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Collinear limits of multi-leg scattering amplitudes

Sebastian Jaskiewicz

Based on **JHEP 02 (2026) 173 (2507.21854)** in collaboration with Claude Duhr, Einan Gardi, Jonas Lübken, and Leonardo Vernazza

LoopFest XXIV
BNL

Upton, NY

27th May 2026

Infrared Singularities

[S. Catani, hep-ph/9802439] [L. Dixon, S. Mert-Aybat, G. Sterman, 0607309]
[T. Becher, M. Neubert, 0903.1126] [E. Gardi, L. Magnea, 0908.3273]
[N. Agarwal, L. Magnea, C. Signorile-Signorile, A. Tripathi, 2112.07099]

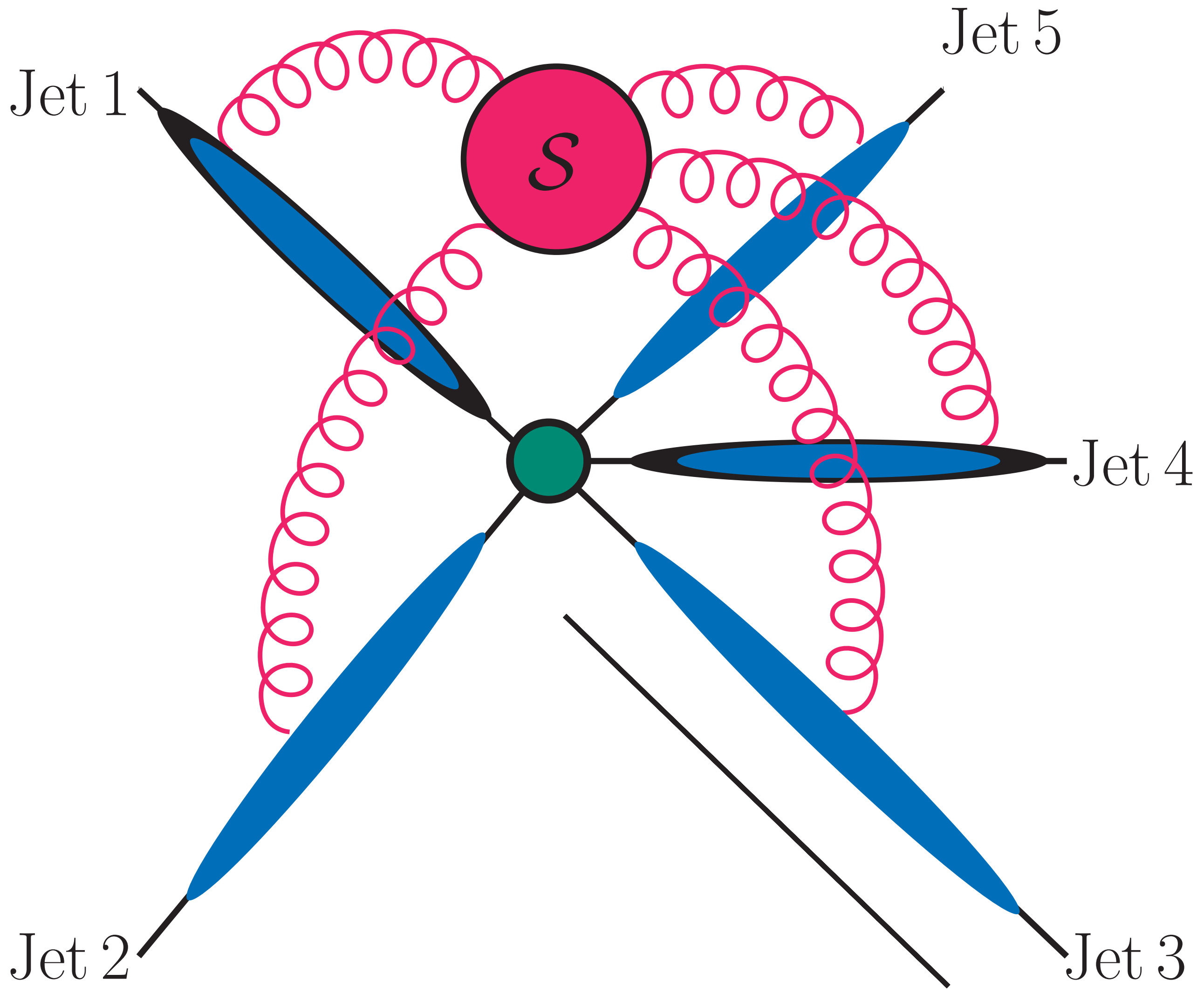
QCD amplitudes develop divergences in soft and collinear limits.

They cancel between different multiplicities - requires sophisticated subtraction techniques. Also give rise to enhanced logarithmic corrections, requiring resummation.

$$\mathscr{M}_n(\{p_i\},\mu,\epsilon) = \mathbf{Z}_n(\{p_i\},\mu,\epsilon)\mathscr{H}_n(\{p_i\},\mu)$$

$$\mathbf{Z}_n\left(\{p_i\},\mu,\epsilon\right) = \mathscr{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s\right)\right\}$$

$\mathbf{\Gamma}_n$ is the Soft Anomalous Dimension, this is the object that governs the IR singularities of amplitudes.



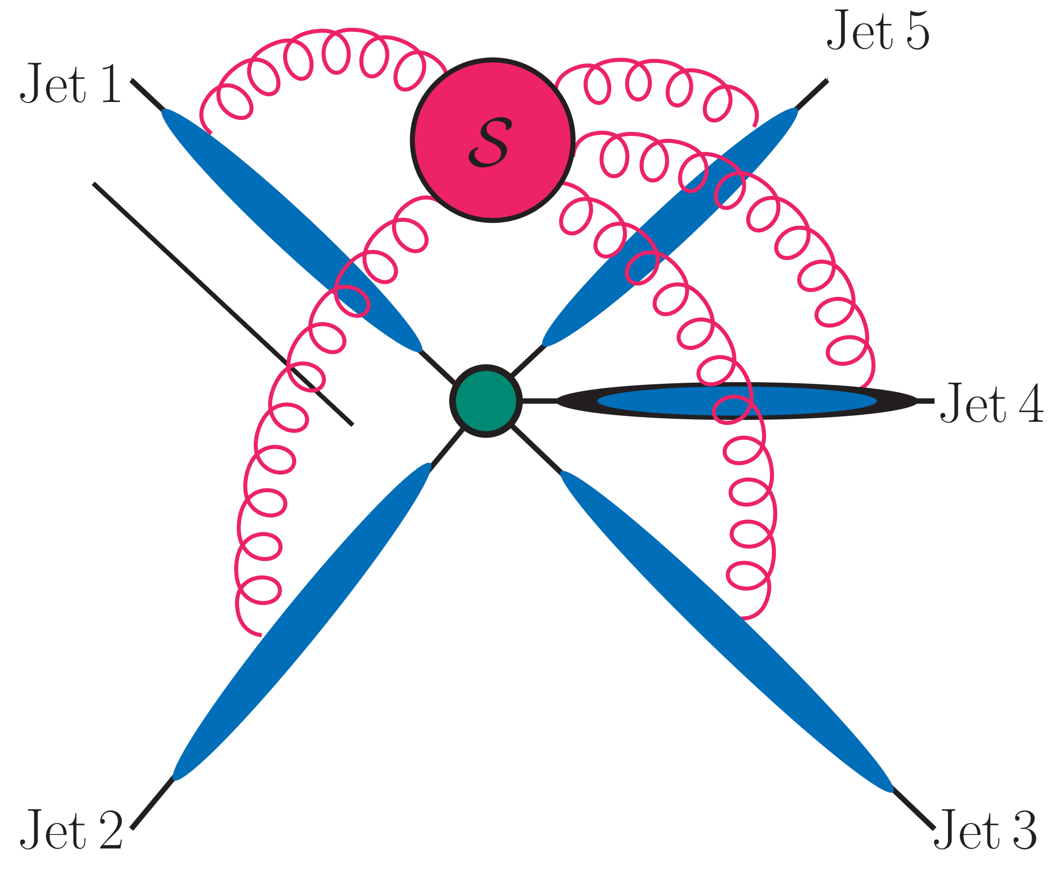
Can be computed with Wilson lines
[Korchemsky, Radyushkin,1986]

$$\mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s)\sum_{i<j}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,\mathbf{1}$$

$$(-s_{ij}) = 2|p_i\cdot p_j|e^{-i\pi\lambda_{ij}}$$

$\lambda_{ij} = 1$ if partons i and j both belong to either the initial or the final state, and 0 otherwise.

Soft Anomalous Dimension



Corrections to the dipole formula of the soft anomalous dimension have been studied intensively. The structure through four-loops is

$$\boldsymbol{\Gamma}_n\left(\{p_i\}, \lambda, \alpha_s(\lambda^2)\right) = \boldsymbol{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\}, \lambda, \alpha_s\right) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\}, \alpha_s)$$

$$+\boldsymbol{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\}, \lambda, \alpha_s) + \boldsymbol{\Gamma}_{n,\text{Q4T}-\text{4L}}(\{\beta_{ijkl}\}, \alpha_s)$$

$$+\boldsymbol{\Gamma}_{n,\text{5T}-\text{4L}}(\{\beta_{ijkl}\}, \alpha_s) + \boldsymbol{\Gamma}_{n,\text{5T}-\text{5L}}(\{\beta_{ijkl}\}, \alpha_s) + \mathcal{O}(\alpha_s^5)$$

Three-loop correction

Calculated explicitly in

[Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

Four-loop correction

The four-loop cusp anomalous dimension has been computed [J. Henn, G. Korchemsky, B. Mistlberger, 1911.10174]

Constraints on the functional form have also been determined using Regge and two-particle collinear limits

[A. Vladimirov 1707.07606] [T. Becher and M. Neubert, 1908.11379]

[G. Falcioni, E. Gardi, N. Maher, C. Milloy, L. Vernazza, 2111.10664]

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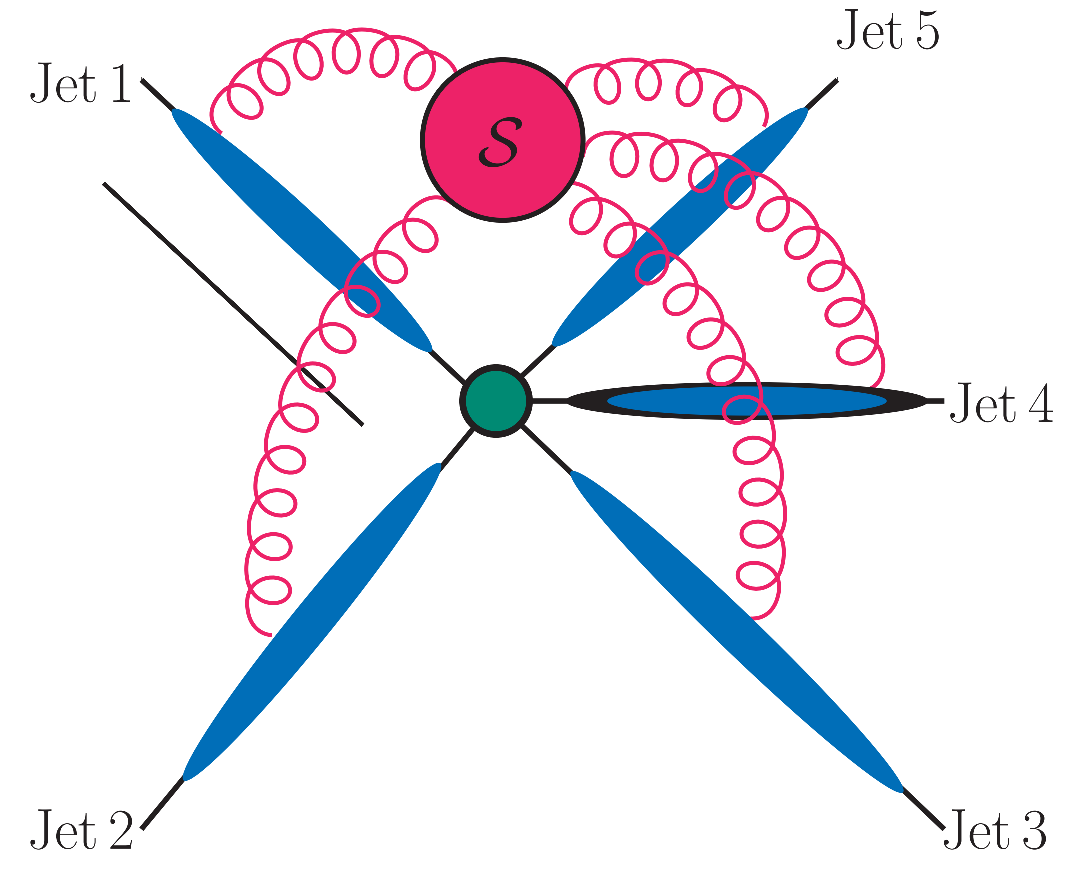

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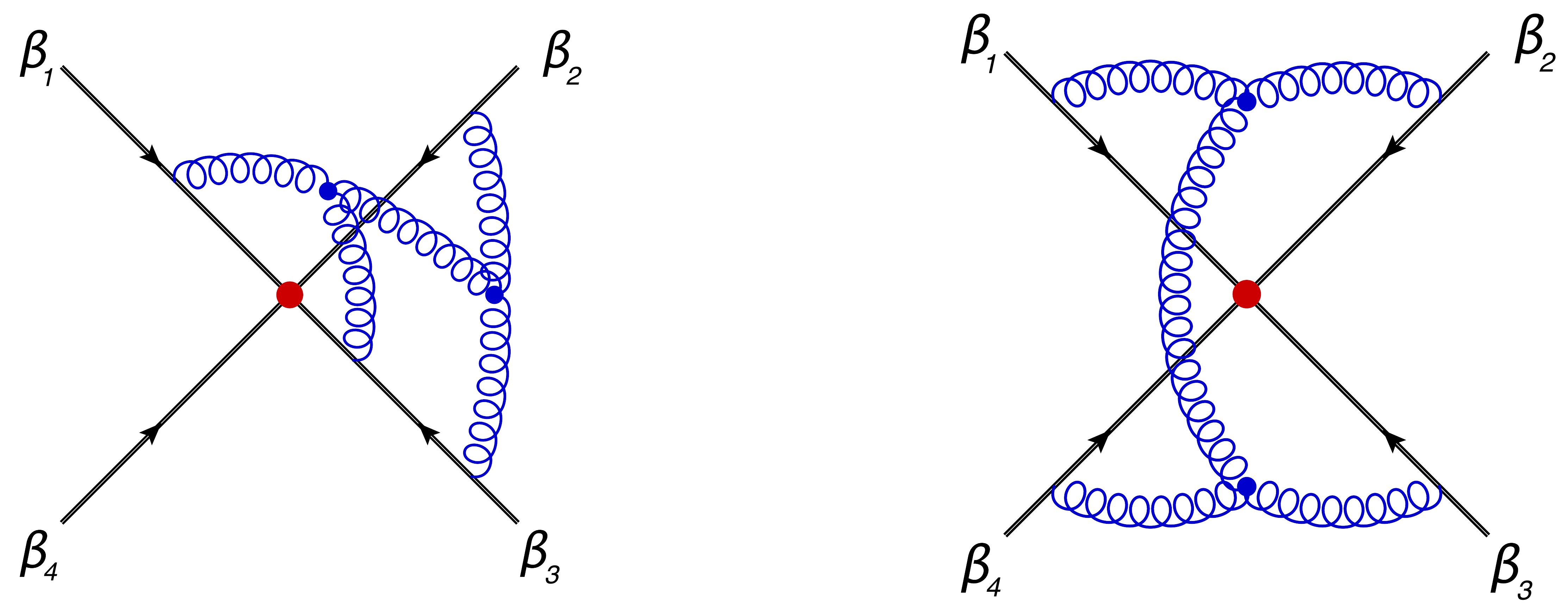
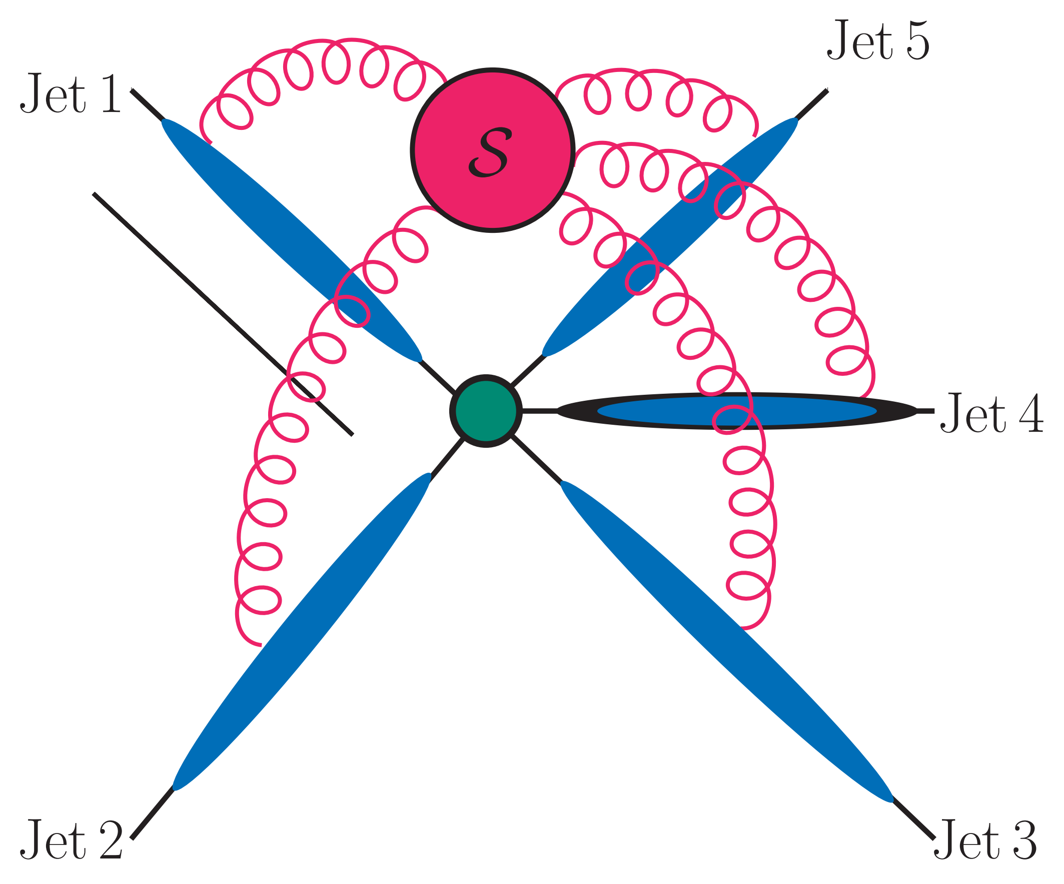


$$\boldsymbol{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n}\left[f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_l^c\mathbf{T}_j^d\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + \ldots\right.$$

Soft Anomalous Dimension



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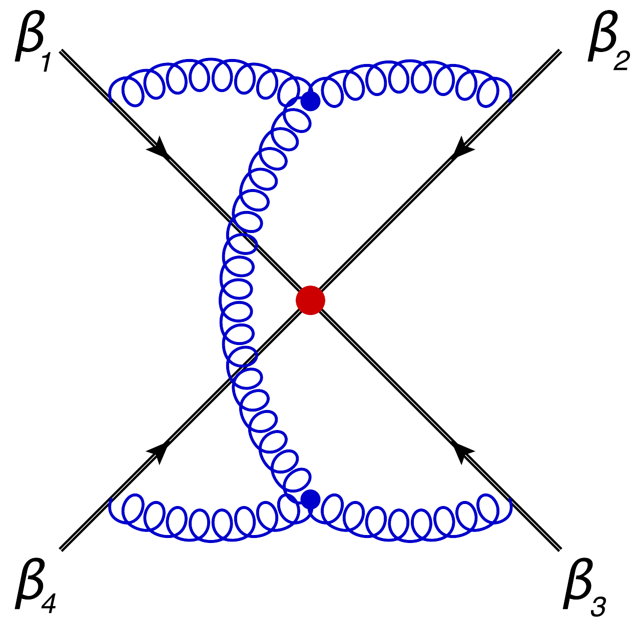
[Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

Functional form also obtained by bootstrap techniques

[Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]



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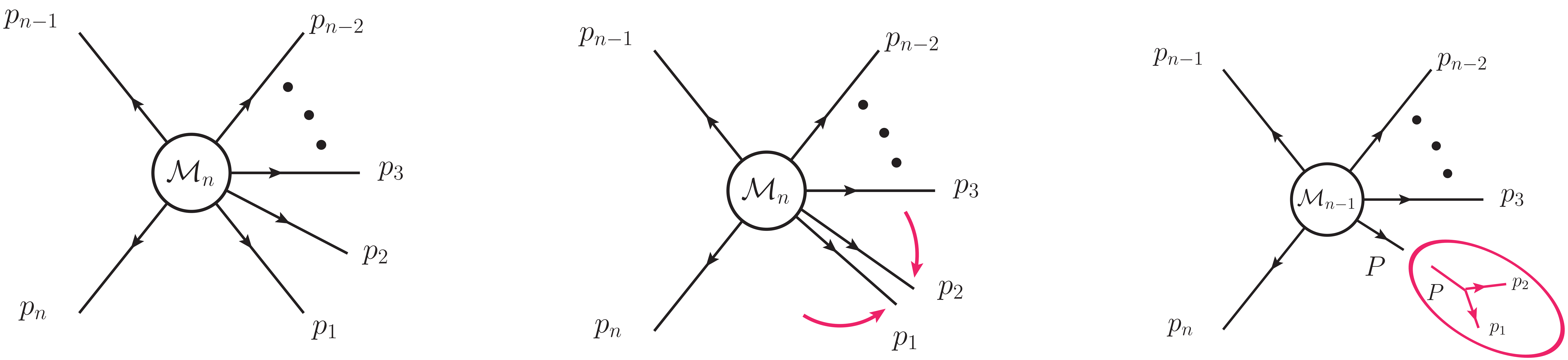
[G. Falcioni, E. Gardi, N. Maher, C. Milloy, L. Vernazza, 2111.10664]

Difficult to compute, but insights can be gained from investigating limits. We will focus on collinear limits.

Collinear limits

Relax the wide-angle condition and allow now $p_1 \cdot p_2 \to 0$



$$\mathscr{M}_n(p_1, p_2, \{p_i\}_{\text{rest}}; \mu) \xrightarrow{p_1 \parallel p_2} \mathbf{Sp}_2(p_1, p_2; \mu)\, \mathscr{M}_{n-1}(P, \{p_i\}_{\text{rest}}; \mu)$$

[F. Berends, W. Giele, 1989]
[Z. Bern, G. Chalmers, 9503236]
[D. Kosower, 9901201]
[M. Mangano,S. Parke, 0509223]
[I. Feige, M. Schwartz, 1403.6472]

$\mathbf{Sp}_2$ depends only on the degrees of freedom of particles 1 and 2 $\to$ **Strict collinear factorisation**

[S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

Splitting amplitude soft anomalous dimension

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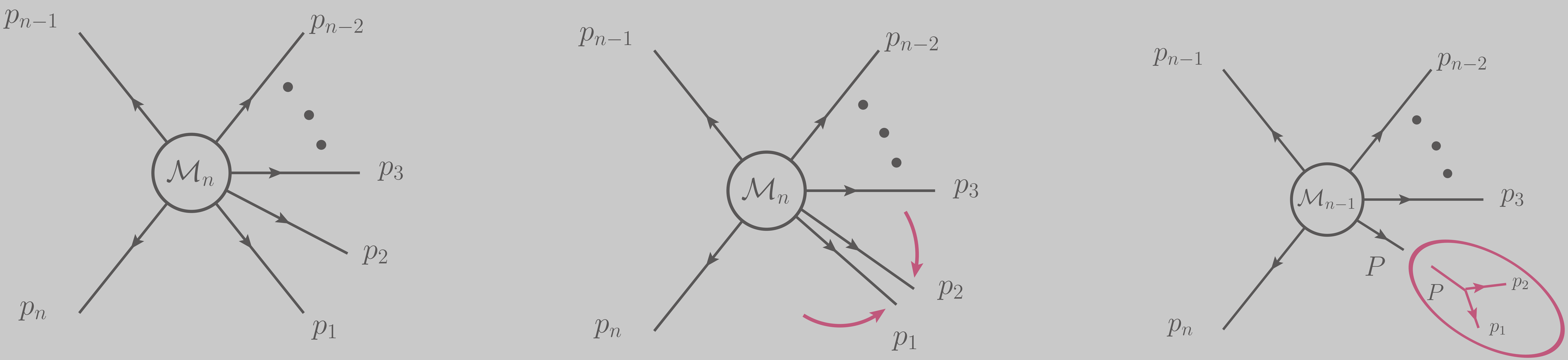


$$\frac{d}{d\ln\mu}\mathbf{Sp}_{\mathscr{H},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu)\,\mathbf{Sp}_{\mathscr{H},2}(p_1,p_2;\mu)$$

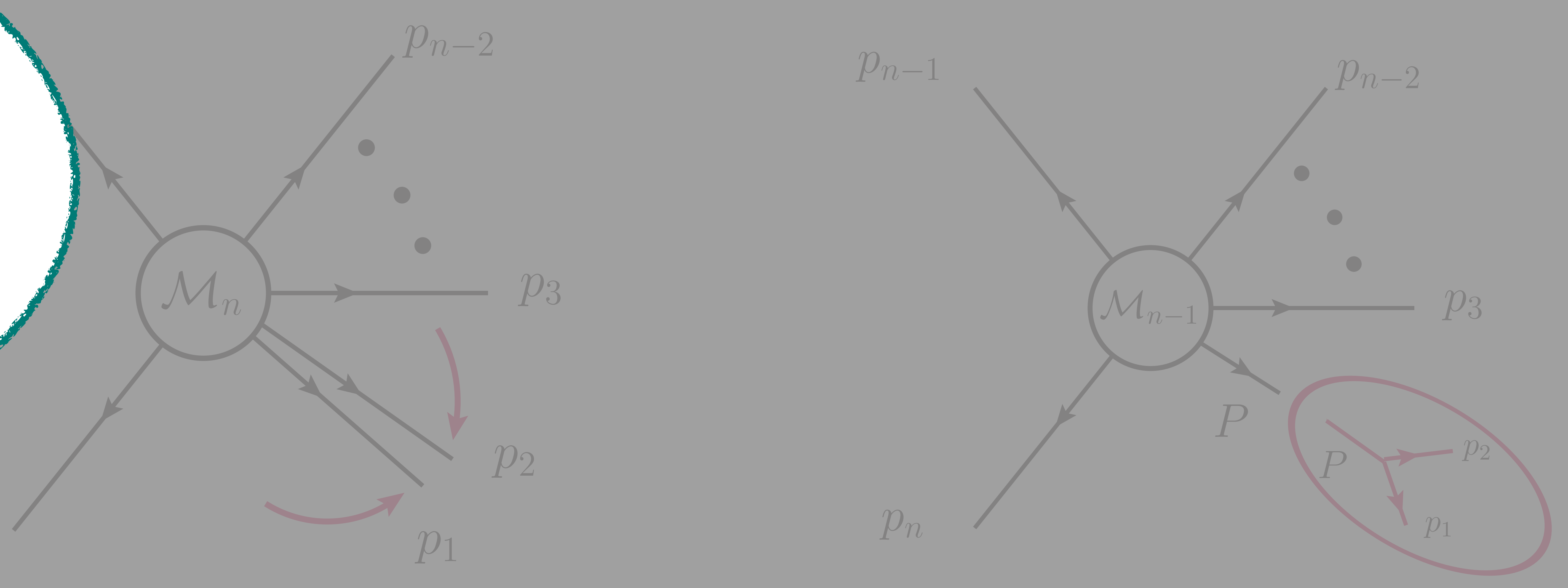
$$\mathbf{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_n(p_1,p_2,p_3,\ldots p_n;\mu) - \mathbf{\Gamma}_{n-1}(P,p_3,\ldots p_n;\mu)\,\Big|_{\mathbf{T}_P\to\sum_{i=1}^{2}\mathbf{T}_i}$$

Central object in the investigation.

Remarkable cancellations need to occur on the right-hand side!

Aims, method, and different perspectives

Relax the wide-angle co… allow now $p_1 \cdot p_2 \to 0$



Check of expectations

Up to two loops it is straightforward to show how this is satisfied.

Complicated colour structures at higher loops. Is strict collinear factorization satisfied? How?

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Constraints/Bootstrap

Imposing collinear factorisation gives constraints on the form of the allowed structures.

E.g. Constraints from 2-particle collinear limits used in the 3-loop bootstrap [Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]

$$\mathcal{F}^{\mathrm{S}}_{1234}(\{\beta\})\Big|_{p_1||p_2} = f(\alpha_s)$$

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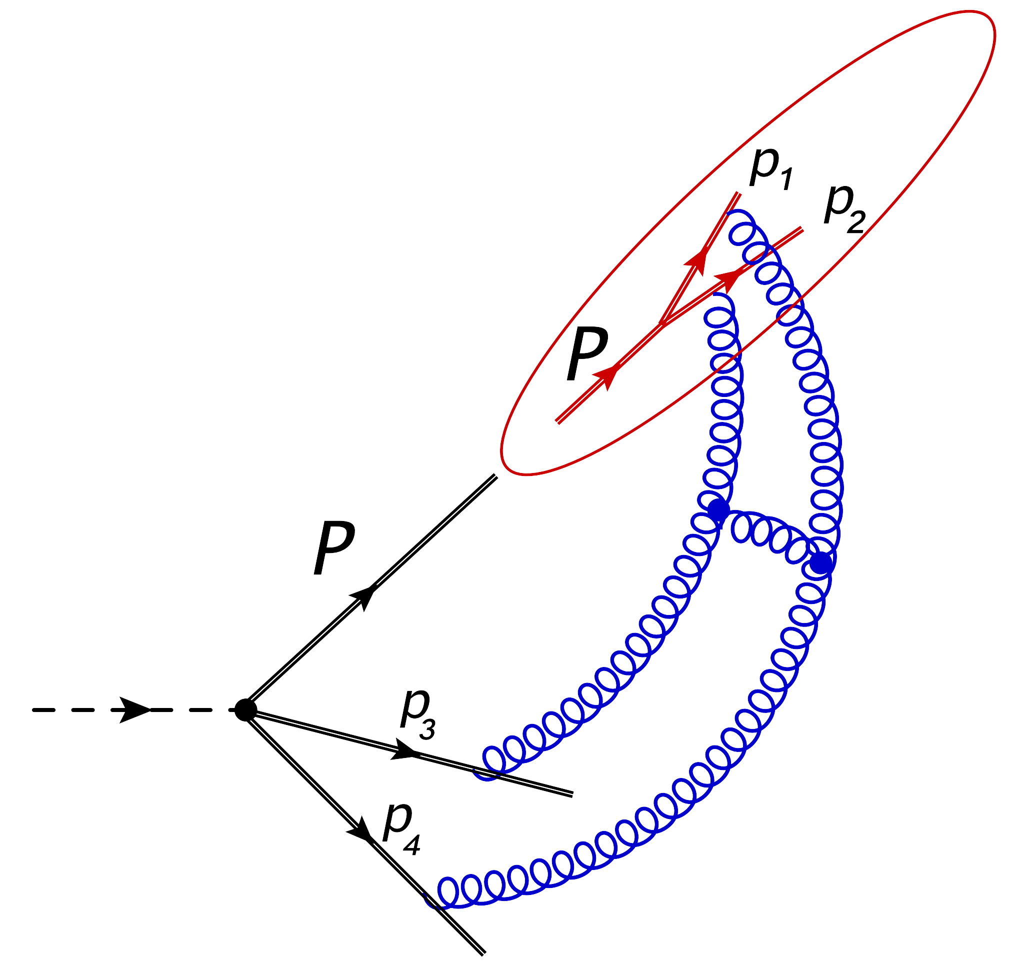

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- **Check of expectations**
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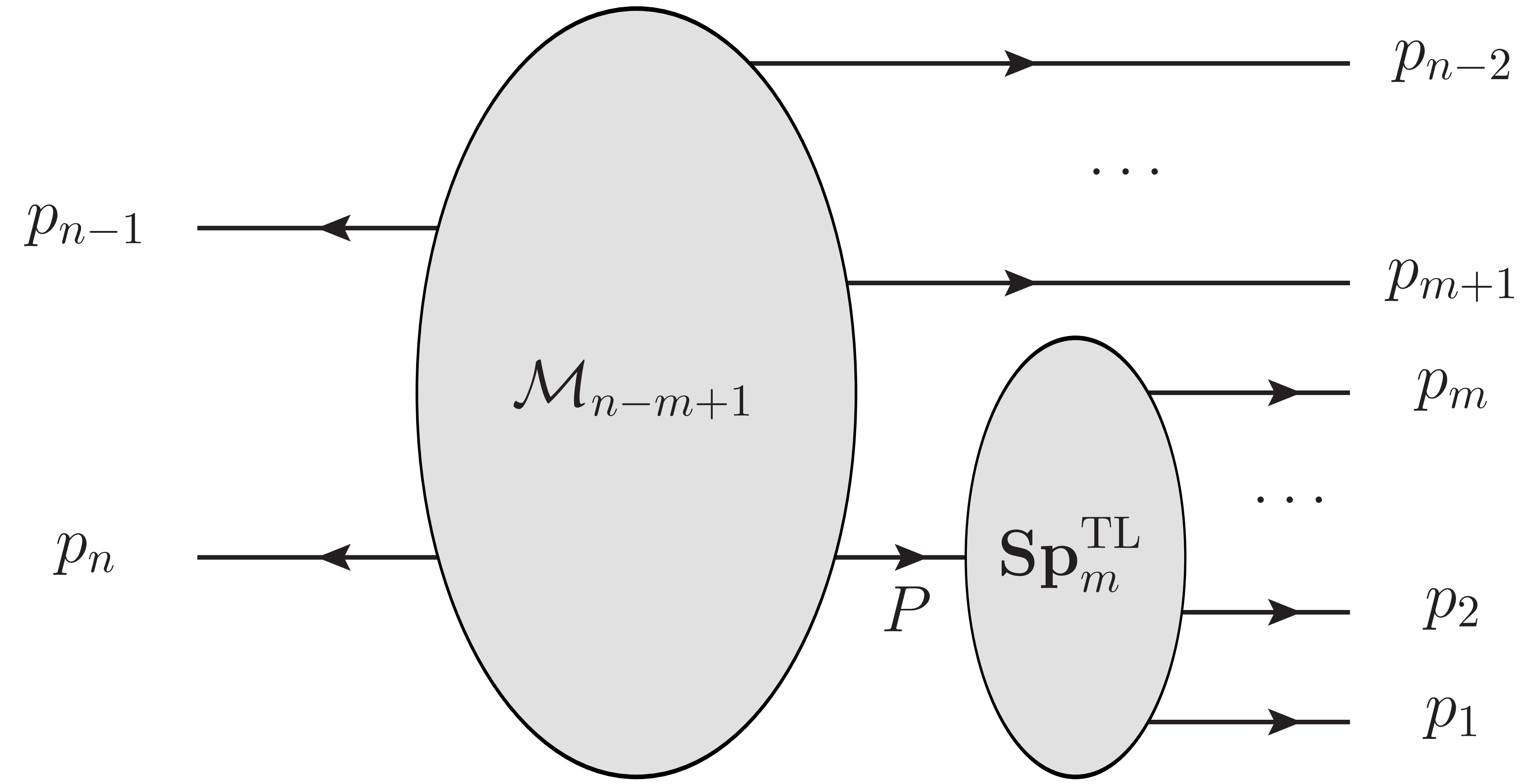

$$\frac{d}{d\ln\mu}\mathbf{Sp}_{\mathscr{H},m}(p_1,\ldots,p_m;\mu) = \boldsymbol{\Gamma}_{\mathbf{Sp},m}(p_1,\ldots,p_m;\mu)\,\mathbf{Sp}_{\mathscr{H},m}(p_1,\ldots,p_m;\mu)$$

$$\boldsymbol{\Gamma}_{\mathbf{Sp},m}(p_1,\ldots,p_m;\mu) = \boldsymbol{\Gamma}_n(p_1,p_2,p_3,\ldots p_n;\mu) - \boldsymbol{\Gamma}_{n-m+1}(P,p_{m+1},\ldots p_n;\mu)\,\Big|_{\mathbf{T}_P\to\sum_{i=1}^m \mathbf{T}_i}$$

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What can we learn?

[C. Duhr, E. Gardi, SJ, J. Lübken, L. Vernazza, **2507.21854**]

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Universality of two-particle collinear limit constraints (m=2). Do we gain anything from considering e.g. n = 8 and n = 7? (on top of n = 4 and n = 3)

Do multi-collinear ($m \geq 3$) limits yield additional constraints on top of the ones obtained in the two-particle collinear limit?

Does anything change if we consider the anomalous dimensions with massive partons in the external states?

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Does anything change if we consider the anomalous dimensions with massive partons in the external states?

✓ **Yes, new constraints do arise in the three-particle collinear limit at the three loop order, for amplitudes containing one massive particle on top of two-particle collinear limit constraints. (Identical to constraint from the small-mass expansion)**

$\Gamma_{\mathbf{Sp},2}$ at the dipole level

We now do a first example using the dipole contribution:

$$\Gamma_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s)\sum_{i<j}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,\mathbf{1}$$

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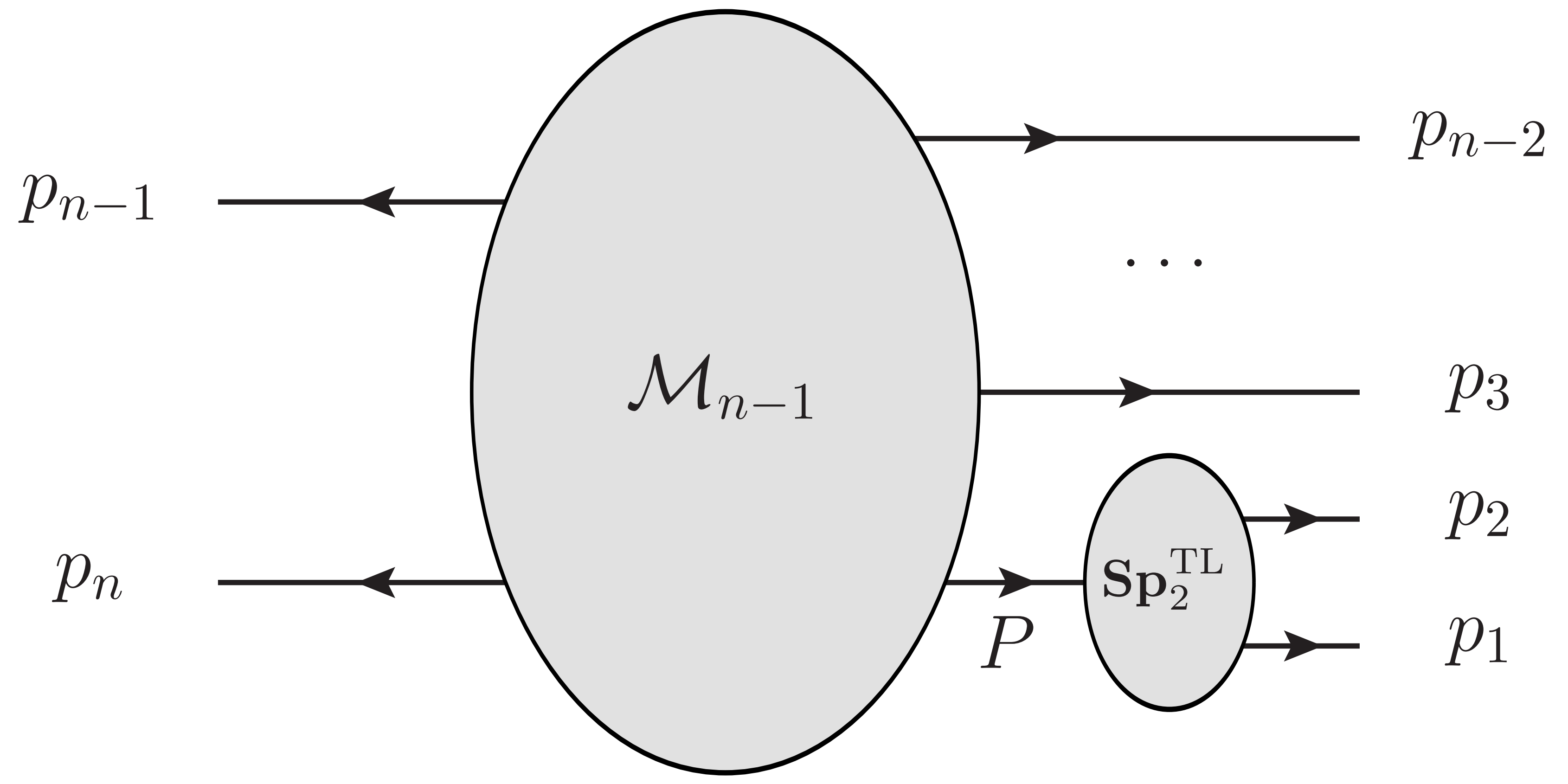

Kinematics and colour charges

$$p_1^\mu = x_1P^\mu + k^\mu \qquad x_1 + x_2 = 1$$

$$p_2^\mu = x_2P^\mu - k^\mu \qquad \mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

where k^μ represents a small residual (transverse) momentum, $k \sim \lambda P$ with $\lambda \ll 1$

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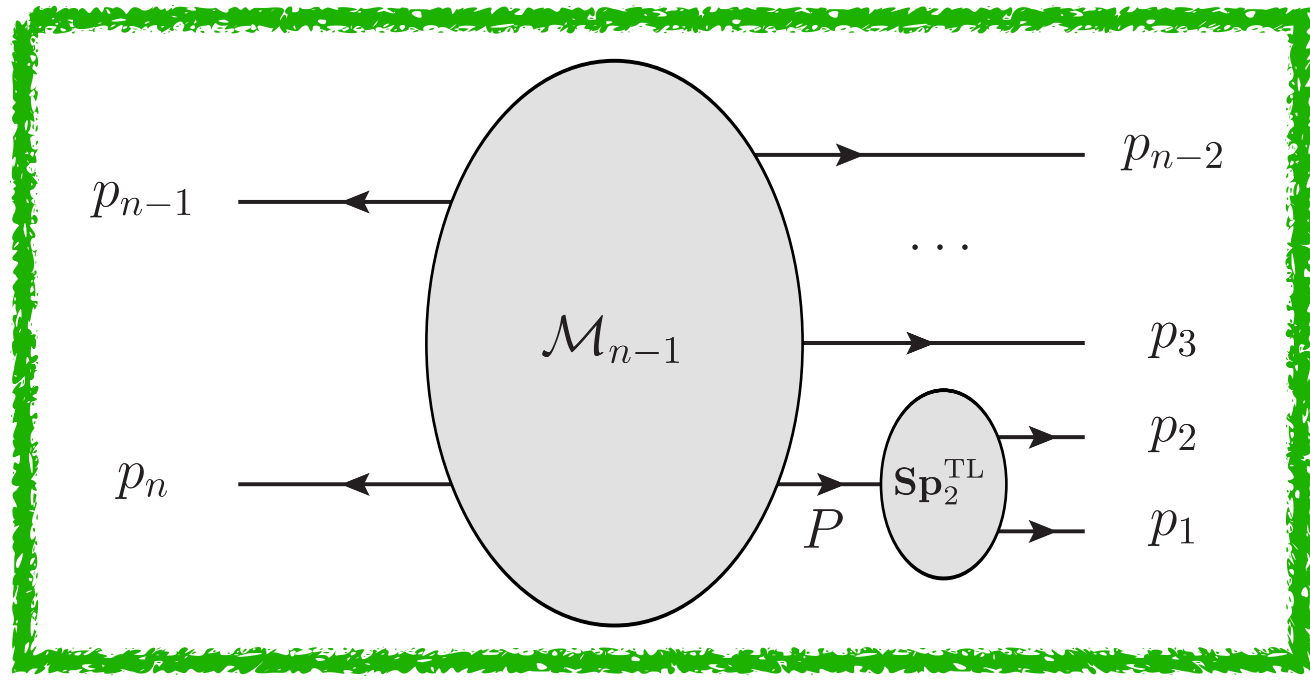

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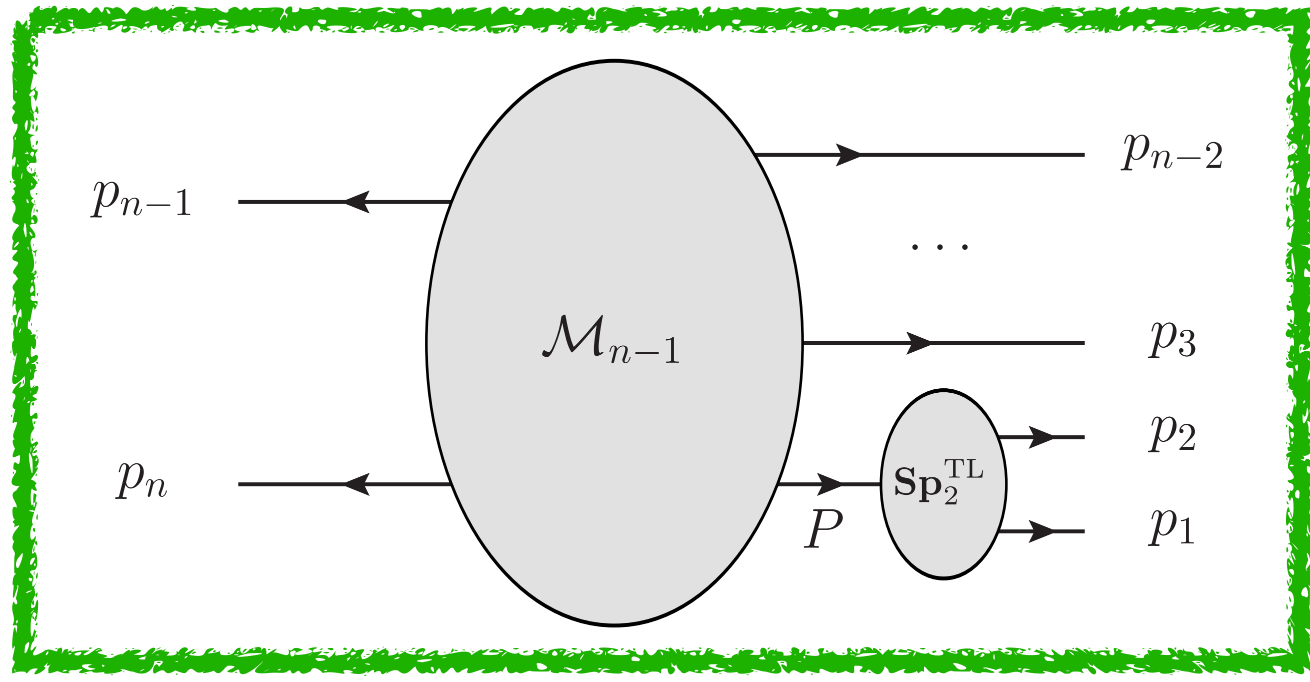

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$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i\gamma_i\mathbf{1}\right]$$

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\mathbf{1}\right],$$

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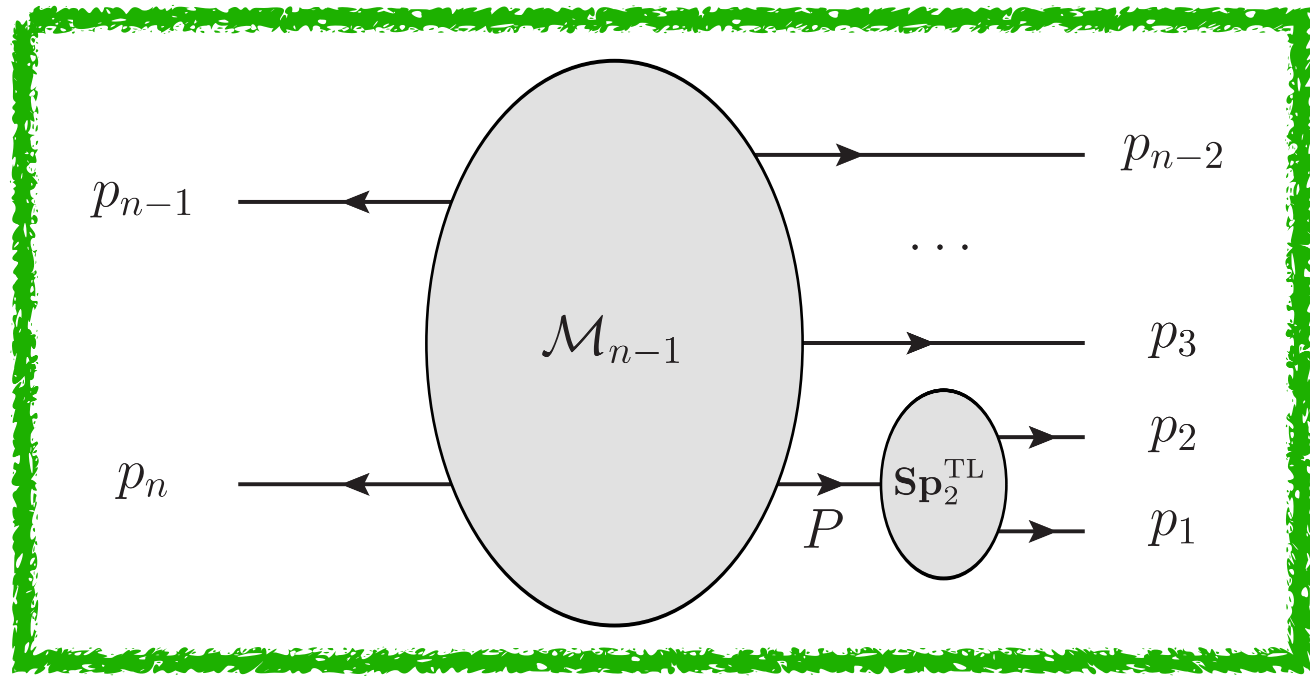

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direct cancellation

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

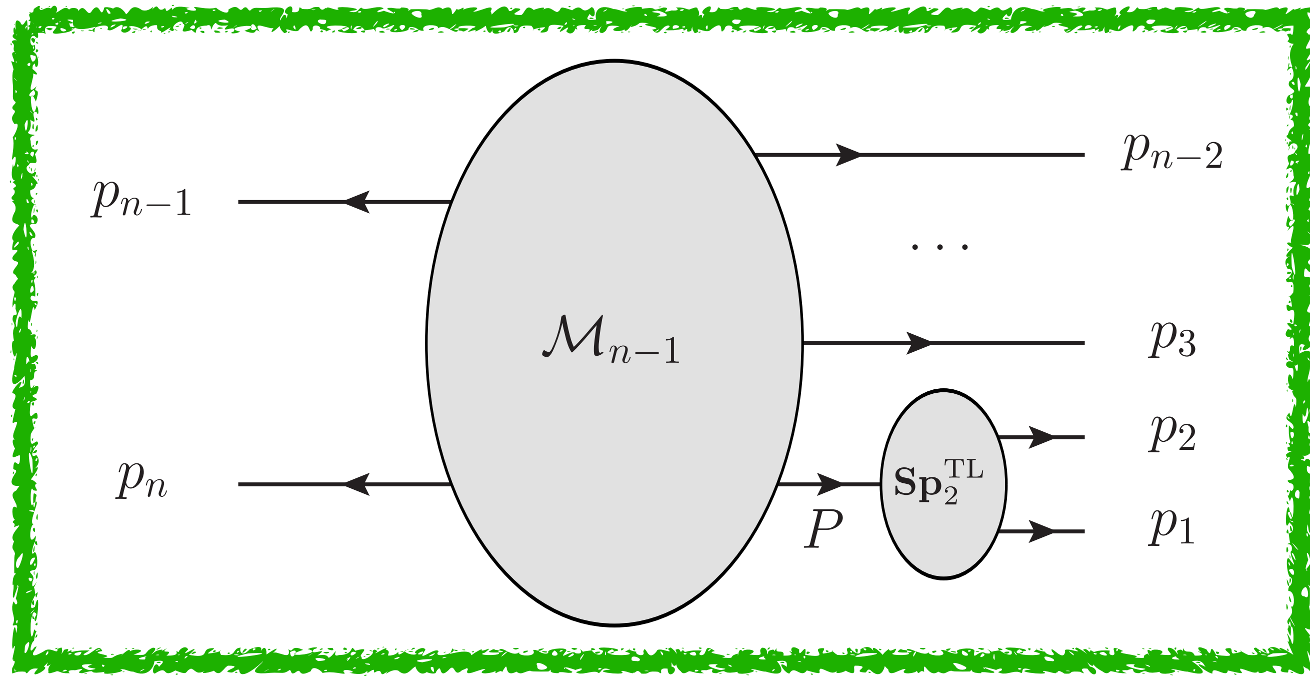
$$(-s_{1j}) = 2\,|p_1\cdot p_j|\,e^{-i\pi\lambda_{1j}}$$

$$= 2x_1\,|P\cdot p_j|\,e^{-i\pi\lambda_{Pj}} = x_1(-s_{Pj})$$

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\mathbf{1}\right],$$

$$= -\frac{1}{2}\gamma_K\ln(x_1)\sum_{3\le j\le n}\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln(x_2)\sum_{3\le j\le n}\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

$\Gamma_{\mathbf{Sp},2}$ at the dipole level



We now do a first example using the dipole contribution:

$$\Gamma_n^{\rm dip.}\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s)\sum_{i<j}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,\mathbf{1}$$

$$\Gamma^{\rm dip.}_{\mathbf{Sp},2}(p_1,p_2) = \Gamma_n^{\rm dip.}(p_1,p_2,p_3,\ldots p_n) - \Gamma^{\rm dip.}_{n-1}(P,p_3,\ldots p_n;\mu_f)\,|_{\mathbf{T}_P\to\mathbf{T}_1+\mathbf{T}_2}$$

$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i\gamma_i\mathbf{1}\right]$$

direct cancellation

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

$$(-s_{1j}) = 2\,|p_1\cdot p_j|\,e^{-i\pi\lambda_{1j}}$$

$$= 2x_1\,|P\cdot p_j|\,e^{-i\pi\lambda_{Pj}} = x_1(-s_{Pj})$$

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\mathbf{1}\right],$$

With colour conservation

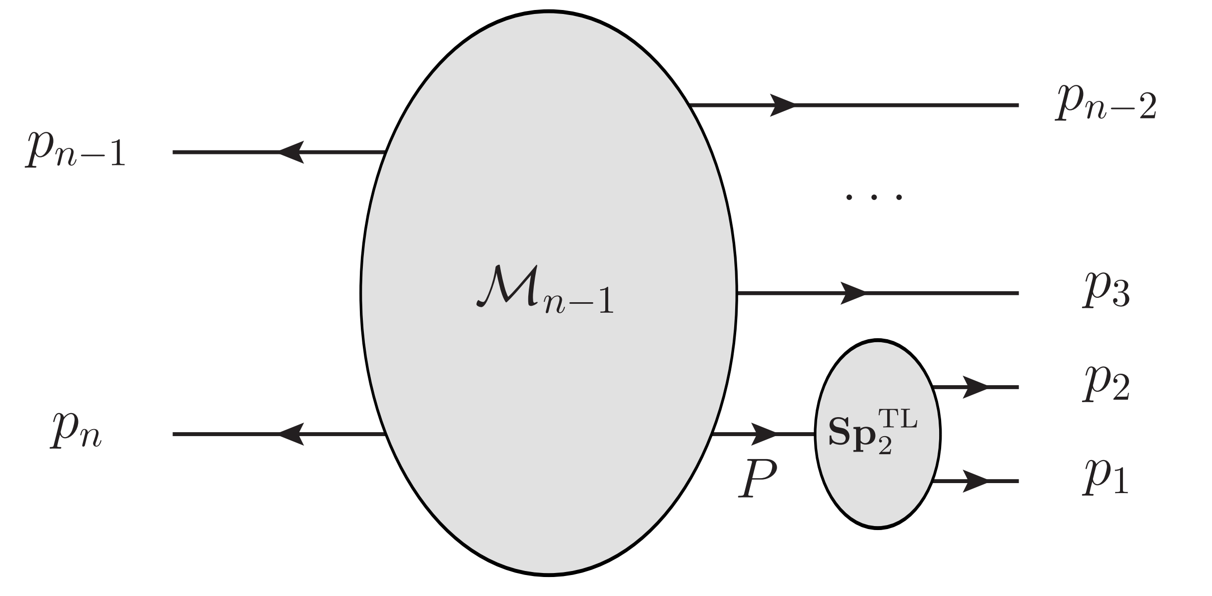
$$= \frac{1}{2}\gamma_K\ln(x_1)\,\mathbf{T}_1\cdot\mathbf{T}_P + \frac{1}{2}\gamma_K\ln(x_2)\,\mathbf{T}_2\cdot\mathbf{T}_P - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

Used $\sum_{i=1}^{n}\mathbf{T}_i = 0$

to rewrite $\sum_{3\le j\le n}\mathbf{T}_j = -\mathbf{T}_P$

Manifestly depends only on the degrees of freedom of particles becoming collinear!

$\Gamma_{\mathbf{Sp},2}$ at the dipole level



We now do a first example using the dipole contribution:

$$\Gamma_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s)\sum_{i<j}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,\mathbf{1}$$

$$\Gamma^{\text{dip.}}_{\mathbf{Sp},2}(p_1,p_2) = \Gamma_n^{\text{dip.}}(p_1,p_2,p_3,\ldots p_n) - \Gamma_{n-1}^{\text{dip.}}(P,p_3,\ldots p_n;\mu_f)\,|_{\mathbf{T}_P\to\mathbf{T}_1+\mathbf{T}_2}$$

$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i\gamma_i\mathbf{1}\right]$$

direct cancellation

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

$$(-s_{1j}) = 2\,|p_1\cdot p_j|\,e^{-i\pi\lambda_{1j}}$$

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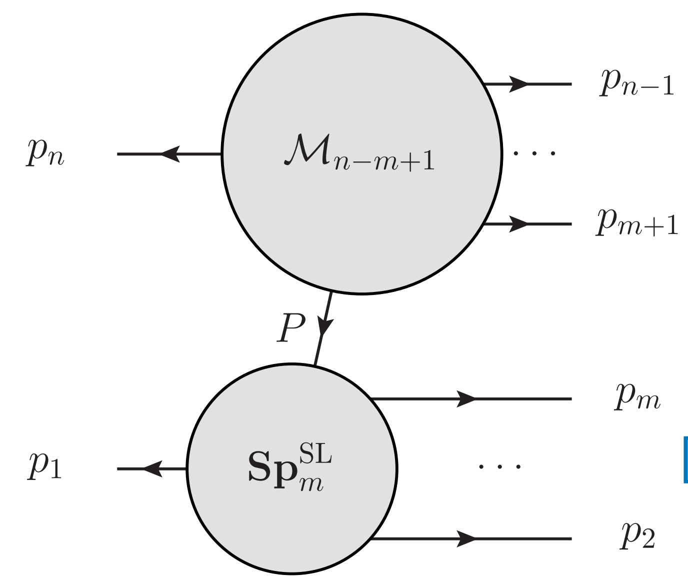
$$= \frac{1}{2}\gamma_K\ln(x_1)\,\mathbf{T}_1\cdot\mathbf{T}_P + \frac{1}{2}\gamma_K\ln(x_2)\,\mathbf{T}_2\cdot\mathbf{T}_P - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$



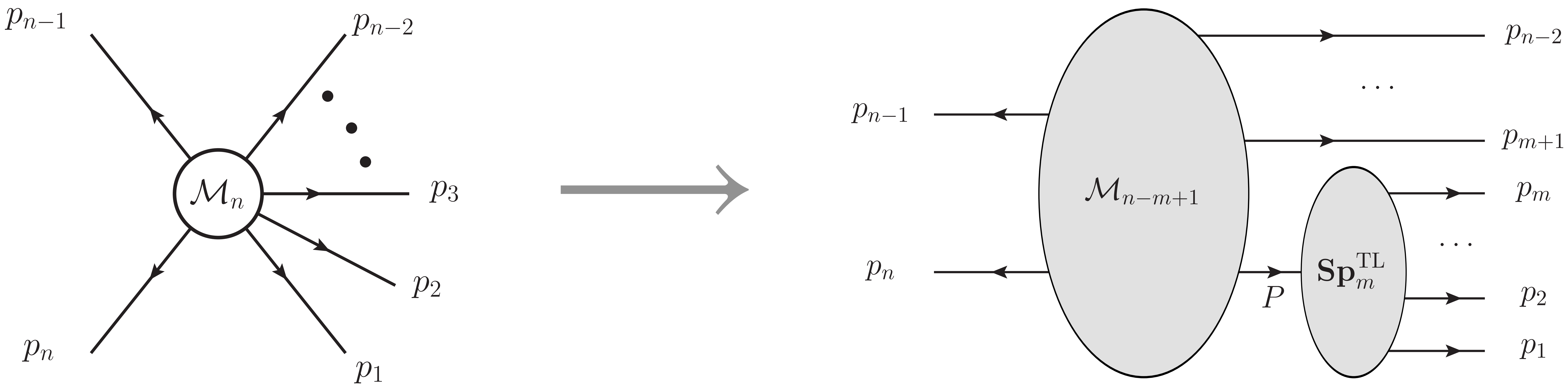
Space-like

$\to\ i\pi\mathbf{T}_n\cdot\mathbf{T}_2$

[S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

See Federico's talk for 2-loop SL

Higher order time-like collinear limit



In the time-like case, we do not expect explicit factorisation violation, but a few questions remain:

- **How is the factorisation realised at higher orders?**

  The three loop contribution to the Soft Anomalous Dimension is much more complicated than the dipole formula, contains quartic interactions, and dependence on non-trivial function of harmonic polylogarithms.

- **Thus far, I've only discussed two-particle collinear limits, what can we learn by studying multi-particle collinear limits?**

The Soft Anomalous Dimension at three-loops

Recall the structure:

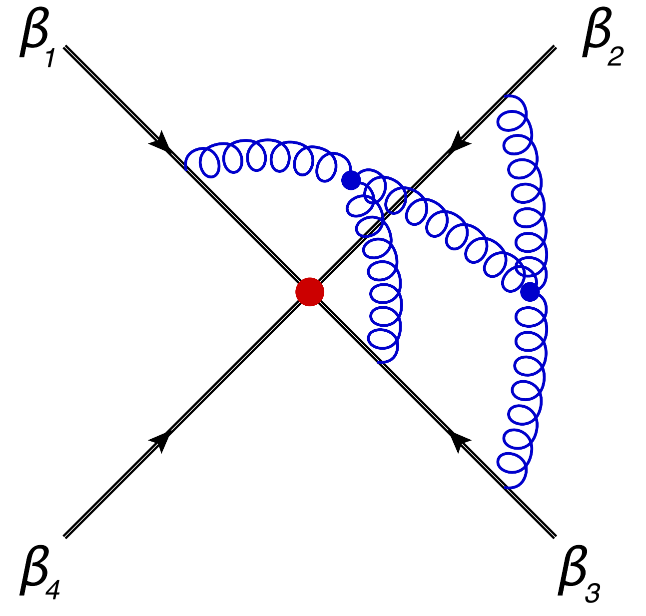
Conformally Invariant Cross Ratios (CICRs)

$$\beta_{ijkl} = \ln \rho_{ijkl} = \ln\left(\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}\right)$$

$$\boldsymbol{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \boldsymbol{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

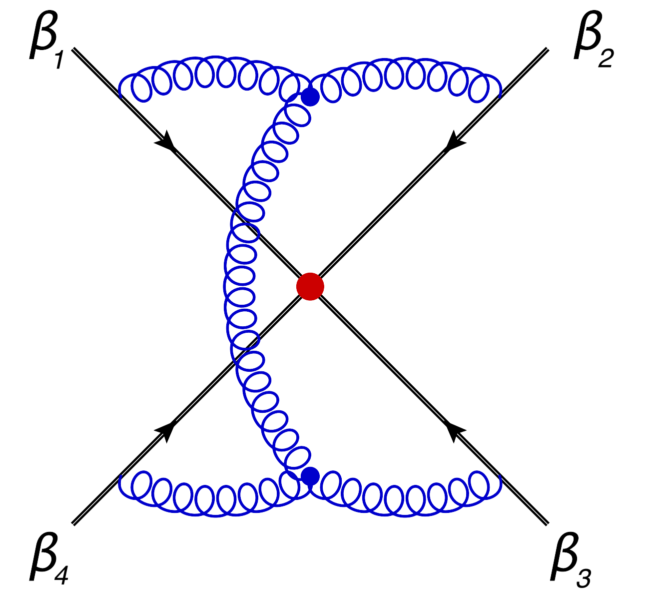
$$+\boldsymbol{\Gamma}_{n,\text{Q4T}-2,\text{3L}}(\{s_{ij}\},\lambda,\alpha_s) + \boldsymbol{\Gamma}_{n,\text{Q4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\boldsymbol{\Gamma}_{n,\text{5T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) + \boldsymbol{\Gamma}_{n,\text{5T}-\text{5L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$



Result of explicit calculation [Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

$$\boldsymbol{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk} = f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} f^{ade}f^{bce}\left\{\mathbf{T}_i^a,\mathbf{T}_i^b\right\}\mathbf{T}_j^c\,\mathbf{T}_k^d$$



$$\boldsymbol{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n}\left[T_{iklj}\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj})\right]$$

$$T_{ijkl} = f^{ade}f^{bce}\left\{\mathbf{T}_i^a,\mathbf{T}_j^b,\mathbf{T}_k^c,\mathbf{T}_l^d\right\}_+$$

Notice that $\boldsymbol{\Gamma}_{n,\text{4T}-\text{3L}}$ exists already for $n=3$, but the second term from $n=4$

The Soft Anomalous Dimension at three-loops

Recall the structure:

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

Result of explicit calc

$\mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) = 2f($

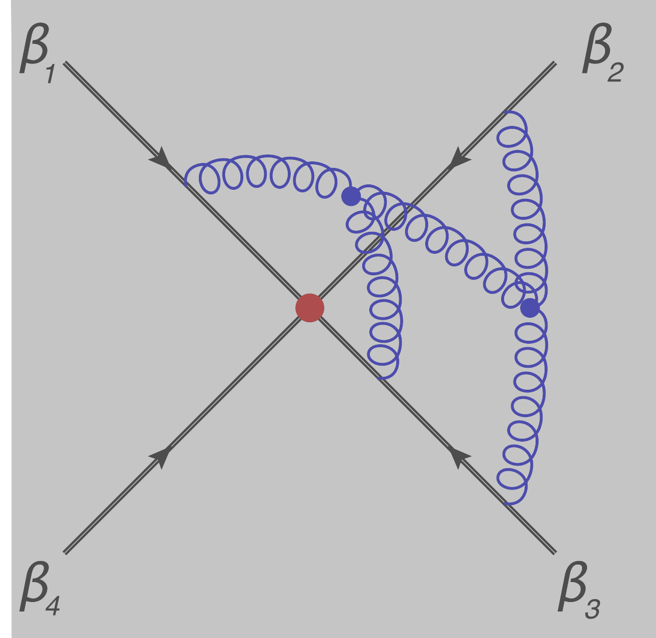
$\mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha$

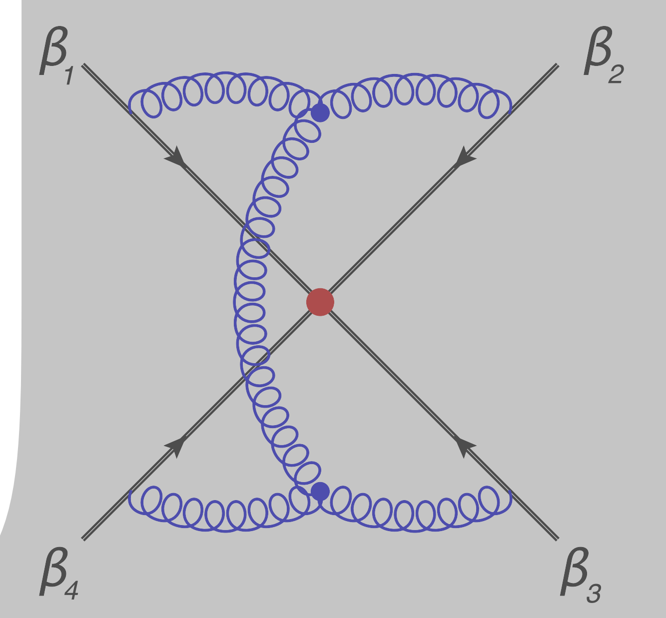
$T_{ijkl} = f^{ade}f^{bce}\left\{\mathbf{T}_i^a, \mathbf{T}_j^b, \dots\right\}_+$ ... $_{n,\text{4T}-\text{3L}}$ ... $n = 4$

$$\mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

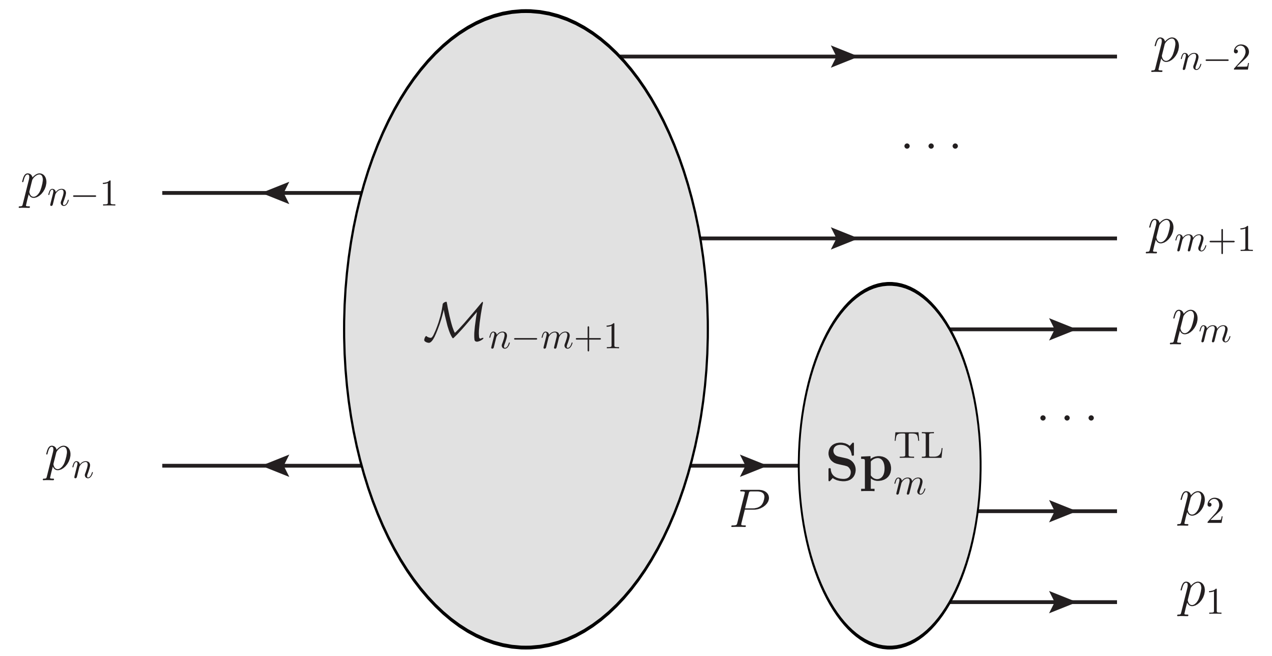
Lots of indices, but we focus on the structure!





Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



The strategy we follow is identical to before, we again split the sums with respect to how many and which collinear particles they involve

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{1jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{2jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{3jkl}(\{\beta\})$$

$$+ \sum_{4\le k<l\le n}\mathbf{a}_{12kl}(\{\beta\}) + \sum_{4\le k<l\le n}\mathbf{a}_{13kl}(\{\beta\}) + \sum_{4\le k<l\le n}\mathbf{a}_{23kl}(\{\beta\}) + \sum_{4\le l\le n}\mathbf{a}_{123l}(\{\beta\})\Bigg]$$

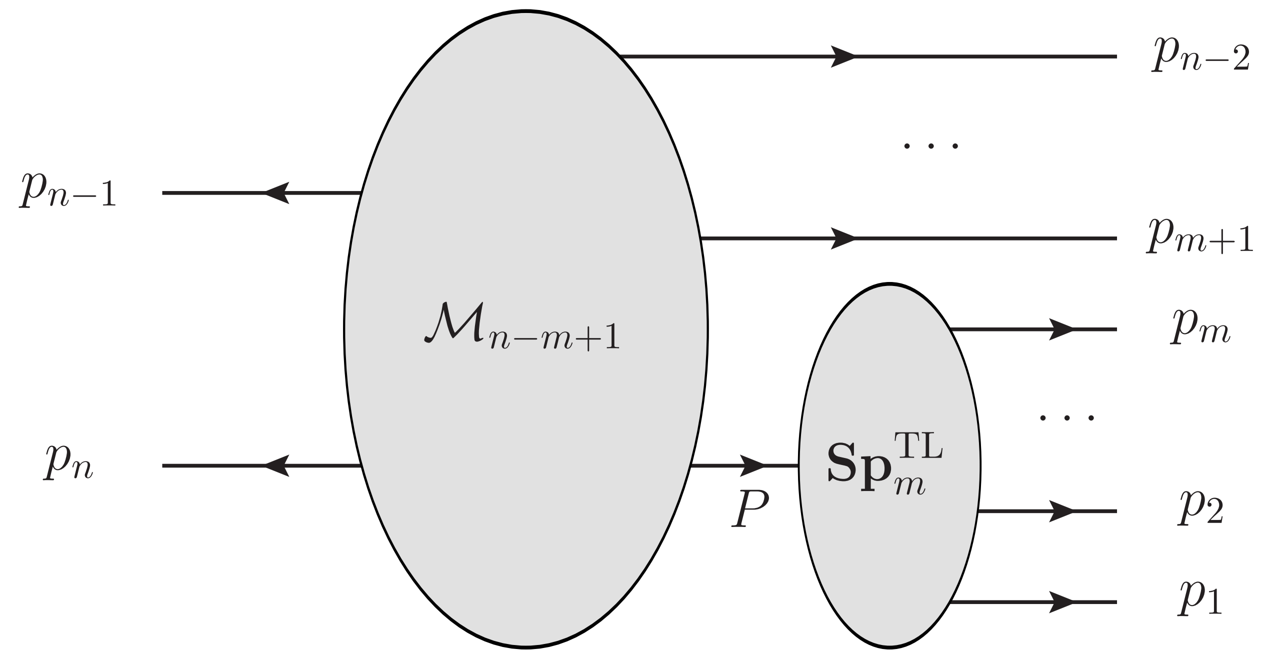
$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{Pjkl}(\{\beta\})\Bigg]$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n-2,4T-4L}(\{\beta_{ijkl}\},\alpha_s)\right)$$



The strategy we follow is identical to before, we again split the sums with respect to how many and which collinear particles they involve. Starting with the kinematics:

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{1jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{2jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{3jkl}(\{\beta\})$$

$$+ \sum_{4\le k<l\le n}\mathbf{a}_{12kl}(\{\beta\}) + \sum_{4\le k<l\le n}\mathbf{a}_{13kl}(\{\beta\}) + \sum_{4\le k<l\le n}\mathbf{a}_{23kl}(\{\beta\}) + \sum_{4\le l\le n}\mathbf{a}_{123l}(\{\beta\})\Bigg]$$

$$\mathbf{\Gamma}_{n,4T-3L}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}_{n-2,4T-4L}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n}\mathbf{a}_{Pjkl}(\{\beta\})\Bigg]$$

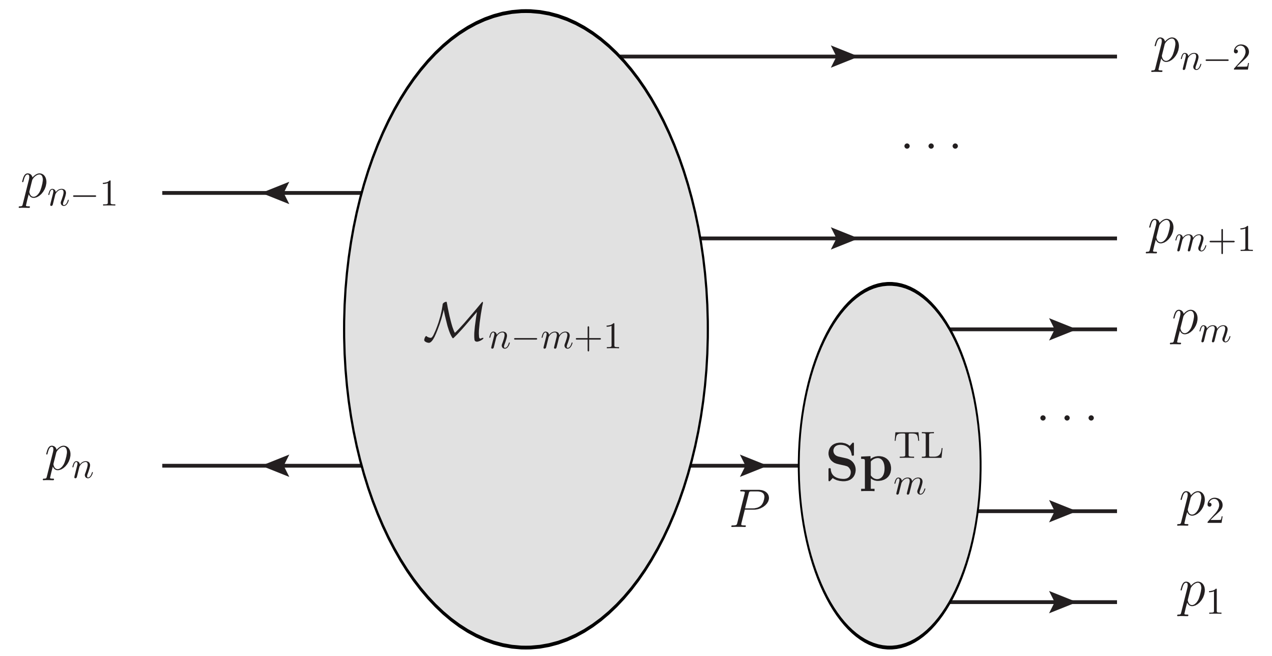
Now, the parent parton is $\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$

We see similar cancellations as before between pink and orange terms.

New type of term with three collinear particles present, marked in green.

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\text{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-2,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



$$\mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

in the end

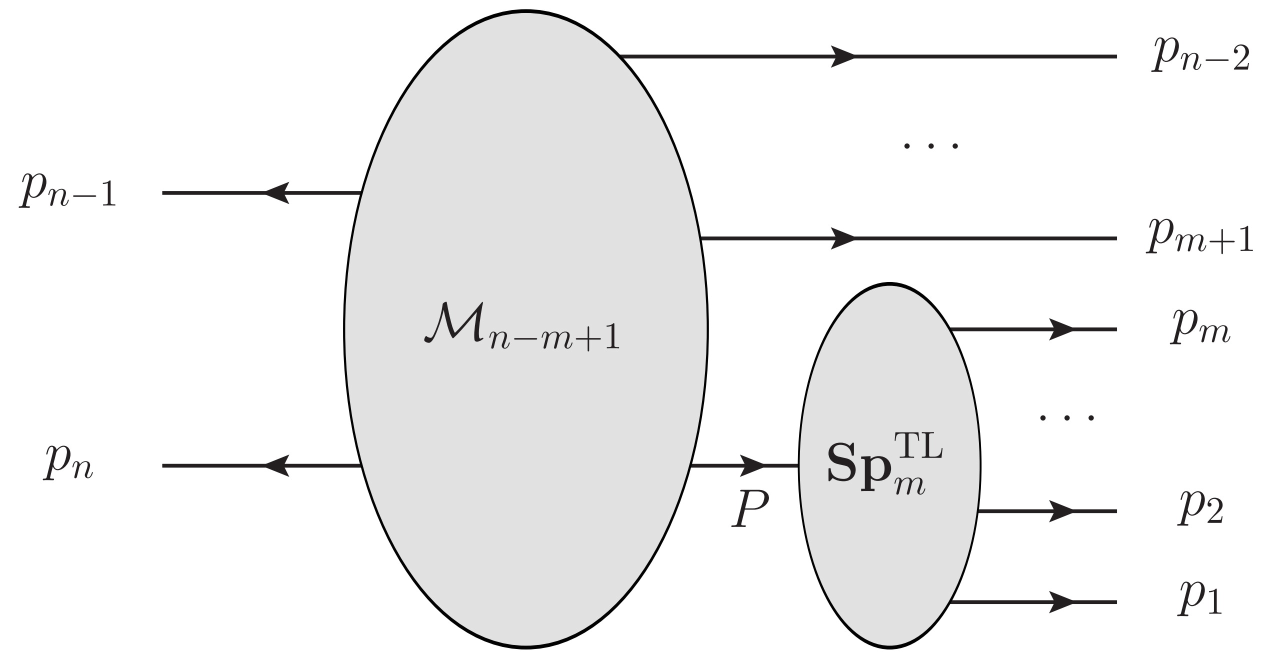
$$\mathbf{\Gamma}^{4\text{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$

$$-4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big]$$

what about the terms in the last line?

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



This is where properties of CICRs come to the rescue, as we have seen, the CICRs with three particles collinear, the dependence on the rest-of-the-process parton scales out.

$$\rho_{123l} \to \frac{x_3(-p_1\cdot p_2)}{x_2(-p_1\cdot p_3)}$$

Then we can apply colour conservation $\displaystyle\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312}$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

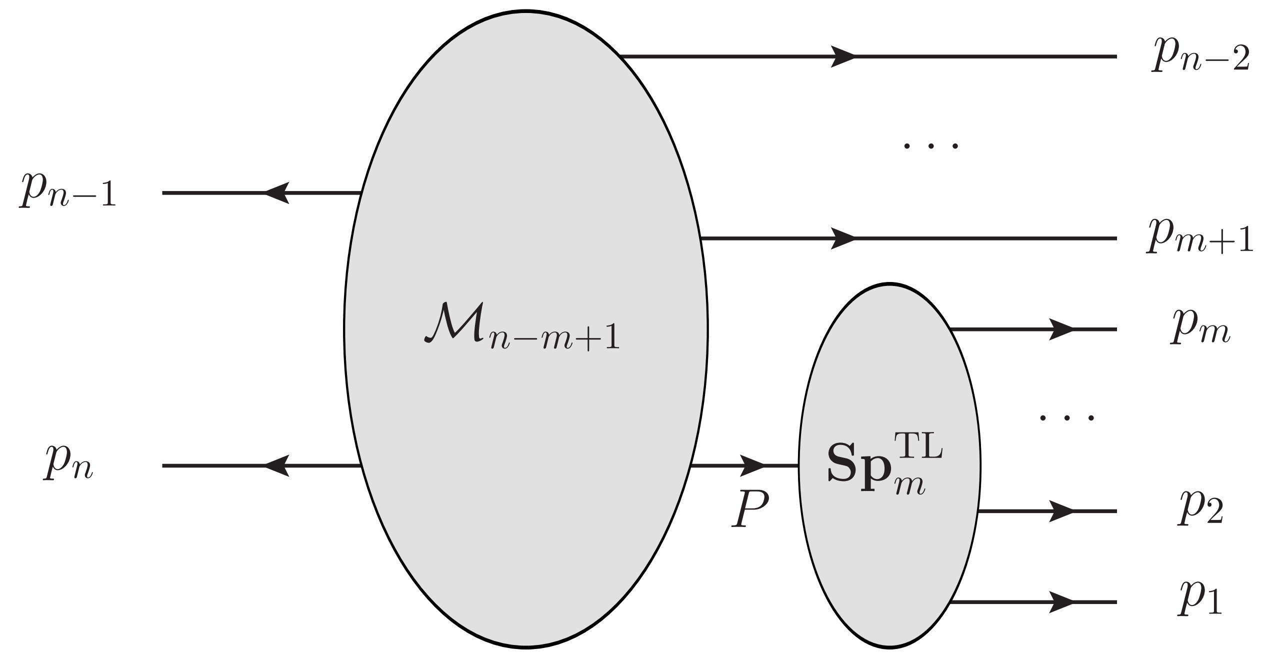
$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$

$$-4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big]$$

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



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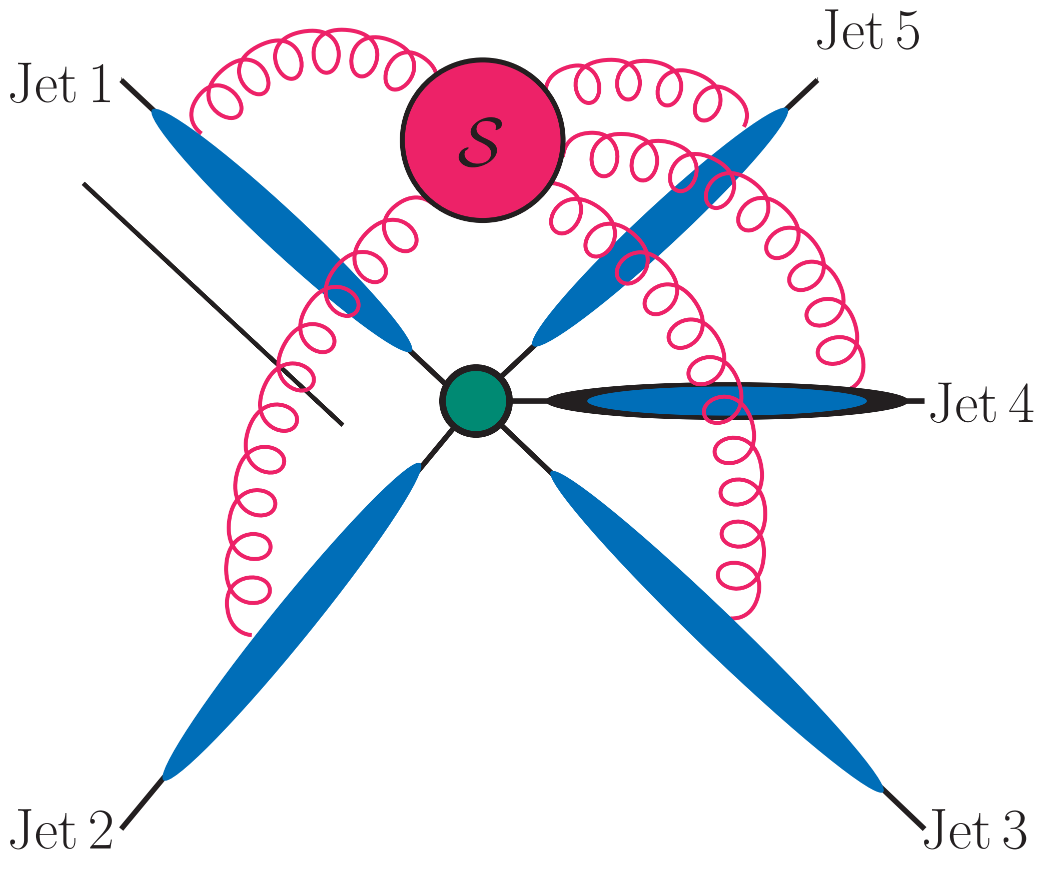
$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$
$$-4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 8\Big(T_{1123}-T_{2213}\Big)\mathcal{F}(\beta_{132l},\beta_{1l23}) + 8\Big(T_{1123}-T_{3312}\Big)\mathcal{F}(\beta_{123l},\beta_{1l32}) + 8\Big(T_{2213}-T_{3312}\Big)\mathcal{F}(\beta_{12l3},\beta_{13l2})$$

Strict collinear factorisation satisfied in the three-particle collinear limit for terms starting at three loops. No additional constraint.

Collinear limit at four-loops



Four-loop correction to the soft anomalous dimension has not been fully explicitly computed. Recall the structure

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$

Form at four loops [T. Becher and M. Neubert, 1908.11379]

$$\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) = \sum_{(i,j,k,l)} T_{ijkli}\,\mathscr{H}_1(\beta_{ijlk},\beta_{iklj};\alpha_s)$$

$$\mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) = \sum_{(i,j,k,l,m)} T_{ijklm}\,\mathscr{H}_2(\beta_{ijkl},\beta_{ijmk},\beta_{ikmj},\beta_{jiml},\beta_{jlmi};\alpha_s)$$

These functions are unknown

The two-particle collinear limit proved useful in providing constraints on the unknown objects
[T. Becher and M. Neubert, 1908.11379]

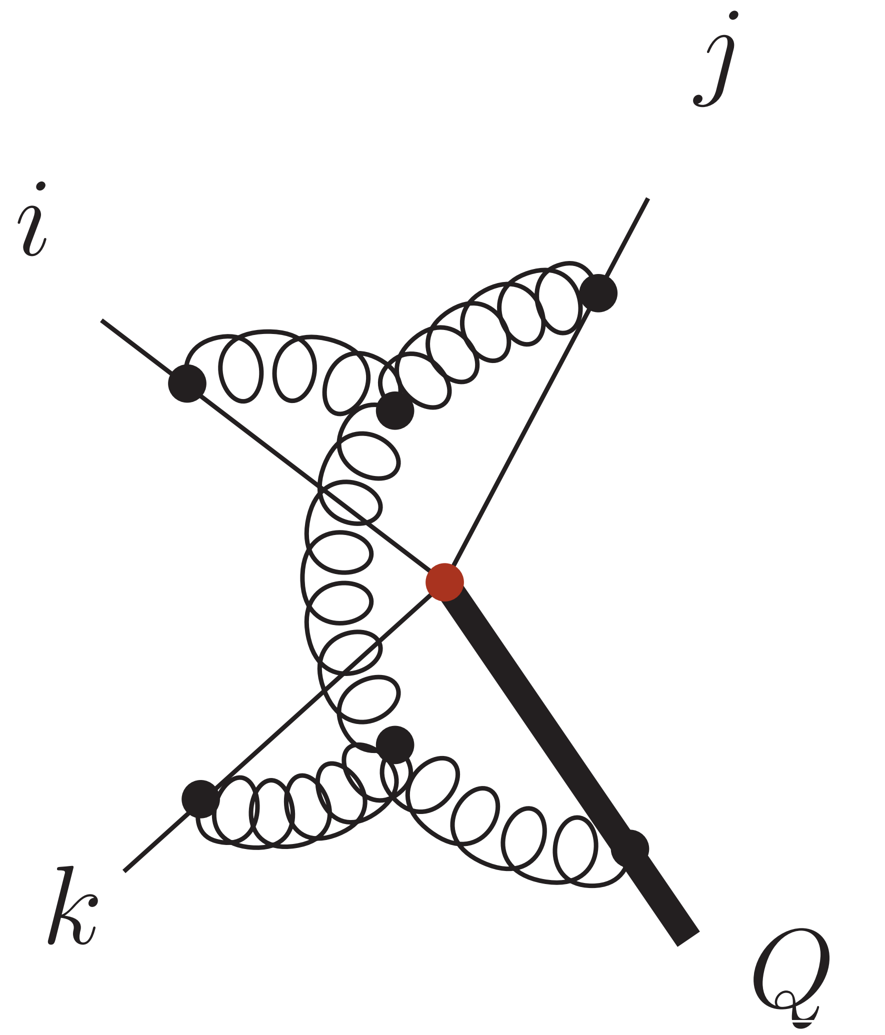
$$\lim_{\beta_{12kl}\to-\infty}\mathscr{H}_1(\beta_{12kl},0) = 0$$

From an explicit calculation we find no new constraint from higher particle collinear limits.

[C. Duhr, E. Gardi, SJ, J. Lübken, L. Vernazza, **2507.21854**]

Collinear limits of amplitudes with a massive leg

The three-loop soft-anomalous dimension with one massive particle leg



Picture from E. Gardi, Z. Zhu, 2510.27567

$$\mathbf{\Gamma}^{(3)}\left(\{p\}, m_I, \lambda, \alpha_s\right) = 2\sum_{1\leq i<j\leq n} T_{IIij}\, F_{\rm h2}\left(r_{ijI}, \alpha_s\right) + \sum_{1\leq i<j<k\leq n} \mathbf{a}^{h}_{ijkI}(\{r\})$$

Where

$$\mathbf{a}^{h}_{ijkI}(\{r\}) = 2\left[T_{ijkI}F_{\rm h3}\left(r_{ijI}, r_{ikI}, r_{jkI}\right) + T_{jikI}F_{\rm h3}\left(r_{jiI}, r_{jkI}, r_{ikI}\right) + T_{kjiI}F_{\rm h3}\left(r_{kjI}, r_{kiI}, r_{jiI}\right)\right]$$

and variables appearing here are

$$r_{ijI} = \frac{p_i \cdot p_j\, p_I^2}{2p_i \cdot p_I\, p_j \cdot p_I}$$

The function $F_{\rm h2}\left(r_{ijI}, \alpha_s\right)$ was computed in [Z. L. Liu, N. Schalch, 2207.02864]

and since recently, $F_{\rm h3}\left(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s\right)$ is known as well [E. Gardi, Z. Zhu, 2510.27567]

Three-particle collinear limit with a massive leg

Going back to the start, we note that functions F_{h3} depends on three massless particles and one massive

$$\mathbf{A}_{n+I}(\{\beta\},\{r\}) = 4 \sum_{1\le i<j<k<l\le n} \mathbf{a}_{ijkl}(\{\beta\}) + 2 \sum_{1\le i<j<k\le n} \mathbf{a}^{h}_{ijkI}(\{r\})$$

$$\mathbf{B}_{n+I}(\{r\}) = 2f(\alpha_s)\sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk} + 2\sum_{1\le i<j\le n} T_{IIij}\, F_{\mathrm{h2}}\left(r_{ijI},\alpha_s\right)$$

Proceeding in the same way as before (already implementing the two-particle collinear limit constraints), we again recover the massless three particle collinear structures, but also extra terms

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3;\mu) = -\frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right) - \frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\right) - \frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\right) - 4f(\alpha_s)\left[T_{1123}+T_{2213}+T_{3312}\right]$$

$$+\,8\sum_{4\le l\le n}\left[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\right]$$

$$+\,2T_{123I}F_{\mathrm{h3}}\left(r_{12I},r_{13I},r_{23I}\right) + 2T_{213I}F_{\mathrm{h3}}\left(r_{21I},r_{23I},r_{13I}\right) + 2T_{312I}F_{\mathrm{h3}}\left(r_{31I},r_{32I},r_{21I}\right)$$

Using colour conservation

$$\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312} - T_{123I}$$

Three-particle collinear limit with a massive leg

Then we find the following expression

$$\Gamma^{4T}_{\mathbf{Sp},3}(p_1,p_2,p_3;\mu) = -\frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_1\cdot\mathbf{T}_2+8T_{1122}\right)-\frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_1\cdot\mathbf{T}_3+8T_{1133}\right)-\frac{3}{4}f(\alpha_s)\left(C_A^2\mathbf{T}_2\cdot\mathbf{T}_3+8T_{2233}\right)-4f(\alpha_s)\left[T_{1123}+T_{2213}+T_{3312}\right]$$

$$+8\left(T_{1123}-T_{2213}\right)\mathcal{F}(\beta_{132l},\beta_{1l23})+8\left(T_{1123}-T_{3312}\right)\mathcal{F}(\beta_{123l},\beta_{1l32})+8\left(T_{2213}-T_{3312}\right)\mathcal{F}(\beta_{12l3},\beta_{13l2})$$

MASSLESS

$$+2T_{123I}\Big(F_{h3}\left(r_{12I},r_{13I},r_{23I},\alpha_s\right)-4\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big)$$

$$+2T_{213I}\Big(F_{h3}\left(r_{21I},r_{23I},r_{13I},\alpha_s\right)-4\mathcal{F}(\beta_{123l},\beta_{1l32})\Big)$$

$$+2T_{312I}\Big(F_{h3}\left(r_{31I},r_{32I},r_{21I},\alpha_s\right)-4\,\mathcal{F}(\beta_{132l},\beta_{1l23})\Big)$$

NEW in the three-particle collinear limit!

Using colour conservation

$$\sum_{4\le l\le n} T_{123l} = T_{2213}-T_{3312}-T_{123I}$$

Three-particle collinear limit with a massive leg

Then we find the following expression

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$$+8\Big(T_{1123}-T_{2213}\Big)\mathcal{F}(\beta_{132l},\beta_{1l23}) + 8\Big(T_{1123}-T_{3312}\Big)\mathcal{F}(\beta_{123l},\beta_{1l32}) + 8\Big(T_{2213}-T_{3312}\Big)\mathcal{F}(\beta_{12l3},\beta_{13l2})$$

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$$+2T_{123I}\Big(F_{\mathrm{h}3}\left(r_{12I},r_{13I},r_{23I},\alpha_s\right) - 4\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big)$$

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NEW in the three-particle collinear limit!

What about the variables?

All $r_{abI} \to 0$ for a and b becoming collinear, and we already know that β_{1l32}s have kinematic structure

Indeed, we need to consider ratios

$$\frac{r_{ijI}}{r_{jkI}} = \frac{p_i\cdot p_j\, p_I^2}{p_i\cdot p_I\, p_j\cdot p_I}\,\frac{p_j\cdot p_I\, p_k\cdot p_I}{p_j\cdot p_k\, p_I^2} = \frac{p_i\cdot p_j\, p_I\cdot p_k}{p_i\cdot p_I\, p_j\cdot p_k} = \rho_{ijIk}$$

$$\frac{r_{ikI}}{r_{jkI}} = \frac{p_i\cdot p_k\, p_I^2}{p_i\cdot p_I\, p_k\cdot p_I}\,\frac{p_j\cdot p_I\, p_k\cdot p_I}{p_j\cdot p_k\, p_I^2} = \frac{p_i\cdot p_k\, p_I\cdot p_j}{p_i\cdot p_I\, p_j\cdot p_k} = \rho_{ikIj}$$

Three-particle collinear limit with a massive leg

We can rewrite F_{h3} in alternative variables

$$F_{\text{h3}}\left(\beta_{abIc},\beta_{acIb}\,;r_{bcI}\right)\equiv F_{\text{h3}}\left(r_{abI},r_{acI},r_{bcI}\right)$$

Which allows us to formulate the constraint

$$\lim_{p_a||p_b||p_c} F_{\text{h3}}\left(r_{abI},r_{acI},r_{bcI}\right)=\lim_{r_{bcI}\to 0} F_{\text{h3}}\left(\beta_{abIc},\beta_{acIb}\,;r_{bcI}\right)=4\mathscr{F}(\beta_{ablc},\beta_{aclb})\Big|_{p_a||p_b||p_c}$$

This is the first instance in which multi-particle collinear limit has yielded an additional constraint.

Three-particle collinear limit with a massive leg

We can rewrite $F_{\rm h3}$ in alternative variables

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This is the first instance in which multi-particle collinear limit has yielded an additional constraint.

Connection with the small-mass limit

In fact, this limit is indistinguishable from the three-particle collinear limit, as also in this case, all the $r_{abI}\to 0$, but for different reasons.

$$r_{abI} = \frac{p_a\cdot p_b\, p_I^2}{2p_a\cdot p_I\, p_b\cdot p_I} \to 0$$

Three-particle collinear

$p_a\cdot p_b \to 0$

small-mass limit

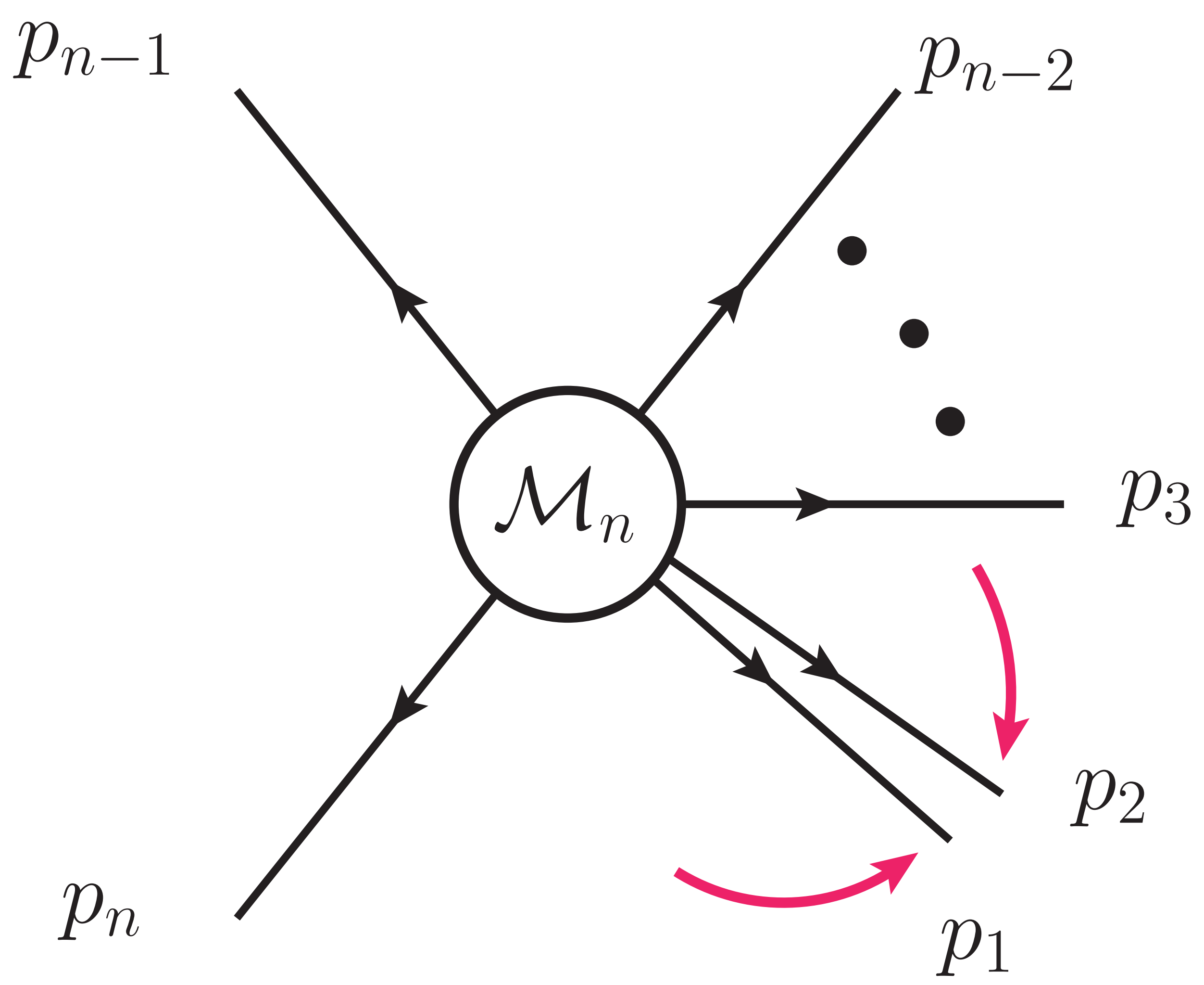
$p_I^2 \to 0$

considered in

[Z. L. Liu, N. Schalch, 2207.02864]

$$\lim_{p_a||p_b||p_c} F_{\rm h3}\left(r_{abI},r_{acI},r_{bcI}\right) = \lim_{p_I^2\to 0} F_{\rm h3}\left(r_{abI},r_{acI},r_{bcI}\right) = 4\mathscr{F}(\beta_{ablc},\beta_{aclb})\Big|_{p_a||p_b||p_c}$$

Summary

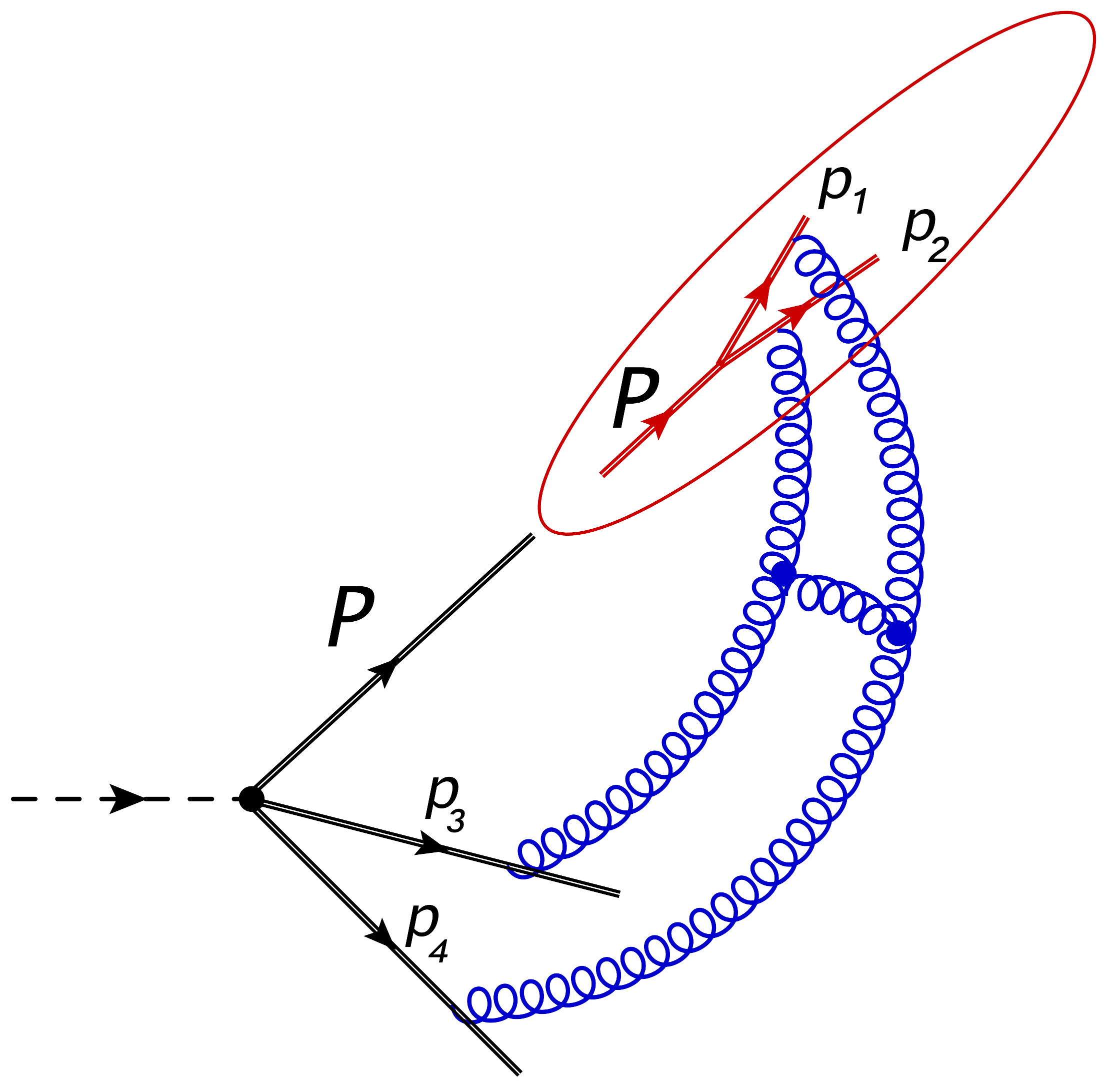


We have considered multi-particle collinear limits of n-point scattering amplitudes with massless external legs at three and four loop order.

No new constraints on kinematic functions appearing in these terms are obtained in the multi-particle collinear limits at three and four loop order, beyond the ones obtained in the two-particle collinear limits.

Working with n and $n-m+1$ particle amplitudes demonstrates universality of the splitting amplitude soft anomalous dimensions through an intricate interplay between the colour and kinematics.

New constraints do arise in the three-particle collinear limit at the three loop order, for amplitudes containing one massive particle.



Summary

We have considered multi-particle collinear limits of n-point scattering amplitudes with massless external legs at three and four loop order.

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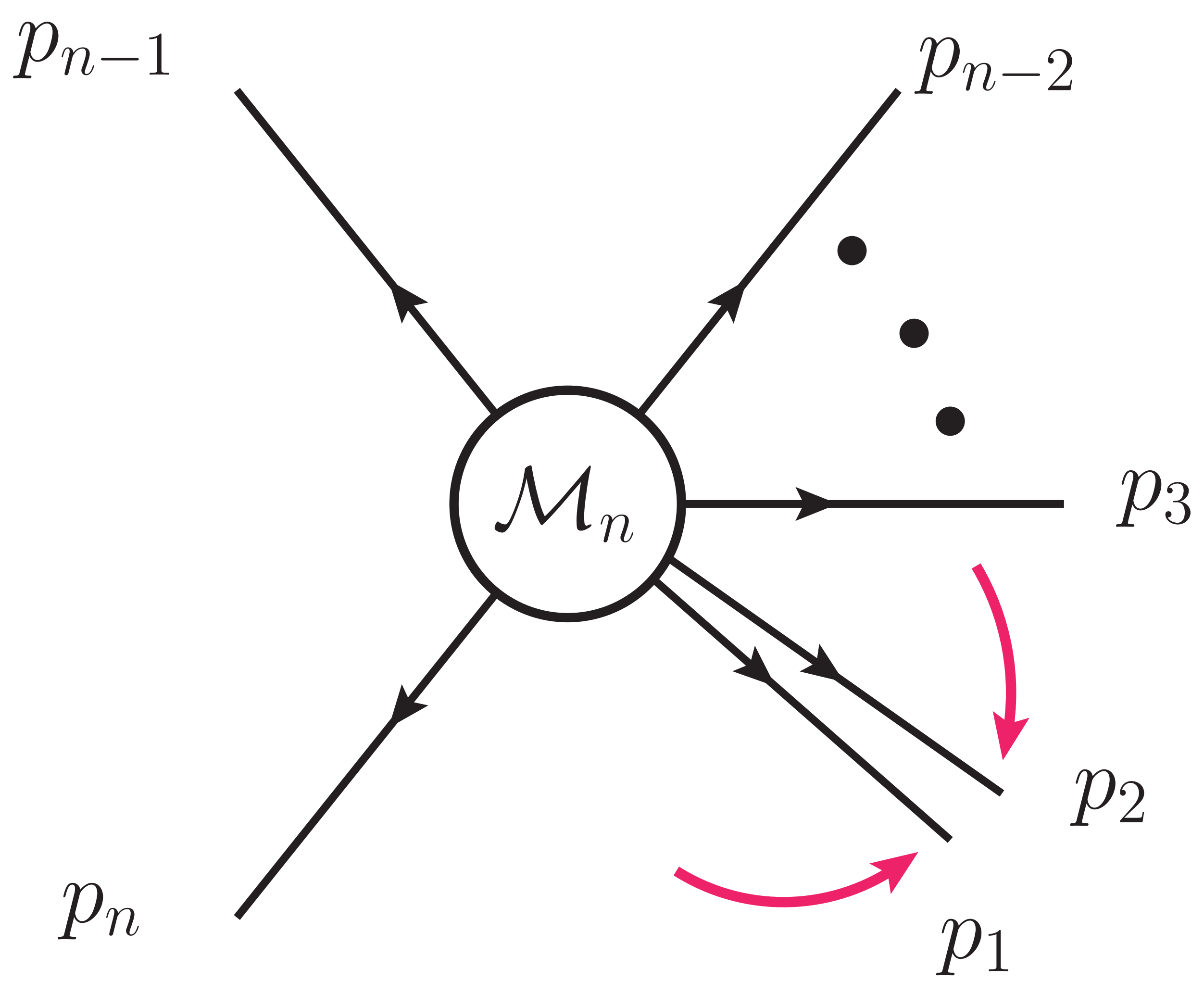
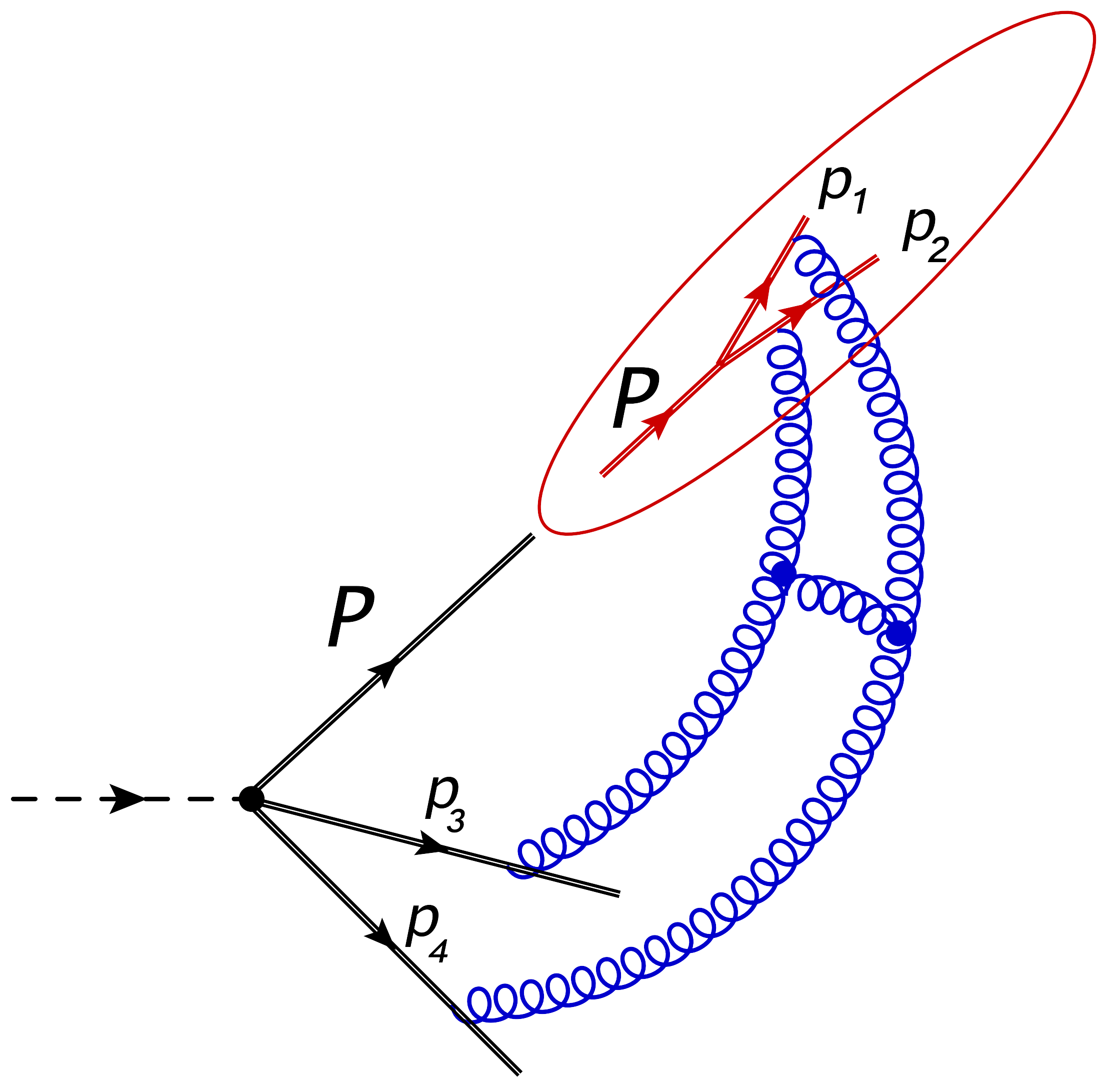
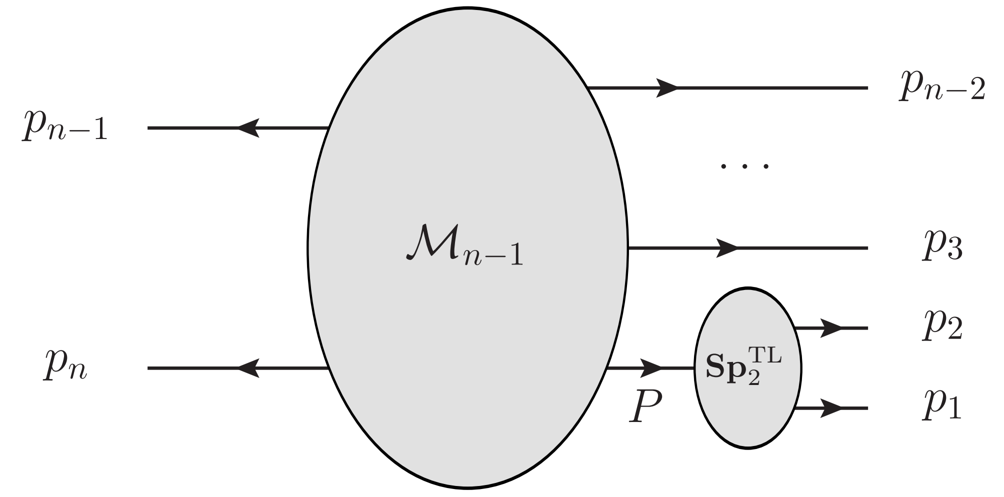




Thank you!

Auxiliary slides

$\boldsymbol{\Gamma}_{\mathbf{Sp},2}$ at the dipole level



We now do a first example using the dipole contribution:

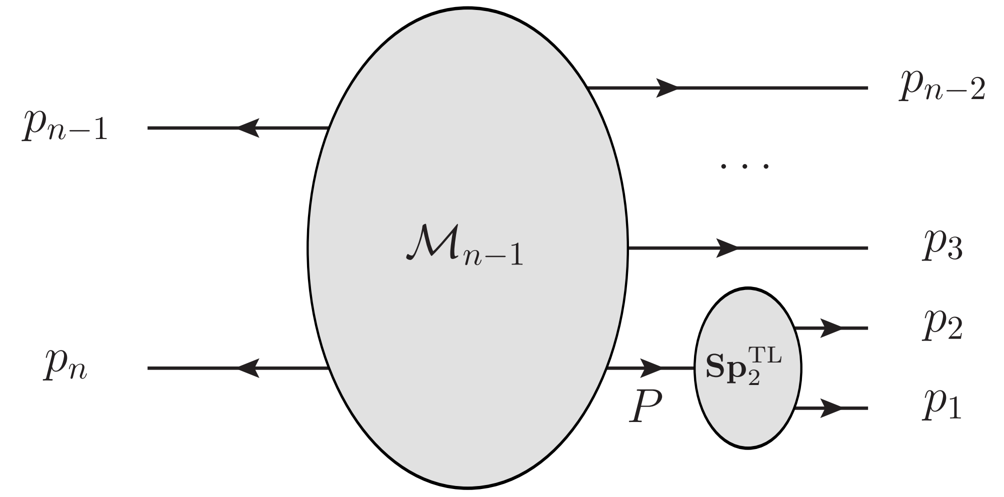
$$\boldsymbol{\Gamma}_n^{\rm dip.}\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s)\sum_{i<j}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,\mathbf{1}$$

$$\boldsymbol{\Gamma}_{\mathbf{Sp},2}^{\rm dip.}(p_1,p_2) = \boldsymbol{\Gamma}_n^{\rm dip.}(p_1,p_2,p_3,\ldots p_n) - \boldsymbol{\Gamma}_{n-1}^{\rm dip.}(P,p_3,\ldots p_n;\mu_f)\,|_{\mathbf{T}_P\to\mathbf{T}_1+\mathbf{T}_2}$$

$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i \gamma_i\,\mathbf{1}\right]$$

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\,\mathbf{1}\right],$$

$\boldsymbol{\Gamma}_{\mathbf{Sp},2}$ at the dipole level



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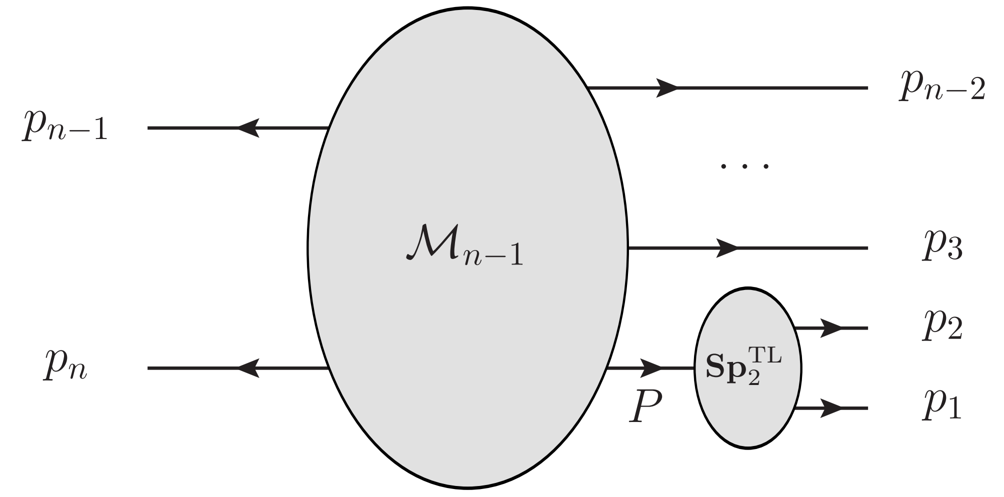
$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i\gamma_i\mathbf{1}\right]$$

direct cancellation

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\mathbf{1}\right],$$

$\Gamma_{\mathbf{Sp},2}$ at the dipole level



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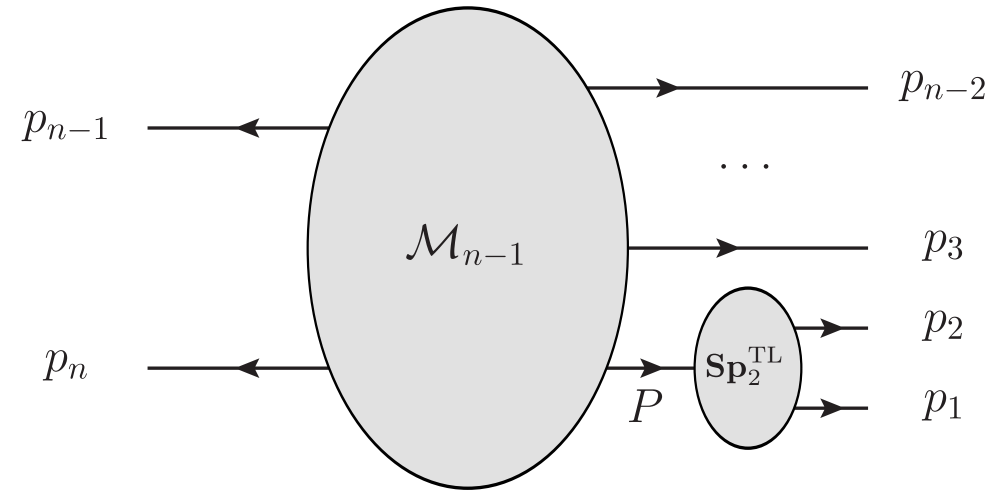
$$= \left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i\gamma_i\mathbf{1}\right]$$

direct cancellation

term already depends on particles 1 and 2

$$-\left[-\frac{1}{2}\gamma_K\sum_{3\le i<j\le n}\ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i\mathbf{1}\right],$$

$\Gamma_{\mathbf{Sp},2}$ at the dipole level



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$$= -\frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(x_1\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\sum_{3\le j\le n}\ln\left(x_2\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

Then, we can apply colour conservation $\sum_{i=1}^{n}\mathbf{T}_i = 0$ to rewrite $\sum_{3\le j\le n}\mathbf{T}_j = -\mathbf{T}_1 - \mathbf{T}_2 = -\mathbf{T}_P$

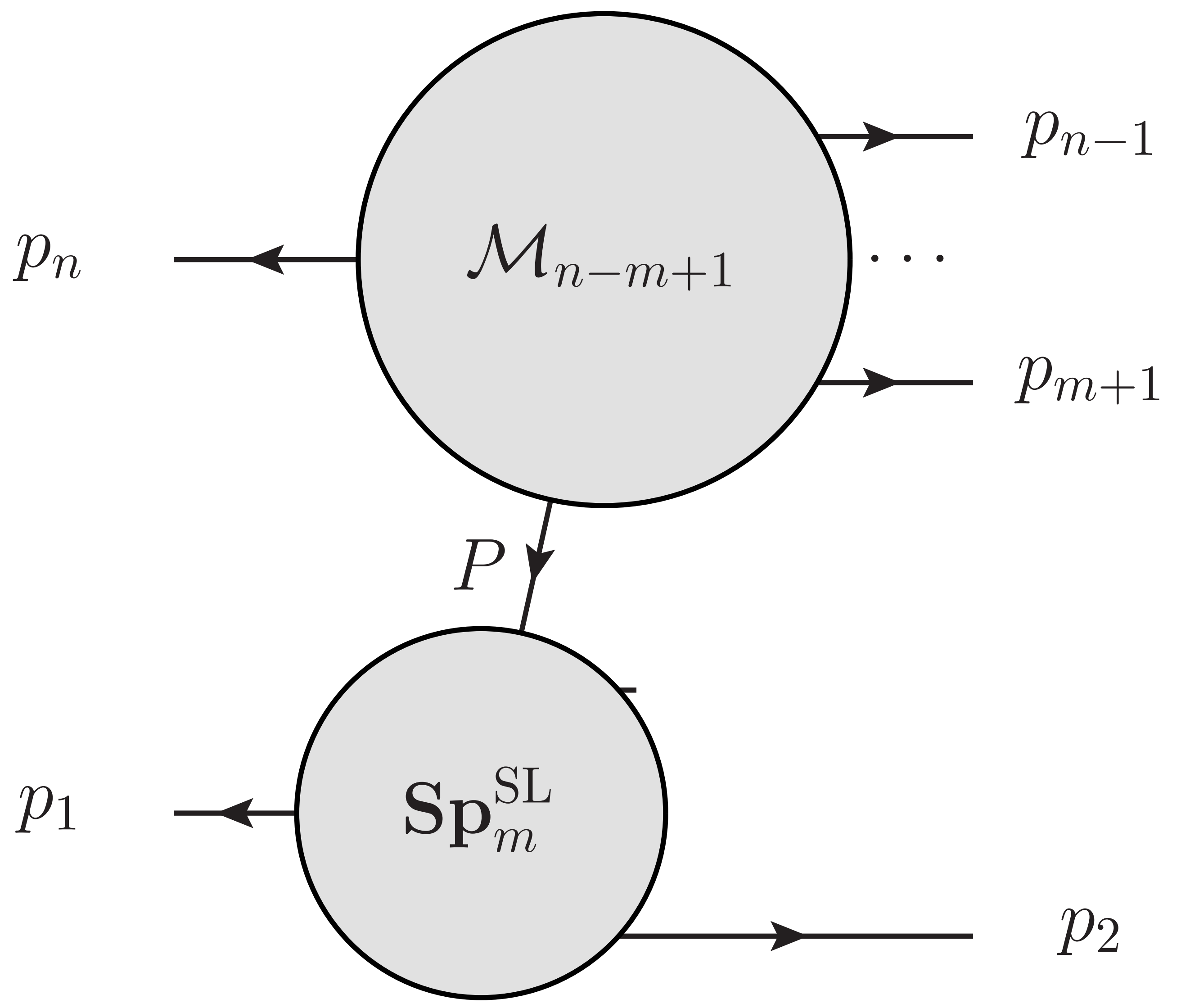
$$\mathbf{\Gamma}_{\mathbf{Sp},2}^{\rm dip.}(p_1,p_2;\mu) = \frac{1}{2}\gamma_K\ln\left(x_1\right)\mathbf{T}_1\cdot\mathbf{T}_P + \frac{1}{2}\gamma_K\ln\left(x_2\right)\mathbf{T}_2\cdot\mathbf{T}_P - \frac{1}{2}\gamma_K\ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

Manifestly depends only on the degrees of freedom of particles becoming collinear!

What happens in the space-like limit?

Investigated by Catani, de Florian, and Rodrigo in 2011. [S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

The set up now is as follows:



Kinematics and colour charges, m=2

$$p_1^\mu = x_1 P^\mu + k^\mu$$

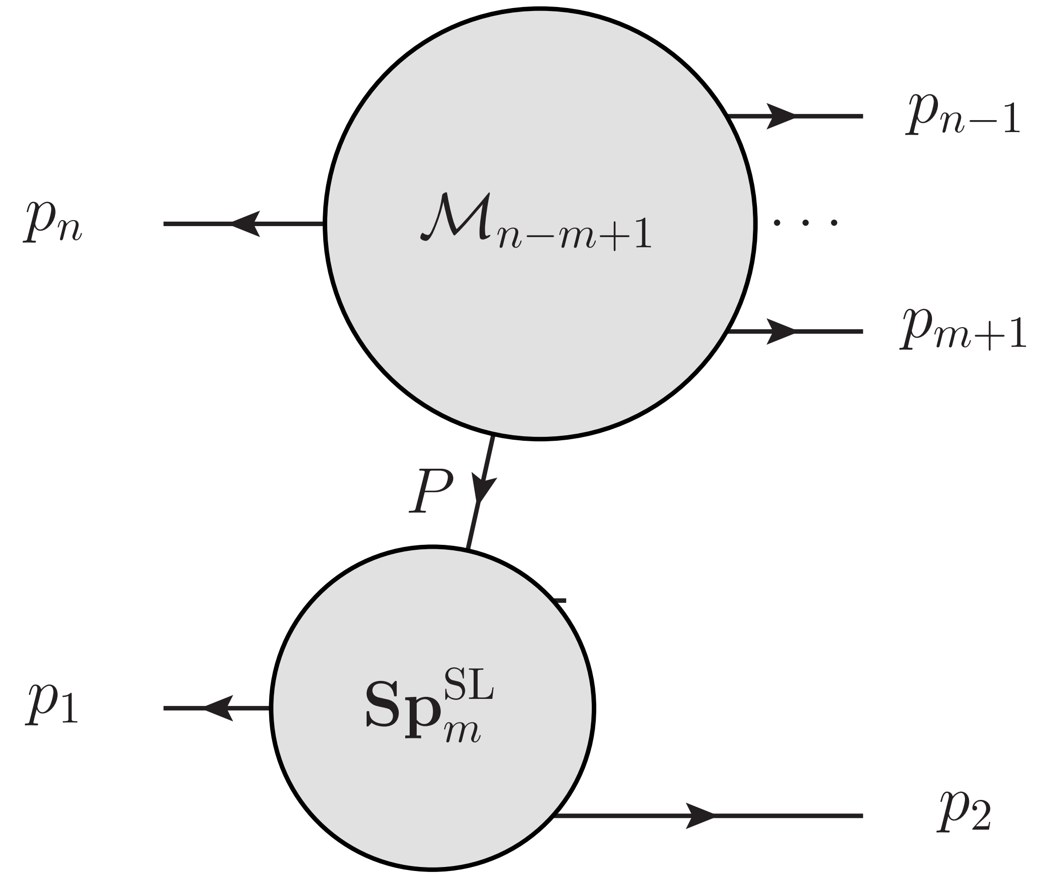
$$p_2^\mu = x_2 P^\mu - k^\mu$$

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

Now x_1 is bigger than 1

where k^μ represents a small residual (transverse) momentum, $k \sim \lambda P$ with $\lambda \ll 1$

What happens in the space-like limit?



We now contrast the cancellation that happened in this step in the time-like limit:

$$\mathbf{\Gamma}^{\mathrm{dip.}}_{\mathbf{Sp},2}(p_1,p_2) = \mathbf{\Gamma}^{\mathrm{dip.}}_{n}(p_1,p_2,p_3,\ldots p_n) - \mathbf{\Gamma}^{\mathrm{dip.}}_{n-1}(P,p_3,\ldots p_n;\mu_f)\,|_{\mathbf{T}_P\to\mathbf{T}_1+\mathbf{T}_2}$$

$$= \Bigg[-\frac{1}{2}\gamma_K \sum_{3\le i<j\le n} \ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i \gamma_i \mathbf{1}\Bigg]$$

$$-\Bigg[-\frac{1}{2}\gamma_K \sum_{3\le i<j\le n} \ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n} \gamma_i \mathbf{1}\Bigg],$$

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

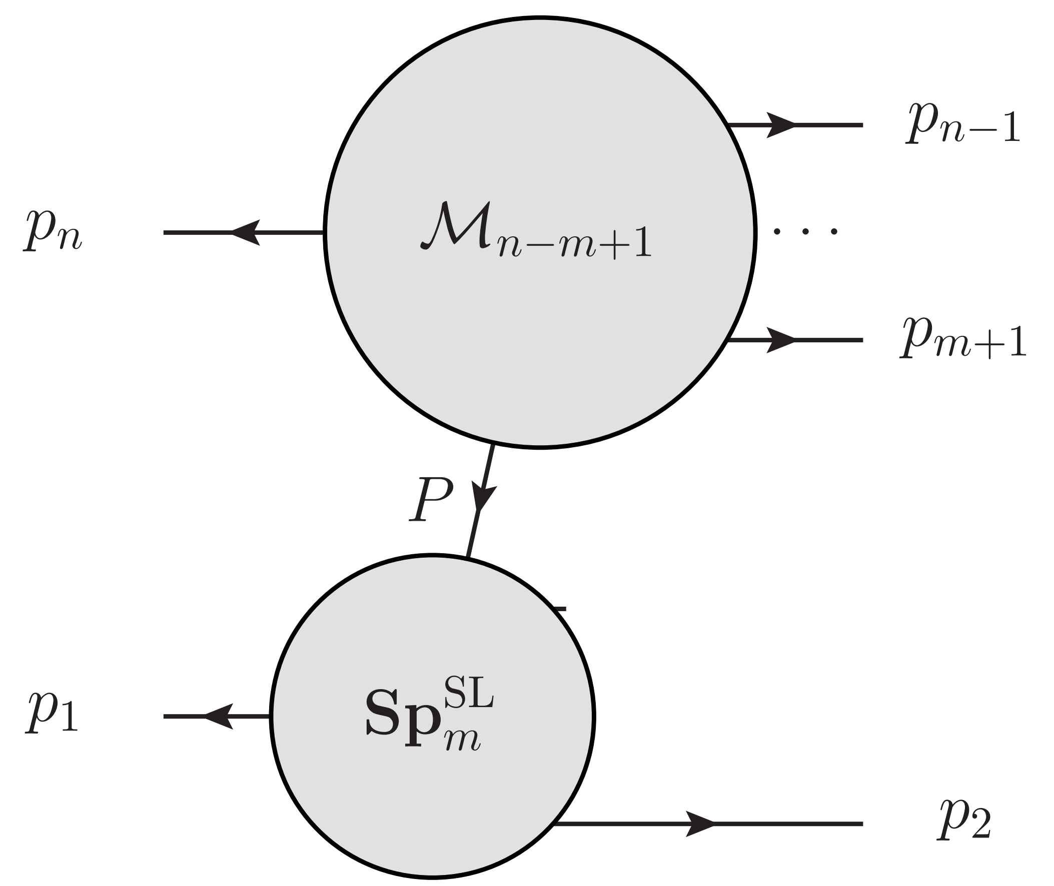
$$(-s_{1j}) = 2\,|p_1\cdot p_j|\,e^{-i\pi\lambda_{1j}}$$

$$= 2x_1\,|P\cdot p_j|\,e^{-i\pi\lambda_{Pj}} = x_1(-s_{Pj})$$

$$= -\frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln(x_1)\,\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln(x_2)\,\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

$$-\frac{1}{2}\gamma_K \sum_{i=3}^{n} \Big((-i\pi\lambda_{i1})\mathbf{T}_i\cdot\mathbf{T}_1 + (-i\pi\lambda_{i2})\mathbf{T}_i\cdot\mathbf{T}_2\Big) + \frac{1}{2}\gamma_K \sum_{i=3}^{n} (-i\pi\lambda_{iP})\mathbf{T}_i\cdot(\mathbf{T}_1+\mathbf{T}_2)$$

What happens in the space-like limit?



We now contrast the cancellation that happened in this step in the time-like limit:

$$\mathbf{\Gamma}^{\rm dip.}_{\mathbf{Sp},2}(p_1,p_2) = \mathbf{\Gamma}^{\rm dip.}_{n}(p_1,p_2,p_3,\ldots p_n) - \mathbf{\Gamma}^{\rm dip.}_{n-1}(P,p_3,\ldots p_n;\mu_f)\,|_{\mathbf{T}_P\to\mathbf{T}_1+\mathbf{T}_2}$$

$$= \left[-\frac{1}{2}\gamma_K \sum_{3\le i<j\le n} \ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{1j}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{2j}}{\mu^2}\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \sum_i \gamma_i \mathbf{1}\right]$$

$$-\left[-\frac{1}{2}\gamma_K \sum_{3\le i<j\le n} \ln\left(\frac{-s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(\frac{-s_{Pj}}{\mu^2}\right)\mathbf{T}_P\cdot\mathbf{T}_j + \sum_{P,3\le i\le n}\gamma_i \mathbf{1}\right],$$

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

$$(-s_{1j}) = 2\,|p_1\cdot p_j|\,e^{-i\pi\lambda_{1j}}$$

$$= 2x_1\,|P\cdot p_j|\,e^{-i\pi\lambda_{Pj}} = x_1(-s_{Pj})$$

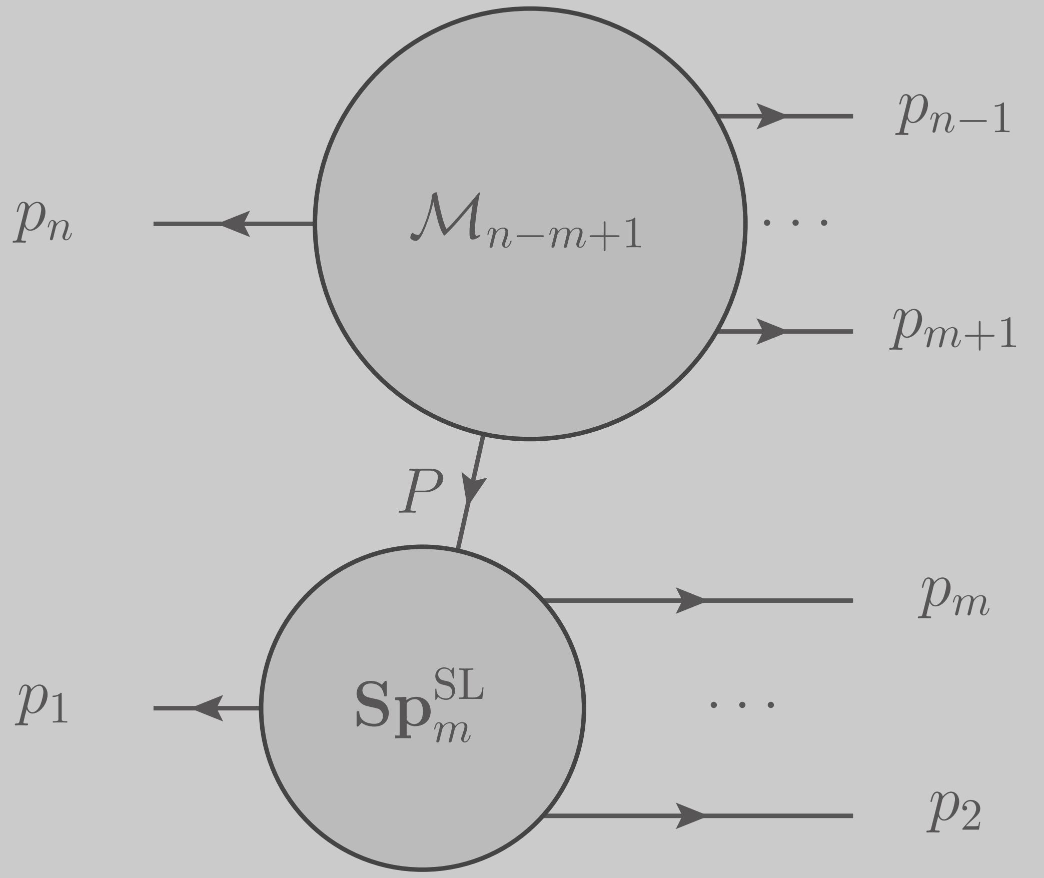
$$= -\frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(x_1\right)\mathbf{T}_1\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \sum_{3\le j\le n} \ln\left(x_2\right)\mathbf{T}_2\cdot\mathbf{T}_j - \frac{1}{2}\gamma_K \ln\left(\frac{-s_{12}}{\mu^2}\right)\mathbf{T}_1\cdot\mathbf{T}_2 + \gamma_1 + \gamma_2 - \gamma_P$$

$$-\frac{1}{2}\gamma_K \sum_{i=3}^{n}\Big((-i\pi\lambda_{i1})\mathbf{T}_i\cdot\mathbf{T}_1 + (-i\pi\lambda_{i2})\mathbf{T}_i\cdot\mathbf{T}_2\Big) + \frac{1}{2}\gamma_K \sum_{i=3}^{n}(-i\pi\lambda_{iP})\mathbf{T}_i\cdot(\mathbf{T}_1+\mathbf{T}_2)$$

$\longrightarrow$ surviving dependence on rest-of-the-process $i\pi\mathbf{T}_n\cdot\mathbf{T}_2$

[S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

What happens in the space-like limit?



Space-like limit is not all bad!

An example from appearance of Super-Leading Logarithms in gap-between jet cross sections, where coherence violating effects give rise to terms with additional powers of logarithms starting at a high loop order.

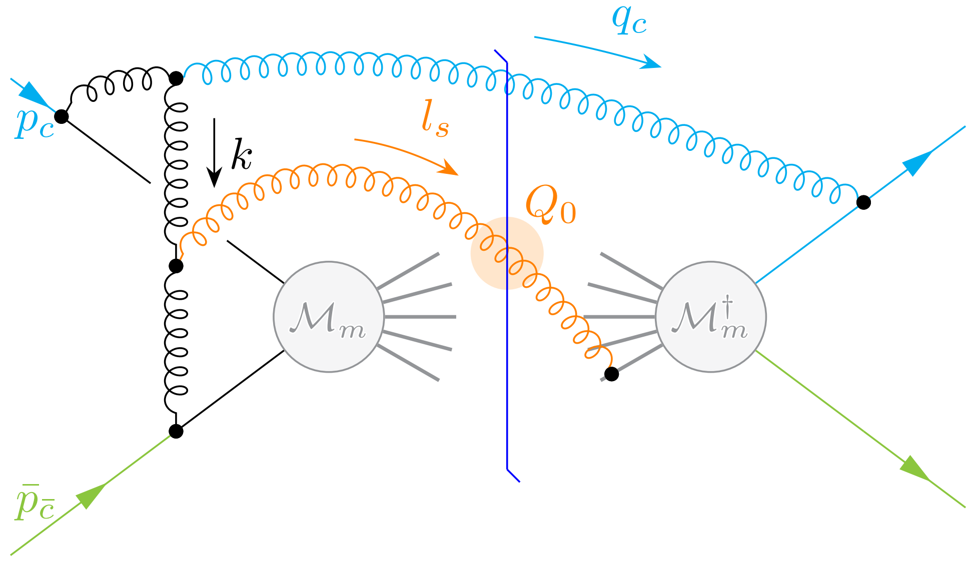
$$\sigma = \sigma_{born}(1 + \alpha_s L + \alpha_s^2 L^2 + \alpha_s^3 L^3 + \alpha_s^4 L^5 + \dots)$$ [J. R. Forshaw, A. Kyrieleis, M. Seymour, 0808.1269]

These can be resummed in an EFT formalism

[T. Becher, M. Neubert, D. Y. Shao, M. Stillger, 2307.06359]

$$\sigma(Q_0) = \sum_{m=m_0}^{\infty} \int d\xi_1 d\xi_2 \langle \mathcal{H}_m(,Q,\xi_1,\xi_2,\mu) \mathcal{W}_m(\{Q_0,\xi_1,\xi_2,\mu) \rangle\,,$$

See related talks of Romy, Josua, Nicolas, and Jürg



And in fact it turns out that its the space-like graphs that give rise to regions in the low-energy matrix element that make it compatible with PDF factorisation.

[T. Becher, P. Hager, SJ, M. Neubert, D Schwienbacher, 2408.10308, 2509.07082]

$$\Big)\ \mathbf{T}_2 \cdot \mathbf{T}_j - \frac{1}{2}\gamma_K \ln\left(\frac{-s_{12}}{\mu^2}\right) \mathbf{T}_1 \cdot \mathbf{T}_2 + \sum_i \gamma_i \mathbf{1}\Big]$$

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

$$(-s_{1j}) = 2\,|p_1 \cdot p_j|\, e^{-i\pi\lambda_{1j}}$$

$$= 2x_1\,|P \cdot p_j|\, e^{-i\pi\lambda_{Pj}} = x_1(-s_{Pj})$$

$$+ \gamma_1 + \gamma_2 - \gamma_P$$

surviving dependence on rest-of-the-process

$$i\pi \mathbf{T}_n \cdot \mathbf{T}_2$$

[S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

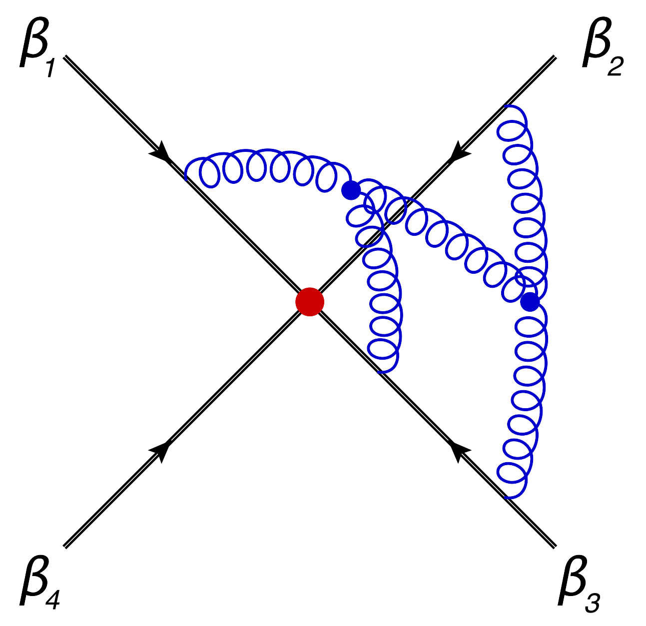
The Soft Anomalous Dimension at three-loops

Recall the structure:

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

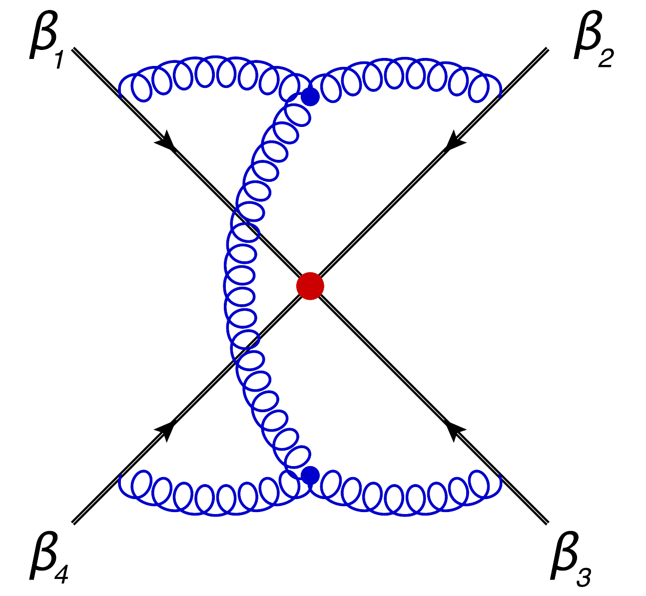
$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,\text{3L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{5T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,\text{5T}-\text{5L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$



Result of explicit calculation [Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

$$\mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk} = f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} f^{ade}f^{bce}\left\{\mathbf{T}_i^a,\mathbf{T}_i^b\right\}\mathbf{T}_j^c\,\mathbf{T}_k^d$$



$$\mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n}\Big[T_{iklj}\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj})\Big]$$

$$T_{ijkl} = f^{ade}f^{bce}\left\{\mathbf{T}_i^a,\mathbf{T}_j^b,\mathbf{T}_k^c,\mathbf{T}_l^d\right\}_+$$

Notice that $\mathbf{\Gamma}_{n,\text{4T}-\text{3L}}$ exists already for $n=3$, but the second term from $n=4$

The Soft Anomalous Dimension at three-loops

A word about the kinematics

$$\Gamma_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8 \sum_{1\leq i<j<k<l\leq n} \Big[T_{iklj}\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj})\Big]$$

$$\mathbf{a}_{ijkl}(\{\beta\})$$

These objects here are logarithms of the *Conformally Invariant Cross Ratios* (CICRs)

$$\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} \qquad \beta_{ijkl} = \ln \rho_{ijkl} = \ln\left(\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}\right)$$

Which can be written in terms of variables $\rho_{ijkl} = z_{ijkl}\bar{z}_{ijkl}\,, \quad \rho_{ilkj} = (1-z_{ijkl})(1-\bar{z}_{ijkl})$ [Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

$$\mathcal{F}^{(3)}(\beta_{ijkl},\beta_{ilkj}) = \frac{1}{32}\Big[F(1-z_{ijkl}) - F(z_{ijkl})\Big] \qquad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\left[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)\right]$$

Single Valued Harmonic Polylogarithms

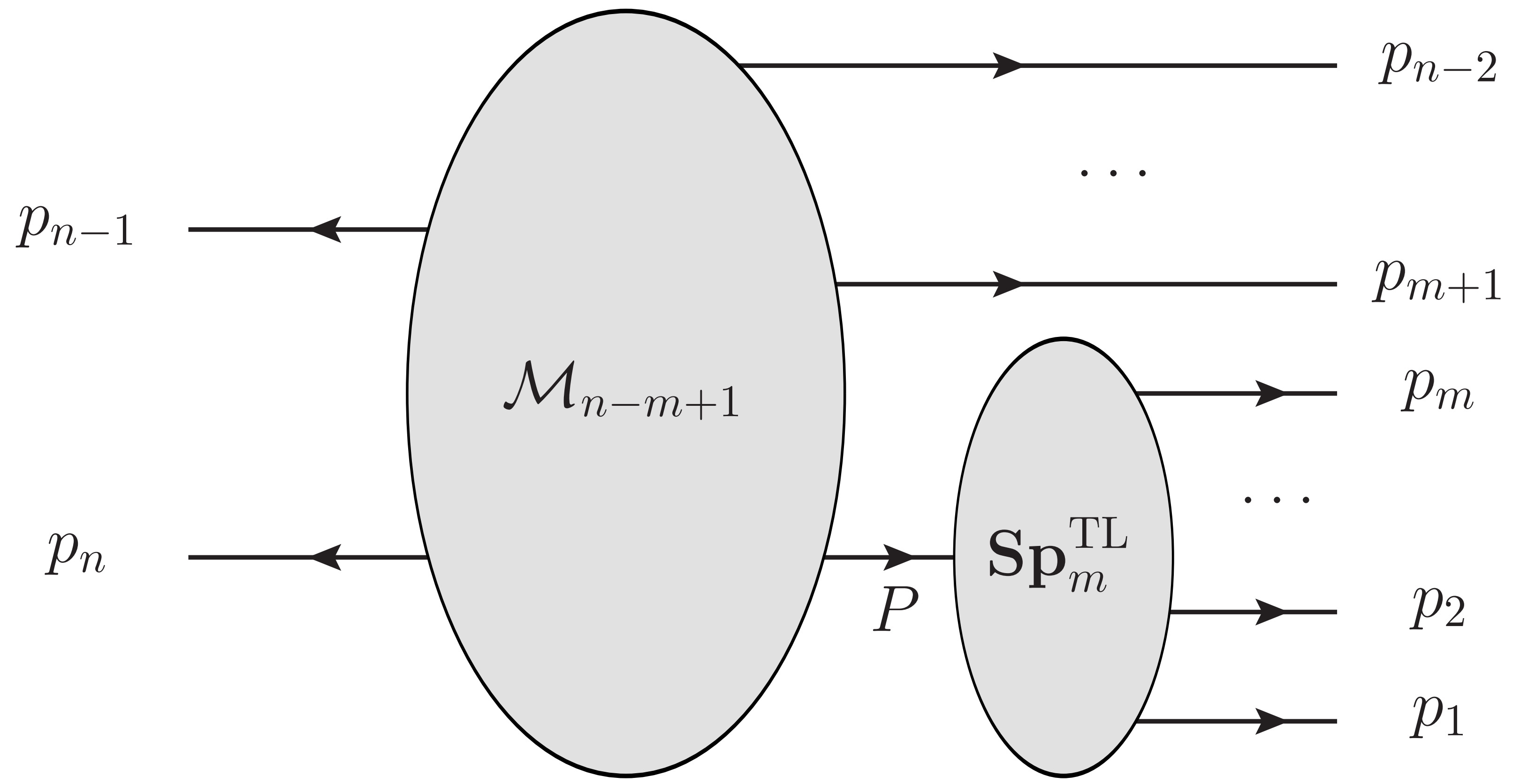
See similar objects Fabian's talk

CICRs in collinear limits

For two-particle collinear limits we need to consider

$$\rho_{12kl} = \frac{(-p_1 \cdot p_2)(-p_k \cdot p_l)}{(-p_1 \cdot p_k)(-p_2 \cdot p_l)} \to 0$$

$$\rho_{1lk2} = \frac{(-p_1 \cdot p_l)(-p_k \cdot p_2)}{(-p_1 \cdot p_k)(-p_l \cdot p_2)} \approx \frac{(-x_1 P \cdot p_l)(-x_2 P \cdot p_k)}{(-x_1 P \cdot p_k)(-x_2 P \cdot p_l)} \to 1$$

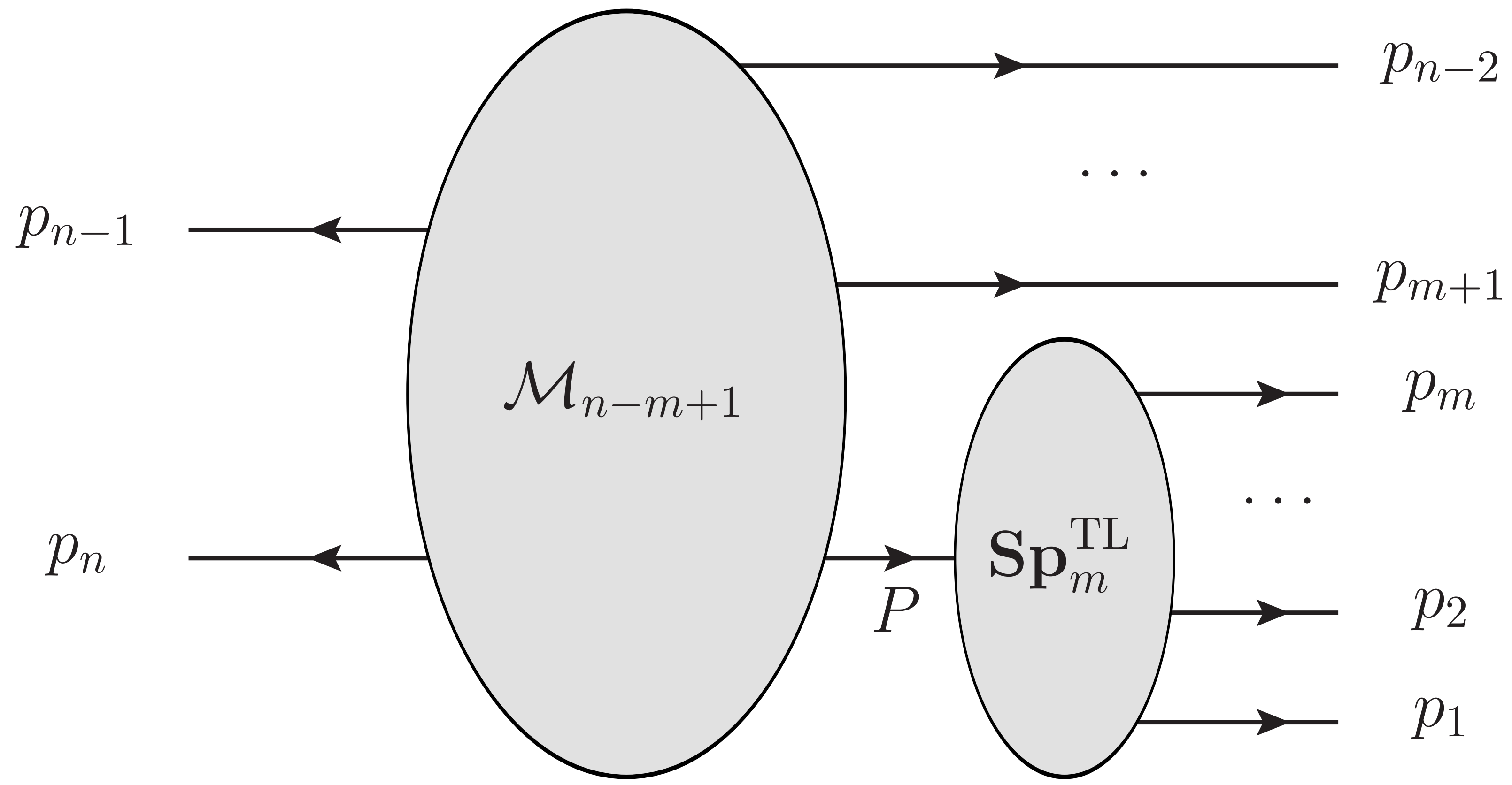


CICRs in collinear limits

For two-particle collinear limits we need to consider

$$\rho_{12kl} = \frac{(-p_1 \cdot p_2)(-p_k \cdot p_l)}{(-p_1 \cdot p_k)(-p_2 \cdot p_l)} \to 0$$

$$\rho_{1lk2} = \frac{(-p_1 \cdot p_l)(-p_k \cdot p_2)}{(-p_1 \cdot p_k)(-p_l \cdot p_2)} \approx \frac{(-x_1 P \cdot p_l)(-x_2 P \cdot p_k)}{(-x_1 P \cdot p_k)(-x_2 P \cdot p_l)} \to 1$$



In the three-particle collinear limit on the other hand

$$\rho_{123l} = \frac{(-p_1 \cdot p_2)(-p_3 \cdot p_l)}{(-p_1 \cdot p_3)(-p_2 \cdot p_l)} \approx \frac{(-p_1 \cdot p_2)(-x_3\, P \cdot p_l)}{(-p_1 \cdot p_3)(-x_2\, P \cdot p_l)}$$

$$\rho_{123l} \to \frac{x_3(-p_1 \cdot p_2)}{x_2(-p_1 \cdot p_3)}$$

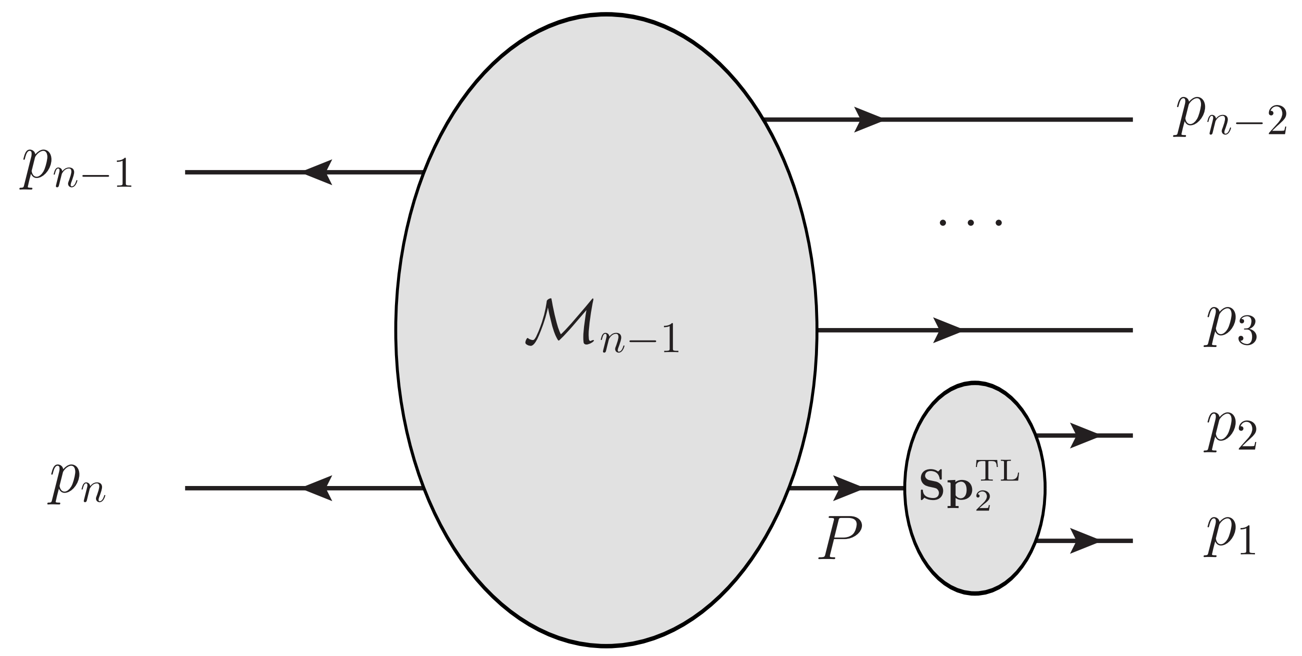
$$\rho_{1l32} = \frac{(-p_1 \cdot p_l)(-p_3 \cdot p_2)}{(-p_1 \cdot p_3)(-p_l \cdot p_2)} \approx \frac{x_1(-p_3 \cdot p_2)}{x_2(-p_1 \cdot p_3)}$$

Do retain kinematic dependence, but on the collinear partons only!
The rest-of-the-process dependence scales out.

The four-particle collinear limit saturates the CICR and no expansions to be done

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Can we pick n=3?

If we do select $n = 3$, most terms above cannot contribute, only:

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{3,4\mathrm{T}-3\mathrm{L}}(\alpha_s)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

Explicitly

$$\mathbf{\Gamma}_{3,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\left(T_{1123}+T_{2213}+T_{3312}\right)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

With colour conservation: $\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3 = 0$

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2} = -\frac{3}{4}f(\alpha_s)\left(C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right)$$

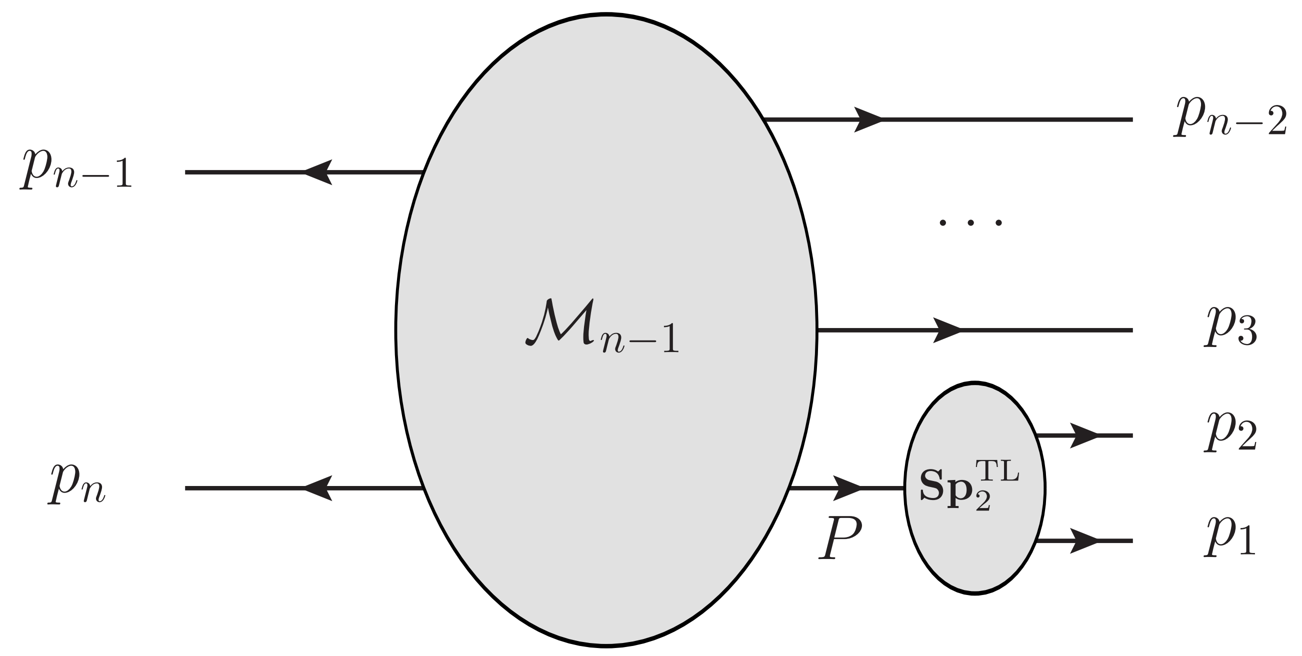
But thus far, we have not made use of universality of this object, the left-hand side is independent of n .

Manifestly depending only on particles 1 and 2, good!

When we consider a higher point amplitude, more structures enter. Importantly, the kinematic ones, but the result should be the same!

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Just as in the dipole case, split the sums!

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\left[\sum_{3\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{1jkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{2jkl}(\{\beta\}) + \sum_{3\le k<l\le n}\mathbf{a}_{12kl}(\{\beta\})\right]$$

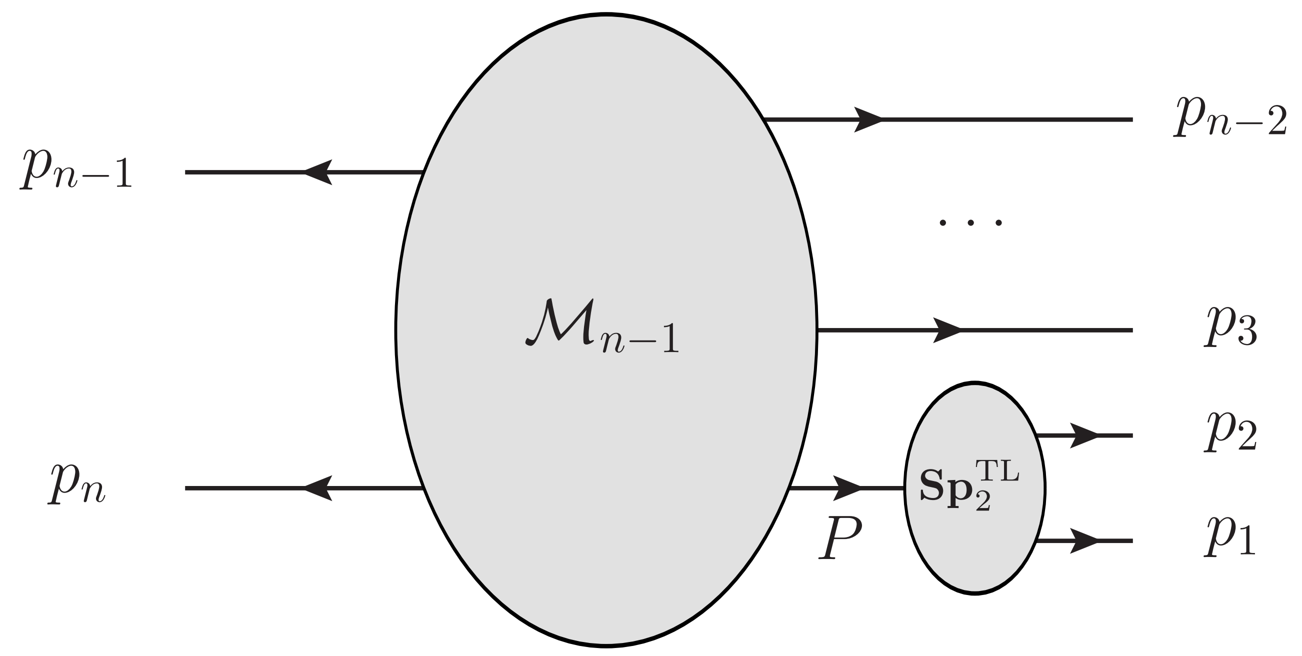
$$\mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\left[\sum_{3\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{Pjkl}(\{\beta\})\right]$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T-3L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T-3L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Just as in the dipole case, split the sums!

$$\mathbf{\Gamma}_{n,4\mathrm{T-4L}}(\{\beta\}) = 4\Bigg[\sum_{3\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{1jkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{2jkl}(\{\beta\}) + \sum_{3\le k<l\le n}\mathbf{a}_{12kl}(\{\beta\})\Bigg]$$

$$\mathbf{\Gamma}_{n-1,4\mathrm{T-4L}}(\{\beta\}) = 4\Bigg[\sum_{3\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + \sum_{3\le j<k<l\le n}\mathbf{a}_{Pjkl}(\{\beta\})\Bigg]$$

$$\mathbf{\Gamma}_{n,4\mathrm{T-3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

The structure of these terms is for example like this: $f^{ade}f^{bce}\mathbf{T}_1^a\mathbf{T}_k^b\mathbf{T}_l^c\,\mathbf{T}_j^d\,\mathcal{F}(\beta_{1kjl},\beta_{1ljk})$

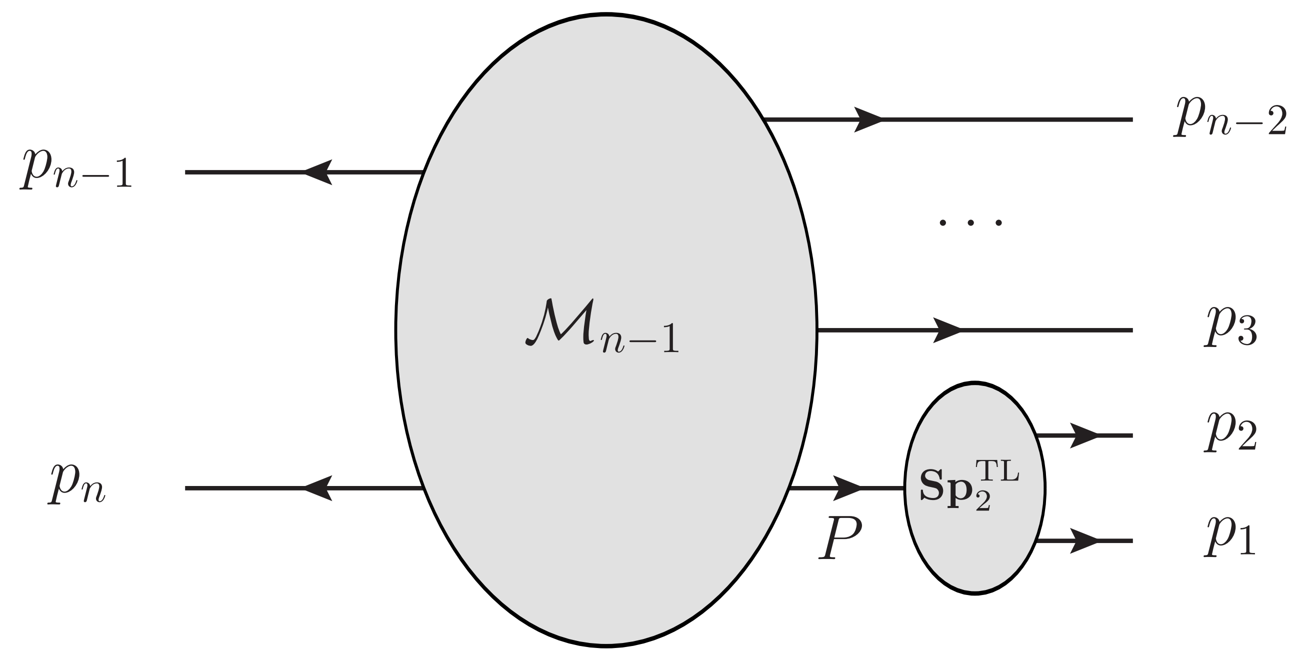
and colour too

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

Kinematics corresponds to each other $\quad \rho_{1ljk} = \dfrac{(-p_1\cdot p_l)(-p_j\cdot p_k)}{(-p_1\cdot p_j)(-p_l\cdot p_k)} \approx \dfrac{(-x_1 P\cdot p_l)(-p_k\cdot p_j)}{(-x_1 P\cdot p_j)(-p_k\cdot p_l)} = \rho_{Pljk}$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Next, consider the constant piece $\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\left\{\sum_{i=3}^{n}\sum_{\substack{3\le k<l\le n\\ k,l\neq i}}\mathcal{T}_{iikl} + \sum_{3\le k<l\le n}\left(\mathcal{T}_{11kl}+\mathcal{T}_{22kl}\right) + \sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\neq i}}^{n}\left(T_{ii1k}+T_{ii2k}\right) + \sum_{i=3}^{n}\left(T_{112i}+T_{221i}+T_{ii12}\right)\right\}$$

$$\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\left\{\sum_{i=3}^{n}\sum_{\substack{3\le k<l\le n\\ k,l\neq i}}T_{iikl} + \sum_{3\le k<l\le n}T_{PPkl} + \sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\neq i}}^{n}T_{iiPk}\right\}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}}T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n}a_{ijkl}(\beta)$$

These cancel exactly as before using

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

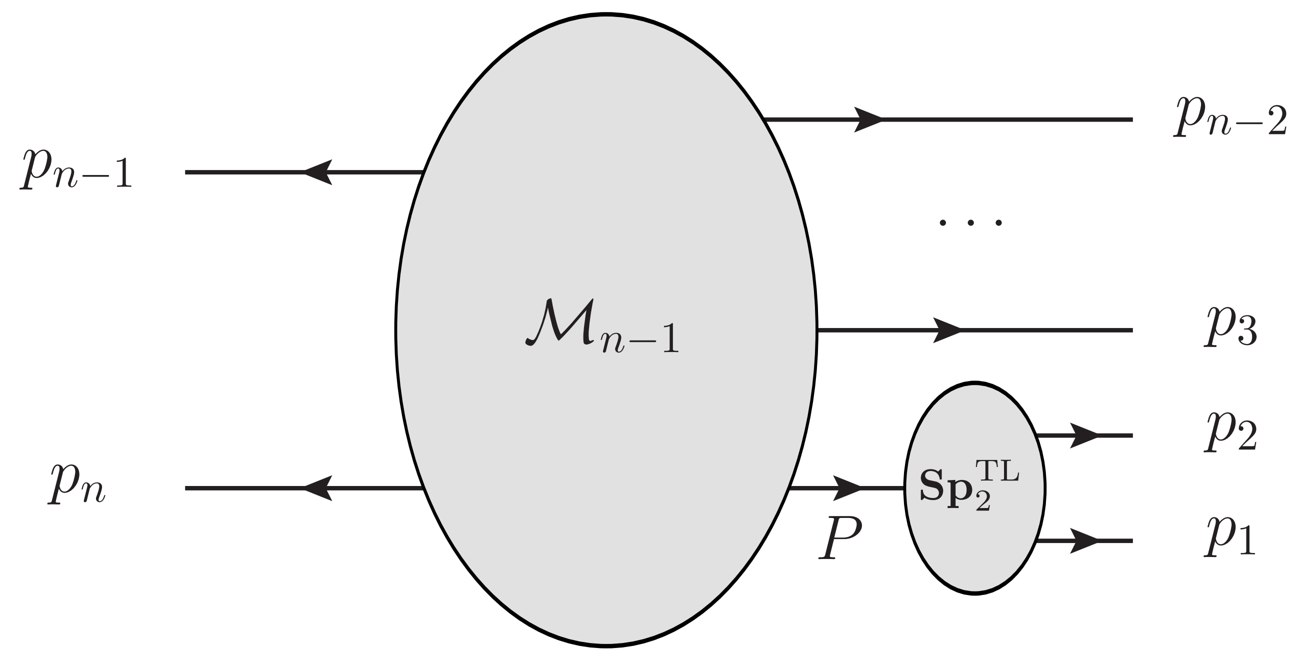
The blue terms do not cancel exactly, because of the anticommutator, there are cross terms

$$f^{ade}f^{bce}\left\{\mathbf{T}_P^a,\mathbf{T}_P^b\right\}\mathbf{T}_k^c\,\mathbf{T}_l^d$$

$$T_{PPjk} = T_{11jk} + T_{22jk} + T_{12jk} + T_{12kj}$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Now, we combine the constant $\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}$ with the kinematic part

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n}\left[4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \left(4\mathcal{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\right)\left(T_{12lk}+T_{12kl}\right)\right]$$

$$+2f(\alpha_s)\sum_{i=3}^{n}\left(T_{ii12}+T_{112i}+T_{221i}\right)$$

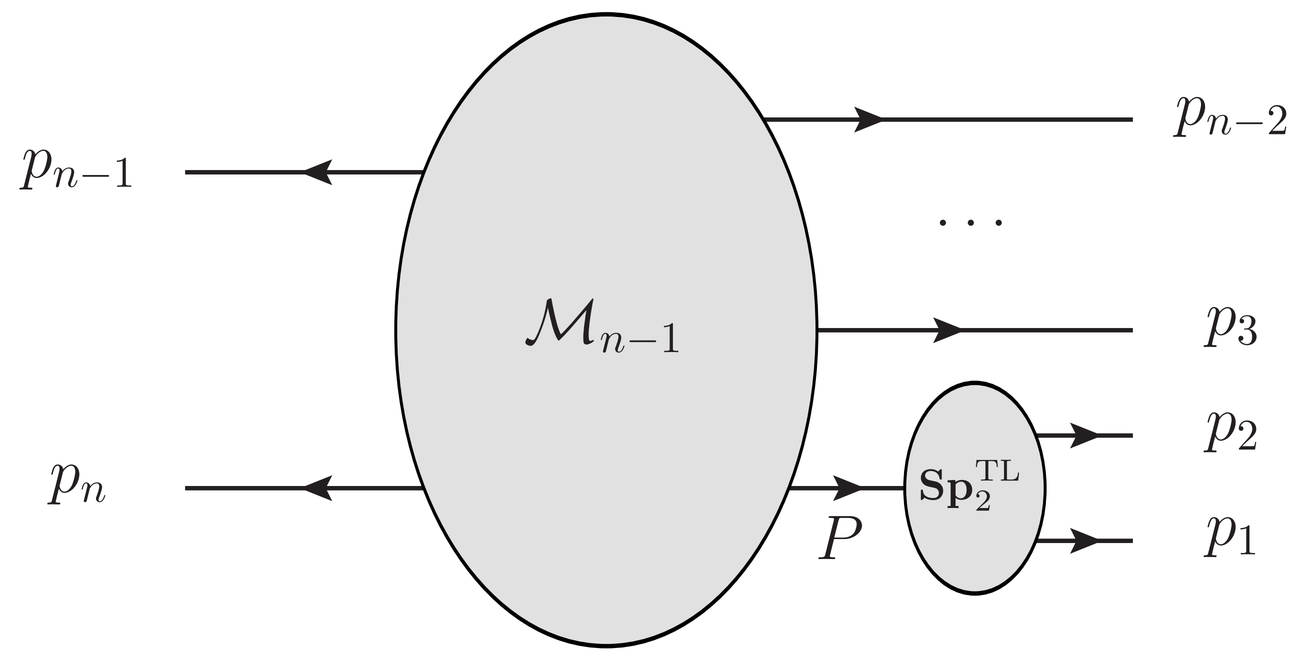
$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Thinking back to the $n=3$ case, this just appears in the last line, we can recover it easily, and apply colour conservation.

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Now, we combine the constant $\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}$ with the kinematic part

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n}\left[4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \left(4\mathcal{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\right)\left(T_{12lk} + T_{12kl}\right)\right]$$

$$+2f(\alpha_s)\sum_{i=3}^{n}\left(T_{ii12} + T_{112i} + T_{221i}\right)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

The first term is seemingly problematic:

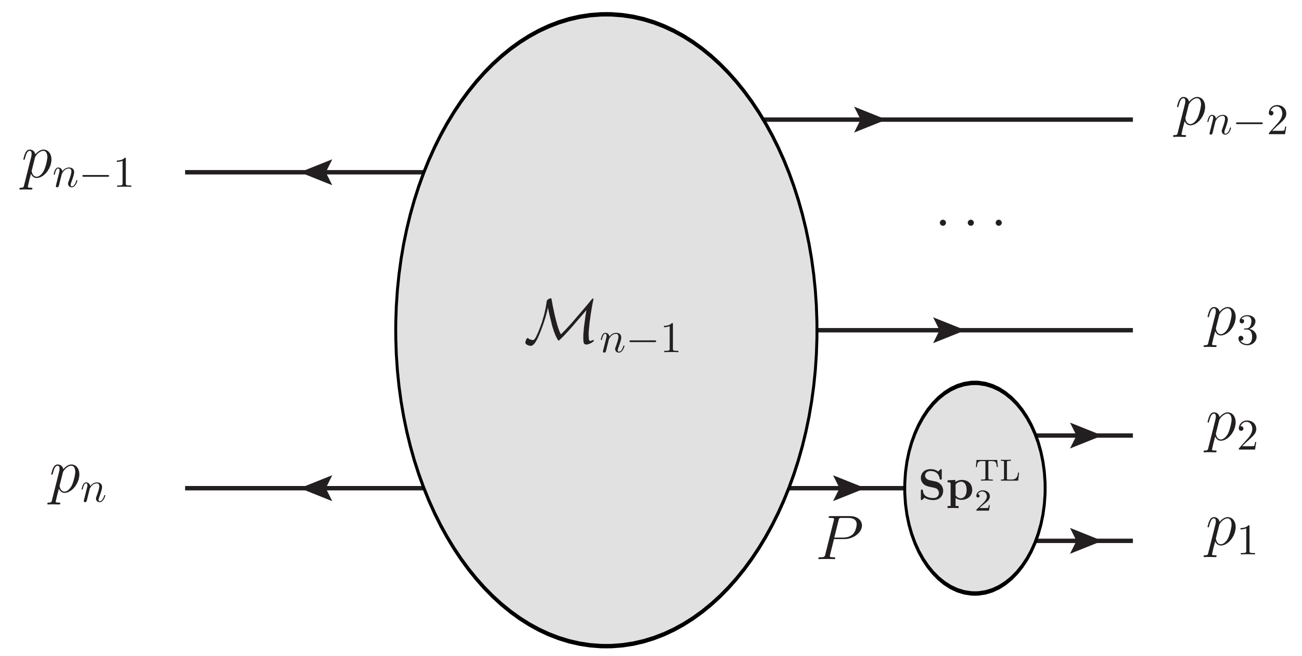
$$f^{ade}f^{bce}\left\{\mathbf{T}^a_3,\mathbf{T}^b_3\right\}\mathbf{T}^c_1\,\mathbf{T}^d_2 + f^{ade}f^{bce}\left\{\mathbf{T}^a_4,\mathbf{T}^b_4\right\}\mathbf{T}^c_1\,\mathbf{T}^d_2 + f^{ade}f^{bce}\left\{\mathbf{T}^a_5,\mathbf{T}^b_5\right\}\mathbf{T}^c_1\,\mathbf{T}^d_2 + \ldots.$$

Difficult to see how these terms alone can be collected together to make them depend only on particles 1 and 2.

Indeed, we need the top line!!

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T-3L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T-3L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Now, we combine the constant $\mathbf{\Gamma}_{n,4\mathrm{T-3L}}$ with the kinematic part

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n}\left[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \left(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\right)\left(T_{12lk}+T_{12kl}\right)\right]$$

$$+2f(\alpha_s)\sum_{i=3}^{n}\left(T_{ii12}+T_{112i}+T_{221i}\right)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T-3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T-4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

For $n \geq 4$ we need non trivial interplay of the kinematics and constant part, conspiring to produce the same result as $n=3$

Notice the following relation

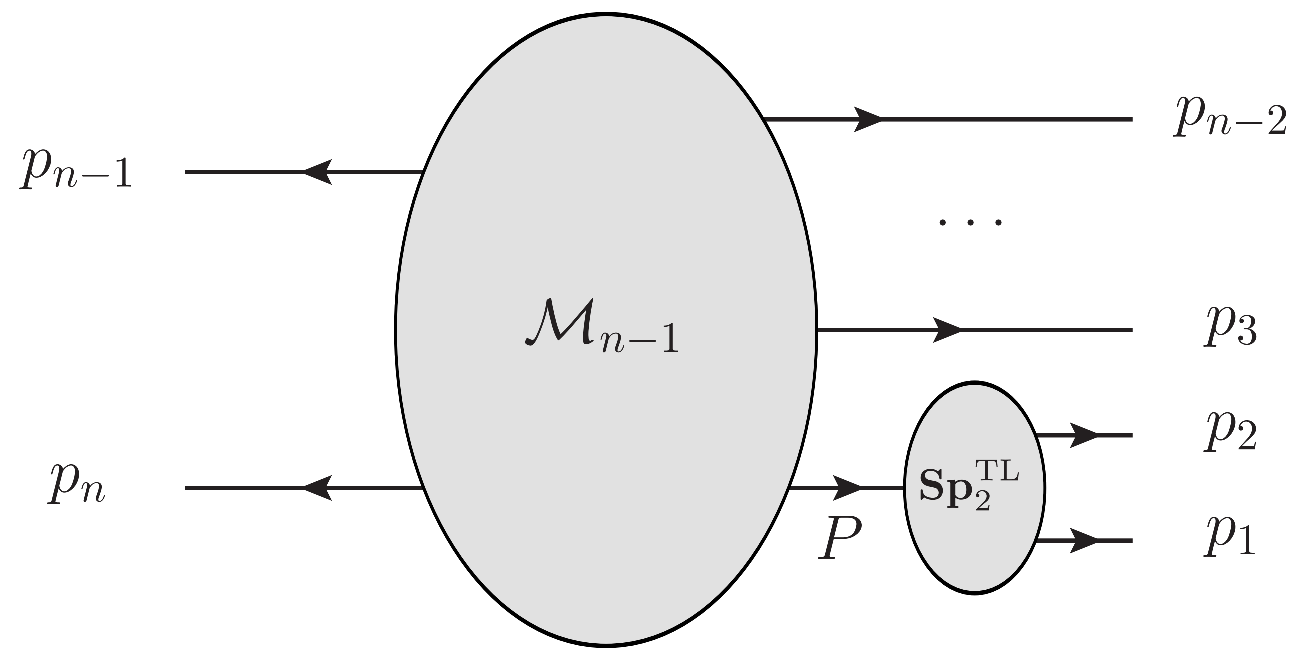
$$2\sum_{3\le k<l\le n}\left(T_{12lk}+T_{12kl}\right) + 2\sum_{i=3}^{n}T_{ii12} = \sum_{k=3}^{n}\sum_{l=3}^{n}\left(T_{12lk}+T_{12kl}\right)$$

On the right-hand side, colour conservation can be applied in turn!

$$f^{ade}f^{bce}\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_k^c(\mathbf{T}_3^d+\mathbf{T}_4^d+\ldots) = -f^{ade}f^{bce}\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_k^c(\mathbf{T}_1^d+\mathbf{T}_2^d)$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Then applying colour conservation, gives a familiar result (of n=3 case)

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},\mathbf{2}}(p_1,p_2;\mu) = -\frac{3}{4}f(\alpha_s)\Big(C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\Big)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

Provided the following condition holds in the two-particle collinear limit:

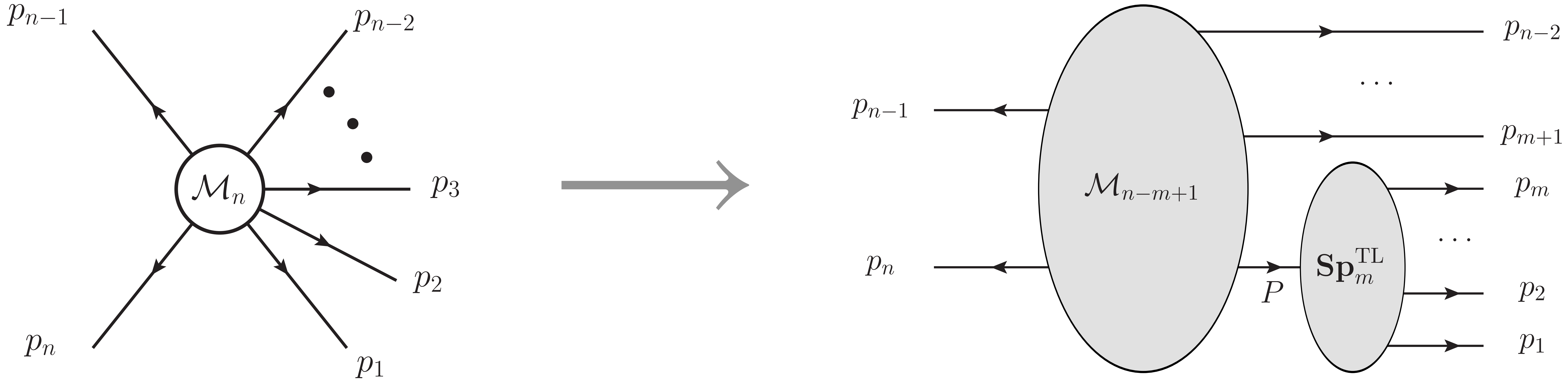
$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathcal{F}^{\mathrm{S}}_{12kl}(\{\beta\})\Big|_{p_1||p_2} = f(\alpha_s) \qquad \mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\Big|_{p_1||p_2} = 0$$

Results for these functions exist, so we can check explicitly these relations, and indeed they are satisfied

No new constraints from considering $n>4$

Multi-particle collinear limits



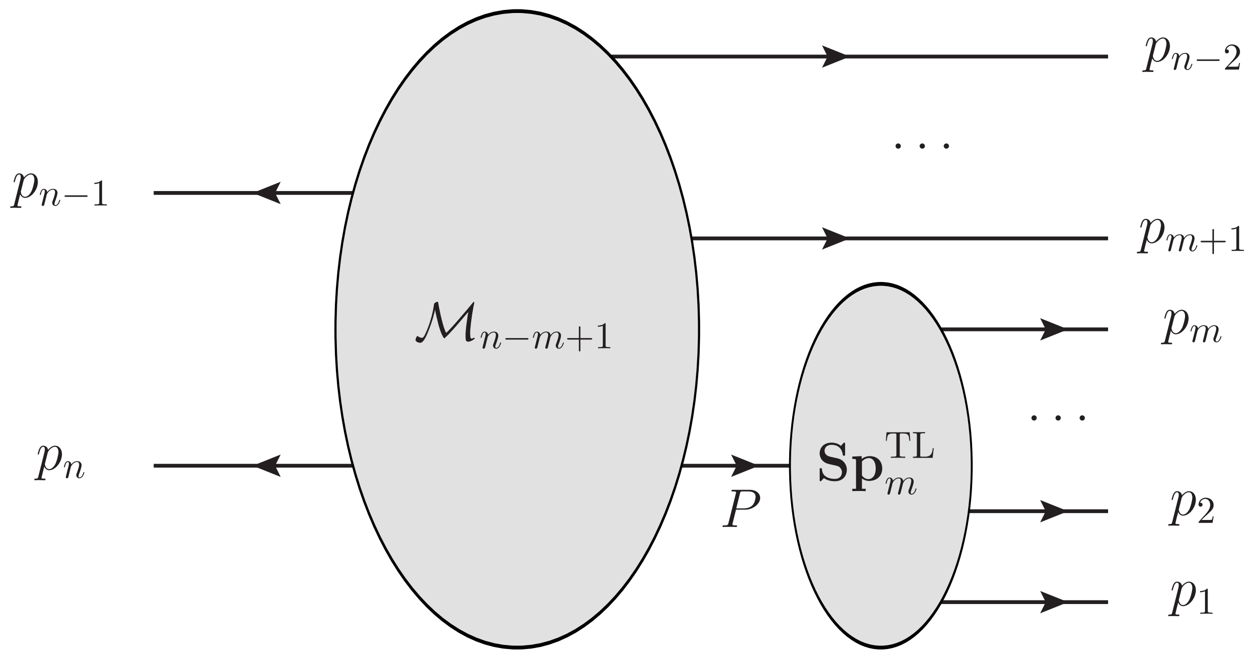
$$\mathscr{M}_n(p_1, \ldots p_m, \{p_i\}_{\mathrm{rest}}; \mu) \xrightarrow{p_1 \parallel p_2 \parallel \ldots \parallel p_m} \mathbf{Sp}_m(p_1, \ldots p_m; \mu)\, \mathscr{M}_{n-m+1}(P, \{p_i\}_{\mathrm{rest}}; \mu)$$

The object of interest is

$$\boldsymbol{\Gamma}_{\mathbf{Sp},m}(p_1, \ldots p_m; \mu) = \boldsymbol{\Gamma}_n(p_1, \ldots p_m, p_{m+1}, \ldots p_n; \mu) - \boldsymbol{\Gamma}_{n-m+1}(P, p_{m+1}, \ldots p_n; \mu)\big|_{\mathbf{T}_P \to \sum_{i=1}^m \mathbf{T}_i}$$

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



The strategy we follow is identical to before, we again split the sums with respect to how many and which collinear particles they involve

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n} \mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n} \mathbf{a}_{1jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n} \mathbf{a}_{2jkl}(\{\beta\}) + \sum_{4\le j<k<l\le n} \mathbf{a}_{3jkl}(\{\beta\})$$
$$+ \sum_{4\le k<l\le n} \mathbf{a}_{12kl}(\{\beta\}) + \sum_{4\le k<l\le n} \mathbf{a}_{13kl}(\{\beta\}) + \sum_{4\le k<l\le n} \mathbf{a}_{23kl}(\{\beta\}) + \sum_{4\le l\le n} \mathbf{a}_{123l}(\{\beta\})\Bigg]$$

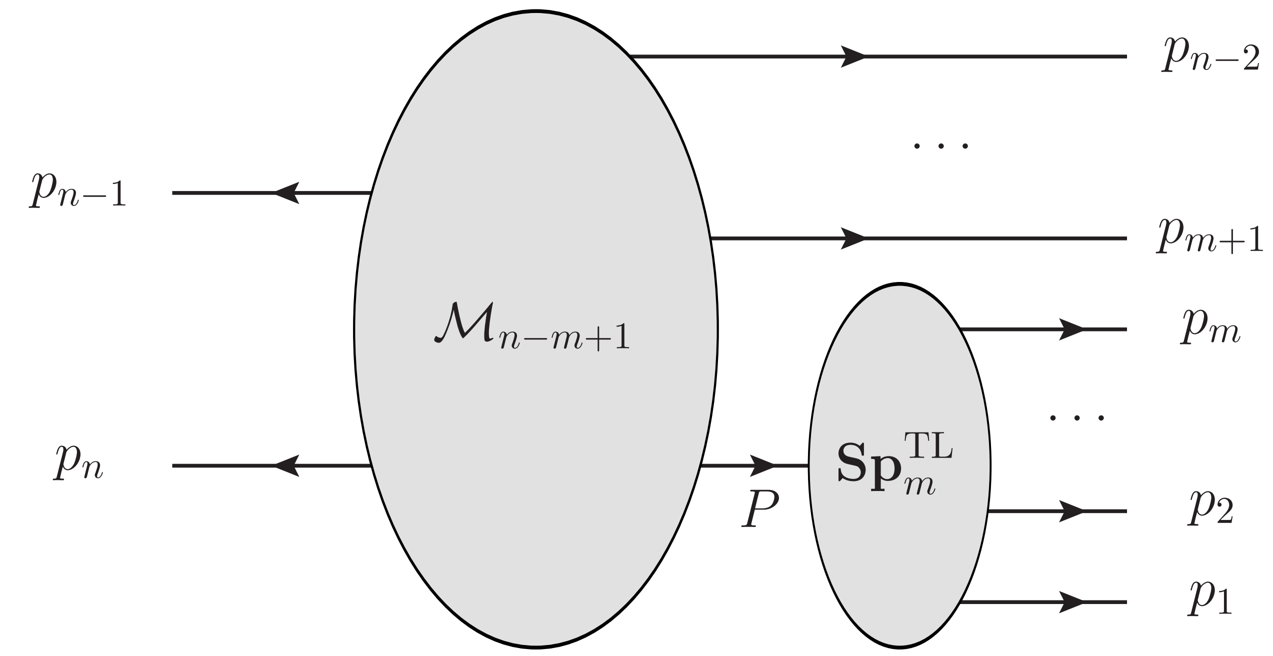
$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta\}) = 4\Bigg[\sum_{4\le i<j<k<l\le n} \mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\le j<k<l\le n} \mathbf{a}_{Pjkl}(\{\beta\})\Bigg]$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1, p_2; \mu) = \mathbf{\Gamma}_{n,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\}, \alpha_s) - \left(\mathbf{\Gamma}_{n-2,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n-2,4T-4L}(\{\beta_{ijkl}\}, \alpha_s)\right)$$



The strategy we follow is identical to before, we again split the sums with respect to how many and which collinear particles they involve. Starting with the kinematics:

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta\}) = 4\Bigg[\sum_{4\leq i<j<k<l\leq n} \mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\leq j<k<l\leq n} \mathbf{a}_{1jkl}(\{\beta\}) + \sum_{4\leq j<k<l\leq n} \mathbf{a}_{2jkl}(\{\beta\}) + \sum_{4\leq j<k<l\leq n} \mathbf{a}_{3jkl}(\{\beta\})$$

$$+ \sum_{4\leq k<l\leq n} \mathbf{a}_{12kl}(\{\beta\}) + \sum_{4\leq k<l\leq n} \mathbf{a}_{13kl}(\{\beta\}) + \sum_{4\leq k<l\leq n} \mathbf{a}_{23kl}(\{\beta\}) + \sum_{4\leq l\leq n} \mathbf{a}_{123l}(\{\beta\})\Bigg]$$

$$\mathbf{\Gamma}_{n,4T-3L}(\alpha_s) = 2f(\alpha_s) \sum_{i=1}^{n} \sum_{\substack{1 \leq j < k \leq n, \\ j,k \neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\}, \alpha_s) = 8 \sum_{1\leq i<j<k<l\leq n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}_{n-2,4T-4L}(\{\beta\}) = 4\Bigg[\sum_{4\leq i<j<k<l\leq n} \mathbf{a}_{ijkl}(\{\beta\}) + \sum_{4\leq j<k<l\leq n} \mathbf{a}_{Pjkl}(\{\beta\})\Bigg]$$

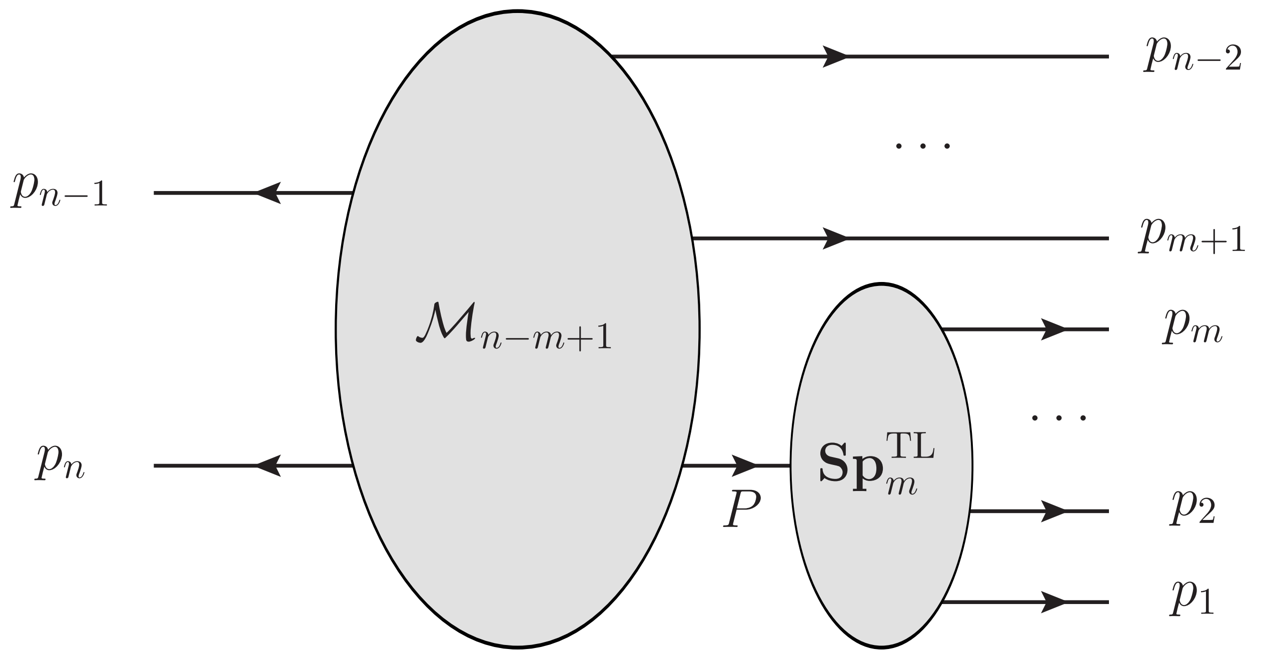
Now, the parent parton is $\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$

We see similar cancellations as before between pink and orange terms.

New type of term with three collinear particles present, marked in green.

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Now the constant part

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\Bigg\{\sum_{i=4}^{n}\sum_{\substack{4\le j<k\le n\\ j,k\neq i}} T_{iijk} + \sum_{4\le j<k\le n}\left(T_{11jk}+T_{22jk}+T_{33jk}\right)$$

$$+\sum_{i=4}^{n}\sum_{\substack{k=4,\\ k\neq i}}^{n}\left(T_{ii1k}+T_{ii2k}+T_{ii3k}\right)$$

$$+\sum_{i=4}^{n}\left(T_{112i}+T_{221i}+T_{ii12}+T_{113i}+T_{331i}+T_{ii13}+T_{223i}+T_{332i}+T_{ii23}\right)$$

$$+T_{1123}+T_{2213}+T_{3312}\Bigg\}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Terms of this type

$$f^{ade}f^{bce}\left\{\mathbf{T}^a_P,\mathbf{T}^b_P\right\}\mathbf{T}^c_k\mathbf{T}^d_l$$

with

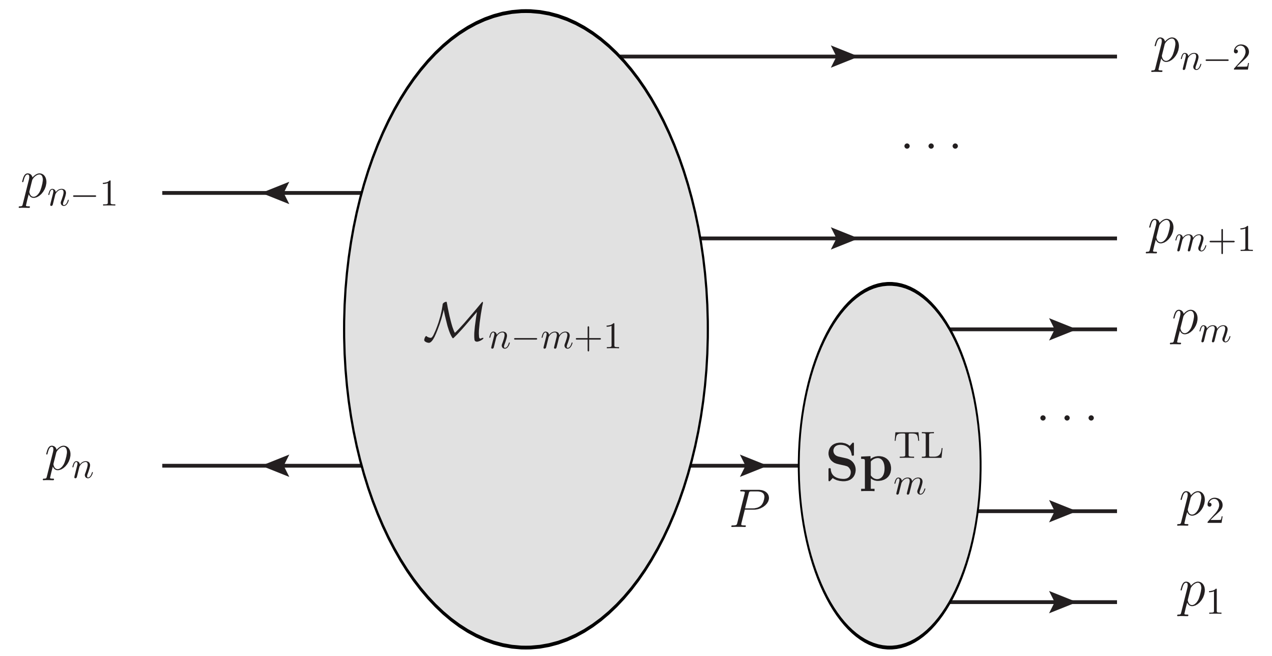
$$\mathbf{T}_P = \mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3$$

$$\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\Bigg\{\sum_{i=4}^{n}\sum_{\substack{4\le j<k\le n\\ j,k\neq i}} T_{iijk} + \sum_{4\le j<k\le n} T_{PPjk} + \sum_{i=4}^{n}\sum_{\substack{k=4,\\ k\neq i}}^{n} T_{iiPk}\Bigg\}$$

$$T_{PPjk} = T_{11jk}+T_{22jk}+T_{33jk}+T_{12jk}+T_{12kj}+T_{13jk}+T_{13kj}+T_{23jk}+T_{23kj}$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Combining terms we find

$$\begin{aligned}
\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = &\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \Big(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{12lk}+T_{12kl}\Big)\Big]\\
&+\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{13kl}(\{\beta\})\,T_{1kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{13kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{13lk}+T_{13kl}\Big)\Big]\\
&+\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{23kl}(\{\beta\})\,T_{2kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{23kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{23lk}+T_{23kl}\Big)\Big]\\
&+2f(\alpha_s)\Big[\sum_{i=4}^{n}\Big(T_{112i}+T_{221i}+T_{ii12}+T_{113i}+T_{331i}+T_{ii13}+T_{223i}+T_{332i}+T_{ii23}\Big)\Big]\\
&+2f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 4\sum_{4\le l\le n}\mathbf{a}_{123l}(\{\beta\})
\end{aligned}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

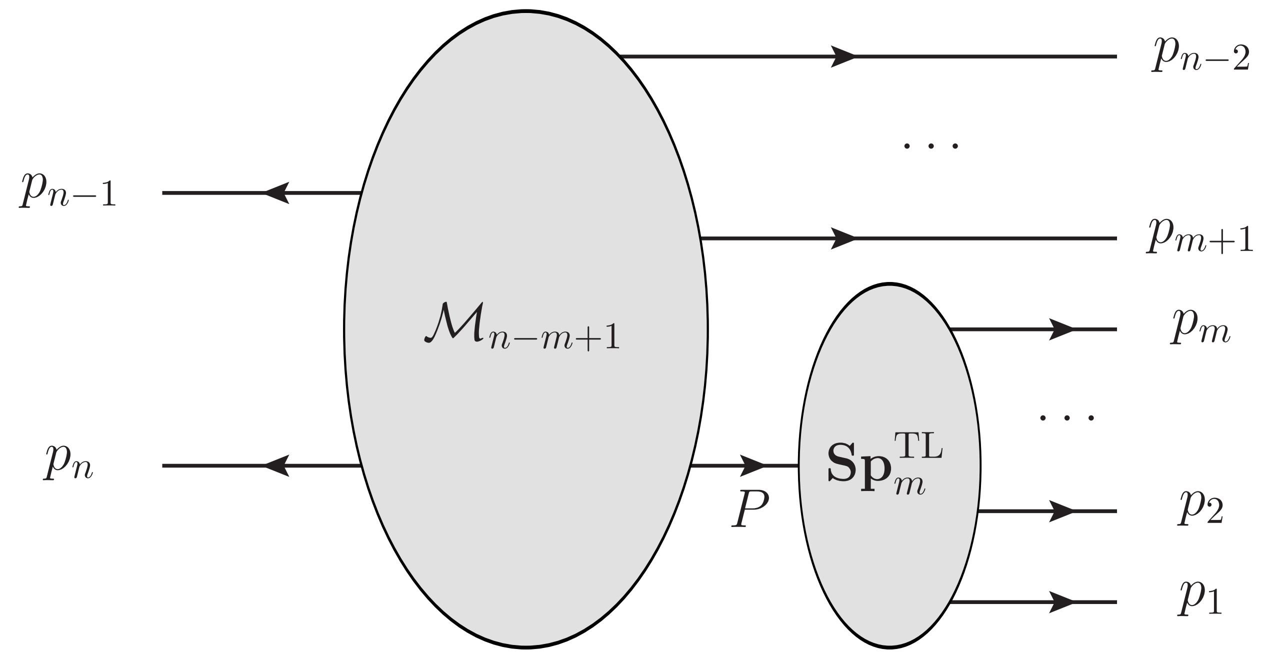
$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Three copies of the two-particle collinear limit, between each pair of the three becoming collinear.

And new terms!

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

We then find:

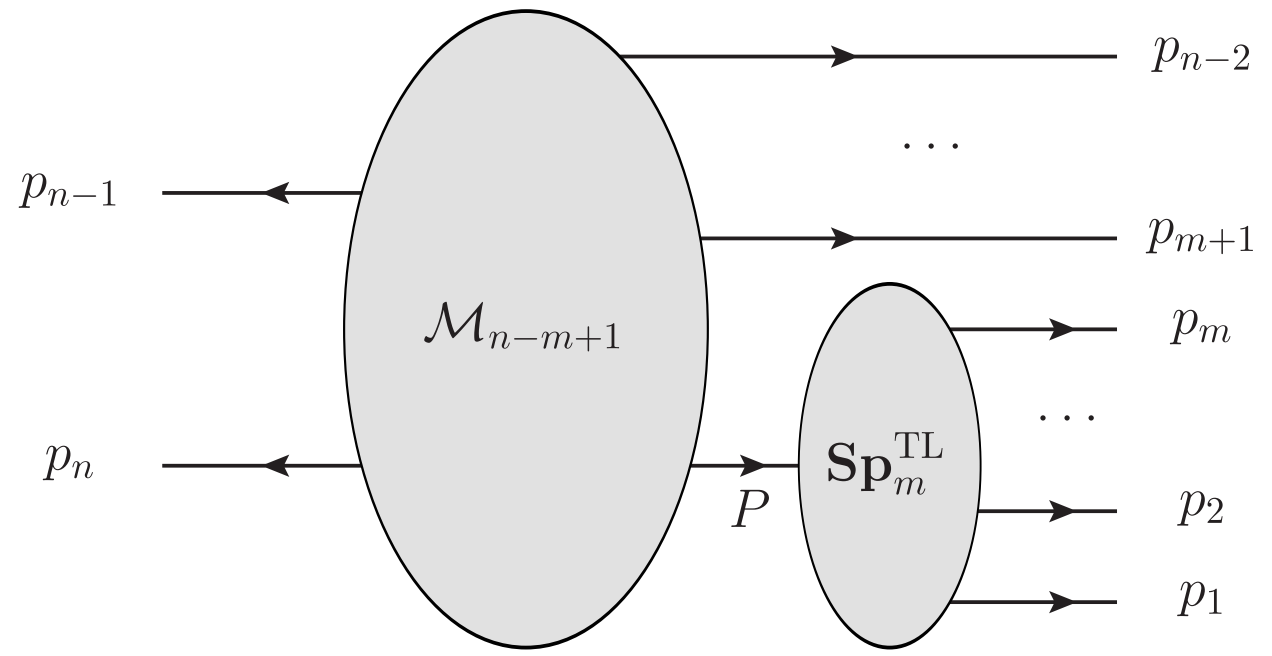
$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$

$$-4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big]$$

what about the terms in the last line?

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n-2,4T-4L}(\{\beta_{ijkl}\},\alpha_s)\right)$$



This is where properties of CICRs come to the rescue, as we have seen, the CICRs with three particles collinear, the dependence on the rest-of-the-process parton scales out.

$$\rho_{123l} \to \frac{x_3(-p_1\cdot p_2)}{x_2(-p_1\cdot p_3)}$$

Then we can apply colour conservation $\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312}$

$$\mathbf{\Gamma}_{n,4T-3L}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

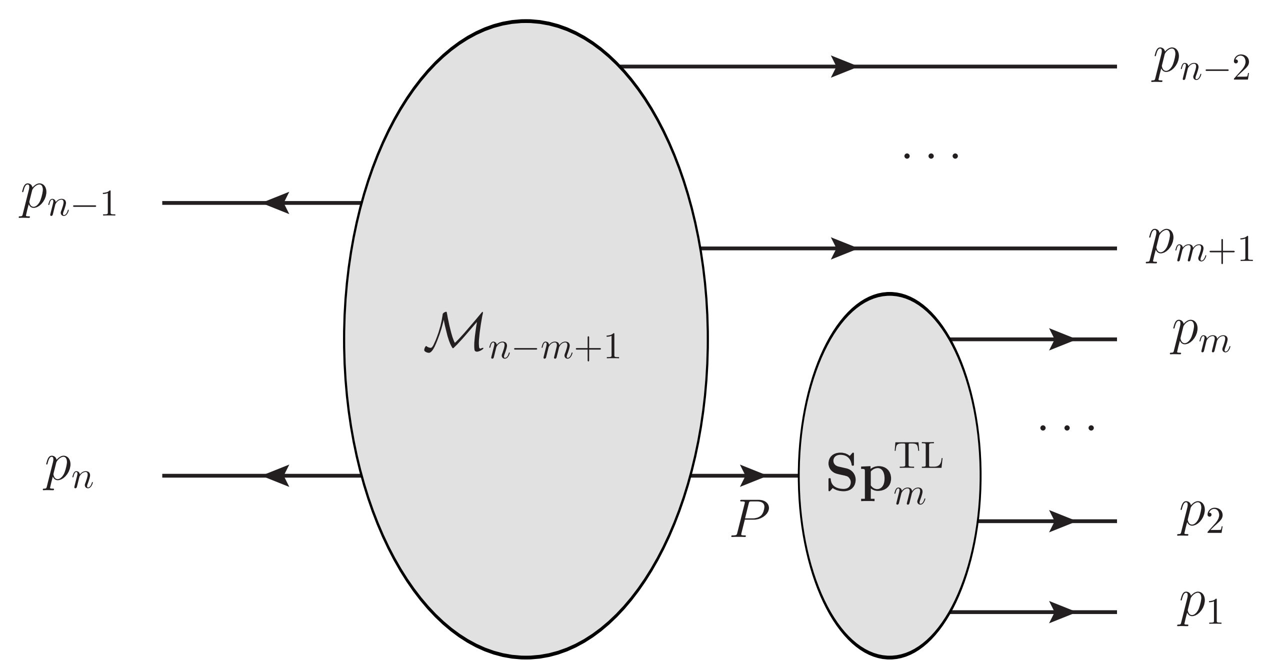
$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\right]$$

$$-4f(\alpha_s)\left[T_{1123}+T_{2213}+T_{3312}\right] + 8\sum_{4\le l\le n}\left[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\right]$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



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Then we can apply colour conservation $\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312}$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

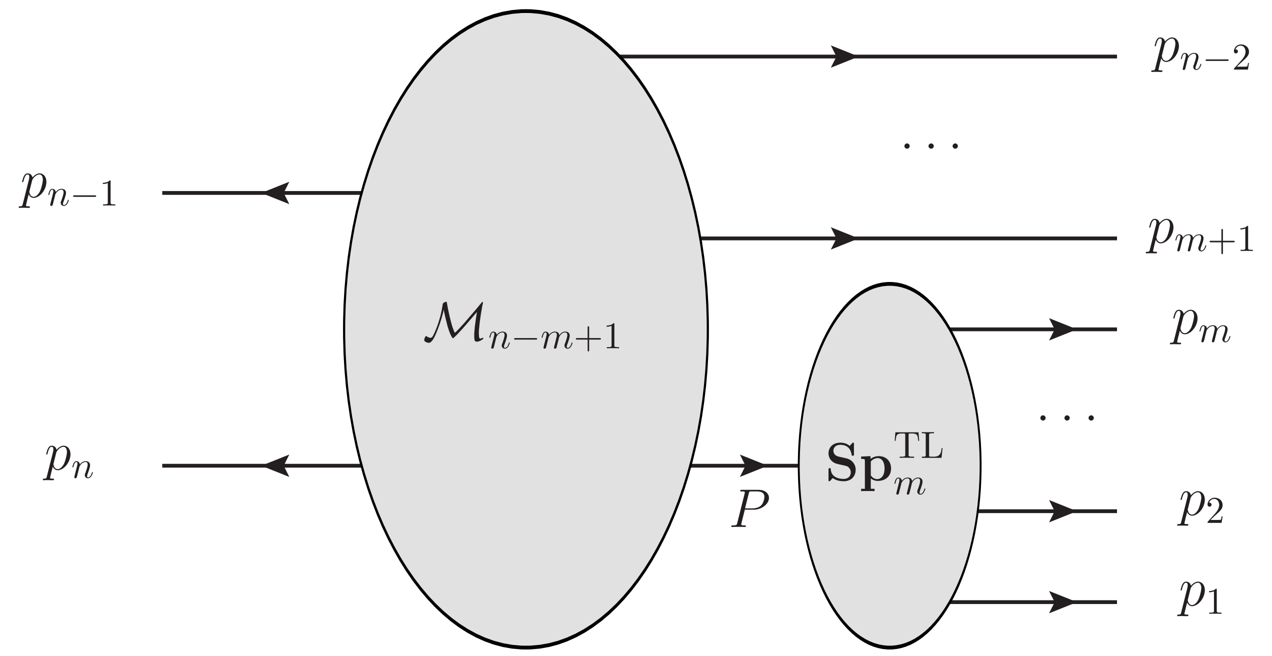
$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\right]$$
$$-4f(\alpha_s)\left[T_{1123}+T_{2213}+T_{3312}\right] + 8\left(T_{1123}-T_{2213}\right)\mathcal{F}(\beta_{132l},\beta_{1l23}) + 8\left(T_{1123}-T_{3312}\right)\mathcal{F}(\beta_{123l},\beta_{1l32}) + 8\left(T_{2213}-T_{3312}\right)\mathcal{F}(\beta_{12l3},\beta_{13l2})$$

Strict collinear factorisation satisfied in the three-particle collinear limit for terms starting at three loops. No additional constraint.

Four-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},4}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-3,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-3,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



The strategy for this calculation is the same as for the previous cases.

We again recover lower particle collinear limits between subsets of particles becoming collinear.

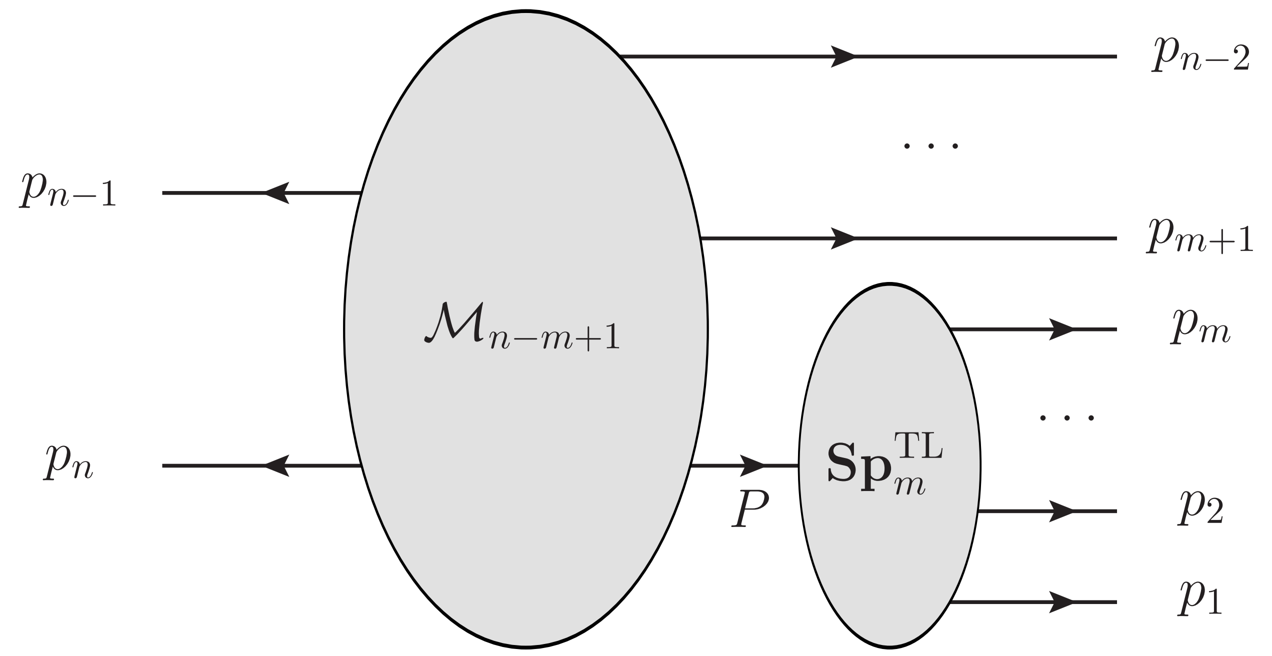
New structures are the kinematic functions that now depend on all four particles becoming collinear. But this naturally satisfies the strict collinear factorisation.

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Four-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},4}(p_1, p_2; \mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s) - \Big(\mathbf{\Gamma}_{n-3,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-3,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s)\Big)$$



The strategy for this calculation is the same as for the previous cases.

We again recover lower particle collinear limits between subsets of particles becoming collinear.

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s) \sum_{i=1}^{n} \sum_{\substack{1 \le j < k \le n, \\ j,k \neq i}} T_{iijk}$$

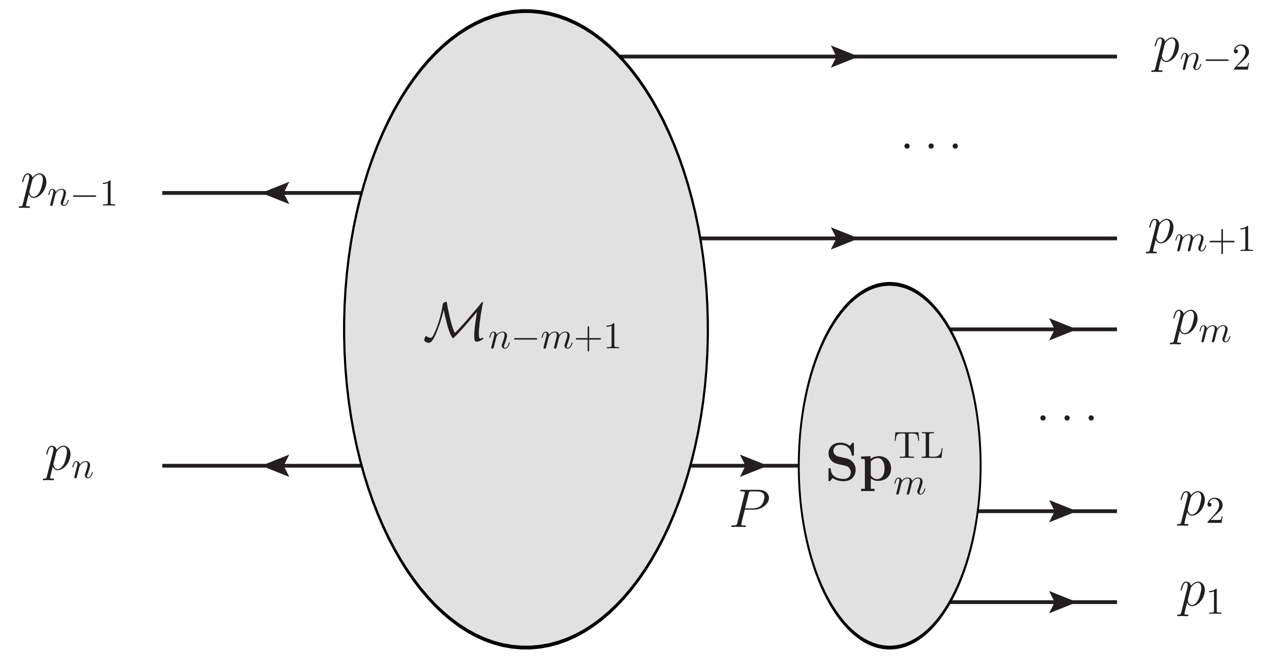
New structures are the kinematic functions that now depend on all four particles becoming collinear. But this naturally satisfies the strict collinear factorisation.

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s) = 8 \sum_{1 \le i < j < k < l \le n} a_{ijkl}(\beta)$$

We will spare the details of the calculation…

Four-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},4}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-3,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n-3,4T-4L}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



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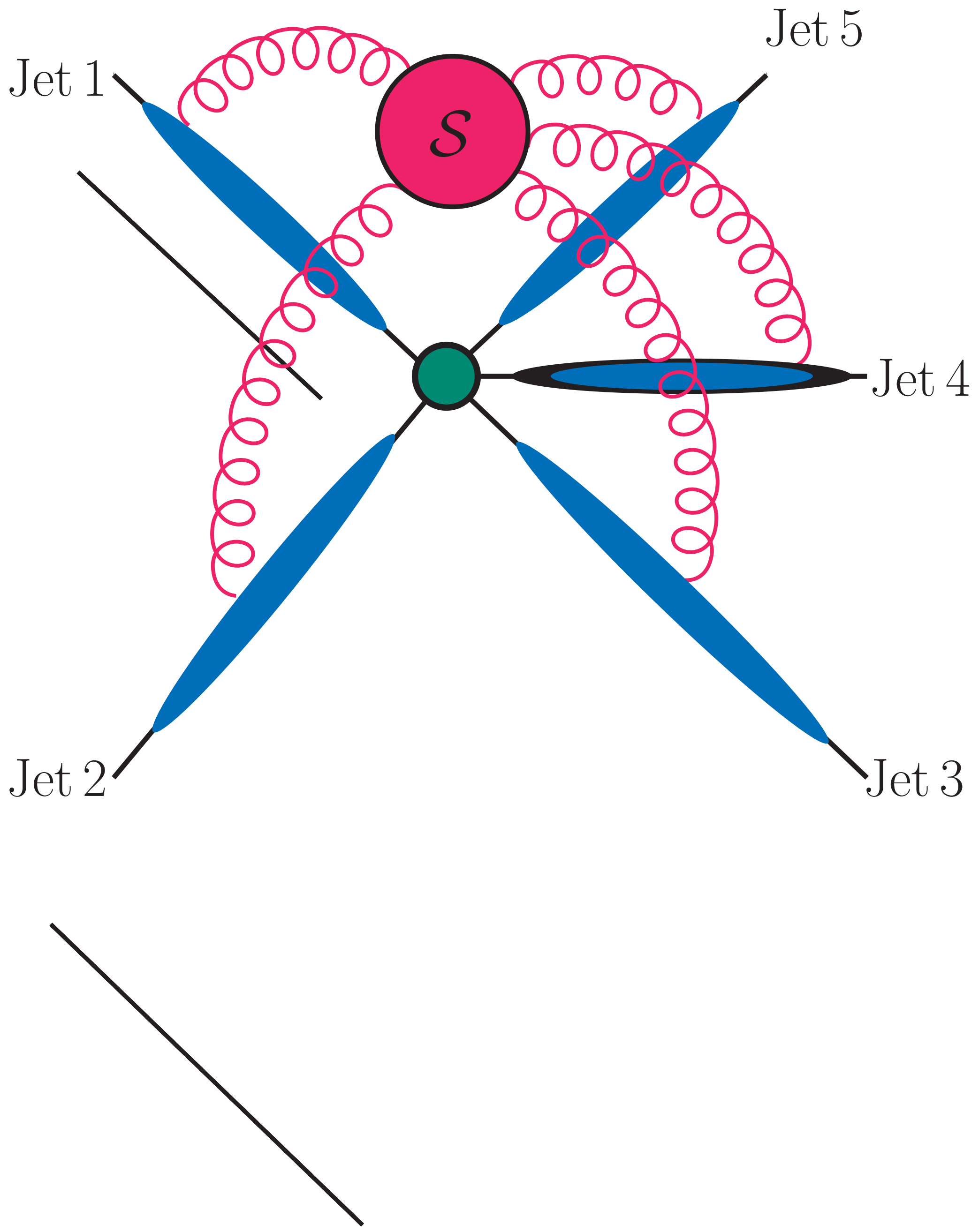
New structures are the kinematic functions that now depend on all four particles becoming collinear. But this naturally satisfies the strict collinear factorisation.

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

We will spare the details of the calculation… How about at four loops?

Collinear limit at four-loops



Four-loop correction to the soft anomalous dimension has not been fully explicitly computed. Recall the structure

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$

Form at four loops [T. Becher and M. Neubert, 1908.11379]

$$\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) = -\sum_R g_R(\alpha_s)\left[\sum_{1\le i<j\le n}\left(\mathcal{D}^R_{iijj}+\mathcal{D}^R_{iiij}+\mathcal{D}^R_{jjji}\right)\ell_{ij} + \sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\mathcal{D}^R_{jkii}\ell_{jk}\right]$$

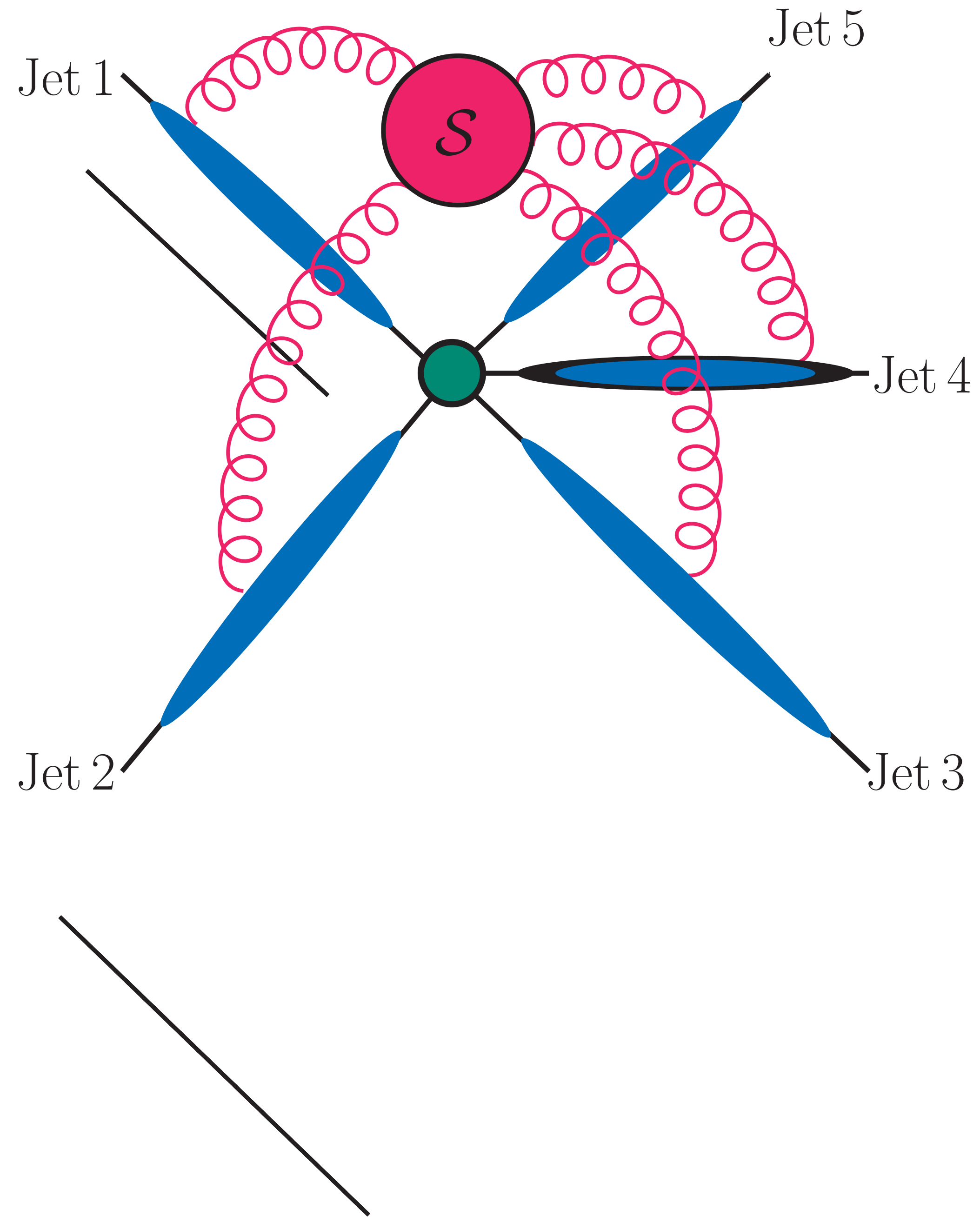
$$\ell_{ij} = \ln\left(\frac{-s_{ij}}{\lambda^2}\right)$$

The g_R has been computed by [J. Henn, G. Korchemsky, B. Mistlberger, 1911.10174]

$$\mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) = 24\sum_R\sum_{1\le i<j<k<l\le n}\mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\beta_{ijlk},\beta_{iklj})$$

This function is unknown

Collinear limit at four-loops



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$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$

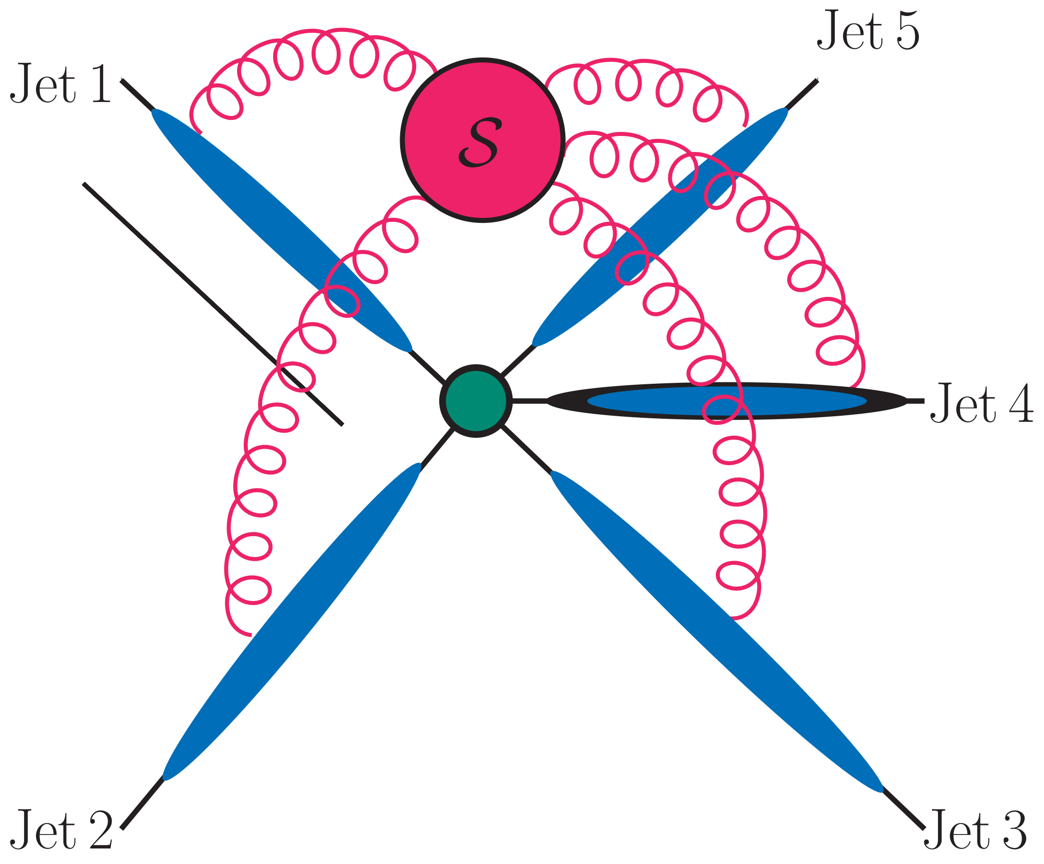
Form at four loops [T. Becher and M. Neubert, 1908.11379]

$$\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) = \sum_{(i,j,k,l)} T_{ijkli}\,\mathcal{H}_1(\beta_{ijlk},\beta_{iklj};\alpha_s)$$

$$\mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) = \sum_{(i,j,k,l,m)} T_{ijklm}\,\mathcal{H}_2(\beta_{ijkl},\beta_{ijmk},\beta_{ikmj},\beta_{jiml},\beta_{jlmi};\alpha_s)$$

These functions are unknown

Collinear limit at four-loops



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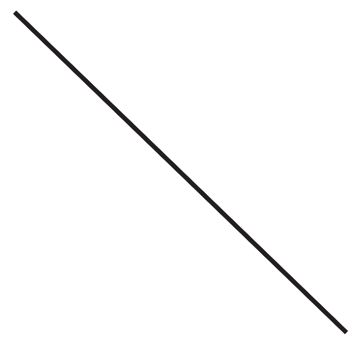
$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$



The two-particle collinear limit proved useful in providing constraints on the unknown objects [T. Becher and M. Neubert, 1908.11379]

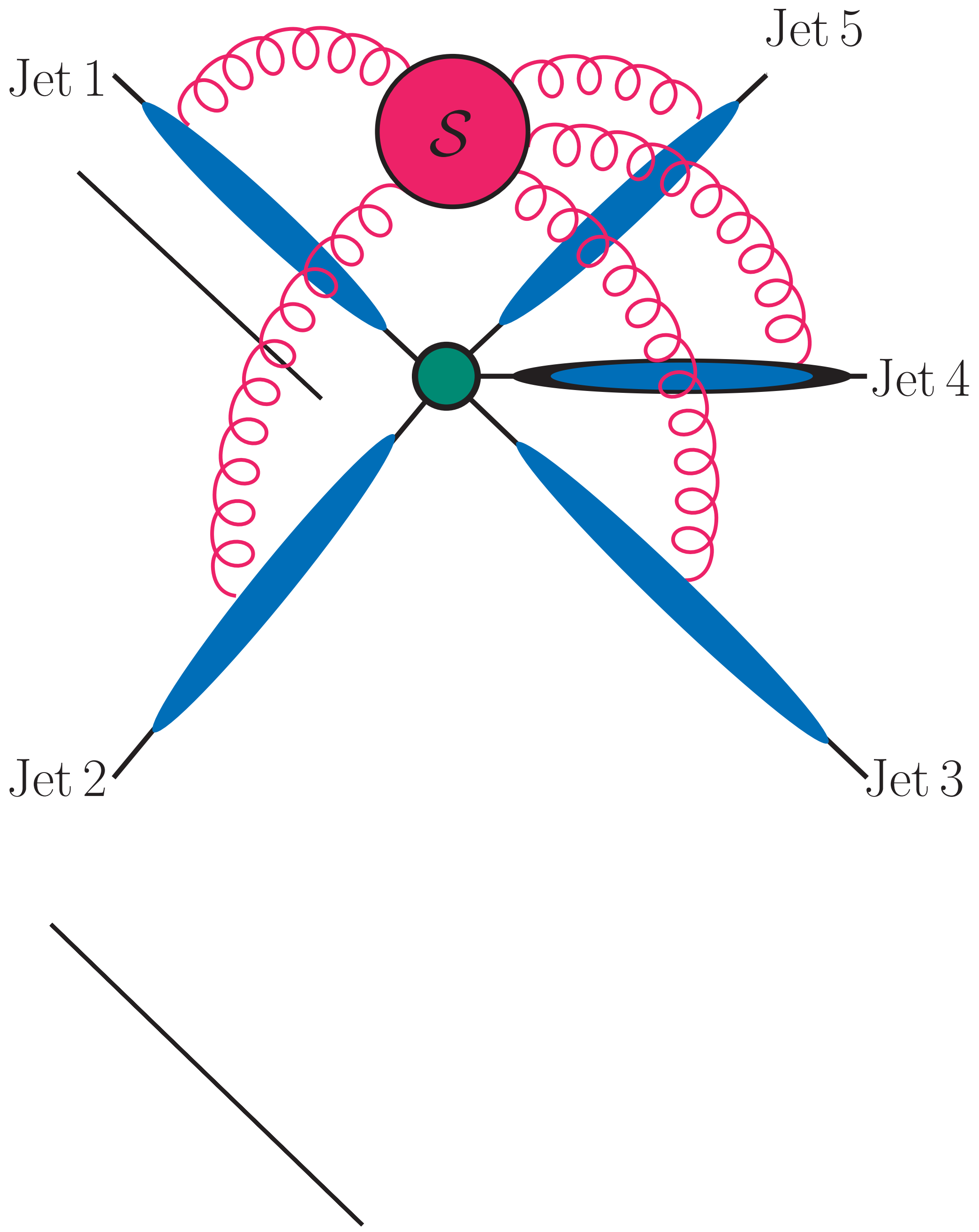
$$\lim_{\beta_{12kl}\to-\infty} \mathcal{G}_R(\beta_{12kl},0) = -\frac{g_R}{12}\beta_{12kl} \qquad \lim_{\beta_{12kl}\to-\infty} \mathcal{H}_1(\beta_{12kl},0) = 0$$

And the combination

$$\Big(4\mathcal{H}_2(-\beta_{1kml},-\beta_{12kl}-\beta_{1lmk},-\beta_{12kl}-\beta_{1lmk},0,\beta_{12kl}+\beta_{1kml}) - \mathcal{H}_1(-\beta_{1lmk},\beta_{1kml}-\beta_{1lmk}) + 2\mathcal{H}_2(\beta_{12kl},\beta_{12kl}+\beta_{1lmk},0,\beta_{12kl}+\beta_{1kml},0)\Big)$$

has to be symmetric, since it multiplies $\left[T_{12klm}-T_{12lkm}\right]$

Collinear limit at four-loops



Four-loop correction to the soft anomalous dimension has not been fully explicitly computed. Recall the structure

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$

We **investigated the three-particle collinear limit,** here is one place where we could envisage multi-particle collinear limit to provide additional constraints.

but in the end, once the two-particle collinear limit constraint on the functions is implemented, the seemingly new constraint is satisfied.

Details in [C. Duhr, E. Gardi, SJ, J. Lübken, L. Vernazza, **2507.21854**]

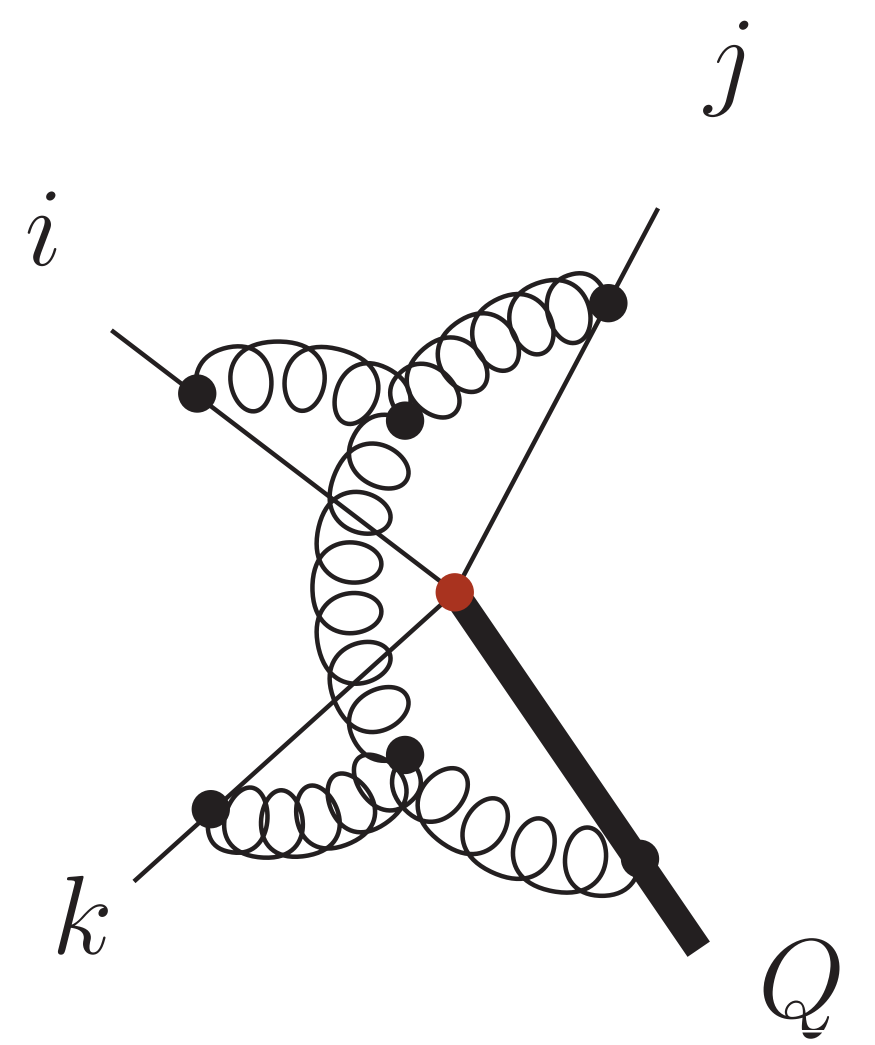
Collinear limits of amplitudes with a massive leg

This far, we have considered only collinear limits of n-point scattering amplitudes with massless external legs.

We found no additional constraints beyond the ones arising from the two-particle collinear limit.

Can we learn something extra from multi-particle collinear limits of amplitudes with a massive leg?

To this end, add these terms to the anomalous dimension



Picture from E. Gardi, Z. Zhu, 2510.27567

$$\mathbf{\Gamma}^{(3)}\left(\{p\}, m_I, \lambda, \alpha_s\right) = 2 \sum_{1\le i<j\le n} T_{IIij} F_{\mathrm{h2}}\left(r_{ijI}, \alpha_s\right) + \sum_{1\le i<j<k\le n} \mathbf{a}^h_{ijkI}(\{r\})$$

Where

$$\mathbf{a}^h_{ijkI}(\{r\}) = 2\left[T_{ijkI}F_{\mathrm{h3}}\left(r_{ijI}, r_{ikI}, r_{jkI}\right) + T_{jikI}F_{\mathrm{h3}}\left(r_{jiI}, r_{jkI}, r_{ikI}\right) + T_{kjiI}F_{\mathrm{h3}}\left(r_{kjI}, r_{kiI}, r_{jiI}\right)\right]$$

and variables appearing here are

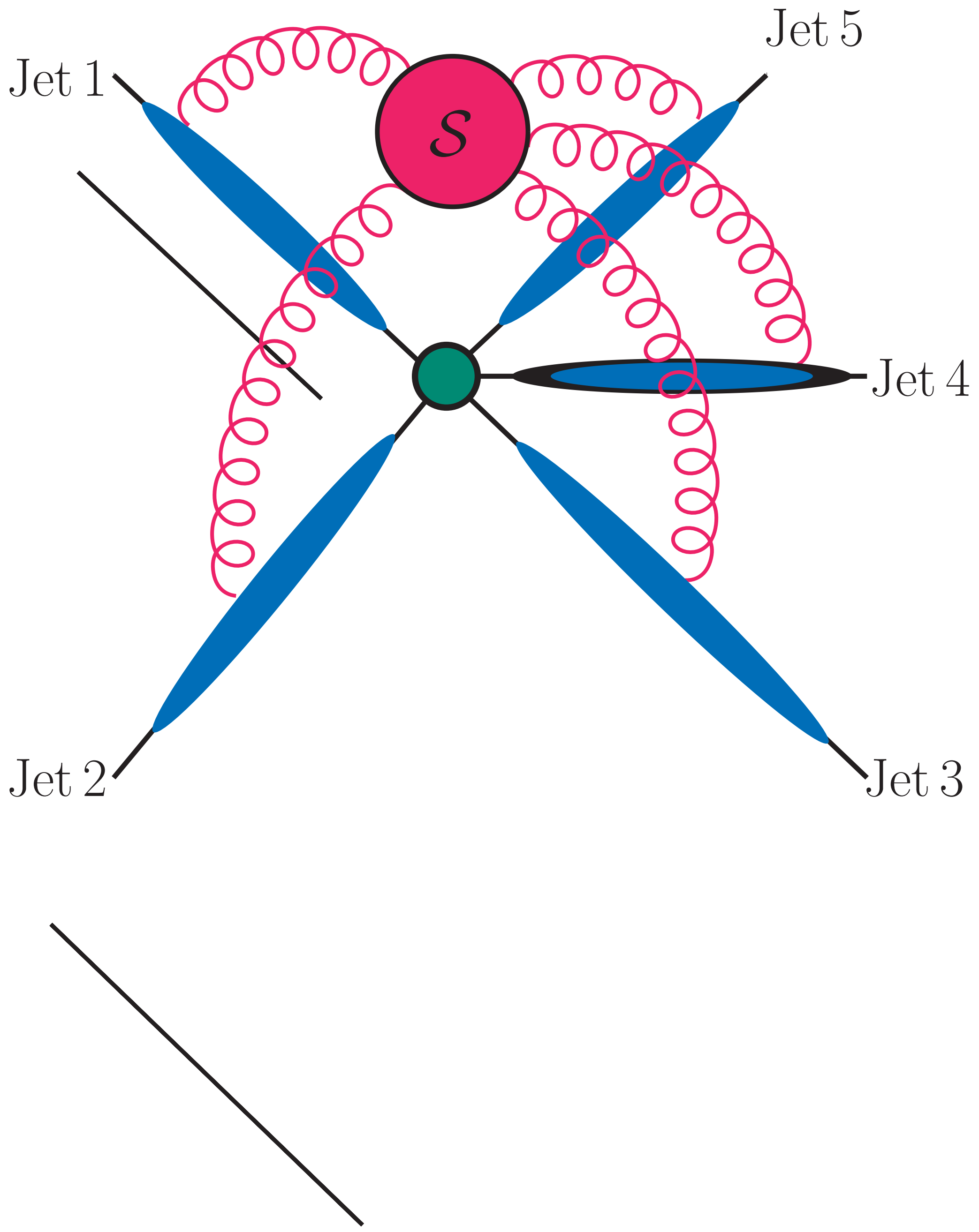
$$r_{ijI} = \frac{p_i \cdot p_j \, p_I^2}{2p_i \cdot p_I \, p_j \cdot p_I}$$

The function $F_{\mathrm{h2}}\left(r_{ijI}, \alpha_s\right)$ was computed in [Z. L. Liu, N. Schalch, 2207.02864]

and since recently, $F_{\mathrm{h3}}\left(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s\right)$ is known as well [E. Gardi, Z. Zhu, 2510.27567]

Auxiliary slides

Soft Anomalous Dimension



Corrections to the dipole formula of the soft anomalous dimension have been studied intensively. The structure through four-loops is

$$\boldsymbol{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \boldsymbol{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\boldsymbol{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \boldsymbol{\Gamma}_{n,\text{Q4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\boldsymbol{\Gamma}_{n,\text{5T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) + \boldsymbol{\Gamma}_{n,\text{5T}-\text{5L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$

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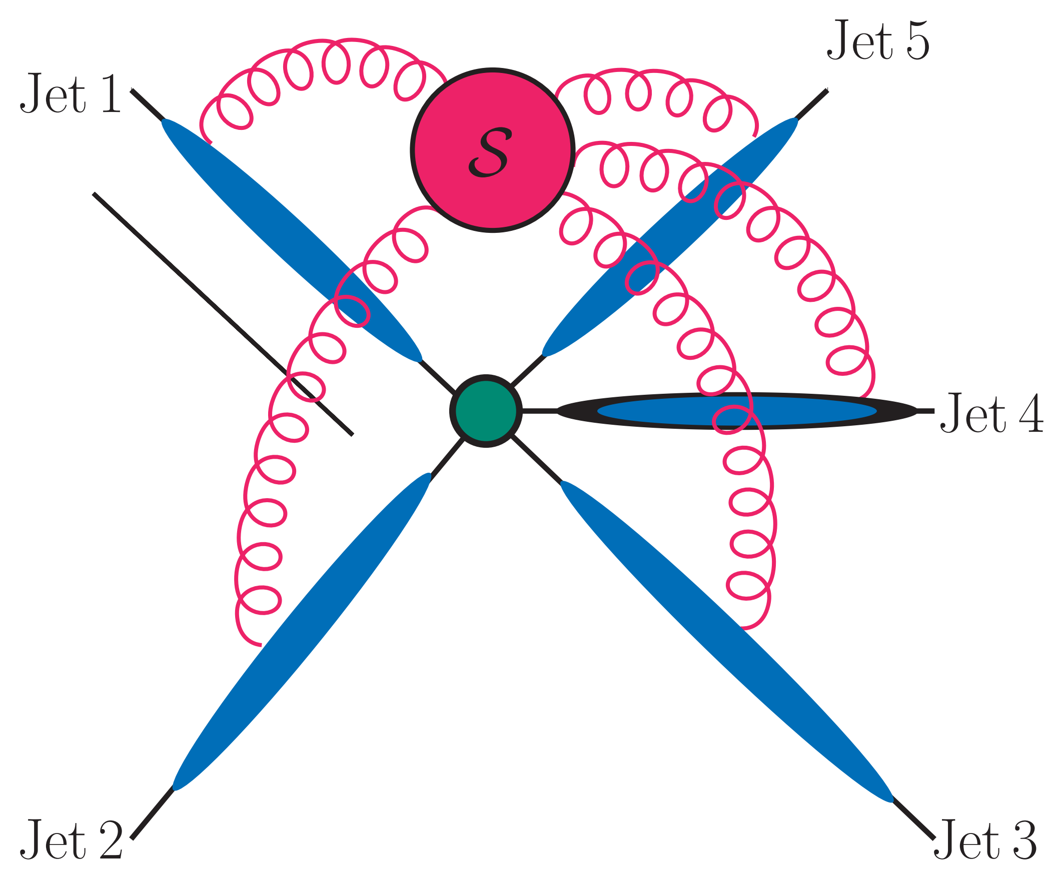


Three-loop correction

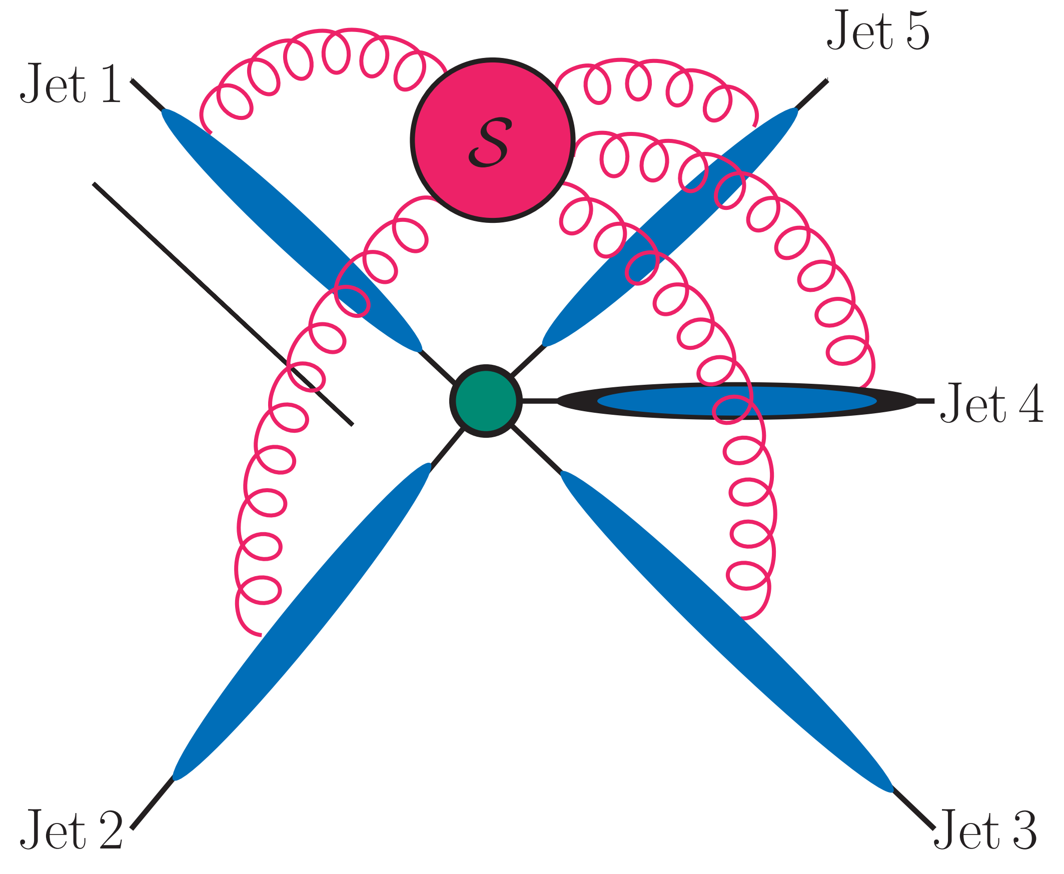
Calculated explicitly in

[Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

Functional form also obtained by bootstrap techniques

[Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]

Soft Anomalous Dimension



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Four-loop correction

The four-loop cusp anomalous dimension has been computed [J. Henn, G. Korchemsky, B. Mistlberger, 1911.10174]

Some constraints on the functional form have also been determined using Regge and two-particle collinear limits

[A. Vladimirov 1707.07606] [T. Becher and M. Neubert, 1908.11379]

[G. Falcioni, E. Gardi, N. Maher, C. Milloy, L. Vernazza, 2111.10664]

Splitting amplitude soft anomalous dimension

The infrared singularities of the timelike splitting amplitude are given to all orders in terms of the so-called splitting amplitude soft anomalous dimension

$$\mathcal{M}_n(\{p_i\},\mu,\epsilon) = \mathbf{Z}_n(\{p_i\},\mu,\epsilon)\mathcal{H}_n(\{p_i\},\mu)$$

$$\mathcal{M}_{n-1}(\{p_i\},\mu,\epsilon) = \mathbf{Z}_{n-1}(\{p_i\},\mu,\epsilon)\mathcal{H}_{n-1}(\{p_i\},\mu)$$

Collinear factorisation also applies to the hard function

$$\mathcal{H}_n(p_1,p_2,\{p_i\}_{\text{rest}},\mu) \longrightarrow \mathbf{Sp}_{\mathcal{H},2}(p_1,p_2;\mu)\,\mathcal{H}_{n-1}(P,\{p_i\}_{\text{rest}},\mu)$$

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$$\frac{d}{d\ln\mu}\mathbf{Sp}_{\mathscr{H},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu)\,\mathbf{Sp}_{\mathscr{H},2}(p_1,p_2;\mu)$$

$$\mathbf{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_n(p_1,p_2,p_3,\ldots p_n;\mu) - \mathbf{\Gamma}_{n-1}(P,p_3,\ldots p_n;\mu)\,|_{\mathbf{T}_P\to\sum_{i=1}^m\mathbf{T}_i}$$

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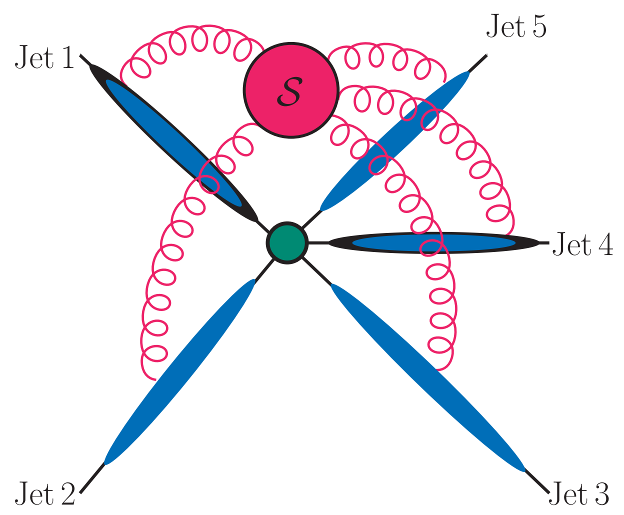

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$$\mathcal{H}_n(p_1,p_2,\{p_i\}_{\text{rest}},\mu) \longrightarrow \mathbf{Sp}_{\mathcal{H},2}(p_1,p_2;\mu)\,\mathcal{H}_{n-1}(P,\{p_i\}_{\text{rest}},\mu)$$

Central object in the investigation. Remarkable cancellations need to occur on the right-hand side!

$$\frac{d}{d\ln\mu}\mathbf{Sp}_{\mathcal{H},2}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu)\,\mathbf{Sp}_{\mathcal{H},2}(p_1,p_2;\mu)$$

$$\boldsymbol{\Gamma}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_n(p_1,p_2,p_3,\ldots p_n;\mu) - \boldsymbol{\Gamma}_{n-1}(P,p_3,\ldots p_n;\mu)\,|_{\mathbf{T}_P\to\sum_{i=1}^m\mathbf{T}_i}$$

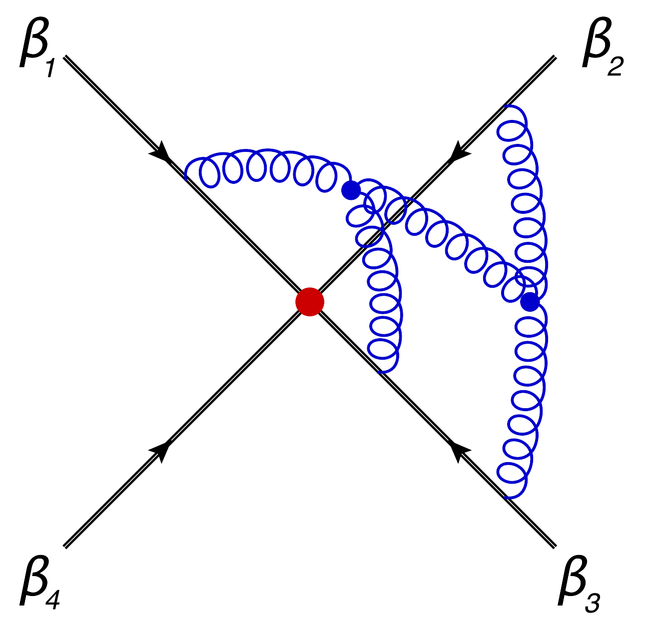
The Soft Anomalous Dimension at three-loops

Recall the structure:

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) + \mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

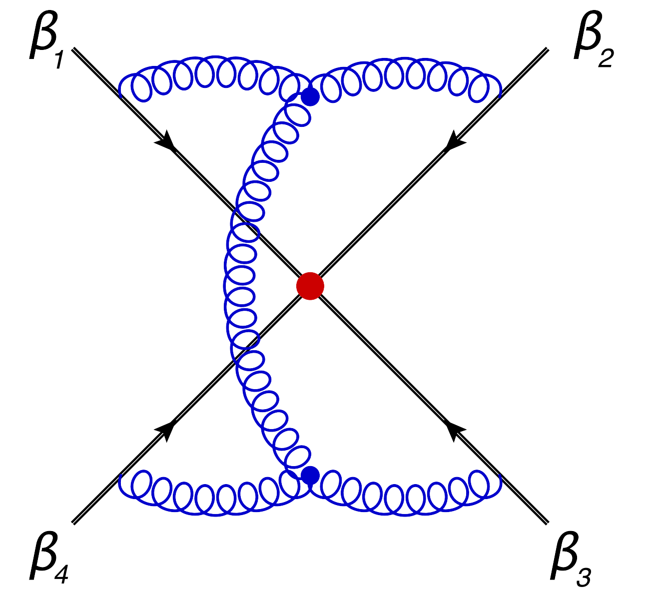
$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,\text{3L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{5T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,\text{5T}-\text{5L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$



Result of explicit calculation [Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

$$\mathbf{\Gamma}_{n,\text{4T}-\text{3L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$



$$\mathbf{\Gamma}_{n,\text{4T}-\text{4L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n}\Big[T_{iklj}\,\mathscr{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathscr{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathscr{F}(\beta_{ijlk},\beta_{iklj})\Big]$$

$$T_{ijkl} = f^{ade}f^{bce}\left\{\mathbf{T}_i^a,\mathbf{T}_j^b,\mathbf{T}_k^c,\mathbf{T}_l^d\right\}_+$$

The Soft Anomalous Dimension at three-loops

A word about the kinematics

$$\Gamma_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8 \sum_{1\le i<j<k<l\le n} \left[T_{iklj}\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj})\right]$$

Bose Symmetry:

$$\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) = -\,\mathcal{F}(\beta_{ilkj},\beta_{ijkl})$$

$$T_{ijlk} = f^{ade} f^{bce} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_l^c \mathbf{T}_k^d$$

These objects here are logarithms of the *Conformally Invariant Cross Ratios* (CICRs)

$$\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} \qquad \beta_{ijkl} = \ln \rho_{ijkl} = \ln\left(\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}\right)$$

Which can be written in terms of variables $\rho_{ijkl} = z_{ijkl}\bar{z}_{ijkl}\,, \quad \rho_{ilkj} = (1-z_{ijkl})(1-\bar{z}_{ijkl})$

$$\mathcal{F}^{(3)}(\beta_{ijkl},\beta_{ilkj}) = \frac{1}{32}\left[F(1-z_{ijkl}) - F(z_{ijkl})\right] \qquad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\left[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)\right]$$

← **Single Valued Harmonic Polylogarithms**

[Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

The Soft Anomalous Dimension at three-loops

A word about the kinematics

$$\Gamma_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8 \sum_{1\le i<j<k<l\le n} \Big[T_{iklj}\,\mathcal{F}(\beta_{ikjl},\beta_{iljk}) + T_{ijlk}\,\mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + T_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj}) \Big]$$

$$\mathbf{a}_{ijkl}(\{\beta\})$$

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$$T_{ijlk} = f^{ade} f^{bce} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_l^c \mathbf{T}_k^d$$

For future reference, we can also rewrite this in terms of two terms

$$f^{ace} f^{bde} = f^{abe} f^{cde} + f^{ade} f^{bce}$$

$$\mathbf{a}_{ijkl}(\{\beta\}) = \mathcal{F}^{\mathrm{A}}_{ijkl}(\{\beta\})\, T_{iklj} + \mathcal{F}^{\mathrm{S}}_{ijkl}(\{\beta\}) \Big(T_{ijlk} + T_{ijkl} \Big)$$

where

$$\mathcal{F}^{\mathrm{A}}_{ijkl}(\{\beta\}) = \mathcal{F}(\beta_{ijkl},\beta_{ilkj}) - \mathcal{F}(\beta_{ijlk},\beta_{iklj}) + 2\mathcal{F}(\beta_{ikjl},\beta_{iljk})$$

$$\mathcal{F}^{\mathrm{S}}_{ijkl}(\{\beta\}) = \mathcal{F}(\beta_{ijkl},\beta_{ilkj}) + \mathcal{F}(\beta_{ijlk},\beta_{iklj})$$

Single Valued Harmonic Polylogarithms

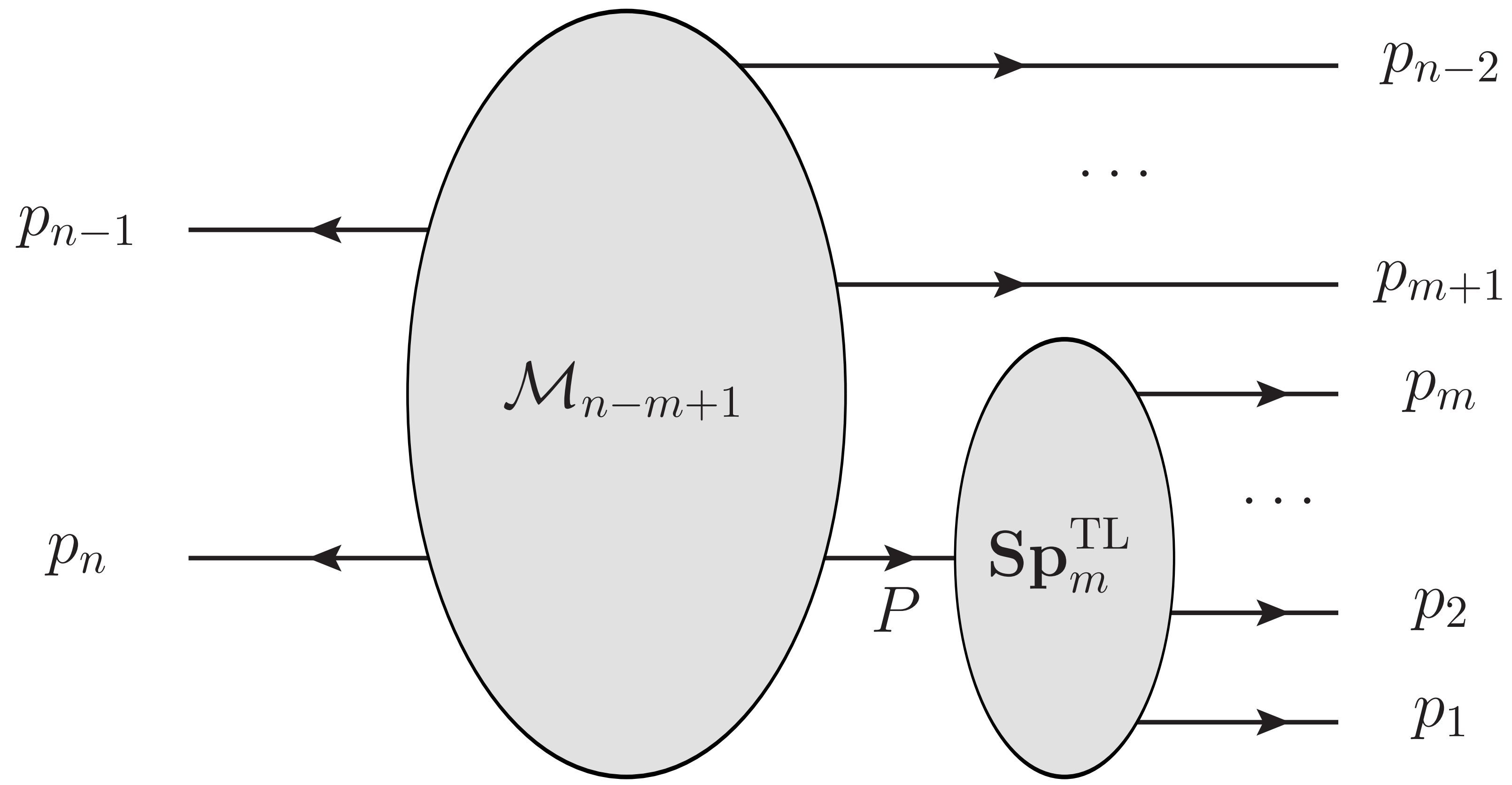
[Ø. Almelid, C. Duhr, E. Gardi, 1507.00047]

CICRs in collinear limits

For two-particle collinear limits we need to consider

$$\rho_{12kl} = \frac{(-p_1 \cdot p_2)(-p_k \cdot p_l)}{(-p_1 \cdot p_k)(-p_2 \cdot p_l)} \to 0$$

$$\rho_{1lk2} = \frac{(-p_1 \cdot p_l)(-p_k \cdot p_2)}{(-p_1 \cdot p_k)(-p_l \cdot p_2)} \approx \frac{(-x_1 P \cdot p_l)(-x_2 P \cdot p_k)}{(-x_1 P \cdot p_k)(-x_2 P \cdot p_l)} \to 1$$



In the three-particle collinear limit on the other hand

$$\rho_{123l} = \frac{(-p_1 \cdot p_2)(-p_3 \cdot p_l)}{(-p_1 \cdot p_3)(-p_2 \cdot p_l)} \approx \frac{(-p_1 \cdot p_2)(-x_3 P \cdot p_l)}{(-p_1 \cdot p_3)(-x_2 P \cdot p_l)}$$

$$\rho_{123l} \to \frac{x_3(-p_1 \cdot p_2)}{x_2(-p_1 \cdot p_3)}$$

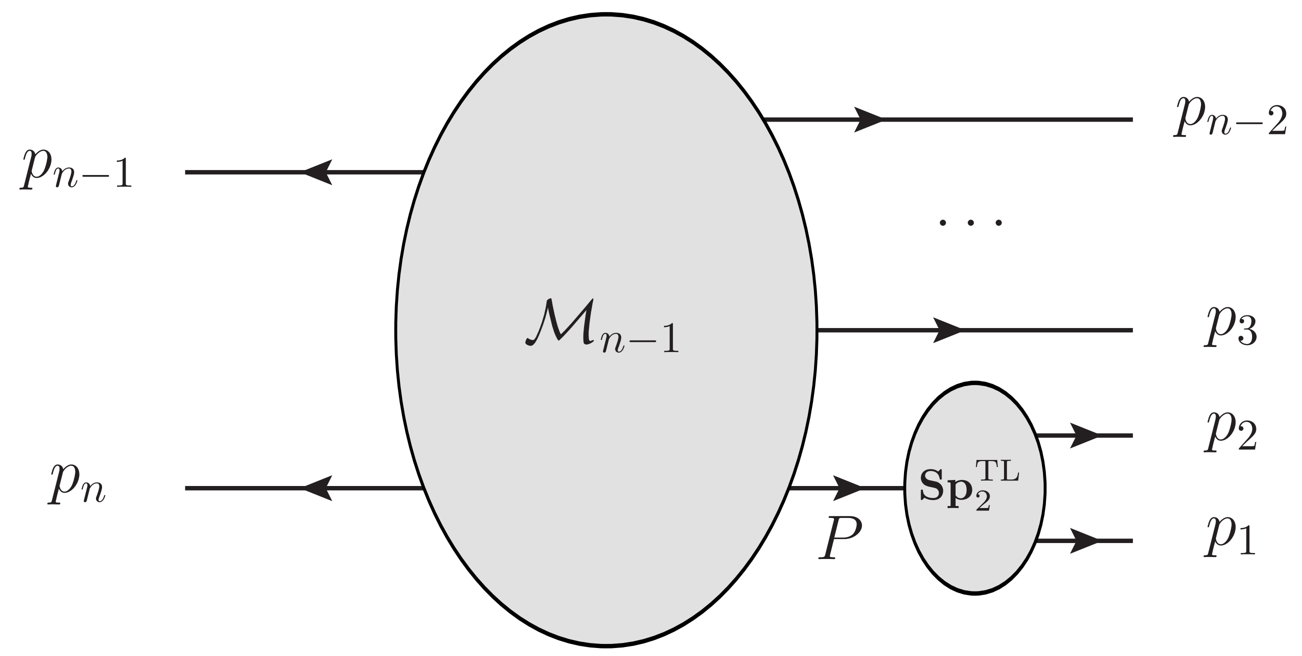
$$\rho_{1l32} = \frac{(-p_1 \cdot p_l)(-p_3 \cdot p_2)}{(-p_1 \cdot p_3)(-p_l \cdot p_2)} \approx \frac{x_1(-p_3 \cdot p_2)}{x_2(-p_1 \cdot p_3)}$$

Do retain kinematic dependence, but on the collinear partons only!
The rest-of-the-process dependence scales out.

Two-particle collinear limit at three-loops

Motivation here is two-fold:

- The result at three-loops is known, so we can study how strict collinear factorisation is realised.
- If it was unknown, assuming strict collinear factorisation holds, are there constraints we can derive from this limit.



Considerations along these lines already in bootstrap approach

[Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]

Can we improve on this?

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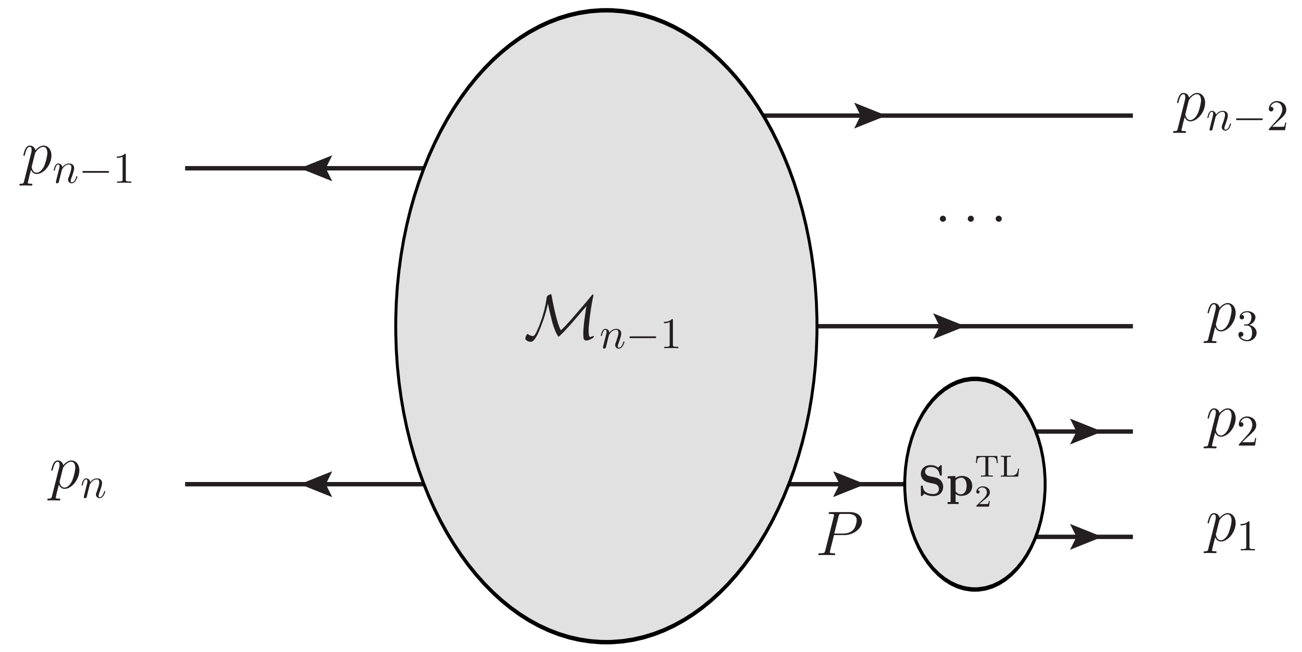


Considerations along these lines already in bootstrap approach

[Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]

Can we improve on this?

Starting at three-loops, we had two contributions

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s) \sum_{i=1}^{n} \sum_{\substack{1 \le j < k \le n, \\ j,k \ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s) = 8 \sum_{1 \le i<j<k<l \le n} \left[T_{iklj}\,\mathscr{F}(\beta_{ikjl}, \beta_{iljk}) + T_{ijlk}\,\mathscr{F}(\beta_{ijkl}, \beta_{ilkj}) + T_{ijkl}\,\mathscr{F}(\beta_{ijlk}, \beta_{iklj}) \right]$$

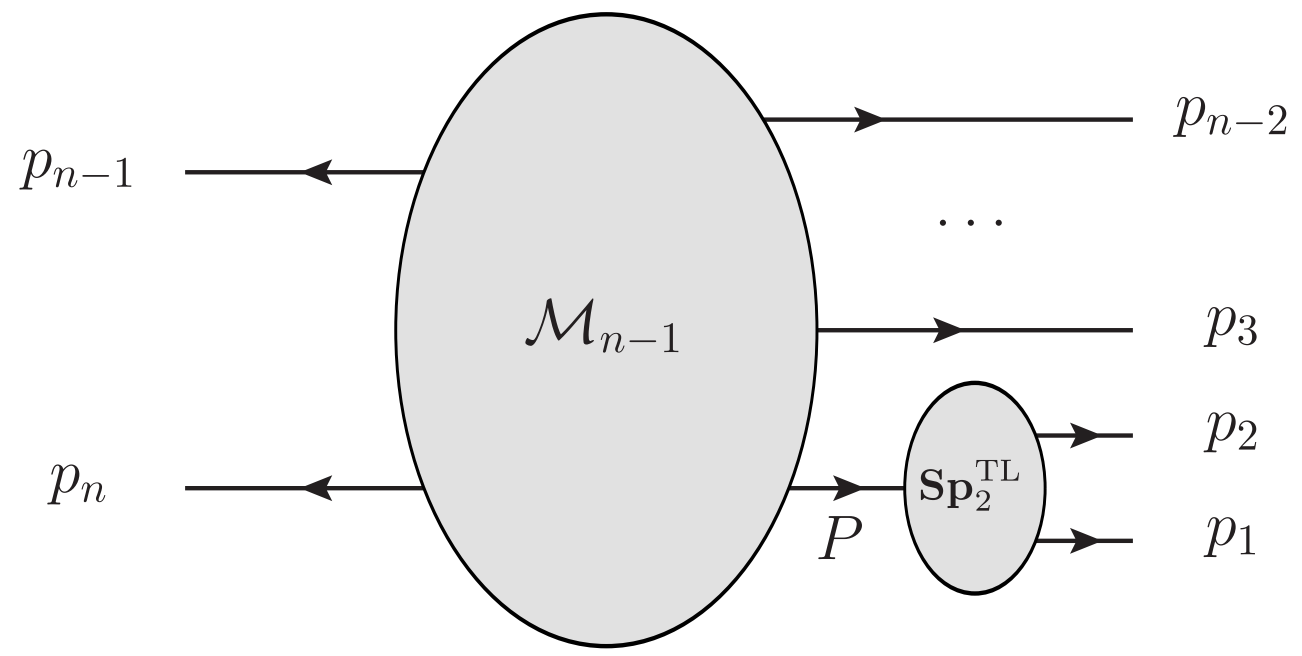
And the object of interest is

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1, p_2; \mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\}, \alpha_s) \right)$$

Can we pick n=3?

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Selecting $n = 3$, most terms above cannot contribute, only:

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{3,4\mathrm{T}-3\mathrm{L}}(\alpha_s)$$

Explicitly

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{3,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\left(T_{1123} + T_{2213} + T_{3312}\right)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Remember, these terms have the following structure

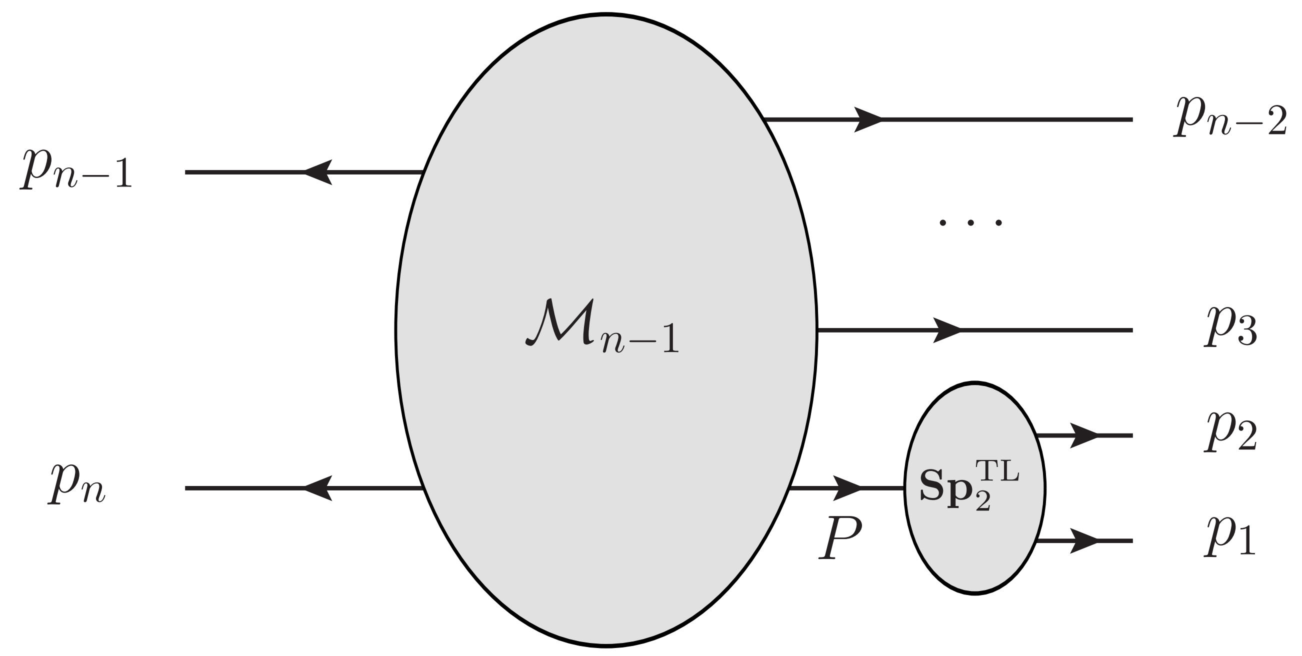
$$f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c\,\mathbf{T}_3^d$$

Since we only have three particles, we can directly apply colour conservation $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 0$

$$f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c\,\mathbf{T}_3^d = -f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c\,\mathbf{T}_1^d - f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c\,\mathbf{T}_2^d$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



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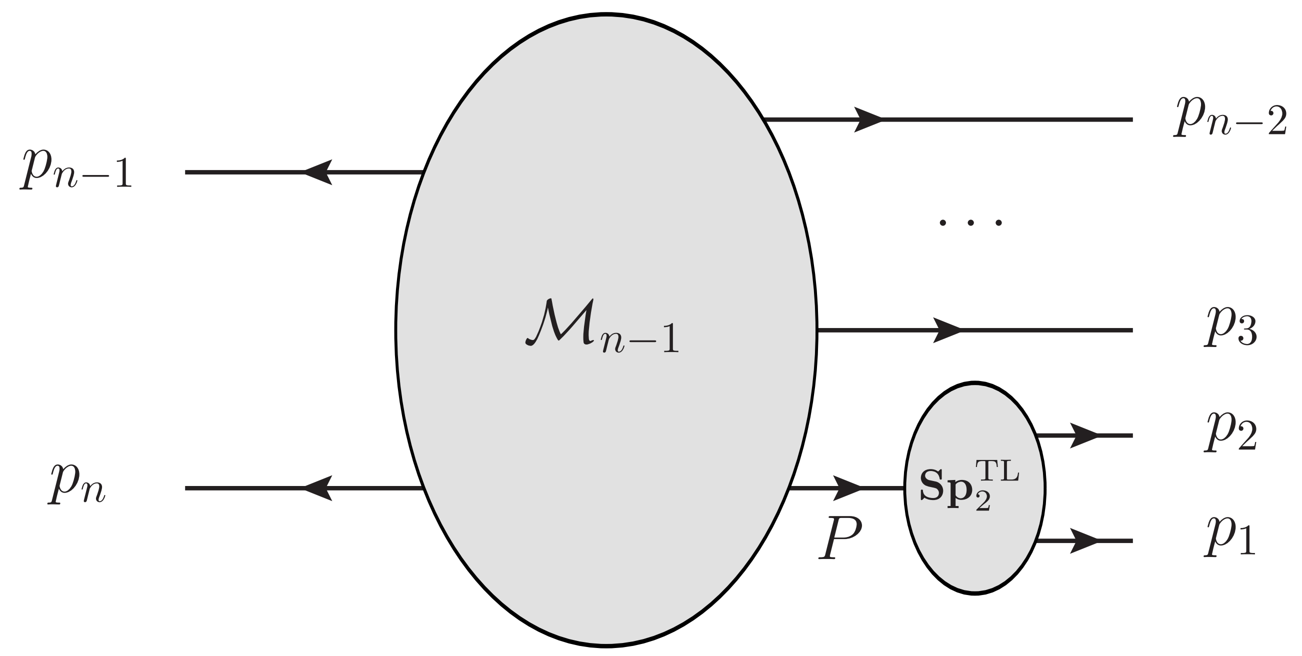
Even the last term is manageable

$$f^{ade}f^{bce}\left\{\mathbf{T}_3^a,\mathbf{T}_3^b\right\}\mathbf{T}_1^c\,\mathbf{T}_2^d$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



But thus far, we have not made use of universality of this object, the left-hand side is independent of n .

When we consider a higher point amplitude, more structures enter. Importantly, the kinematic ones, but the result should be the same!

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

Precisely this consideration for $n=4$ was used in
[Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, C. White, 1706.10162]

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Two-particle collinear limit at three-loops

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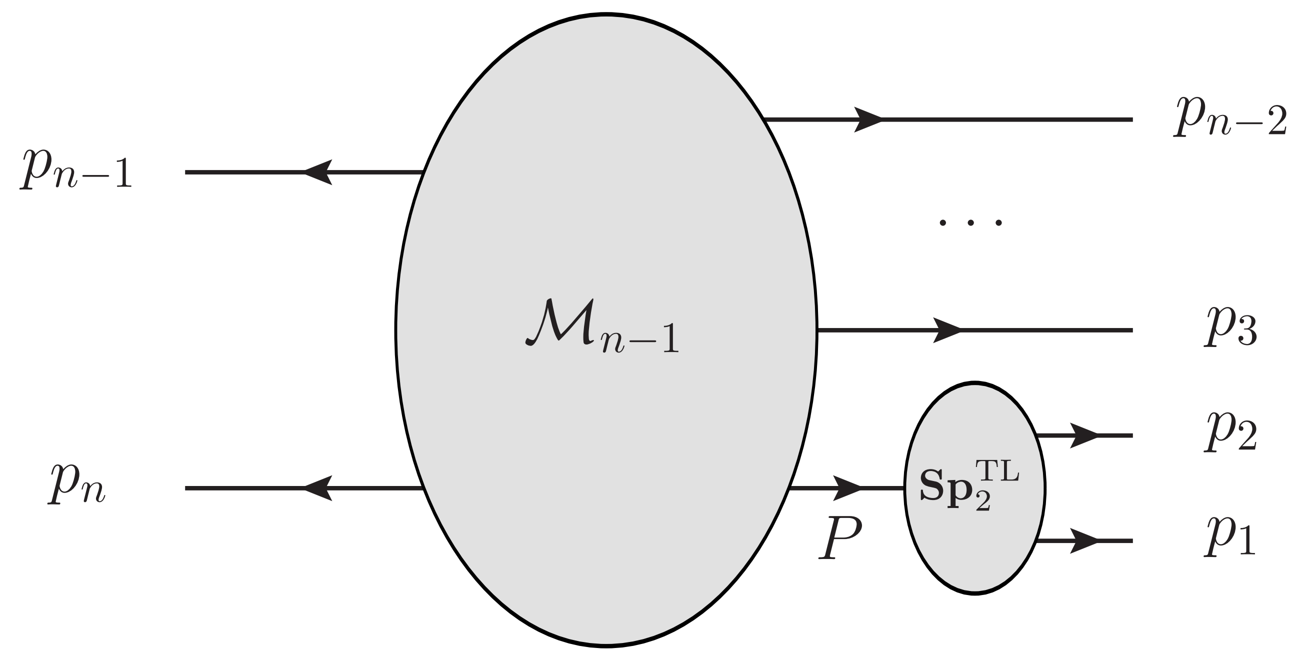

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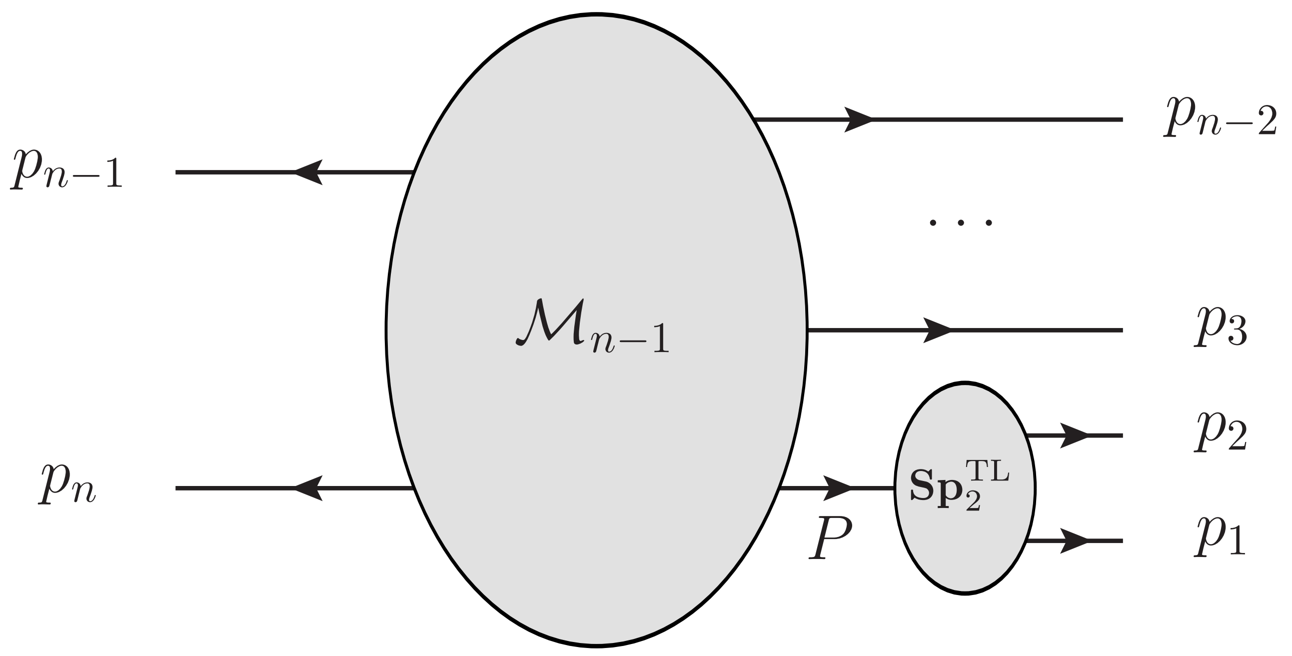
Next, we will work with general n to show that this is the only constraint in the two-particle collinear limit.

We start with $\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}$ and we will work with it as in the dipole case, selecting particles 1 and 2 to be collinear and separating the sum.

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Next, consider the constant piece $\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\Bigg\{\sum_{i=3}^{n}\sum_{\substack{3\le k<l\le n\\ k,l\neq i}}\mathcal{T}_{iikl} + \sum_{3\le k<l\le n}\Big(\mathcal{T}_{11kl}+\mathcal{T}_{22kl}\Big)$$

$$+\sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\neq i}}^{n}\Big(T_{ii1k}+T_{ii2k}\Big) + \sum_{i=3}^{n}\Big(T_{112i}+T_{221i}+T_{ii12}\Big)\Bigg\}$$

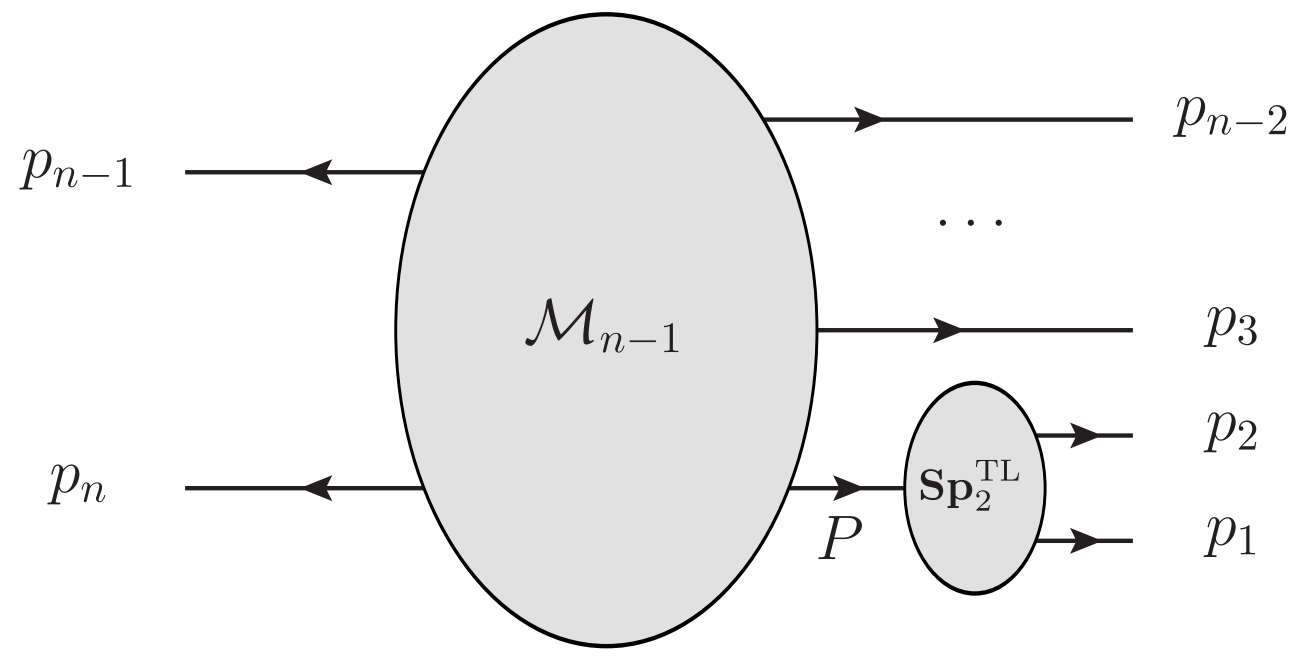
$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\Bigg\{\sum_{i=3}^{n}\sum_{\substack{3\le k<l\le n\\ k,l\neq i}} T_{iikl} + \sum_{3\le k<l\le n} T_{PPkl} + \sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\neq i}}^{n} T_{iiPk}\Bigg\}$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4T}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4T-3L}(\alpha_s) + \mathbf{\Gamma}_{n-1,4T-4L}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



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$$+\sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\ne i}}^{n}\left(T_{ii1k}+T_{ii2k}\right) + \sum_{i=3}^{n}\left(T_{112i}+T_{221i}+T_{ii12}\right)\Bigg\}$$

$$\mathbf{\Gamma}_{n,4T-3L}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

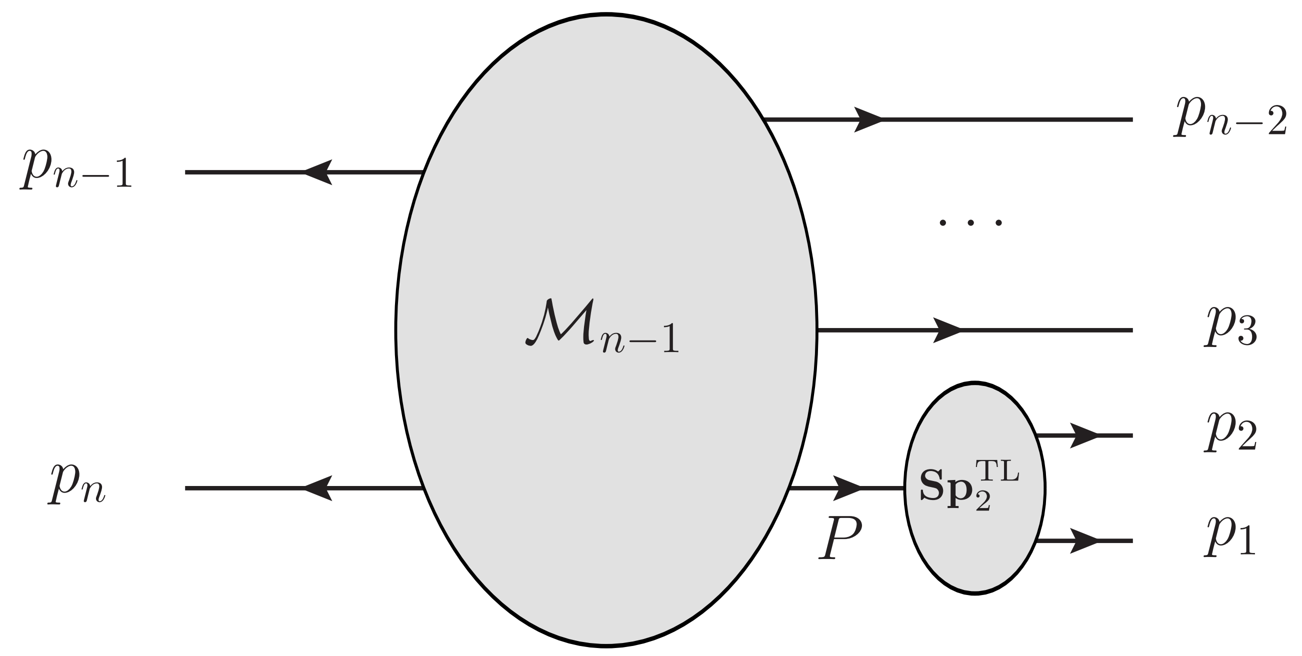
$$\mathbf{\Gamma}_{n-1,4T-3L}(\alpha_s) = 2f(\alpha_s)\Bigg\{\sum_{i=3}^{n}\sum_{\substack{3\le k<l\le n\\ k,l\ne i}} T_{iikl} + \sum_{3\le k<l\le n} T_{PPkl} + \sum_{i=3}^{n}\sum_{\substack{k=3,\\ k\ne i}}^{n} T_{iiPk}\Bigg\}$$

These cancel exactly as before using

$$\mathbf{T}_P = \mathbf{T}_1 + \mathbf{T}_2$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



Now, we combine the constant $\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}$ with the kinematic part

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n}\left[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \left(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\right)\left(T_{12lk}+T_{12kl}\right)\right]$$

$$+2f(\alpha_s)\sum_{i=3}^{n}\left(T_{ii12}+T_{112i}+T_{221i}\right)$$

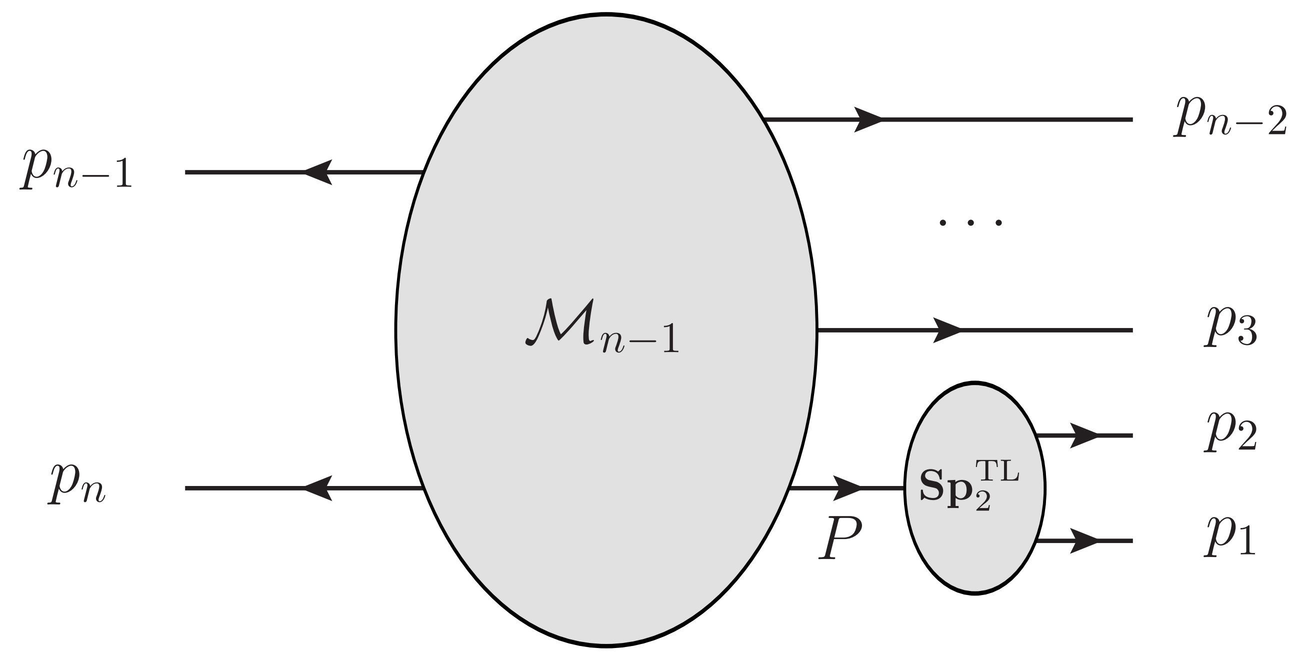
Thinking back to the $n=3$ case, this just appears in the last line, we can recover it easily, and apply colour conservation.

But we want to keep n general, the latter two terms are not an issue, colour conservation can be applied for any n

$$f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c(\mathbf{T}_3^d+\mathbf{T}_4^d+\ldots) = -f^{ade}f^{bce}\left\{\mathbf{T}_1^a,\mathbf{T}_1^b\right\}\mathbf{T}_2^c(\mathbf{T}_1^d+\mathbf{T}_2^d)$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Now the bottom and the top line can be combined

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{S}P,2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n} 4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + f(\alpha_s)\sum_{k=3}^{n}\sum_{l=3}^{n}\Big(T_{12lk} + T_{12kl}\Big) + 2f(\alpha_s)\sum_{i=3}^{n}\big(T_{112i} + T_{221i}\big)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

Provided the following condition holds in the two-particle collinear limit:

$$\Big(4\mathcal{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big|_{p_1||p_2} = 2f(\alpha_s)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Then applying colour conservation, gives a familiar result

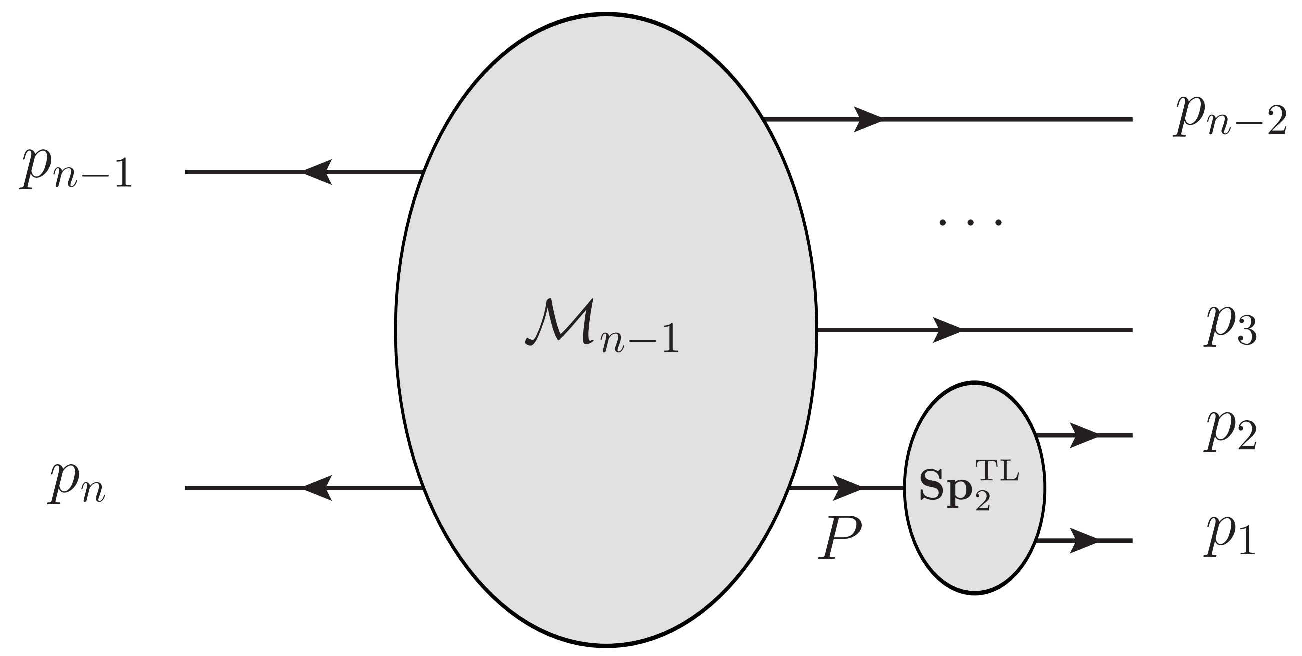
$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp,2}}(p_1,p_2;\mu) = \sum_{3\le k<l\le n} 4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} - \frac{3}{4}f(\alpha_s)\Big(C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\Big)$$

Which coincides with what we had earlier, as long as

$$\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\Big|_{p_1||p_2} = 0$$

Two-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-1,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-1,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Now the bottom and the top line can be combined

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{S}P,2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n} 4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + f(\alpha_s)\sum_{k=3}^{n}\sum_{l=3}^{n}\Big(T_{12lk}+T_{12kl}\Big) + 2f(\alpha_s)\sum_{i=3}^{n}\big(T_{112i}+T_{221i}\big)$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

Provided the following condition holds in the two-particle collinear limit:

$$\Big(4\mathcal{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big|_{p_1||p_2} = 2f(\alpha_s)$$

Results for these functions exist, so we can check explicitly these relations, and indeed they are satisfied

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

No new constraints from considering $n>4$

Then applying colour conservation, gives a familiar result

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},\mathbf{2}}(p_1,p_2;\mu) = \sum_{3\le k<l\le n} 4\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} - \frac{3}{4}f(\alpha_s)\Big(C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\Big)$$

Which coincides with what we had earlier, as long as

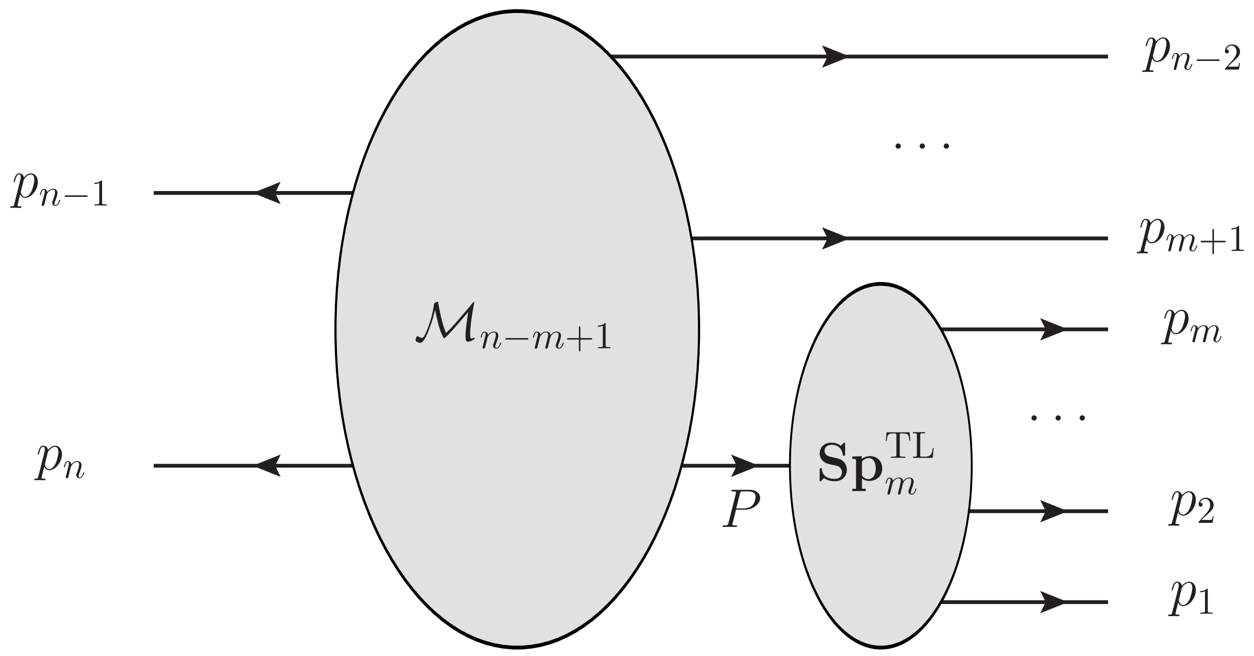
$$\mathcal{F}^{\mathrm{A}}_{12kl}(\{\beta\})\Big|_{p_1||p_2} = 0$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$

Combining terms we find

$$\begin{aligned}
\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = &\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \Big(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{12lk}+T_{12kl}\Big)\Big] \\
&+\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{13kl}(\{\beta\})\,T_{1kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{13kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{13lk}+T_{13kl}\Big)\Big] \\
&+\sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{23kl}(\{\beta\})\,T_{2kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{23kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{23lk}+T_{23kl}\Big)\Big] \\
&+2f(\alpha_s)\Big[\sum_{i=4}^{n}\Big(T_{112i}+T_{221i}+T_{ii12}+T_{113i}+T_{331i}+T_{ii13}+T_{223i}+T_{332i}+T_{ii23}\Big)\Big] \\
&+2f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 4\sum_{4\le l\le n}\mathbf{a}_{123l}(\{\beta\})
\end{aligned}$$

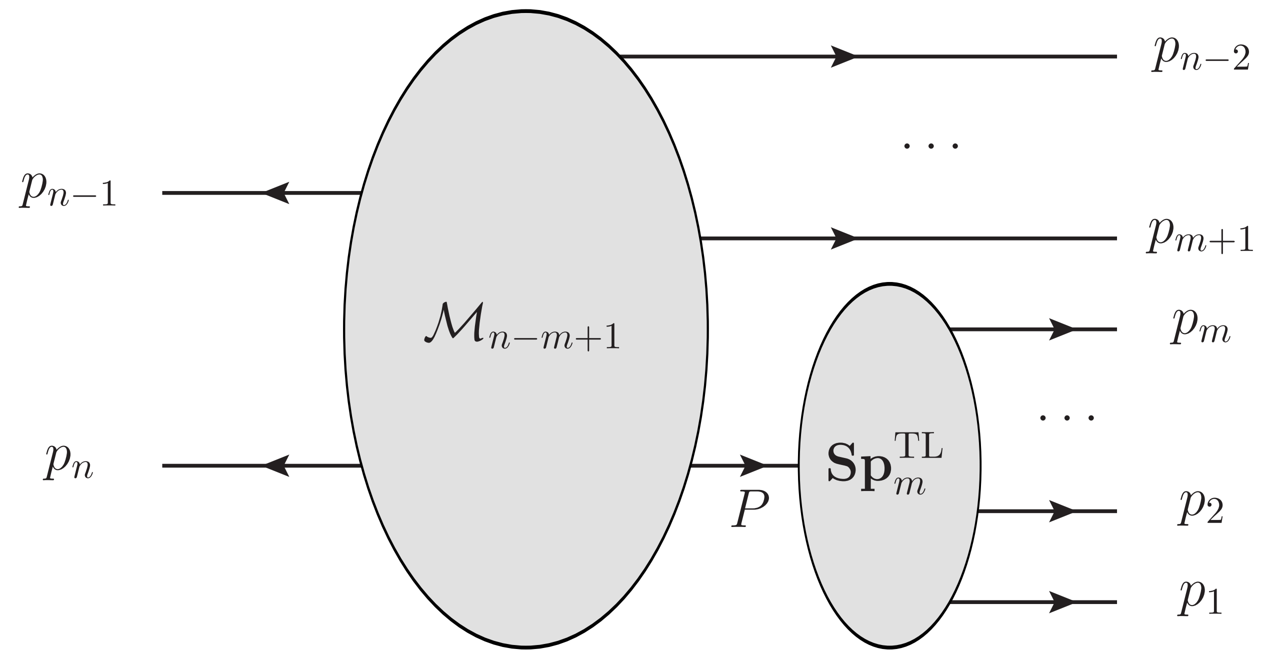


$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Combining terms we find

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = \sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \Big(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{12lk}+T_{12kl}\Big)\Big]$$
$$+ \sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{13kl}(\{\beta\})\,T_{1kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{13kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{13lk}+T_{13kl}\Big)\Big]$$
$$+ \sum_{4\le k<l\le n}\Big[4\mathscr{F}^{\mathrm{A}}_{23kl}(\{\beta\})\,T_{2kl3} + \Big(4\mathscr{F}^{\mathrm{S}}_{23kl}(\{\beta\}) - 2f(\alpha_s)\Big)\Big(T_{23lk}+T_{23kl}\Big)\Big]$$
$$+2f(\alpha_s)\Big[\sum_{i=4}^{n}\Big(T_{112i}+T_{221i}+T_{ii12}+T_{113i}+T_{331i}+T_{ii13}+T_{223i}+T_{332i}+T_{ii23}\Big)\Big]$$
$$+2f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 4\sum_{4\le l\le n}\mathbf{a}_{123l}(\{\beta\})$$

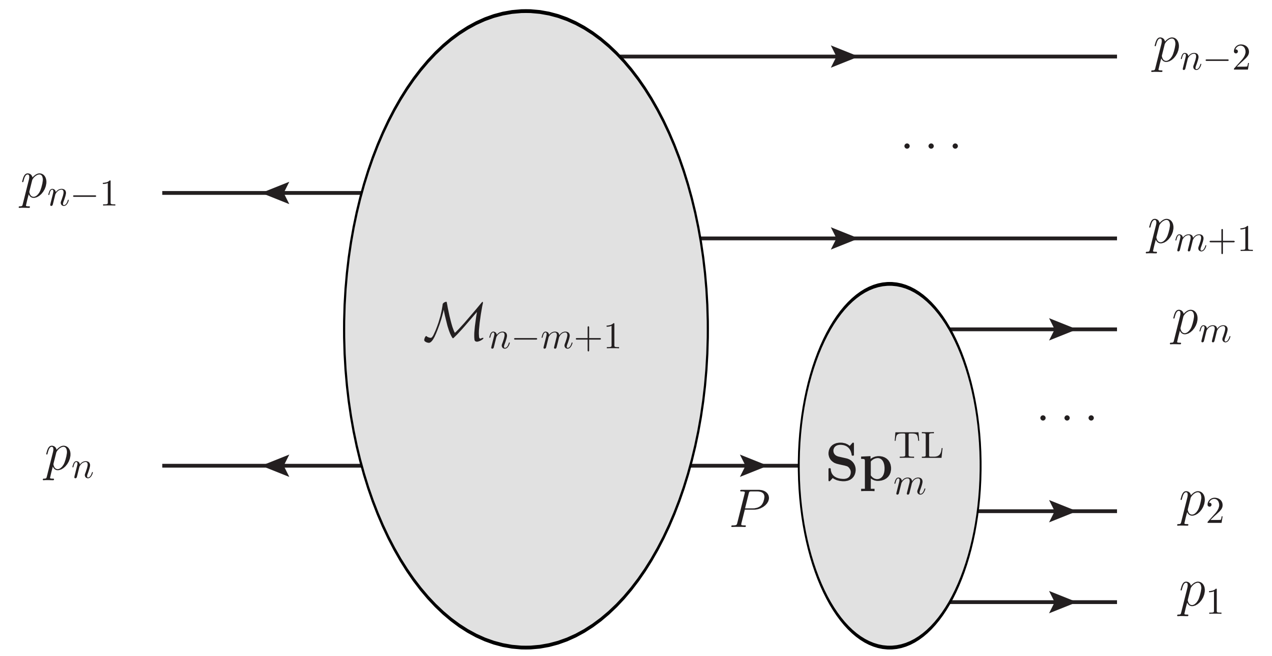
$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

Three copies of the two-particle collinear limit, between each pair of the three becoming collinear.

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



Imposing the two-particle collinear limit constraints:

$$\Big(4\mathcal{F}^{\mathrm{S}}_{abkl}(\{\rho\}) - 2f(\alpha_s)\Big)\Big|_{p_a||p_b} = 2f(\alpha_s)\,, \qquad (a,b) = (1,2)\,,\ (1,3)\,,\ \text{or}\ (2,3)$$

And being careful with colour conservation, for example:

$$\sum_{i=4}^{n} T_{112i} = -\frac{1}{8}C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 - T_{1122} - T_{1123}$$

Additional term with respect to the two-particle collinear limit!

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

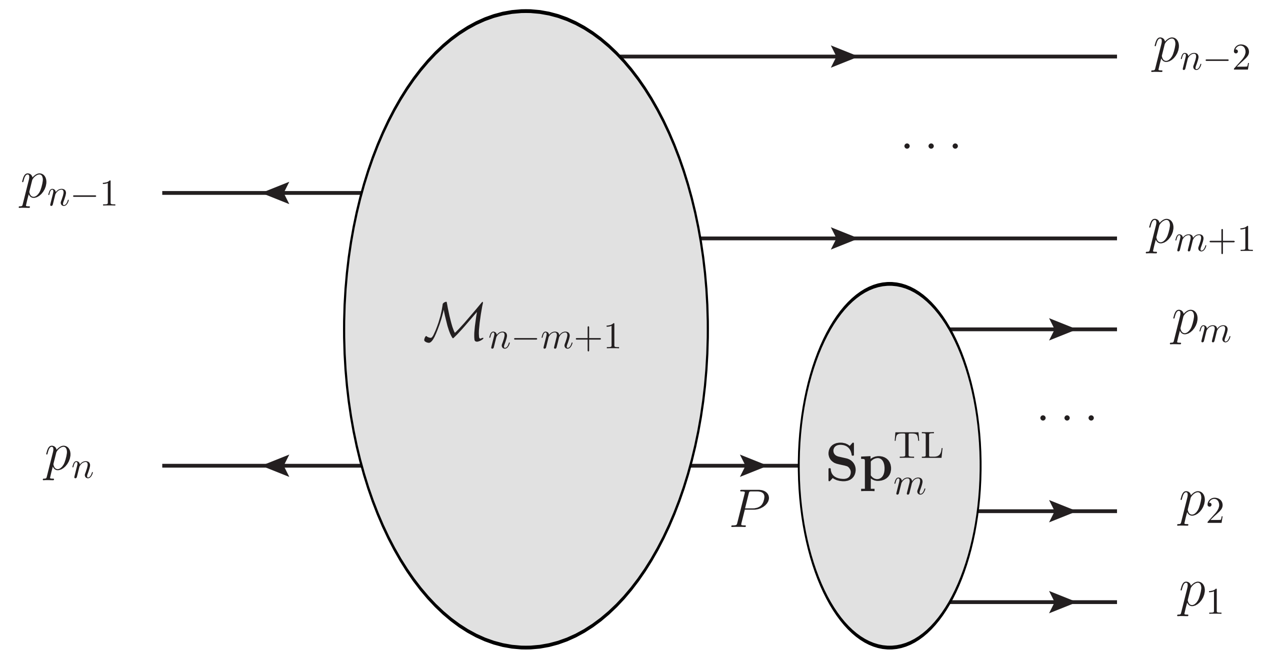
We then find:

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$

$$-4f(\alpha_s)\Big[T_{1123} + T_{2213} + T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big]$$

Three-particle collinear limit at three-loops

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \left(\mathbf{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \mathbf{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\right)$$



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Additional term with respect to the two-particle collinear limit!

$$\mathbf{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} T_{iijk}$$

$$\mathbf{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

We then find:

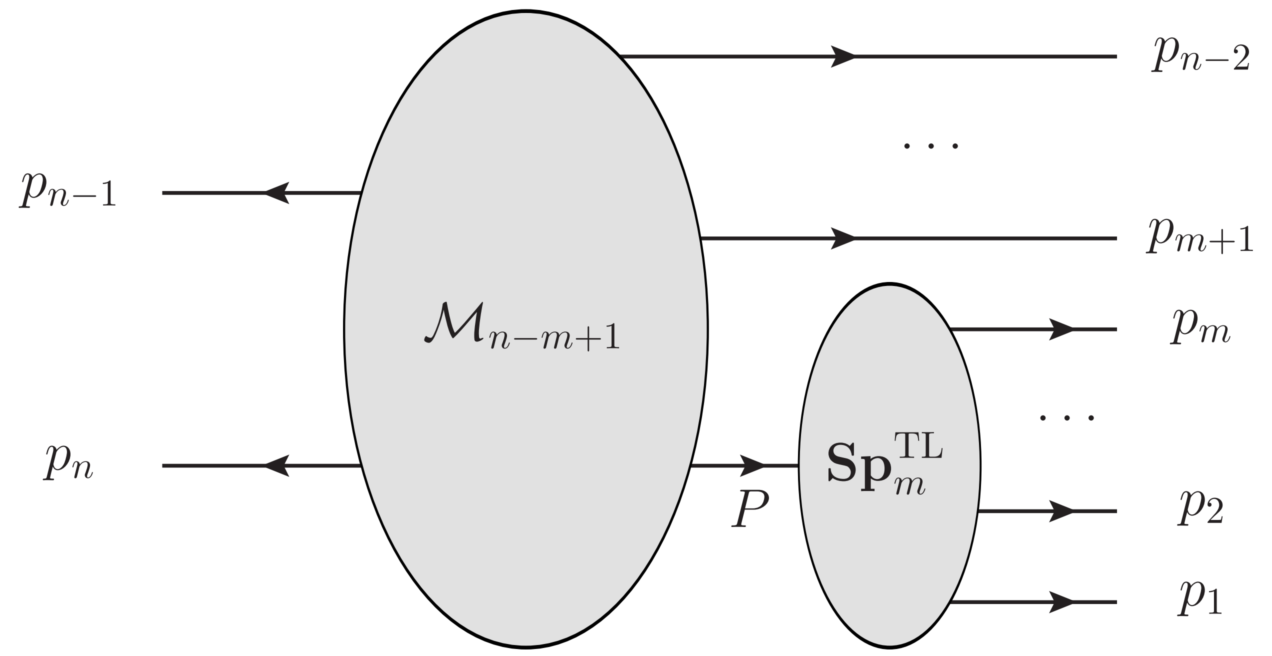
$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\right]$$

$$-4f(\alpha_s)\Big[T_{1123} + T_{2213} + T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathscr{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathscr{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathscr{F}(\beta_{12l3},\beta_{13l2})\Big]$$

what about the terms in the last line?

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) - \Big(\boldsymbol{\Gamma}_{n-2,4\mathrm{T}-3\mathrm{L}}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s)\Big)$$



This is where properties of CICRs come to the rescue, as we have seen, the CICRs with three particles collinear, the dependence on the rest-of-the-process parton scales out.

$$\rho_{123l} \to \frac{x_3(-p_1\cdot p_2)}{x_2(-p_1\cdot p_3)}$$

Then we can apply colour conservation $\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312}$

$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-3\mathrm{L}}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

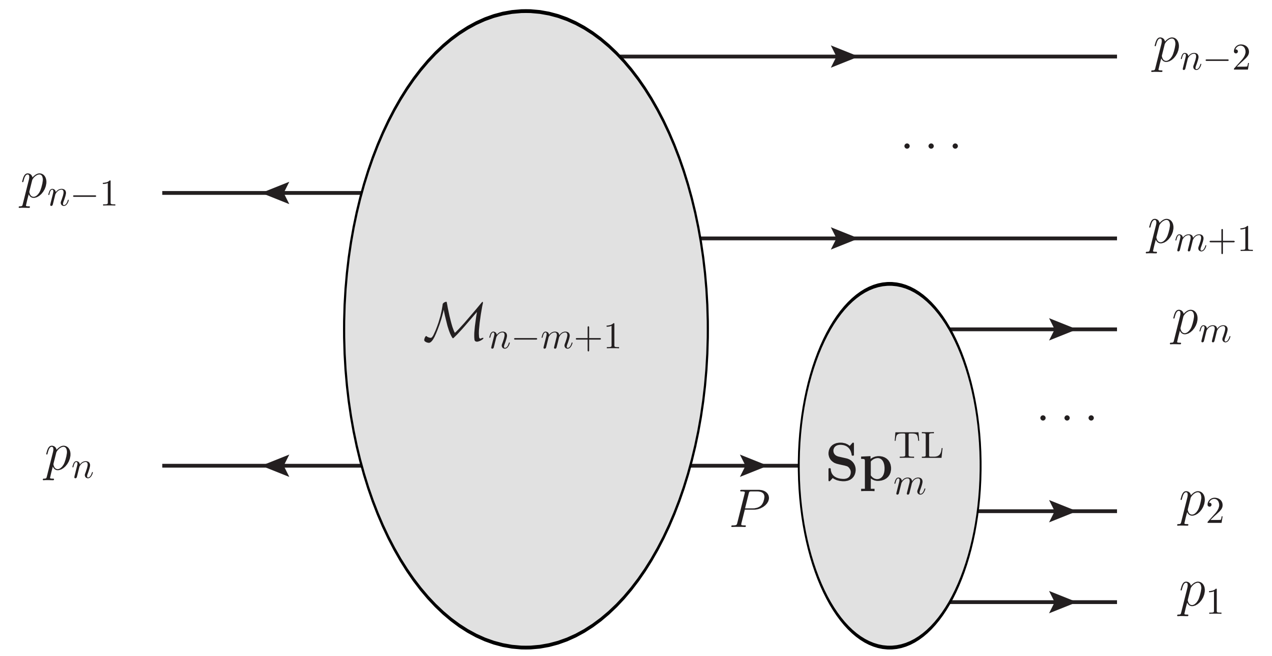
$$\boldsymbol{\Gamma}_{n,4\mathrm{T}-4\mathrm{L}}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\boldsymbol{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\big] - \frac{3}{4}f(\alpha_s)\big[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\big]$$

$$-4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big] + 8\sum_{4\le l\le n}\Big[T_{13l2}\,\mathcal{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\,\mathcal{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\,\mathcal{F}(\beta_{12l3},\beta_{13l2})\Big]$$

Three-particle collinear limit at three-loops

$$\boldsymbol{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1,p_2;\mu) = \boldsymbol{\Gamma}_{n,4T-3L}(\alpha_s) + \boldsymbol{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) - \left(\boldsymbol{\Gamma}_{n-2,4T-3L}(\alpha_s) + \boldsymbol{\Gamma}_{n-2,4T-4L}(\{\beta_{ijkl}\},\alpha_s)\right)$$



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Then we can apply colour conservation $\sum_{4\le l\le n} T_{123l} = T_{2213} - T_{3312}$

$$\boldsymbol{\Gamma}_{n,4T-3L}(\alpha_s) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk}$$

$$\boldsymbol{\Gamma}_{n,4T-4L}(\{\beta_{ijkl}\},\alpha_s) = 8\sum_{1\le i<j<k<l\le n} a_{ijkl}(\beta)$$

$$\boldsymbol{\Gamma}^{4T}_{\mathbf{Sp},3}(p_1,p_2,p_3) = -\frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\right] - \frac{3}{4}f(\alpha_s)\left[C_A^2\,\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\right]$$
$$-4f(\alpha_s)\left[T_{1123}+T_{2213}+T_{3312}\right] + 8\left(T_{1123}-T_{2213}\right)\mathcal{F}(\beta_{132l},\beta_{1l23}) + 8\left(T_{1123}-T_{3312}\right)\mathcal{F}(\beta_{123l},\beta_{1l32}) + 8\left(T_{2213}-T_{3312}\right)\mathcal{F}(\beta_{12l3},\beta_{13l2})$$

Strict collinear factorisation satisfied in the three-particle collinear limit for terms starting at three loops. No additional constraint.

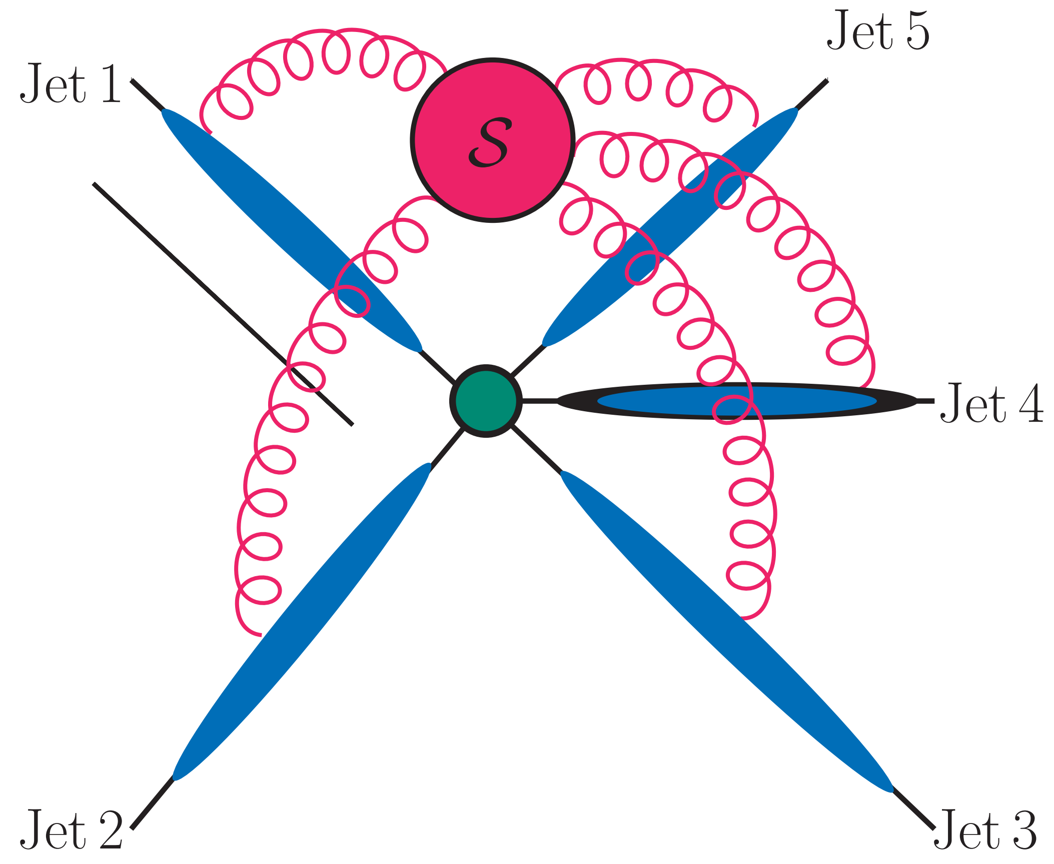
Collinear limit at four-loops

Four-loop correction to the soft anomalous dimension has not been fully explicitly computed. Recall the structure

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{s_{ij}\},\lambda,\alpha_s\right) + \mathbf{\Gamma}_{n,4\text{T}-3\text{L}}(\alpha_s) + \mathbf{\Gamma}_{n,4\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,\text{Q4T}-2,3\text{L}}(\{s_{ij}\},\lambda,\alpha_s) + \mathbf{\Gamma}_{n,\text{Q4T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s)$$

$$+\mathbf{\Gamma}_{n,5\text{T}-4\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathbf{\Gamma}_{n,5\text{T}-5\text{L}}(\{\beta_{ijkl}\},\alpha_s) + \mathcal{O}(\alpha_s^5)$$



Collinear limits of amplitudes with a massive leg

We combine the anomalous dimension of the massless and massive partons

$$\mathbf{A}_{n+I}(\{\beta\},\{r\}) = 4\sum_{1\le i<j<k<l\le n} \mathbf{a}_{ijkl}(\{\beta\}) + 2\sum_{1\le i<j<k\le n} \mathbf{a}^{h}_{ijkI}(\{r\})$$

$$\mathbf{B}_{n+I}(\{r\}) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk} + 2\sum_{1\le i<j\le n} T_{IIij}\,F_{\mathrm{h2}}\left(r_{ijI},\alpha_s\right)$$

And we make the same considerations as before. Starting with two-particle collinear limit, split the sums..

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1,p_2;\mu) = \sum_{3\le k<l\le n}\left[4\mathscr{F}^{\mathrm{A}}_{12kl}(\{\beta\})\,T_{1kl2} + \left(4\mathscr{F}^{\mathrm{S}}_{12kl}(\{\beta\}) - 2f(\alpha_s)\right)\left(T_{12lk}+T_{12kl}\right)\right]$$

$$+2f(\alpha_s)\sum_{i=3}^{n}\left(T_{ii12}+T_{112i}+T_{221i}\right) + 2\sum_{3\le k\le n}\mathbf{a}^{h}_{12kI} + 2T_{II12}F_{\mathrm{h2}}\left(r_{12I},\alpha_s\right)$$

Massless part

Again leads to same constraints as before on the massless kinematic functions

Additional massive part

Collinear limits of amplitudes with a massive leg

Then application of colour conservation must also include the massive coloured final leg. So the example relation changes to

$$\sum_{i=3}^{n} T_{112i} = -\frac{1}{8} C_A^2 \mathbf{T_1} \cdot \mathbf{T_2} - T_{1122} - T_{112I}$$

In the end we find

Massless part

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},2}(p_1, p_2; \mu) = -\frac{3}{4} f(\alpha_s)\Big(C_A^2 \mathbf{T}_1 \cdot \mathbf{T}_2 + 8\, T_{1122}\Big) - 6 f(\alpha_s) T_{12II} + 2 T_{12II} F_{\mathrm{h}2}\left(r_{12I}, \alpha_s\right)$$

$$+ 2 \sum_{k=3}^{n} T_{12kI} F_{\mathrm{h}3}\left(r_{12I}, r_{1kI}, r_{2kI}, \alpha_s\right) + 2 \sum_{k=3}^{n} T_{21kI} F_{\mathrm{h}3}\left(r_{21I}, r_{2kI}, r_{1kI}, \alpha_s\right) - 4 f(\alpha_s) \sum_{k=3}^{n} \left(T_{12kI} + T_{21kI}\right)$$

Demanding strict collinear factorisation, leads to constraints: [Z. L. Liu, N. Schalch, 2207.02864]

$$F_{\mathrm{h}2}\left(r_{12I}, \alpha_s\right)\Big|_{p_1 || p_2} = 3 f(\alpha_s)\,, \qquad F_{\mathrm{h}3}\left(r_{12I}, r_{1kI}, r_{2kI}, \alpha_s\right)\Big|_{p_1 || p_2} = 2 f(\alpha_s)$$

Three-particle collinear limit with a massive leg

Going back to the start, we note that functions F_{h3} depends on three massless particles and one massive

$$\mathbf{A}_{n+I}(\{\beta\},\{r\}) = 4\sum_{1\le i<j<k<l\le n}\mathbf{a}_{ijkl}(\{\beta\}) + 2\sum_{1\le i<j<k\le n}\mathbf{a}^{h}_{ijkI}(\{r\})$$

$$\mathbf{B}_{n+I}(\{r\}) = 2f(\alpha_s)\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} T_{iijk} + 2\sum_{1\le i<j\le n} T_{IIij}F_{\mathrm{h2}}\left(r_{ijI},\alpha_s\right)$$

Proceeding in the same way as before (already implementing the two-particle collinear limit constraints), we again recover the massless three particle collinear structures, but also extra terms

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3;\mu) = -\frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\Big) - \frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\Big) - \frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\Big) - 4f(\alpha_s)\Big[T_{1123}+T_{2213}+T_{3312}\Big]$$

$$+8\sum_{4\le l\le n}\Big[T_{13l2}\mathscr{F}(\beta_{132l},\beta_{1l23}) + T_{12l3}\mathscr{F}(\beta_{123l},\beta_{1l32}) + T_{123l}\mathscr{F}(\beta_{12l3},\beta_{13l2})\Big]$$

$$+2T_{123I}F_{\mathrm{h3}}\left(r_{12I},r_{13I},r_{23I}\right) + 2T_{213I}F_{\mathrm{h3}}\left(r_{21I},r_{23I},r_{13I}\right) + 2T_{312I}F_{\mathrm{h3}}\left(r_{31I},r_{32I},r_{21I}\right)$$

Three-particle collinear limit with a massive leg

Then we find the following expression

$$\mathbf{\Gamma}^{4\mathrm{T}}_{\mathbf{Sp},3}(p_1,p_2,p_3;\mu) = -\frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_1\cdot\mathbf{T}_2 + 8T_{1122}\Big) - \frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_1\cdot\mathbf{T}_3 + 8T_{1133}\Big) - \frac{3}{4}f(\alpha_s)\Big(C_A^2\mathbf{T}_2\cdot\mathbf{T}_3 + 8T_{2233}\Big) - 4f(\alpha_s)\Big[T_{1123} + T_{2213} + T_{3312}\Big]$$

$$+\,8\left(T_{1123} - T_{2213}\right)\mathscr{F}(\beta_{132l},\beta_{1l23}) + 8\left(T_{1123} - T_{3312}\right)\mathscr{F}(\beta_{123l},\beta_{1l32}) + 8\Big(T_{2213} - T_{3312}\Big)\mathscr{F}(\beta_{12l3},\beta_{13l2})$$

$$+2T_{123I}\Big(F_{\mathrm{h3}}\left(r_{12I},r_{13I},r_{23I},\alpha_s\right) - 4\mathscr{F}(\beta_{12l3},\beta_{13l2})\Big)$$

$$+2T_{213I}\Big(F_{\mathrm{h3}}\left(r_{21I},r_{23I},r_{13I},\alpha_s\right) - 4\mathscr{F}(\beta_{123l},\beta_{1l32})\Big)$$

$$+2T_{312I}\Big(F_{\mathrm{h3}}\left(r_{31I},r_{32I},r_{21I},\alpha_s\right) - 4\,\mathscr{F}(\beta_{132l},\beta_{1l23})\Big)$$

Auxiliary slides

Quartic Casimir

$$\mathscr{D}^R_{ijkl} = \frac{1}{4!} \sum_{\sigma \in S_4} \mathrm{Tr}_R \left(T^{\sigma(a)} T^{\sigma(b)} T^{\sigma(c)} T^{\sigma(d)} \right) \mathbf{T}^a_i \mathbf{T}^b_j \mathbf{T}^c_k \mathbf{T}^d_l$$

Slide Borrowed From Einan Gardi

Collinear constrains on the one-mass soft anomalous dimension

$$
\begin{aligned}
\mathbf{\Gamma} &= \mathbf{\Gamma}_{\text{Dip}} \\
&+ \sum_{i<j<k<q} \sum_{(u,v;w)\in P} \mathbf{T}_{uv;wq} \mathcal{F}_{0,4}(\rho_{uwvq}, \rho_{vwuq}) \\
&+ \mathcal{F}_{0,3} \sum_{i} \sum_{j<k;\, j,k\neq i} \mathbf{T}_{ij;ik} \\
&+ \sum_{i<j<k} \sum_{(u,v;w)\in P} \mathbf{T}_{uv;wQ} \mathcal{F}_{1,3}(r_{uwQ}, r_{vwQ}, r_{uvQ}) \\
&+ \sum_{i<j} \mathbf{T}_{iQ;jQ} \mathcal{F}_{1,2}(r_{ijQ}) + \mathcal{O}(\alpha_s^4),
\end{aligned}
$$

$\beta_i||\beta_j$

$\beta_i||\beta_j$

$\beta_i||\beta_j$

$\beta_i||\beta_j||\beta_k$ OR $\beta_Q^2 \to 0$

$$\mathcal{F}_{0,3}^{(3)} = 32\,[2\zeta(2)\zeta(3) + \zeta(5)]$$

$$r_{ijQ} \equiv \frac{\beta_Q^2 \beta_i \cdot \beta_j}{2\beta_i \cdot \beta_Q \beta_j \cdot \beta_Q}$$

two-particle collinear limits of the one-mass soft anomalous dimension [Liu & Schalch (2022)]

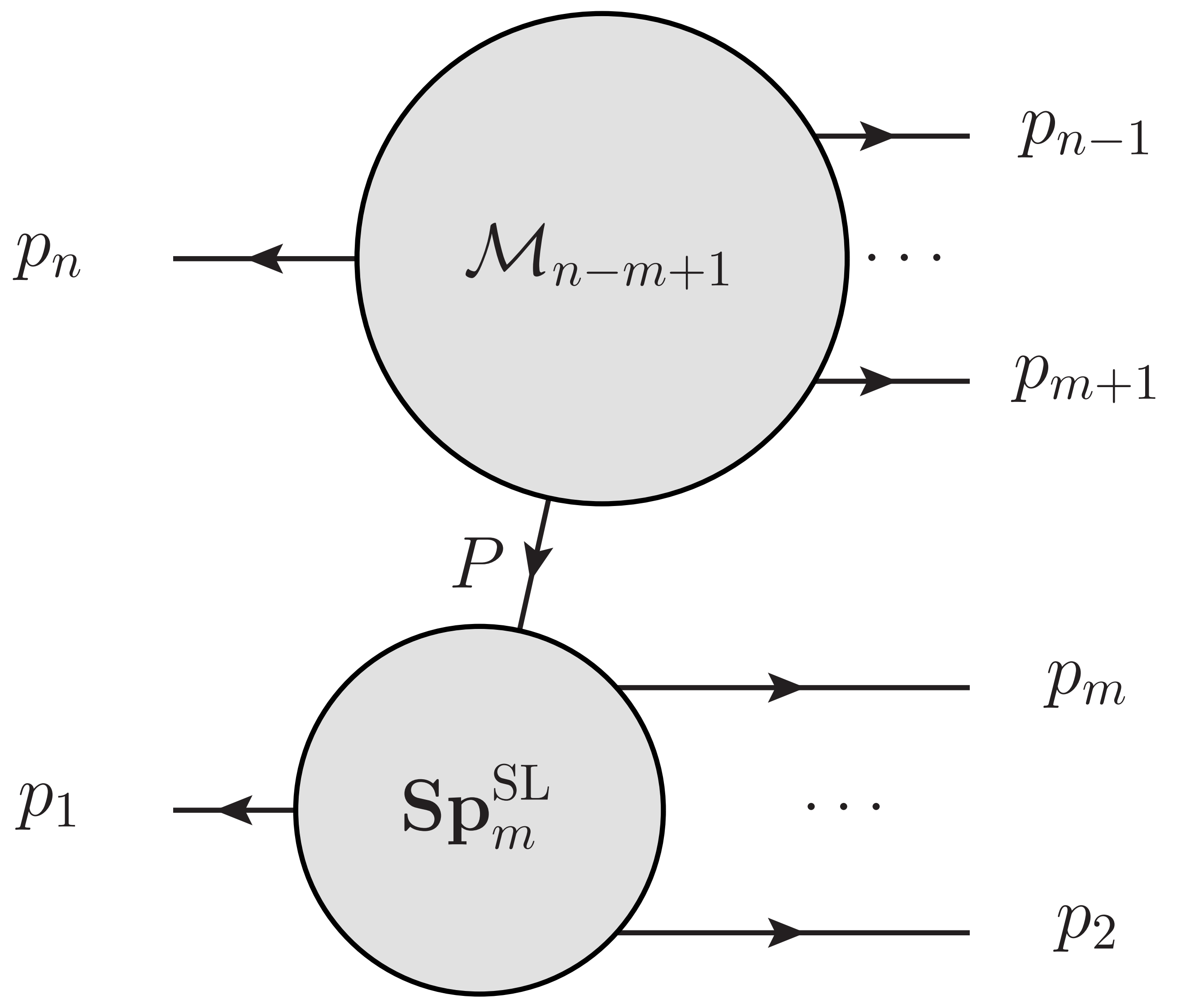
triple collinear limit = massless limit [Duhr, EG, Jaskiewicz, Lübken, Vernazza (2025)]

complementary lightcone expansion [EG, Zehao Zhu 2509.18017 (2025)]

What happens in the space-like limit?

Investigated by Catani, de Florian, and Rodrigo in 2011. [S. Catani, D. de Florian, G. Rodrigo, 1112.4405]

The set up now is as follows:



Kinematics and colour charges, m=2

$$p_1^\mu = \frac{1}{z} P^\mu + k^\mu \qquad p_2^\mu = \frac{(z-1)}{z} P^\mu - k^\mu$$

where k^μ represents a small residual (transverse) momentum, $k \sim \lambda P$ with $\lambda \ll 1$

Splitting amplitude SAD

$$\mathscr{M}_n(p_1, \ldots p_m, \{p_i\}_{\text{rest}}; \mu) \longrightarrow \mathbf{Sp}_m(p_1, \ldots p_m; \mu)\, \mathscr{M}_{n-m+1}(P, \{p_i\}_{\text{rest}}; \mu)$$

$$\mathscr{M}_n\left(p_1, \ldots p_m, \{p_i\}_{\text{rest}}, \mu\right) = \mathbf{Z}_n\left(p_1, \ldots p_m, \{p_i\}_{\text{rest}}, \mu_f\right) \mathscr{H}_n\left(p_1, \ldots p_m, \{p_i\}_{\text{rest}}, \mu_f, \mu\right)$$

$$\mathscr{M}_{n-m+1}\left(P, \{p_i\}_{\text{rest}}, \mu\right) = \mathbf{Z}_{n-m+1}\left(P, \{p_i\}_{\text{rest}}, \mu_f\right) \mathscr{H}_{n-m+1}\left(P, \{p_i\}_{\text{rest}}, \mu_f, \mu\right)$$

$$\mathscr{H}_n(p_1, \ldots p_m, \{p_i\}_{\text{rest}}, \mu_f, \mu) \longrightarrow \mathbf{Sp}_{\mathscr{H},m}(p_1, \ldots p_m; \mu_f, \mu)\, \mathscr{H}_{n-m+1}(P, \{p_i\}_{\text{rest}}, \mu_f, \mu)$$

$$\mathbf{Sp}_{\mathscr{H},m}(p_1, \ldots p_m; \mu_f, \mu) = \mathbf{Z}_n^{-1}(p_1, \ldots p_m, \{p_i\}_{\text{rest}}; \mu_f)\, \mathbf{Sp}_m(p_1, \ldots p_m; \mu) \mathbf{Z}_{n-m+1}(P, \{p_i\}_{\text{rest}}; \mu_f)$$