime(1)}(1) \right] \\
& + \Sigma_f^{(1)}(1) \left[\Sigma_V^{(2)}(1) - 2\Sigma_f^{\prime(2)}(1) \right] + \Sigma_f^{(2)}(1) \left[\Sigma_V^{(1)}(1) - 2\Sigma_f^{\prime(1)}(1) \right] + \\
& - \Sigma_f^{(1)}(1) \left[\Sigma_V^{(1)}(1) - 2\Sigma_f^{\prime(1)}(1) \right]^2 - \Sigma_f^{(1)}(1)^2 \left[\Sigma_f^{\prime(1)}(1) + 2\Sigma_S^{\prime(1)}(1) + 2\Sigma_f^{\prime\prime(1)}(1) \right] \Big\}
\end{aligned}
$$

What’s known out there

	1	α	α^2	α^3
1	✓	✓	🟩 ✓ [3]	
α_s	✓	✓ [2]		
α_s^2	✓ [5]	🟩		
α_s^3	✓ [4]			
α_s^4	🟩 ✓ [1]			
α_s^5				

State of the art is $\mathcal{O}(\alpha_s^4)$ in QCD, $\mathcal{O}(\alpha^2)$ in EW and $O(\alpha_s\alpha)$ in mixed.

The sea-green shading indicates corrections at around 10^{-4} and we see one missing, the topic of today’s talk.

The QED diagrams at $O(\alpha_s^2\alpha)$ Chen, Han, Li, Nieggetiedt (2602.10973) can be obtained as subleading color contributions of $O(\alpha_s^3)$ We’ll focus on exchange of heavy EW bosons.

[1] Marquard, Smirnov, Smirnov, Steinhauser and Wellmann 16

[2] Jegerlehner, Kalmykov, Hu 03; Eiras Steinhauser 05; Kniehl, Pickelner, Veretin 14; S.P. Martin 16

[3] Kniehl, Veretin 14

[4] Chetyrkin, Steinhauser 99

[5] Fleischer, Jegerlehner, Tarasov, Veretin 98

What do we need for our calculation

In order to calculate the relationship between the Pole and MS-bar mass at our desired order we will need :

- One-loop corrections, expanded through $\mathcal{O}(\epsilon^2)$ we will also need the first and second derivatives of the self energy w.r.t the bare mass and the external momentum
- Two-loop correction to the self energy expanded through $\mathcal{O}(\epsilon)$ and first derivatives of the self energy w.r.t. the bare mass and the external momentum
- Three-loop correction to the self-energy expanded through $\mathcal{O}(\epsilon^0)$

Focus of the talk

$$
\begin{aligned}
m_{0,f} = & M_f \Big\{ 1 + \Sigma_f^{(1)}(1) + \Sigma_f^{(2)}(1) + \Sigma_f^{(3)}(1) + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right] \\
& + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(2)}(1) + 2\Sigma_f^{\prime(2)}(1) \right] + \Sigma_f^{(2)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right] + \\
& + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1) \right]^2 + \Sigma_f^{(1)}(1)^2 \left[\Sigma_f^{\prime(1)}(1) + 2\Sigma_S^{\prime(1)}(1) + 2\Sigma_f^{\prime\prime(1)}(1) \right] \Big\}
\end{aligned}
$$

$\Sigma^{(3)}$ CALCULATION SETUP

Topologies and Banana subsectors

Solving canonical integrals as an iterated integral

If we know the differential equation is in canonical form

$$\frac{d}{dx}\mathcal{G}_i^X = \epsilon \mathcal{A}_{ij}^X(x)\mathcal{G}_j^X$$

For us $x = \frac{M_V^2}{m_t^2}$

And we have a boundary point for the integral $G_i(x^0) = G_i^0 + G_i^1\epsilon + \ldots$ we can write down the solution as iterated path integral, from our boundary point to our desired point.

$$\mathcal{G}_i^X(x) = \left(\mathbb{1} + \epsilon \int d\mathcal{A} + \epsilon^2 \int d\mathcal{A} d\mathcal{A} + \ldots\right)\mathcal{G}_i^X(x^0)$$

Feynman Integrals as Iterated Integrals

Differential Equations   Kotiokov, 91   Laporta, 00   Gehrmann, Remiddi, 00

Iterated integrals and ϵ-basis   Henn, 13   Caron-Huot, Henn, 15

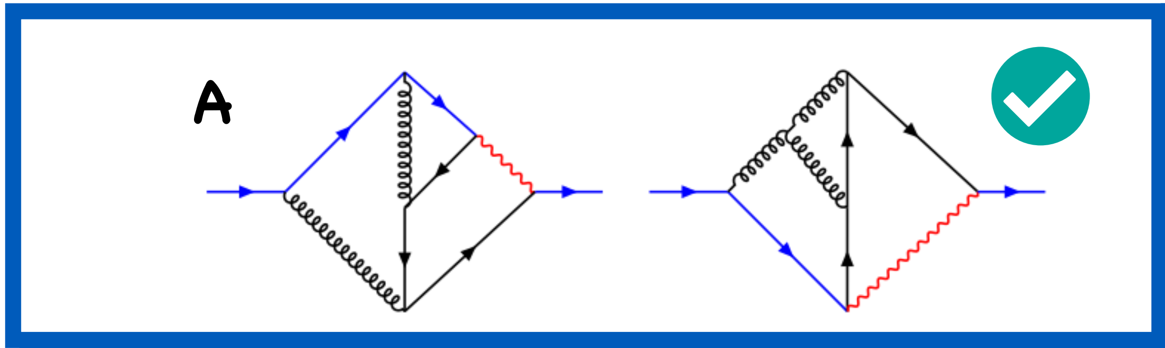
Useful Reviews   Henn (1412.2296)   Weinzierl, 22

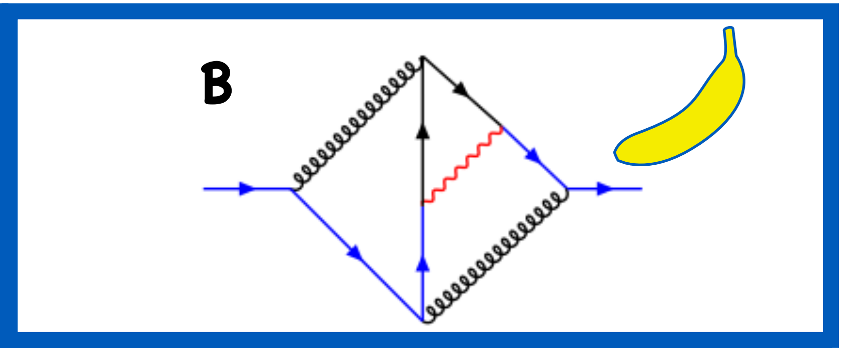
Elliptic Curves and Modular Forms Adams, Bogner, Schweitzer, Weinzierl (1607.01571)
Broedel, Duhr, Dulat, Penante, Tancredi (1804.11144), (1902.09971)   Remiddi, Tancredi (1709.03622)
Giroux, Pokraka (2210.09898)   Müller, Weinzierl (2205.04818)   Weinzierl (2011.07311)

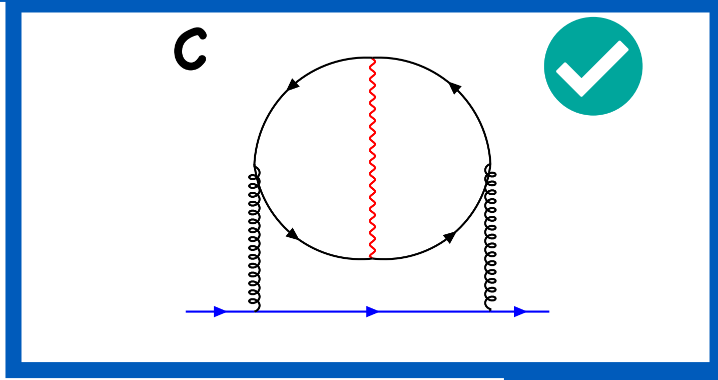
Calabi-Yau Manifolds   Duhr, Klemm, Nega,Tancredi (2212.09550)
Jockers, Kotlewski, Kuusela, McLeod, Pogel, Sarve, Wang, Weinzierl (2412.12057)

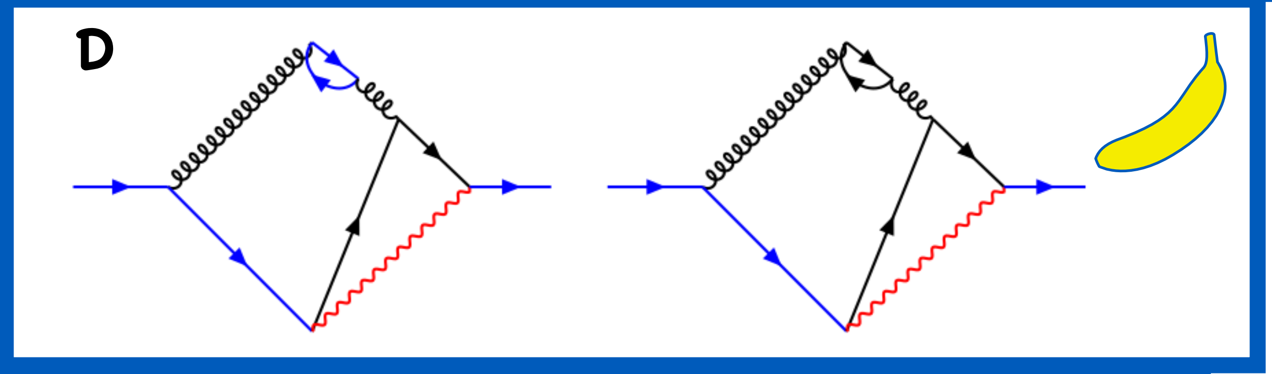
Topologies

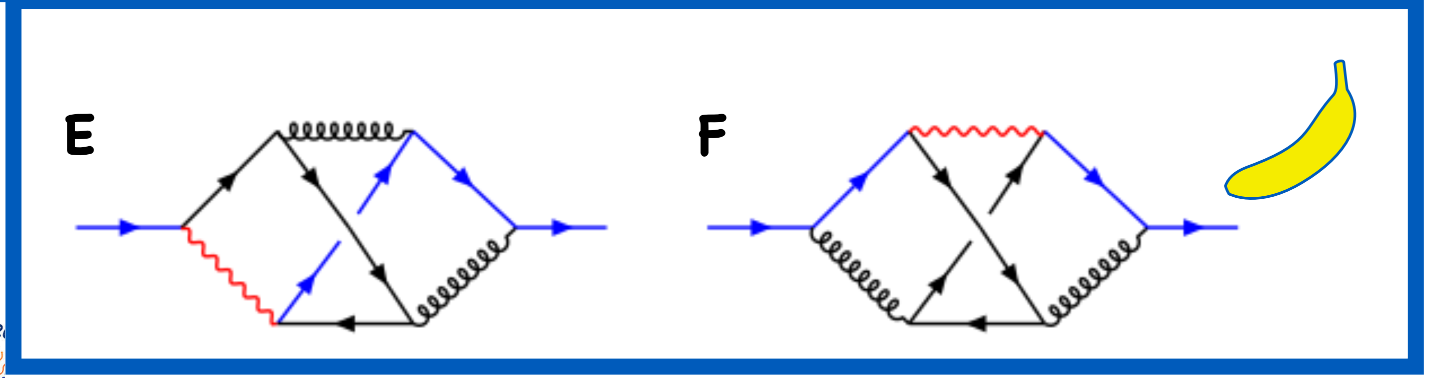
W/Φ



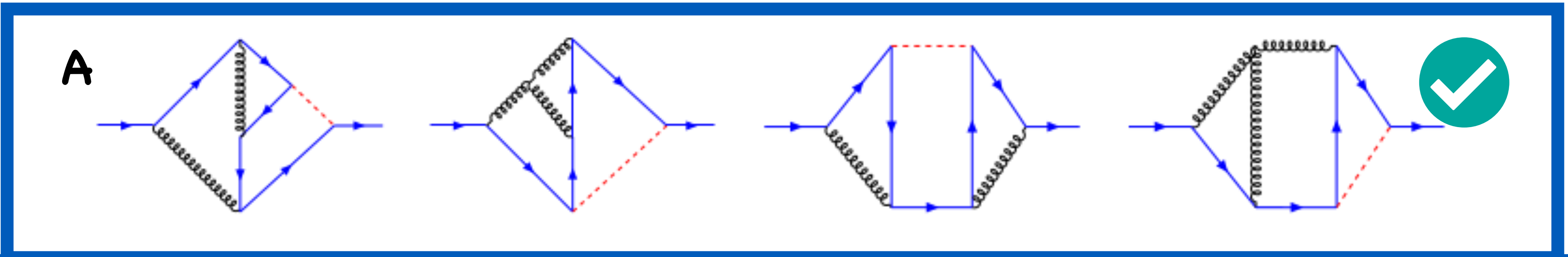


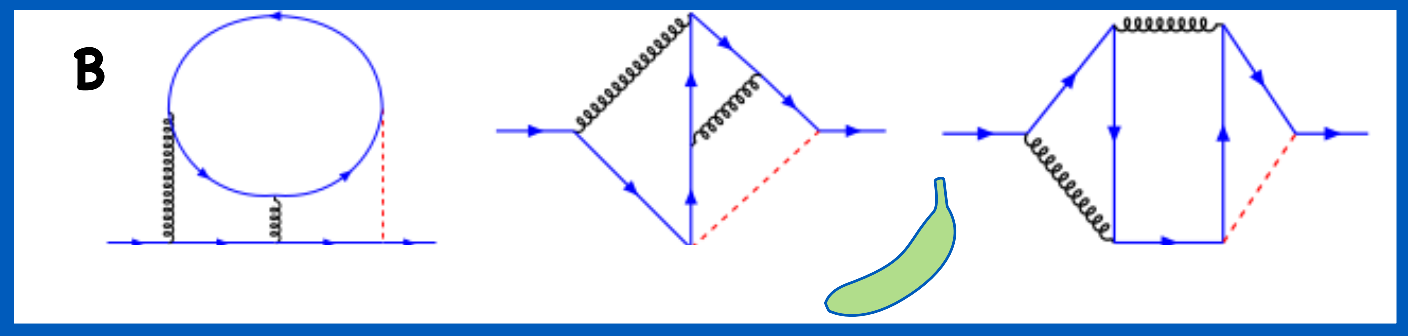


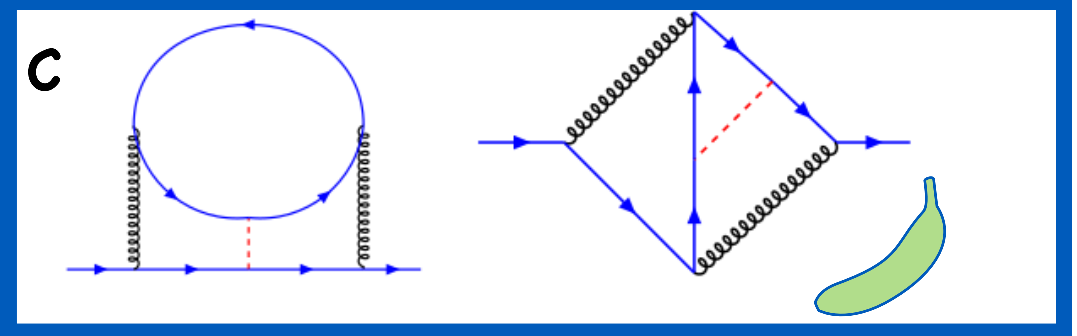


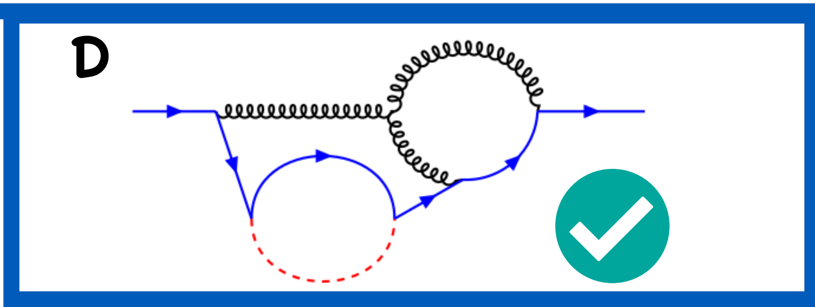


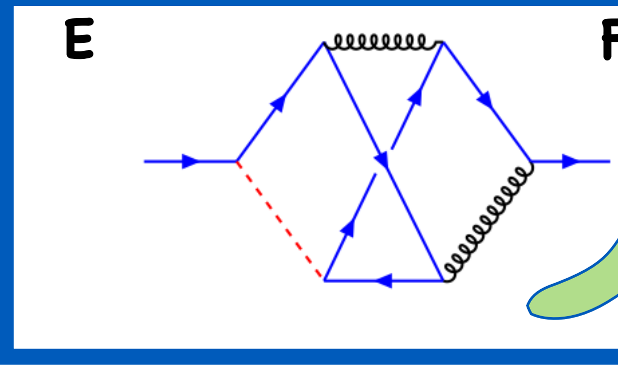
H & Z (χ)

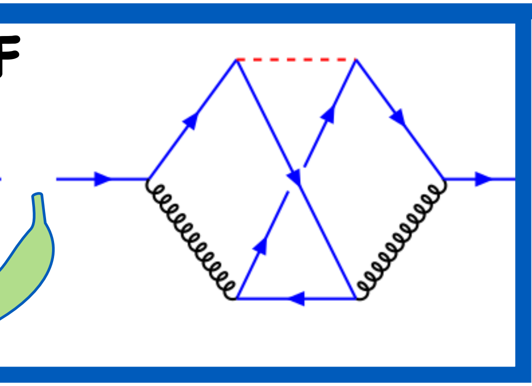










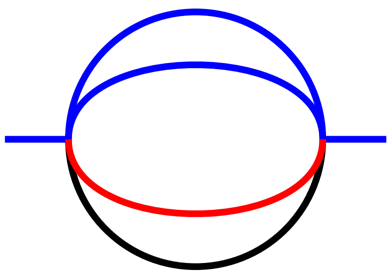


Master Integrals

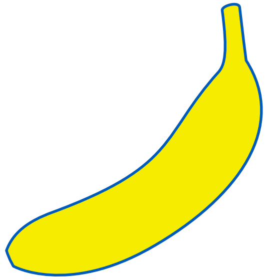
Topology	$n_{W,X}$	Cumulative MIs
A ✓	33	33
B	30	45
C ✓	12	48
D	34	58
E	40	64
F	53	71

Topology	$n_{H,X}$	Original MIs Counter
A ✓	52	52
B	75	88
C	52	102
D ✓	18	104
E	66	109
F	53	112

Light Banana Subsector - Two-loop Sunrise



$B_{11-13}, D_{11-13} E_{11-13}, F_{11-13}$



Hönemann, Tempest and Weinzierl (1811.09308)
Broedel, Duhr, Dulat, Tancredi (1712.07089)
Campert, Moriello, Kotikov (2011.01904)

Two-loop sunrise periods at the x=0 cusp

$$\begin{pmatrix}\psi_{2,0}\\ \psi_{1,0}\end{pmatrix} = \frac{4}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}}\gamma\begin{pmatrix}iK(k')\\ K(k)\end{pmatrix}$$

Canonical basis, integrals 12&13 are obtained raising the power of the red propagator

$$\mathcal{G}_{11}^{W,B} = \frac{\pi}{\psi_1}\mathcal{J}_{11}^{W,B},$$

$$\mathcal{G}_{12}^{W,B} = \frac{\psi_1^2}{2\pi i\epsilon\mathcal{W}}\frac{d}{dx}\mathcal{G}_{11}^{W,B} + \frac{(9-50x+9x^2)\psi_1^2}{24\pi^2}\mathcal{G}_{11}^{W,B},$$

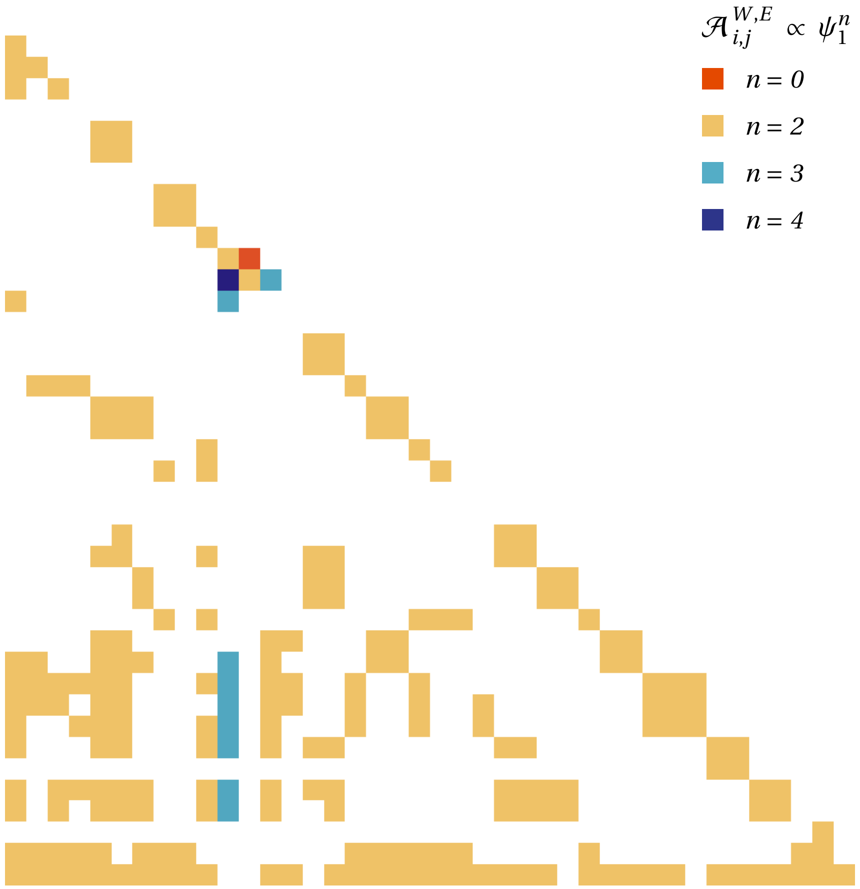
$$\mathcal{G}_{13}^{W,B} = \frac{\pi}{\epsilon\psi_1}\left(\frac{d}{dx}\mathcal{G}_{12}^{W,B} + \frac{(9-50x+9x^2)\psi_1^2}{24\pi^2}\frac{d}{dx}\mathcal{G}_{11}^{W,B}\right) - \frac{\psi_1}{6\pi}(29-13x)\mathcal{G}_{11}^{W,B}$$

K is the elliptic integral of the first kind, $\gamma = 1_2$

Wronskian $$\mathcal{W} = \psi_1\frac{d}{dx}\psi_2 - \psi_2\frac{d}{dx}\psi_1 = -\frac{6\pi i}{x(x-1)(x-9)}\,.$$

We define $$\tau_{1,0} = \frac{\psi_{2,0}}{\psi_{1,0}}, \qquad q_{1,0} = e^{2i\pi\tau_{1,0}}$$

W Differential Equation Matrix



We can visualize the differential equation in $d\tau$ by showing the form of the differential forms.

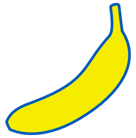
with $$\frac{d\tau_j}{dx} = \frac{1}{2\pi i}\frac{W}{\psi_{1,j}^2}$$

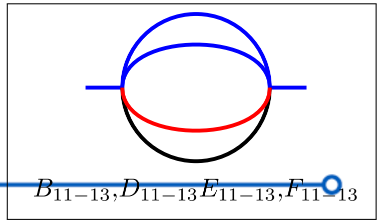
Yellow corresponds to dlog differential forms, where as the other colors demonstrate the presence of elliptic differential forms.

The dlog alphabet is

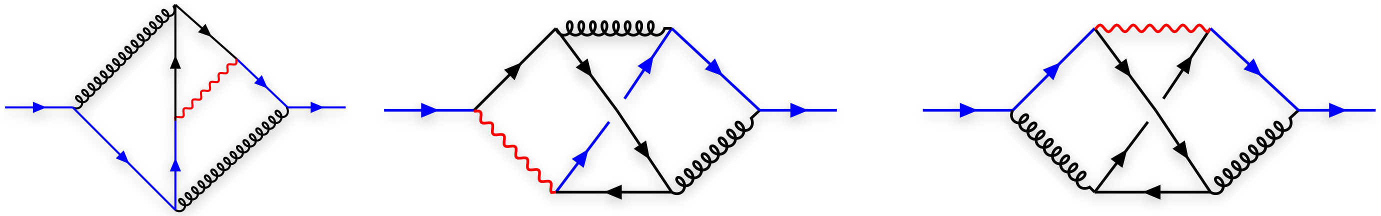
$$\{x, x-1, x+1, \sqrt{x(4-x)}, \sqrt{x(4+x)}\}$$

Elliptic differential forms for W Topology





The integrals needed for our W-initiated loops can be expressed in terms of the following four (elliptic) differential forms, (plus the usual dlog ones)



$$de_0 = d\tau, \quad de_{3,0} = \frac{\psi_1}{\pi}\frac{dx}{x} = \frac{\psi^3}{2\pi i W}d\tau$$

$$de_{3,1} = \frac{\psi_1}{\pi}\frac{dx}{x-1} = \frac{\psi^3}{2\pi i W}\frac{d\tau}{x-1} \quad de_4 = \frac{\psi^2}{2\pi i W}\left(\frac{\psi^2}{\pi^2}\right)\frac{81+1188x-594x^2+372x^3-23x^4}{48x(x-1)(x-9)}d\tau$$

Our differential forms are modular forms of weight 0, 3 3 and 4 respectively

Iterated Integrals of Meromorphic Forms

We encounter meromorphic forms $f_i(x) = P(x)\psi_1(x)^n/Q(x)$, and we get an iterated integral over them

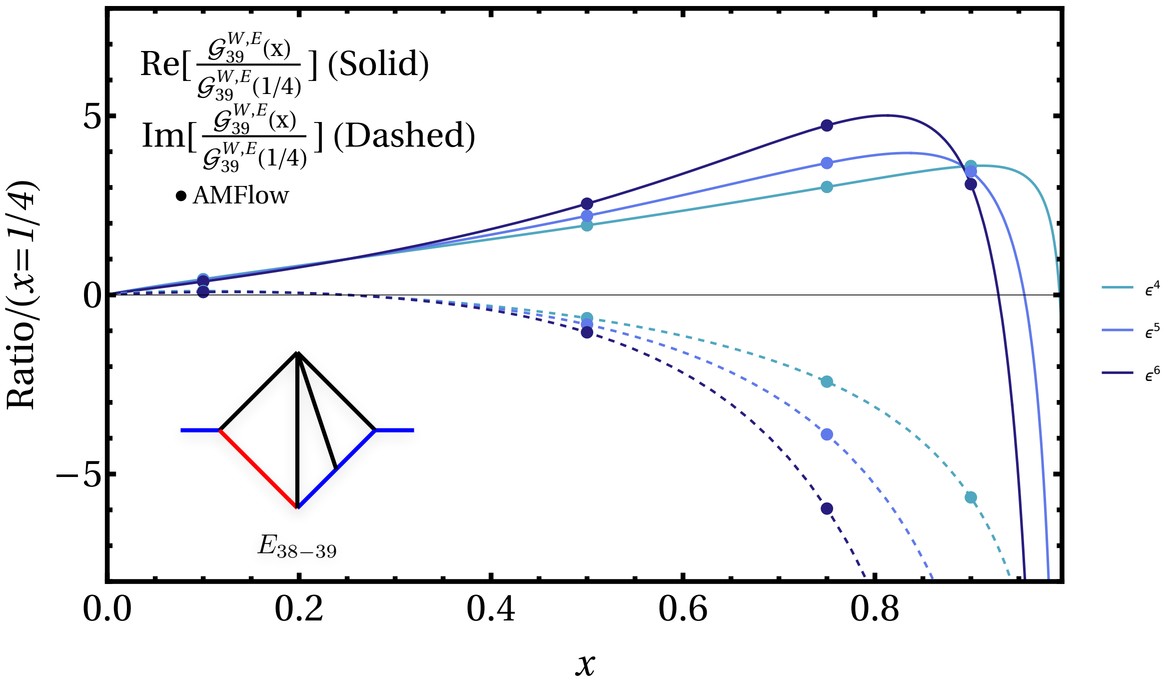
$$I(f_1, f_2 \ldots, f_n; \tau, \tau_0) = (2\pi i)^n \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \cdots \int_{\tau_0}^{\tau_n} d\tau_n f_1(\tau_1) f_2(\tau_2) \ldots f_n(\tau_n)$$

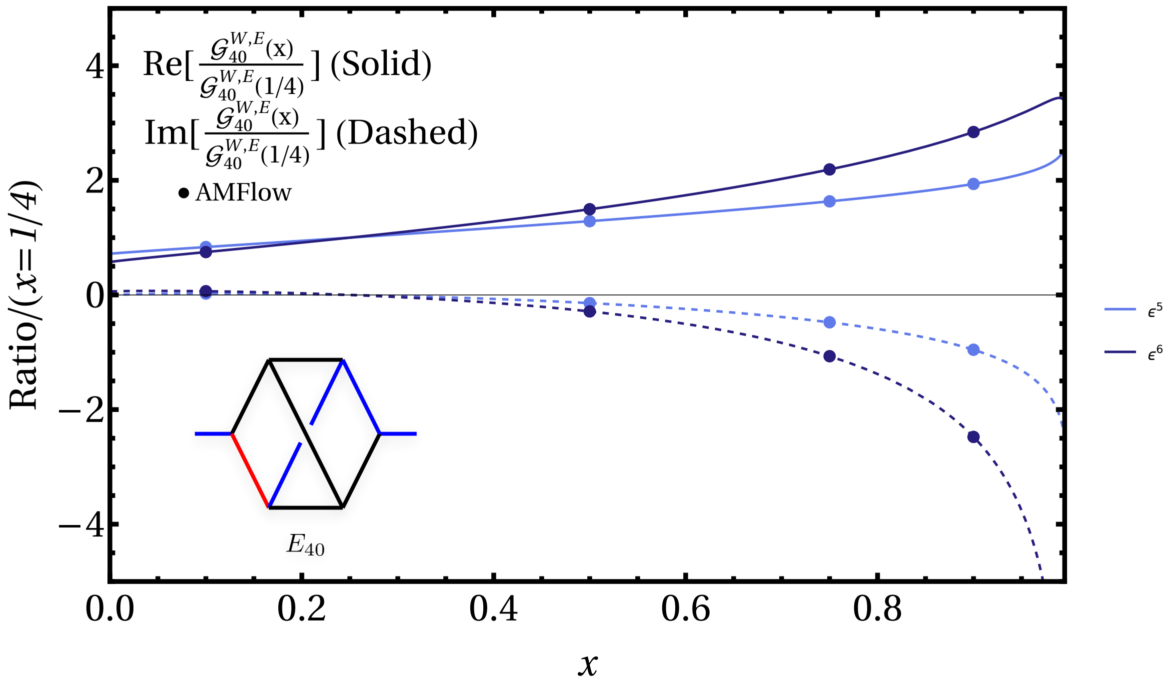
where $\tau_0 = i\infty$; using $q = \exp(2i\pi\tau)$ we can change variables

$$I(f_1, f_2 \ldots, f_n; \tau, \tau_0) = \int_{q_0}^{q} \frac{dq_1}{q_1} \int_{q_0}^{q_1} \frac{dq_2}{q_2} \cdots \int_{q_0}^{q_n} dq_n f_1(q_1) f_2(q_2) \ldots f_n(q_n)$$

Where $q_0 = 0$ at the cusp.

Results - W Non planar topology





We can calculate our results and check against numerical output from `AMFlow` Liu, Ma (2201.11669)

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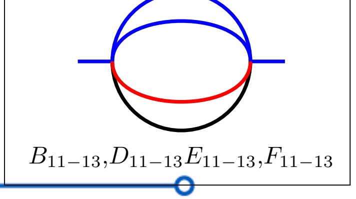
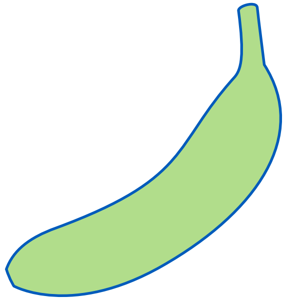
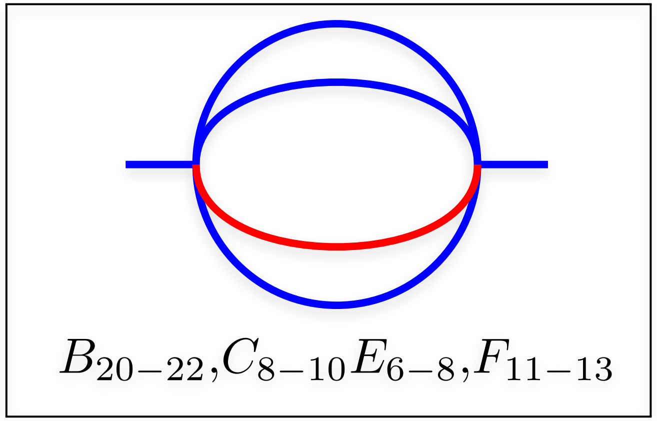
Heavy Banana Subsector - Canonical basis







$$\mathcal{G}_{20}^{H,B} = \frac{\pi^2}{\Psi_1^2}\mathcal{J}_{20}^{H,B},$$

$$\mathcal{G}_{21}^{H,B} = \frac{1}{2i\epsilon}\frac{d}{d\tau}\mathcal{G}_{20}^{H,B} - \left(F_2 + \frac{64-40x+3x^2}{12\sqrt{64-20x+x^2}}\left(\frac{\Psi_1}{\pi}\right)^2\right)\mathcal{G}_{20}^{H,B},$$

$$W = \frac{2\pi i c}{x\sqrt{(4-x)}\sqrt{16-x}}$$

$$\mathcal{G}_{22}^{H,B} = \frac{1}{2i\epsilon}\frac{d}{d\tau}\mathcal{G}_{21}^{H,B} - \left(F_2 + \frac{64-40x+3x^2}{12\sqrt{64-20x+x^2}}\left(\frac{\Psi_1}{\pi}\right)^2\right)\mathcal{G}_{21}^{H,B}$$
$$+ \left(\frac{3}{2}F_2^2 + \frac{\pi^2(8+x)^2(x^2-8x+64)}{8x^2(4-x)(16-x)W^2}\left(\frac{\Psi_1}{\pi}\right)^4\right)\mathcal{G}_{20}^{H,B},$$

We need to define F_2 to cancel the ϵ^0 pieces of the DEQ

$$F_2 = \int_{x_0}^{x}\frac{dz}{z\Psi_1^2\sqrt{64-20z+z^2}}\int_0^z\frac{(-8+w)(8+w)^3\Psi_1^4}{6\pi^2(64-20w+w^2)^2}dw$$

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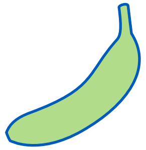
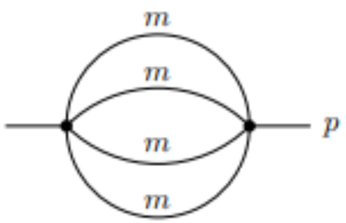
Equal-Mass Banana



Müller-Stach, Weinzierl (1212.4389)
Pogel, Wang and Weinzierl (2207.12893)
Duhr (2502.15325)



$$F_2 = (2\pi i)^2 \int_{i\infty}^{\tau} d\tau_1 \int_{i\infty}^{\tau_1} d\tau_2 \, \frac{x(x-8)(x+8)^3}{864(4-x)^{\frac{3}{2}}(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6,$$

Figure 1: The three-loop banana graph.

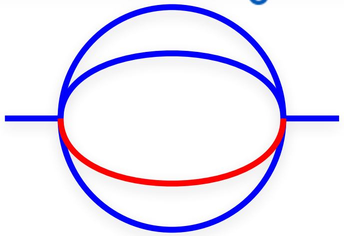
$$x = \frac{p^2}{m^2}$$

$$I_2 = \varepsilon^3 \frac{\pi^2}{\psi_1^2} I_{1111},$$

$$I_3 = \frac{1}{2\pi i\varepsilon} \frac{d}{d\tau} I_2 + \left[F_2 - \frac{\pi i(x-10)}{(x-4)(x-16)W}\left(\frac{\psi_1}{\pi}\right)^2\right] I_2,$$

$$I_4 = \frac{1}{2\pi i\varepsilon} \frac{d}{d\tau} I_3 + \left[\frac{3}{2}F_2^2 + \frac{\pi^2(x+8)^2(x^2-8x+64)}{8x^2(x-4)^2(x-16)^2 W^2}\left(\frac{\psi_1}{\pi}\right)^4\right] I_2$$
$$+ \left[-2F_2 - \frac{\pi i(x-10)}{(x-4)(x-16)W}\left(\frac{\psi_1}{\pi}\right)^2\right] I_3$$

$$x = \frac{M_V^2}{m_t^2}$$



B_{20-22},$C_{8-10}E_{6-8}$,F_{11-13}

$$\mathcal{G}_{20}^{H,B} = \frac{\pi^2}{\Psi_1^2} \mathcal{J}_{20}^{H,B},$$

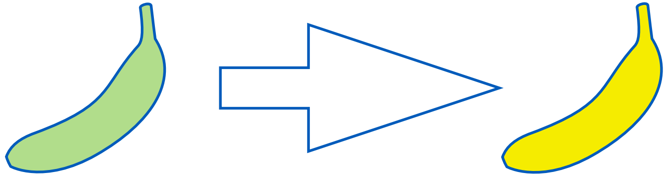
$$\mathcal{G}_{21}^{H,B} = \frac{1}{2i\epsilon} \frac{d}{d\tau} \mathcal{G}_{20}^{H,B} - \left(F_2 + \frac{64 - 40x + 3x^2}{12\sqrt{64 - 20x + x^2}} \left(\frac{\Psi_1}{\pi}\right)^2\right) \mathcal{G}_{20}^{H,B},$$

$$\mathcal{G}_{22}^{H,B} = \frac{1}{2i\epsilon} \frac{d}{d\tau} \mathcal{G}_{21}^{H,B} - \left(F_2 + \frac{64 - 40x + 3x^2}{12\sqrt{64 - 20x + x^2}} \left(\frac{\Psi_1}{\pi}\right)^2\right) \mathcal{G}_{21}^{H,B}$$
$$+ \left(\frac{3}{2}F_2^2 + \frac{\pi^2(8+x)^2(x^2-8x+64)}{8x^2(4-x)(16-x)W^2} \left(\frac{\Psi_1}{\pi}\right)^4\right) \mathcal{G}_{20}^{H,B},$$

$$F_2 = \int_{x_0}^{x} \frac{dz}{z\Psi_1^2\sqrt{64 - 20z + z^2}} \int_0^z \frac{(-8+w)(8+w)^3\Psi_1^4}{6\pi^2(64 - 20w + w^2)^2} dw$$

$$W = \frac{2\pi i c}{x\sqrt{(4-x)}\sqrt{16-x}}$$

Variable change



We make the following variable change (note $x \to 0$ is mapped to $y \to 1$)

$$x = -\frac{(y-1)(y-9)}{y}, \qquad y = \frac{1}{2}\left(10 - x - \sqrt{4-x}\sqrt{16-x}\right)$$

Which rationalizes the Wronksian, finally if we make the definition

$$W = \frac{2\pi i c}{x\sqrt{(4-x)}\sqrt{16-x}}$$

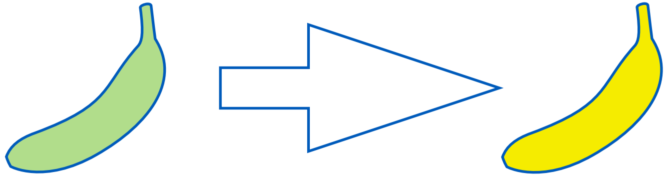
$$\Psi_1 = \sqrt{y}\,\psi_1$$

Then we have

$$\left[\frac{d^2}{dy^2} + \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-9}\right)\frac{d}{dy} + \frac{y-3}{y(y-1)(y-9)}\right]\psi_1 = 0$$

which is the same Pichard-Fuchs operator of the two-loop sunrise!

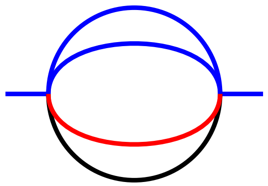
Variable change



Crucially the ratio of the periods is the same in both variables,

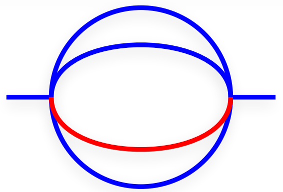
$$\tau = \frac{\Psi_2}{\Psi_1} = \frac{\psi_2}{\psi_1}$$

So we can exploit the same results mapping between the cusps at x=0 and y=1.



$B_{11-13}, D_{11-13} E_{11-13}, F_{11-13}$

$$\tau_{1,0} = \frac{\psi_{2,0}}{\psi_{1,0}}, \qquad q_{1,0} = e^{2i\pi\tau_{1,0}}$$



$B_{20-22}, C_{8-10} E_{6-8}, F_{11-13}$

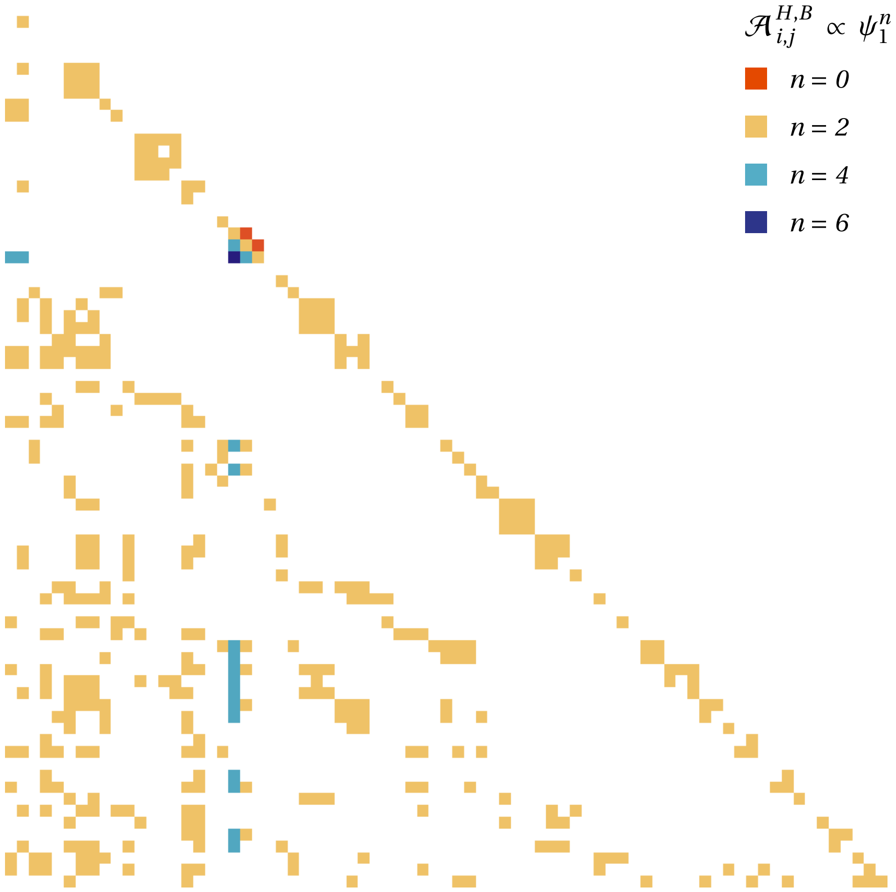
$$\begin{pmatrix} \psi_{2,1} \\ \psi_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{2,0} \\ \psi_{1,0} \end{pmatrix}$$

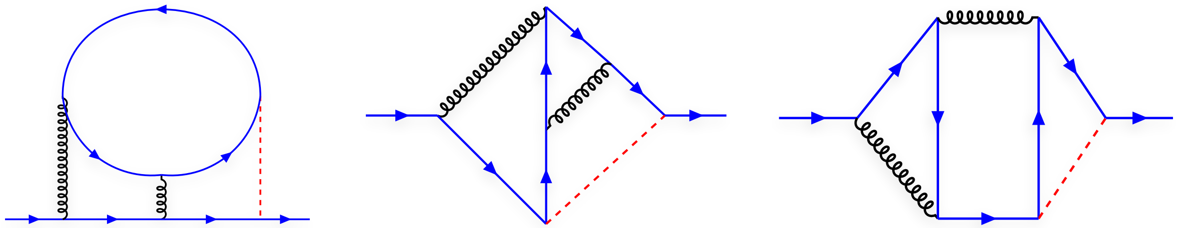
$$\tau_{6,1} = \frac{1}{6} \frac{\psi_{2,1}}{\psi_{1,1}}, \qquad q_{6,1} = e^{2i\pi\tau_{6,1}}$$

$$\psi_{1,0} = \frac{2\pi}{\sqrt{3}} \left(1 + 3q + 3q^2 + 3q^3 + 3q^4 + 3q^6 + 6q^7 + 3q^8 + \dots\right)$$

$$\psi_{1,1} = \frac{i\pi}{2} \left(1 + 2q + 4q^2 + 2q^3 + 2q^4 + 4q^6 + 4q^7 + 4q^8 + \dots\right)$$

H/Z Differential Equation Matrix



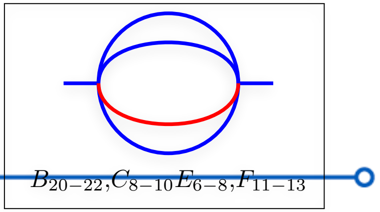


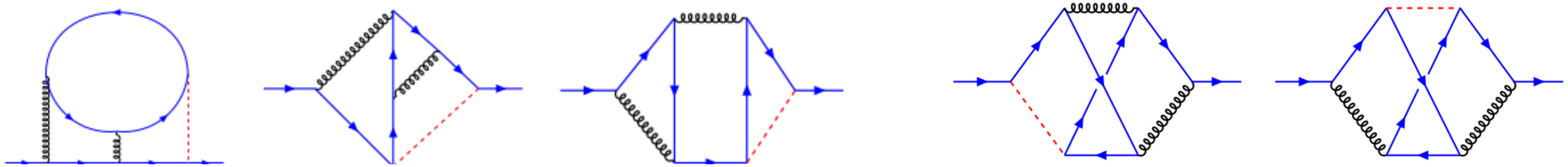
The dlog alphabet is $\{x, x-1, x-4, \sqrt{x(4-x)}\}$

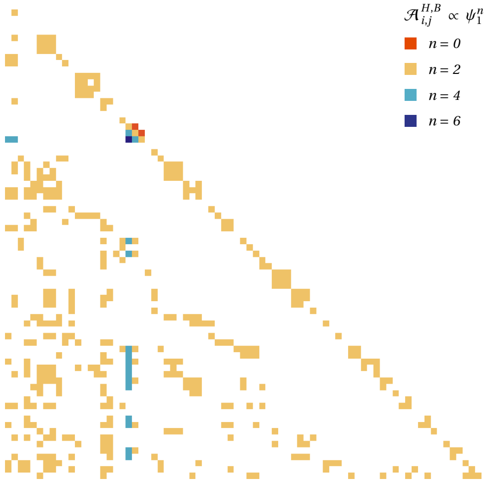
The simplest elliptic differential forms can be written as

$$dE_0 = d\tau \qquad dE_{4,1} = F_2 d\tau \qquad dE_{4,2} = F_3 d\tau$$

Differential forms for H/Z Topology







We need to introduce an additional elliptic function to get our ϵ-basis

$$\boxed{F_3 = -\int_{x_0}^{x} \frac{z(z+2)\psi_1^2(z)}{\pi^2(z(4-z))^{3/2}} dz}$$

This function does not appear in the banana 3x3 sector presented in the literature and it's required to set a canonical basis for the (elliptic) integrals that present a higher number of propagators in our basis

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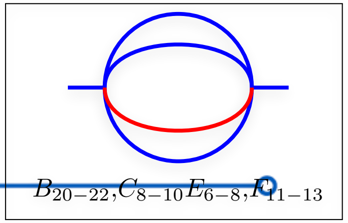
Meromorphic Forms



Power 4 of sunrise period

$$f_4^{(0)} = \frac{(3-y)(3+y)^3}{2}\left(\frac{\psi_1^2}{\pi^2}\right)^2$$

$$f_4^{(1)} = \frac{(9-y^2)(9-10y+y^2)}{2}\left(\frac{\psi_1^2}{\pi^2}\right)^2$$

$$f_4^{(2)} = -9F_2^2 + \frac{(9-18y+y^2)^2(81-108y+102y^2-12y^3+y^4)}{48(y^2-9)^2}\left(\frac{\psi_1^2}{\pi^2}\right)^2$$

$$f_4^{(4)} = \frac{-i\sqrt{(9-10y+y^2)}}{y-3}f_4^{(3)}$$

$$f_4^{(3)} = \frac{(y+3)\left((y^2-10y+9)\,(y+3)^2\psi_1^2 - 4i\pi^2(y-3)\sqrt{(9-10y+y^2)}F_3\right)}{12\pi^2(y-3)}\left(\frac{\psi_1^2}{\pi^2}\right)$$

$$f_4^{(5)} = \frac{\left(i(3+y)^3(9-8y+y^2)\sqrt{9-10y+y^2}\psi_1^2 - 2\pi^2(y^2-9)(27-22y+3y^2)F_3\right)}{12\pi^2(y-3)^2}\left(\frac{\psi_1^2}{\pi^2}\right)$$

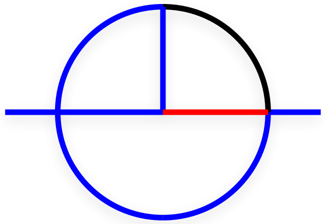
$$f_4^{(6)} = -3F_2F_3 - \frac{F_3\psi_1^2(-3+y)(9-10y+y^2)}{3\pi^2\left(y^2-9\right)} + \frac{i(y+3)\sqrt{y^2-10y+9}\,(8-12y+y^2)\,\psi_1^4}{2\pi^4(y-3)^2}$$

Power 6

$$f_6 = -6F_2^3 + \frac{(81-108y+102y^2-12y^3+y^4)F_2}{24}\left(\frac{(9-18y+y^2)\psi_1^2}{(-9+y^2)\pi^2}\right)^2 + \frac{y(9-y)(1-y)(9-2y+y^2)F_2}{36}\left(\frac{(9-18y+y^2)\psi_1^2}{(-9+y^2)\pi^2}\right)^3$$

$q-$expansion of an example

$$
\begin{aligned}
\mathcal{G}^{H,B}_{60} = {} & \epsilon\Bigg(-\frac{2\pi}{3} + i\Bigg(-\frac{16}{3}q^{1/2} - \frac{16}{9}q^{3/2} - \frac{32}{5}q^{5/2} - \frac{16}{27}q^{9/2} \\
& -\frac{128}{21}q^{7/2} - \frac{64}{11}q^{11/2} + \dots \Bigg)\Bigg) \\
& + \epsilon^2 \Bigg(\frac{4}{3}\pi(-i\pi + \log 2 + \log q) \\
& +40\pi q - 60\pi q^2 + \frac{1048\pi q^3}{3} - 1406\pi q^4 + 6960\pi q^5 - 35468\pi q^6 + \frac{1306688\pi q^7}{7} \\
& -\frac{8}{3}\sqrt{q}(3\pi + 2i(-7 + 9\log 2) + 3i\log q) \\
& -\frac{8}{27}q^{3/2}(-680i + 9\pi + 54i\log 2 + 9i\log q) \\
& -\frac{8}{21}q^{7/2}(-6193i + 24\pi + 144i\log 2 + 24i\log q) \\
& -\frac{8}{75}q^{5/2}(3251i + 90\pi + 540i\log 2 + 90i\log q) \\
& -\frac{4q^{9/2}}{1701}(4092563i + 378\pi + 2268i\log 2 + 378i\log q) \\
& -\frac{4q^{11/2}}{7623}(-94224245i + 16632\pi + 99792i\log 2 + 16632i\log q)) \\
& -\frac{8q^{13/2}}{195195}(6211146037i + 210210\pi + 1261260i\log 2 + 210210i\log q)) \\
& + \dots \Bigg)
\end{aligned}
\tag{4.58}
$$



B_{60-62}

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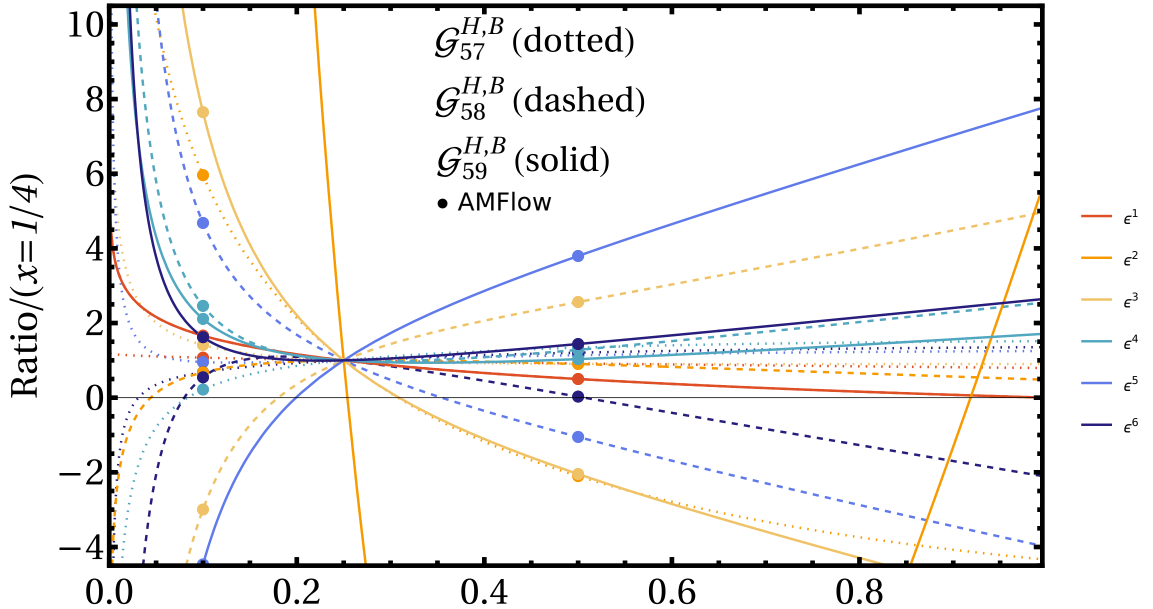
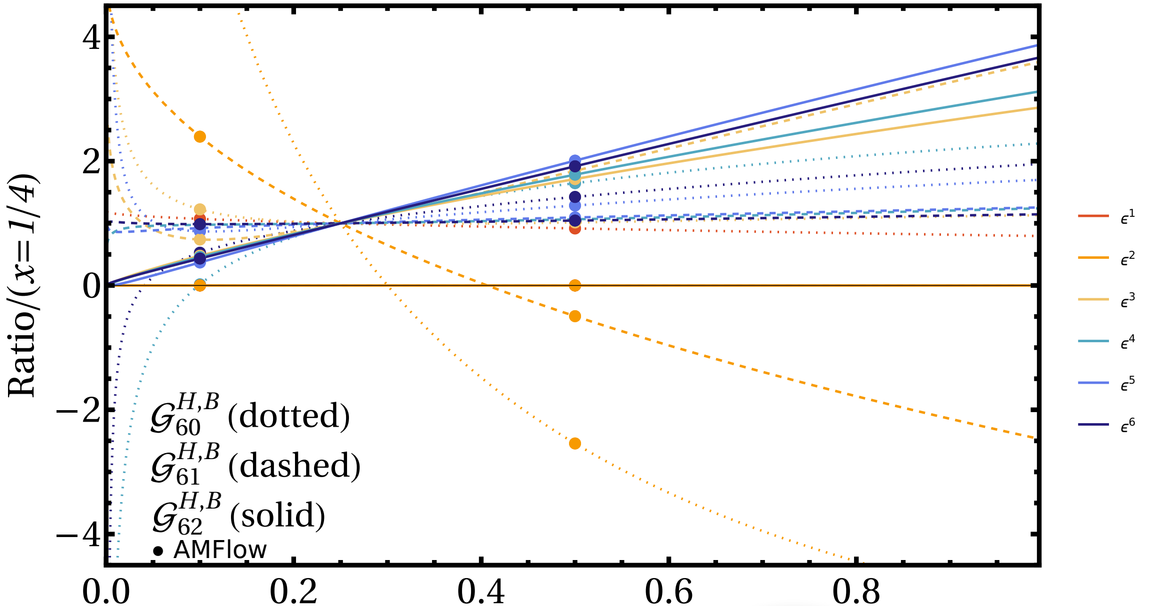
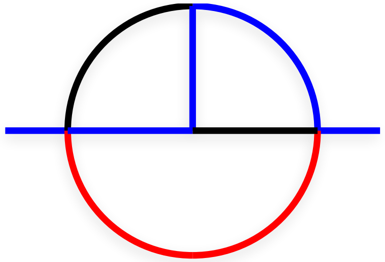
Results H/Z

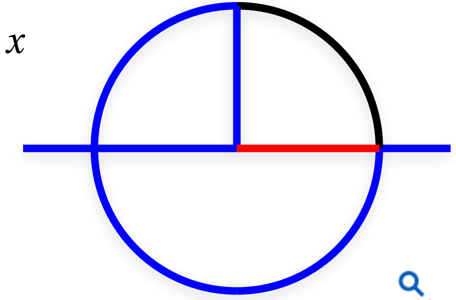






B_{57-59}, F_{46-48}

Again we are able to verify our results against `AMFlow`.



B_{60-62}

RESULTS

$\mathcal{O}(\alpha\alpha_s^2)$ Self-Energy

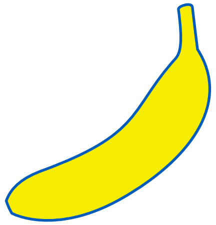
Poles - General Structure

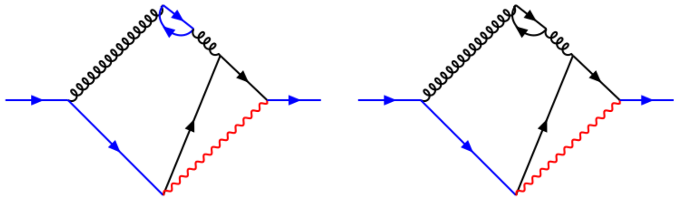
Crucially we have been able to remove any iterated integral which contains an elliptic kernel from the singular pieces. This is crucial since no elliptic kernels appear in any of the lower-loop results. Therefore it must be that they can only effect the **finite** parts of the self energy. Which we have established.

$$
\begin{aligned}
m_{0,f} = & M_f \Big\{ 1 + \Sigma_f^{(1)}(1) + \Sigma_f^{(2)}(1) + \Sigma_f^{(3)}(1) + \Sigma_f^{(1)}(1)\left[\Sigma_S^{(1)}(1) + 2\Sigma_f'^{(1)}(1)\right] \\
& + \Sigma_f^{(1)}(1)\left[\Sigma_S^{(2)}(1) + 2\Sigma_f'^{(2)}(1)\right] + \Sigma_f^{(2)}(1)\left[\Sigma_S^{(1)}(1) + 2\Sigma_f'^{(1)}(1)\right] + \\
& + \Sigma_f^{(1)}(1)\left[\Sigma_S^{(1)}(1) + 2\Sigma_f'^{(1)}(1)\right]^2 + \Sigma_f^{(1)}(1)^2\left[\Sigma_f'^{(1)}(1) + 2\Sigma_S'^{(1)}(1) + 2\Sigma_f''^{(1)}(1)\right] \Big\}
\end{aligned}
$$

$\Sigma_f^{(3)}$ - Cancellation Example of Elliptic Kernels

$$
\begin{aligned}
\Sigma^{(3),D}_{f,W} = &-\frac{1}{\epsilon^6}\frac{384\pi^3\alpha\alpha_S^2 C_F\left(C_A C_F-1\right)}{\sin^2\theta_w C_A}\frac{m_t^4\left(\left(3I^{W,D}_{23,0}+2\boxed{I^{W,D}_{25,0}}\right)m_t^2+3\left(I^{W,D}_{23,0}+\boxed{I^{W,D}_{25,0}}\right)M_W^2\right)}{\left(m_t^2-M_W^2\right)^3}\\
&\frac{1}{\epsilon^5}\frac{4\pi^3\alpha\alpha_S^2 C_F}{\sin^2\theta_w C_A}\frac{1}{m_t^2\left(3m_t^2-2M_W^2\right)\left(m_t^2-M_W^2\right)^2}\Bigg(\Big(3\big(90I^{W,D}_{1,0}-231I^{W,D}_{4,0}+16I^{W,D}_{7,0}+24\boxed{I^{W,D}_{11,0}}\\
&+24\boxed{I^{W,D}_{12,0}}-4\boxed{I^{W,D}_{13,0}}+24I^{W,D}_{16,0}+10I^{W,D}_{17,0}+4I^{W,D}_{20,0}+4I^{W,D}_{21,0}-1344I^{W,D}_{23,0}+288I^{W,D}_{23,1}\\
&-880\boxed{I^{W,D}_{25,0}}-288\log\left(\frac{m_t^2}{\mu^2}\right)\left(3I^{W,D}_{23,0}+2\boxed{I^{W,D}_{25,0}}\right)+192\boxed{I^{W,D}_{25,1}}-18I^{W,D}_{26,0}-31\boxed{I^{W,D}_{27,0}}\\
&+8C_A\Big(C_F\Big(2I^{W,D}_{7,0}-3\boxed{I^{W,D}_{11,0}}-3\boxed{I^{W,D}_{12,0}}+102I^{W,D}_{23,0}-36I^{W,D}_{23,1}+68\boxed{I^{W,D}_{25,0}}\\
&+36\log\left(\frac{m_t^2}{\mu^2}\right)\left(3I^{W,D}_{23,0}+2\boxed{I^{W,D}_{25,0}}\right)-24\boxed{I^{W,D}_{25,1}}+21I^{W,D}_{26,0}+21\boxed{I^{W,D}_{27,0}}\Big)-4\left(I^{W,D}_{26,0}+\boxed{I^{W,D}_{27,0}}\right)\Big)\big)m_t^{10}\\
&-M_W^2\Big(360I^{W,D}_{1,0}-759I^{W,D}_{4,0}+272I^{W,D}_{7,0}+336\boxed{I^{W,D}_{11,0}}-96\boxed{I^{W,D}_{12,0}}-56\boxed{I^{W,D}_{13,0}}+114I^{W,D}_{16,0}-220I^{W,D}_{17,0}\\
&+44I^{W,D}_{20,0}+56I^{W,D}_{21,0}+1488I^{W,D}_{23,0}-288I^{W,D}_{23,1}+288I^{W,D}_{24,0}+2872\boxed{I^{W,D}_{25,0}}\\
&+288\log\left(\frac{m_t^2}{\mu^2}\right)\left(3I^{W,D}_{23,0}+5\boxed{I^{W,D}_{25,0}}\right)-480\boxed{I^{W,D}_{25,1}}+18I^{W,D}_{26,0}-521\boxed{I^{W,D}_{27,0}}+576I^{W,D}_{28,0}+288\boxed{I^{W,D}_{30,0}}\\
&+16C_A\Big(C_F\Big(17I^{W,D}_{7,0}-9\boxed{I^{W,D}_{11,0}}-18\boxed{I^{W,D}_{12,0}}-213I^{W,D}_{23,0}+18I^{W,D}_{23,1}+6I^{W,D}_{24,0}-220\boxed{I^{W,D}_{25,0}}\\
&-18\log\left(\frac{m_t^2}{\mu^2}\right)\left(3I^{W,D}_{23,0}+5\boxed{I^{W,D}_{25,0}}\right)+30\boxed{I^{W,D}_{25,1}}+33I^{W,D}_{26,0}+24\boxed{I^{W,D}_{27,0}}+12I^{W,D}_{28,0}+6\boxed{I^{W,D}_{30,0}}\Big)\\
&-16\left(I^{W,D}_{26,0}+\boxed{I^{W,D}_{27,0}}\right)\Big)\Big)m_t^8+M_W^4\Big(30I^{W,D}_{1,0}-576I^{W,D}_{2,0}+207I^{W,D}_{4,0}-128I^{W,D}_{7,0}+288I^{W,D}_{8,0}+312\boxed{I^{W,D}_{11,0}}\\
&+456\boxed{I^{W,D}_{12,0}}-104\boxed{I^{W,D}_{13,0}}+32I^{W,D}_{16,0}-436I^{W,D}_{17,0}+60I^{W,D}_{20,0}+104I^{W,D}_{21,0}+3216I^{W,D}_{23,0}\\
&-576I^{W,D}_{23,1}-288I^{W,D}_{24,0}+3664\boxed{I^{W,D}_{25,0}}+1728\log\left(\frac{m_t^2}{\mu^2}\right)\left(I^{W,D}_{23,0}+\boxed{I^{W,D}_{25,0}}\right)-576\boxed{I^{W,D}_{25,1}}-222I^{W,D}_{26,0}\\
&-465\boxed{I^{W,D}_{27,0}}+576I^{W,D}_{28,0}+288\boxed{I^{W,D}_{30,0}}-8C_A\Big(28\left(I^{W,D}_{26,0}+\boxed{I^{W,D}_{27,0}}\right)+C_F\Big(72I^{W,D}_{2,0}+16I^{W,D}_{7,0}\\
&-36I^{W,D}_{8,0}-9\boxed{I^{W,D}_{11,0}}-39\boxed{I^{W,D}_{12,0}}+384I^{W,D}_{23,0}-72I^{W,D}_{23,1}+12I^{W,D}_{24,0}+312\boxed{I^{W,D}_{25,0}}+216\log\left(\frac{m_t^2}{\mu^2}\right)\left(I^{W,D}_{23,0}+\boxed{I^{W,D}_{25,0}}\right)\\
&-72\boxed{I^{W,D}_{25,1}}+39I^{W,D}_{26,0}+45\boxed{I^{W,D}_{27,0}}-24I^{W,D}_{28,0}-12\boxed{I^{W,D}_{30,0}}\Big)\Big)\Big)m_t^6\\
&+M_W^6\Big(60I^{W,D}_{1,0}+384I^{W,D}_{2,0}-279I^{W,D}_{4,0}+192I^{W,D}_{7,0}-480I^{W,D}_{8,0}+48\boxed{I^{W,D}_{11,0}}-432\boxed{I^{W,D}_{12,0}}+96\boxed{I^{W,D}_{13,0}}+14I^{W,D}_{16,0}\\
&+184I^{W,D}_{17,0}-36I^{W,D}_{20,0}-96I^{W,D}_{21,0}-288I^{W,D}_{23,0}+576I^{W,D}_{24,0}-456\boxed{I^{W,D}_{25,0}}+282I^{W,D}_{26,0}+27\boxed{I^{W,D}_{27,0}}\\
&+16C_A\Big(4\left(I^{W,D}_{26,0}+\boxed{I^{W,D}_{27,0}}\right)+3C_F\Big(8I^{W,D}_{2,0}+4I^{W,D}_{7,0}-10I^{W,D}_{8,0}-\boxed{I^{W,D}_{11,0}}-7\boxed{I^{W,D}_{12,0}}-4I^{W,D}_{23,0}\\
&+4I^{W,D}_{24,0}-9\boxed{I^{W,D}_{25,0}}+7I^{W,D}_{26,0}+5\boxed{I^{W,D}_{27,0}}\Big)\Big)\Big)m_t^4+2M_W^8\Big(3I^{W,D}_{4,0}+96I^{W,D}_{8,0}-22\boxed{I^{W,D}_{13,0}}-2I^{W,D}_{16,0}+3I^{W,D}_{17,0}\\
&+4I^{W,D}_{20,0}+22I^{W,D}_{21,0}+24\boxed{I^{W,D}_{25,0}}+16C_AC_F\left(6I^{W,D}_{8,0}+\boxed{I^{W,D}_{25,0}}\right)+6I^{W,D}_{26,0}+5\boxed{I^{W,D}_{27,0}}\Big)m_t^2\\
&+4M_W^{10}\left(2\boxed{I^{W,D}_{13,0}}-I^{W,D}_{17,0}-2I^{W,D}_{21,0}\right)\Big)\Bigg)+\dots
\end{aligned}
$$





One expansion of the self-energies is give here in terms of $I_{n,a}^{W,D}$, the $\mathcal{O}(\epsilon^a)$ coefficient of the n-th master integral, and it's a very long expression

The deepest non-zero pole is $1/\epsilon^3$ (for some topologies $1/\epsilon^2$), and all deeper poles do cancel analytically

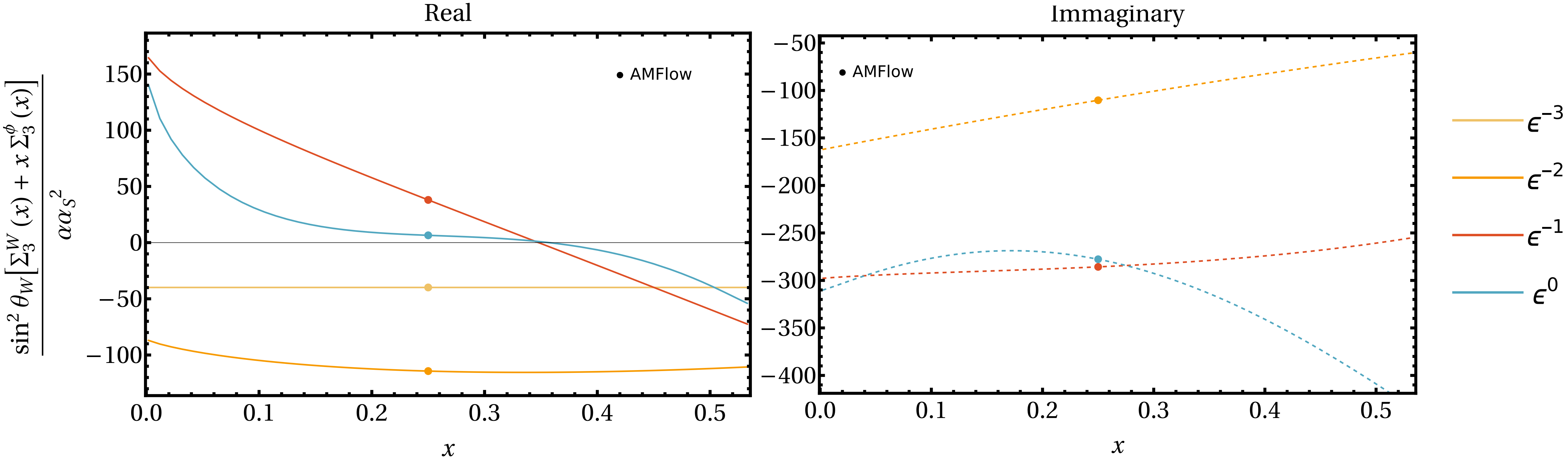
Here we show $1/\epsilon^6$ and $1/\epsilon^5$ poles of the topology D for the W diagrams. The yellow-boxed terms are the elliptic kernels

They do cancel analytically for every topology for any singular terms

$\Sigma_f^{(3)}$- W/Φ

PRELIMINARY

$$x = \frac{M_W^2}{m_t^2}$$



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SUMMARY

Next steps

Summary/Outlook

- We have completed the calculation of the MIs needed to express the three-loop mixed $\alpha_s^2\alpha$ to calculate the top quark mass at this order.
- We have obtained an analytic expression for $\Sigma_f^{(3)}$ in terms of the bare mass, free of elliptic kernels in the singular pieces
- We are assembling the pole mass, and renormalizing

$$\begin{aligned} m_{0,f} = & M_f \Big\{ 1 + \Sigma_f^{(1)}(1) + \Sigma_f^{(2)}(1) + \Sigma_f^{(3)}(1) + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1)\right] \\ & + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(2)}(1) + 2\Sigma_f^{\prime(2)}(1)\right] + \Sigma_f^{(2)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1)\right] + \\ & + \Sigma_f^{(1)}(1) \left[\Sigma_S^{(1)}(1) + 2\Sigma_f^{\prime(1)}(1)\right]^2 + \Sigma_f^{(1)}(1)^2 \left[\Sigma_f^{\prime(1)}(1) + 2\Sigma_S^{\prime(1)}(1) + 2\Sigma_f^{\prime\prime(1)}(1)\right] \Big\} \end{aligned}$$

$$m_{0,t} = Z_m^{\overline{MS}} \overline{m}_t$$

GRAZIE!

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BACKUP

Numerical corrections

	1	α	α^2	α^3
1	✓	✓	✓ [3]	
α_s	✓	✓ [2]		
α_s^2	✓ [5]	▨		
α_s^3	✓ [4]			
α_s^4	✓ [1]			
α_s^5				

[1] Marquard, Smirnov, Smirnov, Steinhauser and Wellmann 16

[2] Jegerlehner, Kalmykov, Hu 03; Eiras Steinhauser 05; Kniehl, Pickelner, Veretin 14; S.P. Martin 16

[3] Kniehl, Veretin 14

[4] Chetyrkin, Steinhauser 99

[5] Fleischer, Jegerlehner, Tarasov, Veretin 98

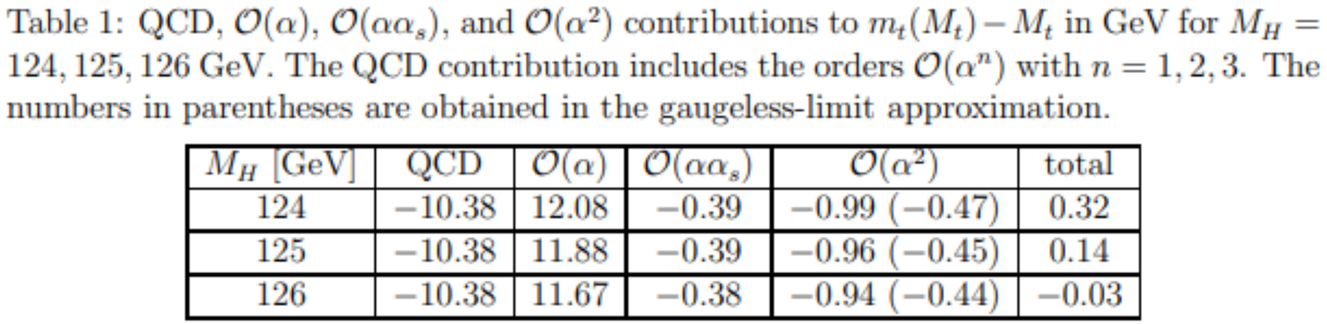
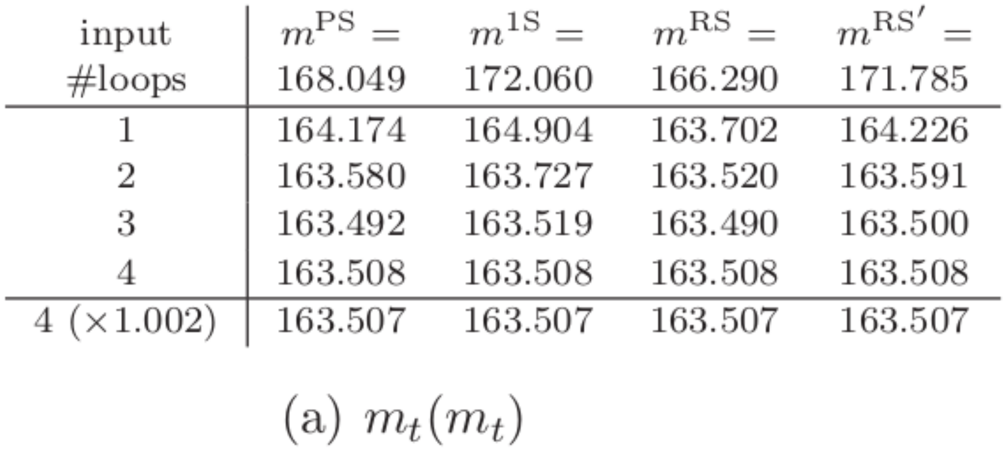


Table 1: QCD, $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha\alpha_s)$, and $\mathcal{O}(\alpha^2)$ contributions to $m_t(M_t) - M_t$ in GeV for $M_H = 124, 125, 126$ GeV. The QCD contribution includes the orders $\mathcal{O}(\alpha^n)$ with $n = 1, 2, 3$. The numbers in parentheses are obtained in the gaugeless-limit approximation.

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
124	−10.38	12.08	−0.39	−0.99 (−0.47)	0.32
125	−10.38	11.88	−0.39	−0.96 (−0.45)	0.14
126	−10.38	11.67	−0.38	−0.94 (−0.44)	−0.03

Kniehl, Pickelner, Veretin (1212.4319v2)



input #loops	$m^{\mathrm{PS}} =$ 168.049	$m^{1\mathrm{S}} =$ 172.060	$m^{\mathrm{RS}} =$ 166.290	$m^{\mathrm{RS}'} =$ 171.785
1	164.174	164.904	163.702	164.226
2	163.580	163.727	163.520	163.591
3	163.492	163.519	163.490	163.500
4	163.508	163.508	163.508	163.508
4 (×1.002)	163.507	163.507	163.507	163.507

(a) $m_t(m_t)$

Marquard, Smirnov, Smirnov, Steinhauser and Wellmann (1606.06754v2)

Renormalon Contributions

Beneke, Marquard, Nason, Steinhauser (1605.03609)

Quark pole masses are not a physical observable due to confinement

The perturbative series has an IR sensitivity coming from the divergence of the running coupling, of order Λ_{QCD}. The top decays before hadronizing, with width larger then Λ_{QCD}, which provides an IR cutoff for perturbation theory

This does not change the effects on IR ambiguity. In the $\overline{MS}$ scheme, at $\mu = m_t$, $\Lambda_{\text{QCD}} \simeq 250$MeV

The impact on the three-loop QCD correction can be estimated as 4%, whereas the $\mathcal{O}(\alpha_s^4)$ contribution is calculated at 5.2%

With the current state of the art results it is possible to estimate the corrections from 5+ loops and the uncertainty due to ambiguous definition of the pole mass

$$\alpha_s(l) = \frac{1}{b_0 \ln l^2/\Lambda_{\text{QCD}}^2} = \frac{\alpha_s(m)}{1 - \alpha_s(m) b_0 \ln m^2/l^2}$$

$$= \sum_1^{\infty} \alpha_s^n(m)\, b_0^n \ln^n \frac{m^2}{l^2}\,.$$

$$m\,(2b_0)^n \alpha_s^{n+1} n! \approx m\,\alpha_s\, n^{-n}\left(\sqrt{2\pi}\, n^{n+1/2} e^{-n}\right)$$

$$\approx m\sqrt{\frac{\pi\alpha_s}{b_0}}\,\exp\left(-\frac{1}{2b_0\alpha_s}\right)$$

$$\approx \sqrt{\frac{\pi\alpha_s}{b_0}}\,\Lambda_{\text{QCD}},$$

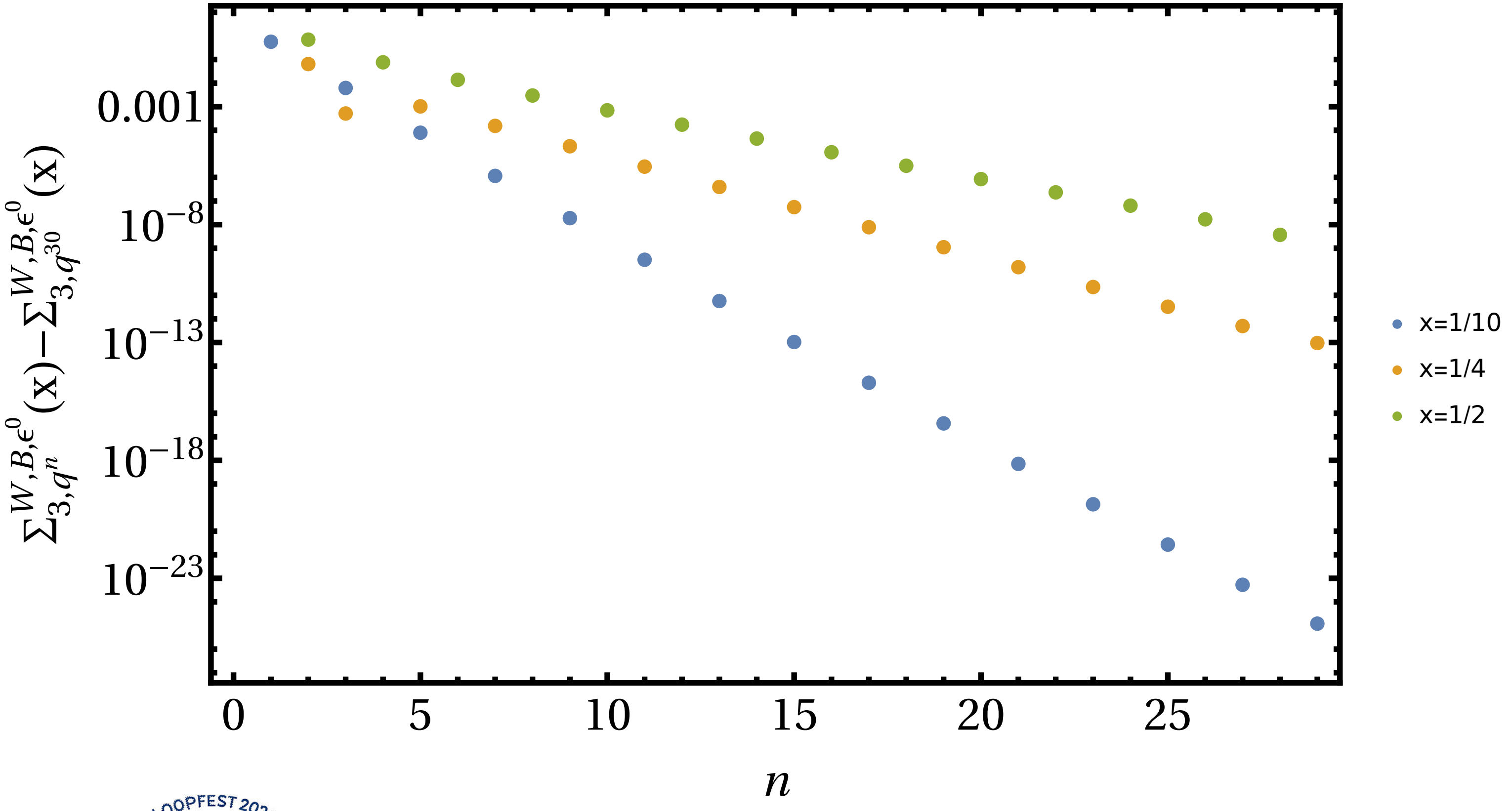
$$\delta^{(5+)} m_P = 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s)$$
$$\pm\, 0.071\,(\text{ambiguity})\,\text{GeV}\,,$$

$$\delta^{(5+)} m_P = 0.272^{+0.016}_{-0.041}\,(N) \pm 0.001\,(c_4) \pm 0.011\,(\alpha_s)$$
$$\pm\, 0.066\,(\text{ambiguity})\,\text{GeV}\,,$$

$$\delta^{(5+)} m_P = 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \pm 0.009\,(\alpha_s)$$
$$\pm\, 0.108\,(\text{ambiguity})\,\text{GeV}\,,$$

Convergence of the q-series

We see how the series behaves by truncating it and taking the difference with the expansion (up to 30 here) we see that series rapidly converges, the faster as x is closer to the cusp



$q_{1/10} = 0.011..$ $q_{1/4} = 0.031..$ $q_{1/3} = 0.0439..$

Renormalization - 2 loop results

Jegerlenher, Kalmykov (0308216v2)

The **bare** to **MS-bar** relationship for the mixed two-loop correction reads

$$\begin{aligned} m_{t,0} \; &= \; m_t(\mu^2)\, Z_t = m_t(\mu^2) \Bigg(1 + \frac{g^2(\mu^2)}{16\pi^2} \frac{1}{\varepsilon} Z_{\alpha}^{(1,1)} + \frac{\alpha_s(\mu^2)}{4\pi} \frac{1}{\varepsilon} Z_{\alpha_s}^{(1,1)} \\ &\quad + \frac{\alpha_s(\mu^2)}{4\pi} \frac{g^2(\mu^2)}{16\pi^2} \left(\frac{1}{\varepsilon} Z_{\alpha\alpha_s}^{(2,1)} + \frac{1}{\varepsilon^2} Z_{\alpha\alpha_s}^{(2,2)} \right) + \mathcal{O}(g^4, \alpha_s^2) \Bigg), \end{aligned}$$

With

$$\begin{aligned} Z_{\alpha}^{(1,1)} \; &= \; \frac{1}{3} - \frac{1}{3}\frac{m_Z^2}{m_W^2} - \frac{3}{4}\frac{m_Z^4}{m_W^2 m_H^2} - \frac{3}{8}\frac{m_H^2}{m_W^2} - \frac{3}{2}\frac{m_W^2}{m_H^2} + \frac{3}{8}\frac{m_t^2}{m_W^2} + N_c \frac{m_t^4}{m_W^2 m_H^2}\,, \\ Z_{\alpha_s}^{(1,1)} \; &= \; -3 C_f . \end{aligned}$$

$$Z_{\alpha\alpha_s}^{(2,2)} = C_f \left[\frac{m_Z^2}{m_W^2} + \frac{9}{4}\frac{m_Z^4}{m_W^2 m_H^2} - 9 N_c \frac{m_t^4}{m_H^2 m_W^2} + \frac{9}{8}\frac{m_H^2}{m_W^2} - \frac{9}{4}\frac{m_t^2}{m_W^2} + \frac{9}{2}\frac{m_W^2}{m_H^2} - 1 \right]$$

$$Z_{\alpha\alpha_s}^{(2,1)} = C_f \left[2 N_c \frac{m_t^4}{m_W^2 m_H^2} + \frac{3}{2}\frac{m_t^2}{m_W^2} + \frac{19}{48}\frac{m_Z^2}{m_W^2} + \frac{31}{24} \right],$$

RGE - 2 loops

Jegerlenher, Kalmykov, Veretin (NuclPhys B641&B658)

It’s possible to get an expression for deepest pole exploiting RGE relationships

$$\left(\gamma_t + \sum_j \beta_{g_j} \frac{\partial}{\partial g_j} + \sum_i \left[\mu^2 \frac{\partial}{\partial \mu^2} m_i^2(\mu^2)\right] \frac{\partial}{\partial m_i^2}\right) Z_t^{(1)} = \frac{1}{2} \sum_j g_j \frac{\partial}{\partial g_j} Z_t^{(2)},$$

$$\gamma_t = \frac{1}{16\pi^2}\left(g^2 Z_\alpha^{(1,1)} + g_s^2 Z_{\alpha_s}^{(1,1)}\right) + \frac{g^2 g_s^2}{(16\pi^2)^2} 2 Z_{\alpha\alpha_s}^{(2,1)} + \mathcal{O}(g^4, \alpha_s^2) ,$$

$$Z_{\alpha\alpha_s}^{(2,2)} = Z_{\alpha_s}^{(1,1)} \left(1 + m_t^2 \frac{\partial}{\partial m_t^2}\right) Z_\alpha^{(1,1)} = Z_{\alpha_s}^{(1,1)} \left(Z_\alpha^{(1,1)} + \frac{3}{8} \frac{m_t^2}{m_W^2} + 2N_c \frac{m_t^4}{m_W^2 m_H^2}\right) .$$

More complicated for $1/\epsilon$

$$m_t^2(\mu^2) = \frac{1}{2} \frac{Y_t^2(\mu^2)}{\lambda(\mu^2)} m^2(\mu^2) ,$$

$$\gamma_t = \gamma_Y + \frac{1}{2}\gamma_{m^2} - \frac{1}{2}\frac{\beta_\lambda}{\lambda} ,$$

$$\gamma_{m^2} \equiv \frac{1}{m^2}\mu^2 \frac{\partial}{\partial \mu^2} m^2 = \frac{1}{16\pi^2}\left[\lambda + 3Y_t^2 - \frac{9}{4}g^2 - \frac{3}{4}g'^2\right] + 20\frac{Y_t^2 g_s^2}{(16\pi^2)^2} + \mathcal{O}(g^4) ,$$

$$\gamma_Y \equiv \frac{1}{Y_t}\mu^2 \frac{\partial}{\partial \mu^2} Y_t = \frac{1}{16\pi^2}\left[\frac{9}{4}Y_t^2 - 4g_s^2 - \frac{9}{8}g^2 - \frac{17}{24}g'^2\right] + \frac{g_s^2}{(16\pi^2)^2}\left[18Y_t^2 + \frac{9}{2}g^2 + \frac{19}{18}g'^2\right] + \mathcal{O}(g^4) ,$$

$$Y_t^2 = \frac{2m_t^2}{v^2} , \quad \lambda = \frac{3m_H^2}{v^2} , \quad g^2 = \frac{4m_W^2}{v^2} , \quad g'^2 = \frac{4(m_Z^2 - m_W^2)}{v^2} ,$$

$$\beta_\lambda \equiv \mu^2 \frac{\partial}{\partial \mu^2}\lambda = \frac{1}{(16\pi^2)}\left[2\lambda^2 + 6\lambda Y_t^2 - 18Y_t^4 - \frac{9}{2}\lambda g^2 - \frac{3}{2}\lambda g'^2 + \frac{27}{8}g^4 + \frac{9}{4}g^2 g'^2 + \frac{9}{8}g'^4\right] + \frac{g_s^2 Y_t^2}{(16\pi^2)^2}\left[40\lambda - 96Y_t^2\right] + \mathcal{O}(g^4) . \tag{4.42}$$

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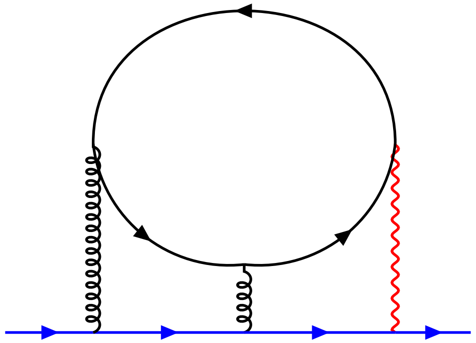
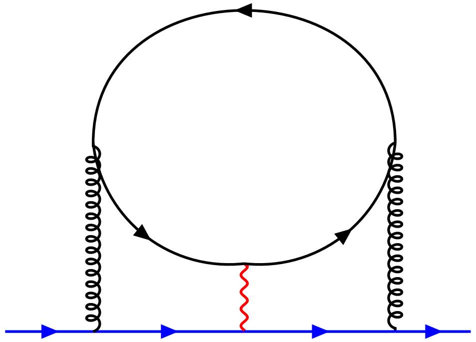
Z only topologies



Topology G: 27 MIs - 11 are new



Falls into Topology D. 17 extra MIs.

Find boundary conditions

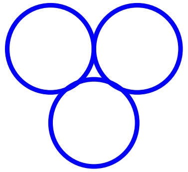
The solutions to our differential equation gives us the x-dependence of our MIs. But if we want to be able to evaluate them we need to have a boundary condition.

One of the most useful ways to do this is to look the singular behavior of our integrals as $x \to \{0,1\}$. Some of our integrals have to be finite in this limit (some even have a stricter condition and vanish at these limits.

By ensuring these integrals are finite we obtain relationships between MIs.

Obviously we want to constrain the most complicated integrals first, and have a few simple lower point integrals to calculate as inputs.

The first thing we do is normalize all our integrals such that the three-loop tadpole is trivial, i.e.



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Boundaries - Elliptics

As already mentioned, we obtain our boundaries by looking at the integrals' behavior near singular limits

$$I^{(2)}_{\ominus,(1)} = -\frac{3}{2} - 6\zeta_2\epsilon^2 - 12\zeta_3\epsilon^3 - 96\zeta_4\epsilon^4$$
$$-(48\zeta_2\zeta_3 + 216\zeta_5)\epsilon^5 - (1374\zeta_6 + 48\zeta_3^2)\epsilon^6 + \mathcal{O}(\epsilon^7)$$

In the elliptic sector we need the two-loop sunrises with one and three masses. The required inputs for the third loop factor onto a tadpole

$$\begin{aligned} I^{(2)}_{\ominus,(3)} = & -\frac{3}{2}\zeta_2\epsilon^2 + \left(\frac{9}{3}\zeta_2 \log 4 - \frac{21}{4}\zeta_3\right)\epsilon^3 \\ & + \left(\frac{93}{4}\zeta_4 + 18\zeta_2 \log^2 2 - \frac{3}{2}\log^4 2 - 36\mathrm{Li}_4\left(\frac{1}{2}\right)\right)\epsilon^4 \\ & + \Bigg(\frac{9}{40}\left(-960\mathrm{Li}_5\left(\frac{1}{2}\right) + 775\zeta_5 + 8\log^5 2\right) \\ & +\pi^2\left(\frac{5\zeta_3}{2} + 6\log^3 2\right) - \frac{31}{20}\pi^4 \log 2\Bigg)\epsilon^5 \\ & + \Bigg(- 540 s_6 - 1296\mathrm{Li}_6\left(\frac{1}{2}\right) + \frac{915\zeta_3^2}{4} - 3\pi^2\left(\zeta_3 \log(32) + 3\log^4 2\right) \\ & + \frac{281\pi^6}{360} - \frac{9\log^6 2}{5} + \frac{93}{20}\pi^4 \log^2 2\Bigg)\epsilon^6 + \mathcal{O}(\epsilon^7). \end{aligned} \tag{4.41}$$

$$s_6 = S(\{-5,-1\},\infty) = \zeta(-5,-1) + \zeta(6).$$

Sunrise Period

Since this a second order differential equation we'll get two solutions $\psi_{1,0}$ and $\psi_{2,0}$.

We can also consider a lattice Λ generated from the solutions

$$\Lambda = \{n_1\psi_{1,0} + n_2\psi_{2,0} \,|\, n_1, n_2 \in \mathbb{Z}\}$$

Provided our solutions are independent, any further pair of solutions related to the first ones via

$$\begin{pmatrix} \psi_{2,j} \\ \psi_{1,j} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_{2,0} \\ \psi_{1,0} \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

Further it can be shown that this lattice is isomorphic to the elliptic curve

$$E \;:\; w^2 - z(z+4)\left[z^2 + 2(1+x)z + (1-x)^2\right] = 0.$$

Expanding around the cusp at $x = 0$

Weinzierl (2201.03593)

For cusp around $x = 0$ we choose

$$\begin{pmatrix} \psi_{2,0} \\ \psi_{1,0} \end{pmatrix} = \frac{4}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}} \gamma \begin{pmatrix} iK(k') \\ K(k) \end{pmatrix},$$

Where k and k' are the modulus and complementary modulus of and $K(k)$ is a complete elliptic integral of the first kind $K(x) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-x^2t^2)}}$

γ is a matrix SL(2,Z) and is chosen to ensure the periods remain smooth as we cross branch cuts, in the region of interest for most of this talk $x \in [0,1]$ and $\gamma = 1_2$

Periods and cusps

The Wronksian is defined as

$$W = \psi_1 \frac{d}{dx} \psi_2 - \psi_1 \frac{d}{dx} \psi_2$$

And we will normalize our periods such that

$$W(x) = -\frac{6\pi i}{x(x-1)(x-9)}$$

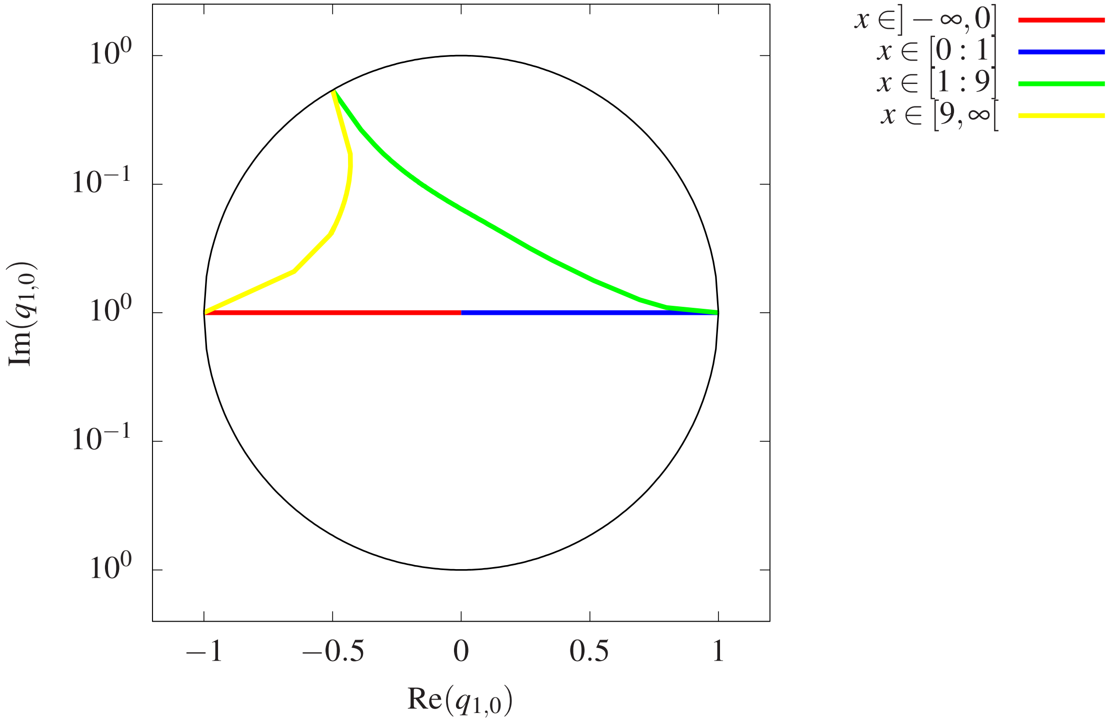
That exhibits cusps at $x = 0,1,9$ and $\pm\infty$)

q as a function of x

Hönemann, Tempest and Weinzierl (1811.09308)

With the periods defined we can plot q as a function of x

We see that at the cusp $x = 0$ $q = 0$ and then at the other three cusps $|q| = 1$



q as a function of x

Hönemann, Tempest and Weinzierl (1811.09308)

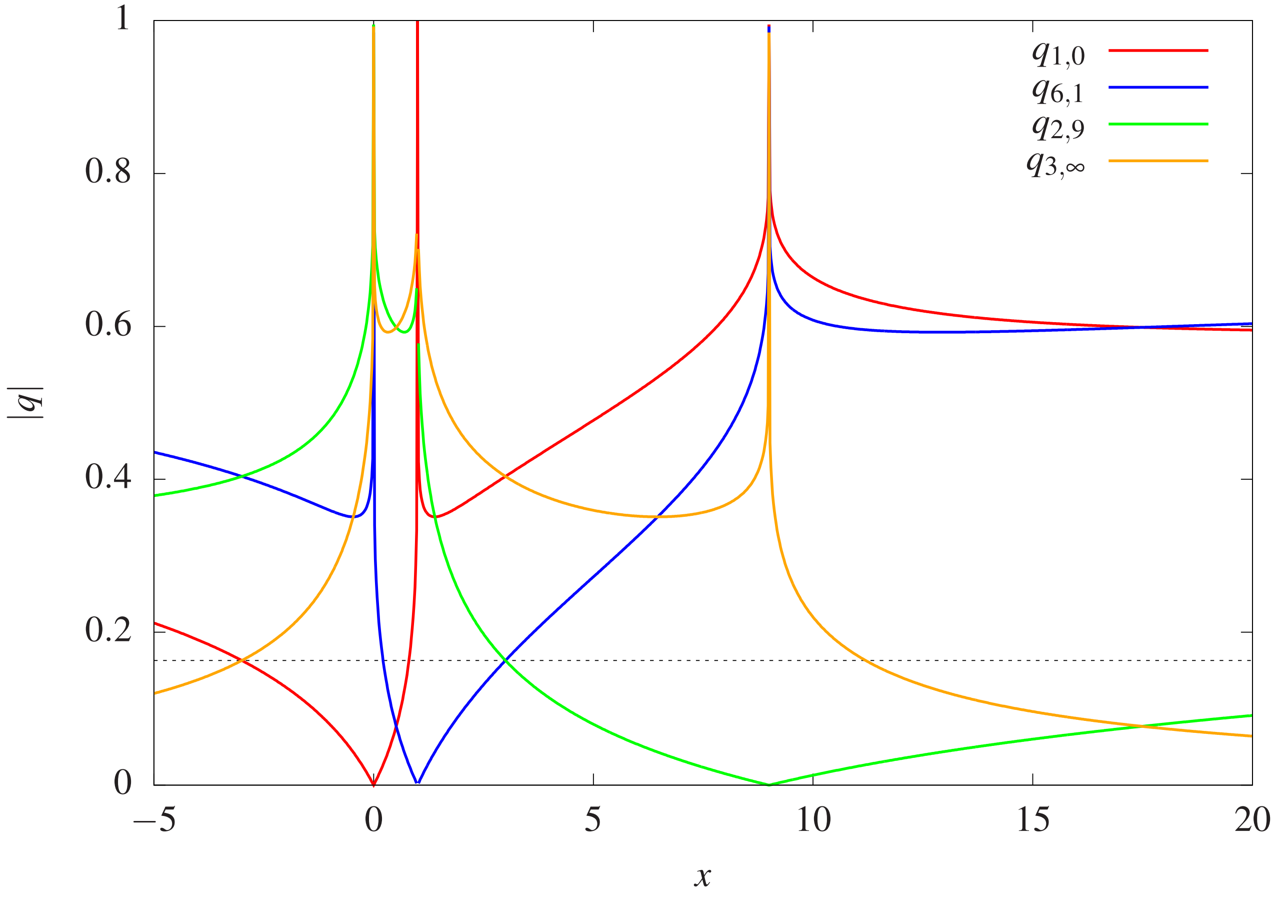
We can do the same analysis, choosing elliptic periods for expansions around the other three cusps.

The question is whether we can always find a cusp around which to ensure a convergent series.

Shown is a plot showing the size of |q| as a function of x

By choosing an appropriate cusp we can always ensure that

$$|q| \leq 0.163$$



For us $x \simeq 0.21$, so we generally expand around the $x = 0$ cusp

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x as a function of τ

Finally, the last thing we need is x as a function of τ. Around the cusp $x = 0$ we have

$$x = \frac{9\eta(\tau)^4\eta(6\tau)^8}{\eta(3\tau)^4\eta(2\tau)^8}$$

Where $\eta(\tau) = e^{\frac{i\pi\tau}{12}}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})$ is the Dedekind eta function.

We now have everything we need to write the differential forms in terms of τ and hence expand the iterated integrals as a $q-$series.

Eisenstein series for the modular forms

Weinzierl (2201.03593)

Since our differential forms have nice modular properties we can express them in a basis of modular weight one forms

$$e_1 = E_1(\tau;\chi_0,\chi_1), \qquad e_2 = E_1(2\tau;\chi_0,\chi_1),$$

Where
$$E_1(\tau;\chi_0,\chi_1) = \frac{1}{6} + \sum_{m=1}^{\infty}\left(\sum_{d|m}\chi_1(d)\right)q^m.$$

And $\chi_{0,1}$ are primitive Dirichlet characters with conductors 1 and 3.

$$\begin{aligned} \chi_0(n) &= 1, \quad \forall n \in \mathbb{Z}, \\ \chi_1(n) &= \begin{cases} 0, & n = 0 \mod 3, \\ 1, & n = 1 \mod 3, \\ -1, & n = 2 \mod 3, \end{cases} \end{aligned}$$

Expanding as a $q-$series
$$E_1(\tau;\chi_0,\chi_1) = \frac{1}{6} + q + q^3 + q^4 + 2q^7 + q^9 + \ldots$$

Modular forms

Although they complicated, the differential forms actual have a nice property that they are all **modular forms.** A (meromorphic) function f is a modular form of weight *k* if under the modular transformation $\tau' = \dfrac{a\tau + b}{c\tau + d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ then f transforms as follows

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau)$$

Our differential forms are modular forms of weight 0, 3 3 and 4 respectively

$$de_0 = d\tau, \quad de_{3,0} = \frac{\psi_1}{\pi}\frac{dx}{x} = \frac{\psi^3}{2\pi i W} d\tau \qquad de_{3,1} = \frac{\psi_1}{\pi}\frac{dx}{x-1} = \frac{\psi^3}{2\pi i W}\frac{d\tau}{x-1} \quad de_4 = \frac{\psi^2}{2\pi i W}\left(\frac{\psi^2}{\pi^2}\right)\frac{81 + 1188x - 594x^2 + 372x^3 - 23x^4}{48x(x-1)(x-9)} d\tau$$

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Eisenstein Series and q-expansion

We now have a convenient series representation for each of our modular forms, for example

$$\frac{\psi^3}{2\pi i W}\frac{1}{x-1} = 36\sqrt{3}(e_1^3 + 2e_1^2 e_2 - e_1 e_2^2 - 2e_2^3)$$

$$= -6\sqrt{3}q(1 + 4q + 9q^2 + 16q^3 + \ldots)$$

We note that $d\log$ letters are naturally modular forms of weight 2, so all of our differential forms can be conveniently written in this manner.

Evaluation of the iterated integral now corresponds to simply performing a series expansion and simple integration in q.

Note that the final expression can be kept analytic until the q-series needs to be performed

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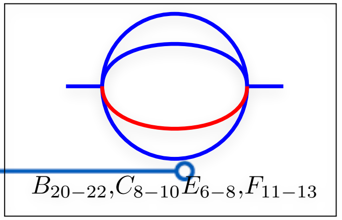
Picard-Fuchs for heavy banana

Müller-Stach, Weinzierl (1212.4389)



The Picard-Fuchs operator for this family of MIs is a third order differential operator

$$L_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2(x-4)} + \frac{3}{2(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2(x-4)(x-16)}\frac{d}{dx} + \frac{1}{x^2(x-16)}$$

However, $L_3^{(0)}$ is the symmetric product of $L_2^{(0)}$ (Verrill 96) where

$$L_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2(x-4)} + \frac{1}{2(x-16)}\right]\frac{d}{dx} + \frac{x-8}{4x(x-4)(x-16)}$$

Then we define $L_2^{(0)}\Psi_i = 0 \quad i = 1,2$ And $\omega_1 = c_1\Psi_1^2 + c_2\Psi_1\Psi_2 + c_3\Psi_2^2$

Solves $L_0^{(3)}\omega_1 = 0.$ We'll choose $c_1 = 1 \quad c_2 = c_3 = 0$

Tangential Base Point

Not all the forms vanish at $q_0 \to 0$, but we can regularize them with a small q_0

$$I(f_1, f_2 \ldots, f_n; \tau, \tau_0) = \lim_{q_0 \to 0} R\left[\int_{q_0}^{q} \frac{dq_1}{q_1} \int_{q_0}^{q_1} \frac{dq_2}{q_2} \cdots \int_{q_0}^{q_n} dq_n f_1(q_1) f_2(q_2) \ldots f_n(q_n)\right]$$

They will manifest as $\log(q_0)$ pieces. The R operator removes them. The final result is free of singularities when combined with the boundary terms, that will also contain such divergent pieces.

In our calculation we keep all of them separately and cancel them in the final stage