INTEGRAND ANALYSIS, LEADING SINGULARITIES AND CANONICAL BASES BEYOND POLYLOGARITHMS

Loopfest XXIV
28.05.2026

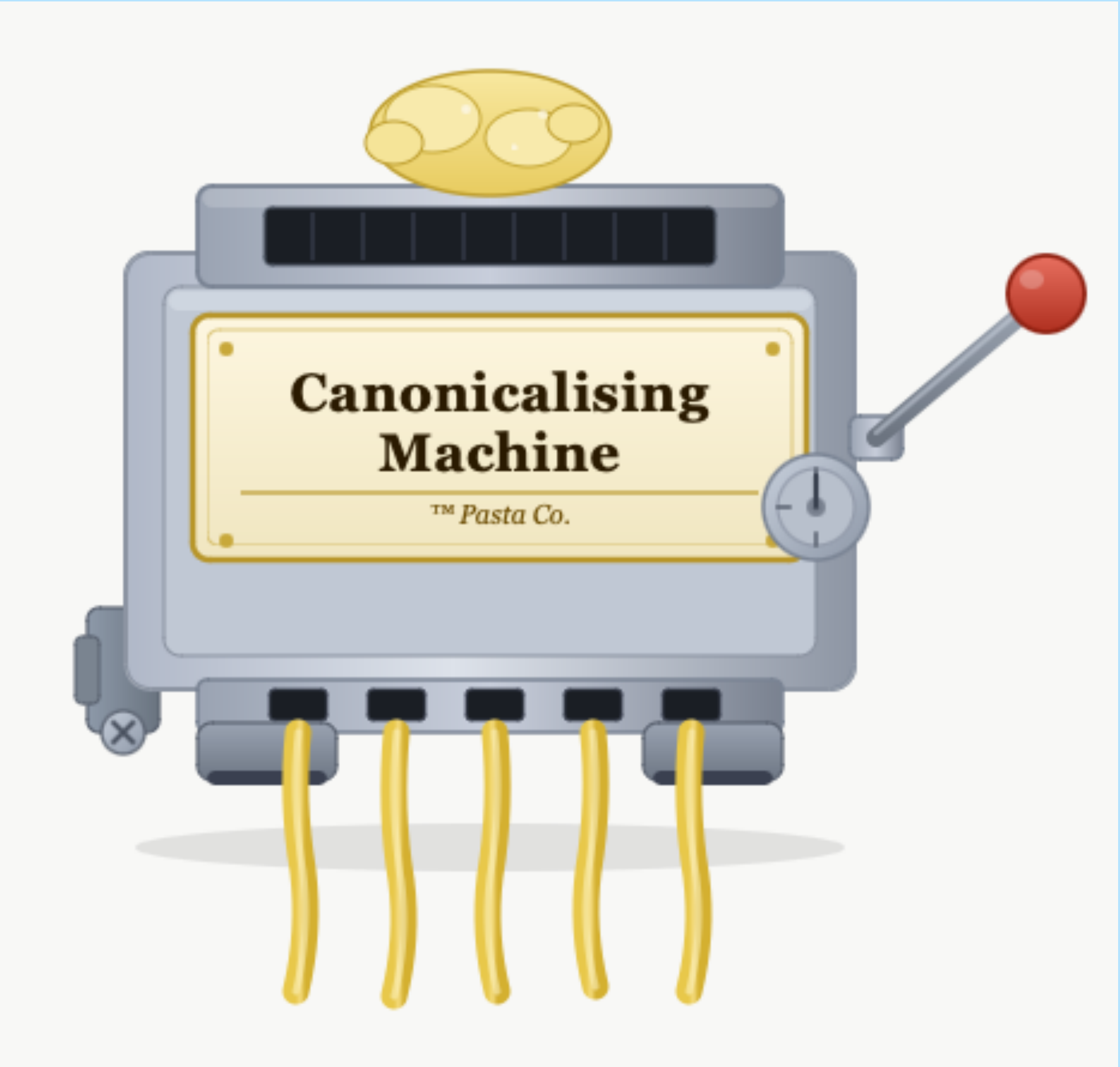
Based on 2604.25270, with Felix Forner, Christoph Nega, Fabian Wagner and Lorenzo Tancredi

THE GOAL OF THIS TALK

The quest for Canonical Bases

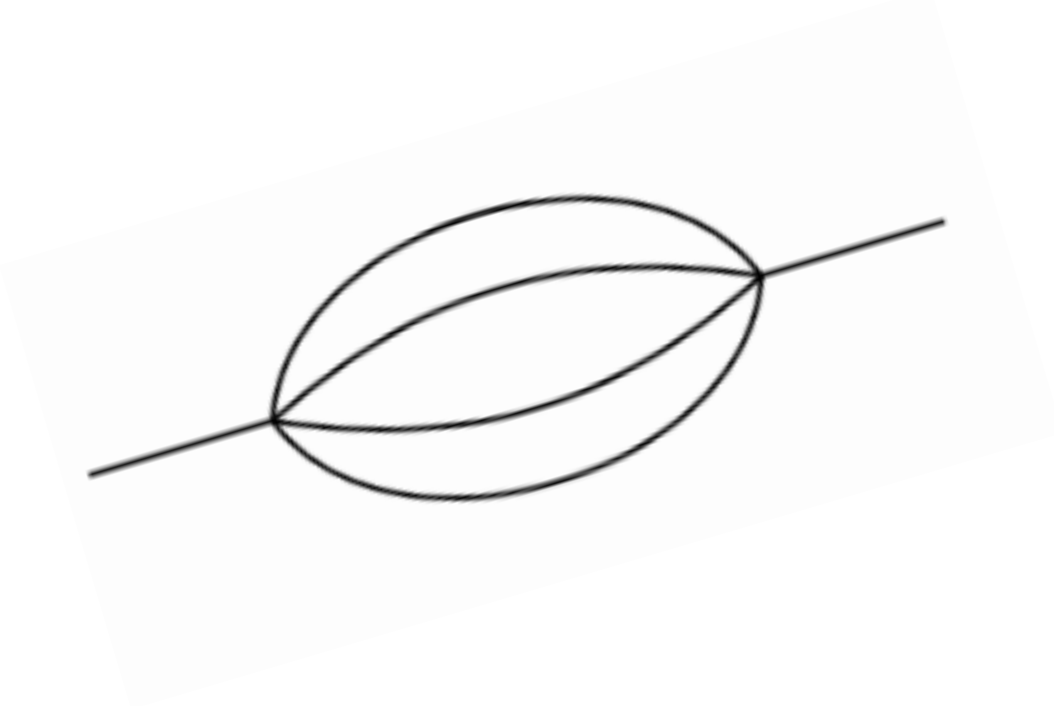
$$d\vec{I}(s_{ij}) = M(\epsilon, s_{ij}) \cdot \vec{I}$$

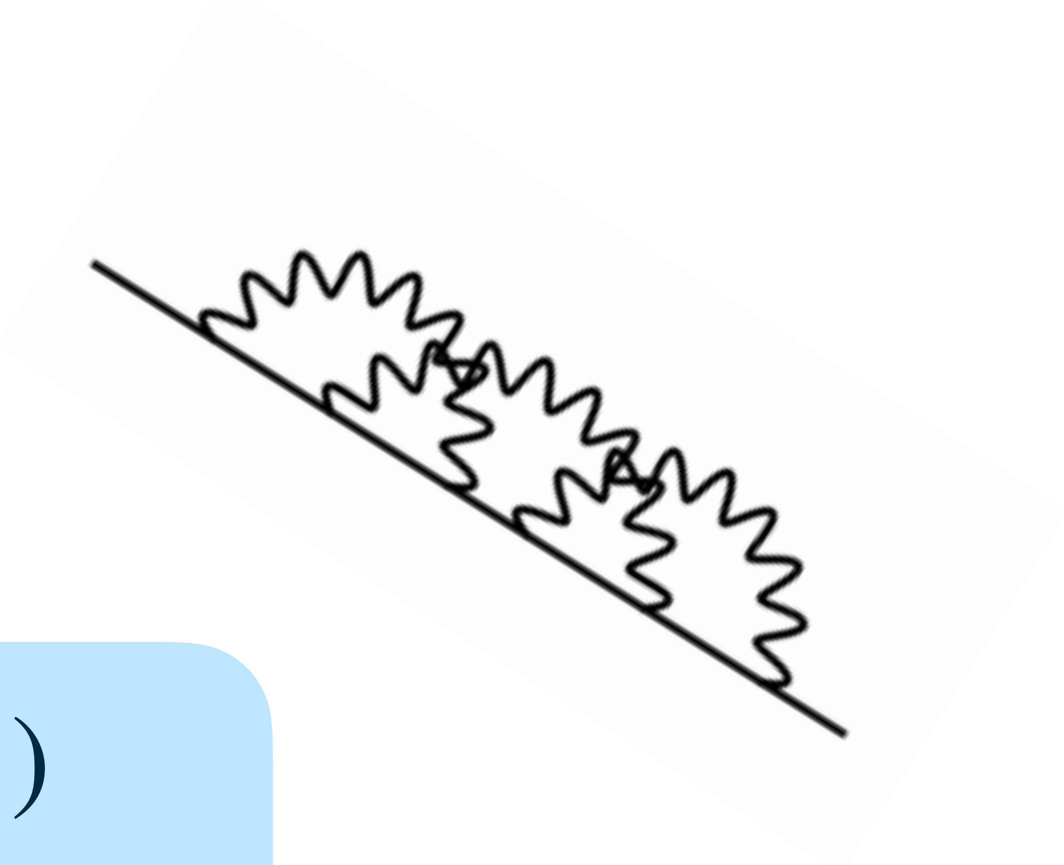
[Kotikov, Gehrmann, Remiddi]



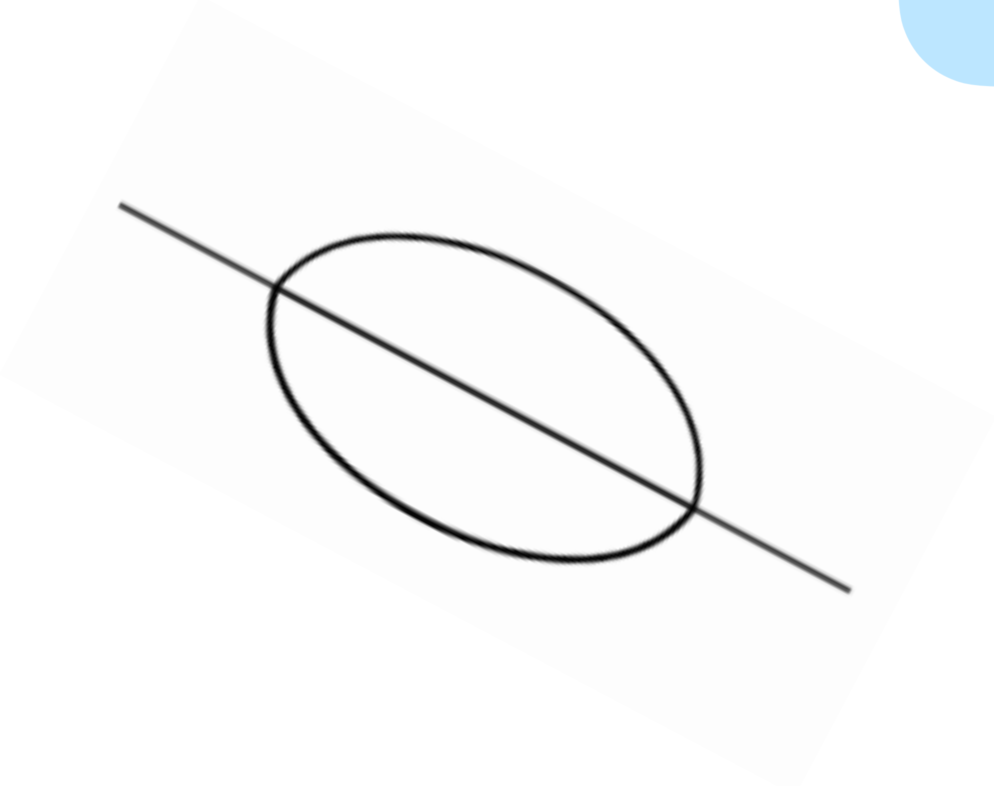
$$d\vec{I}(s_{ij}) = \epsilon\, M(s_{ij}) \cdot \vec{I}$$

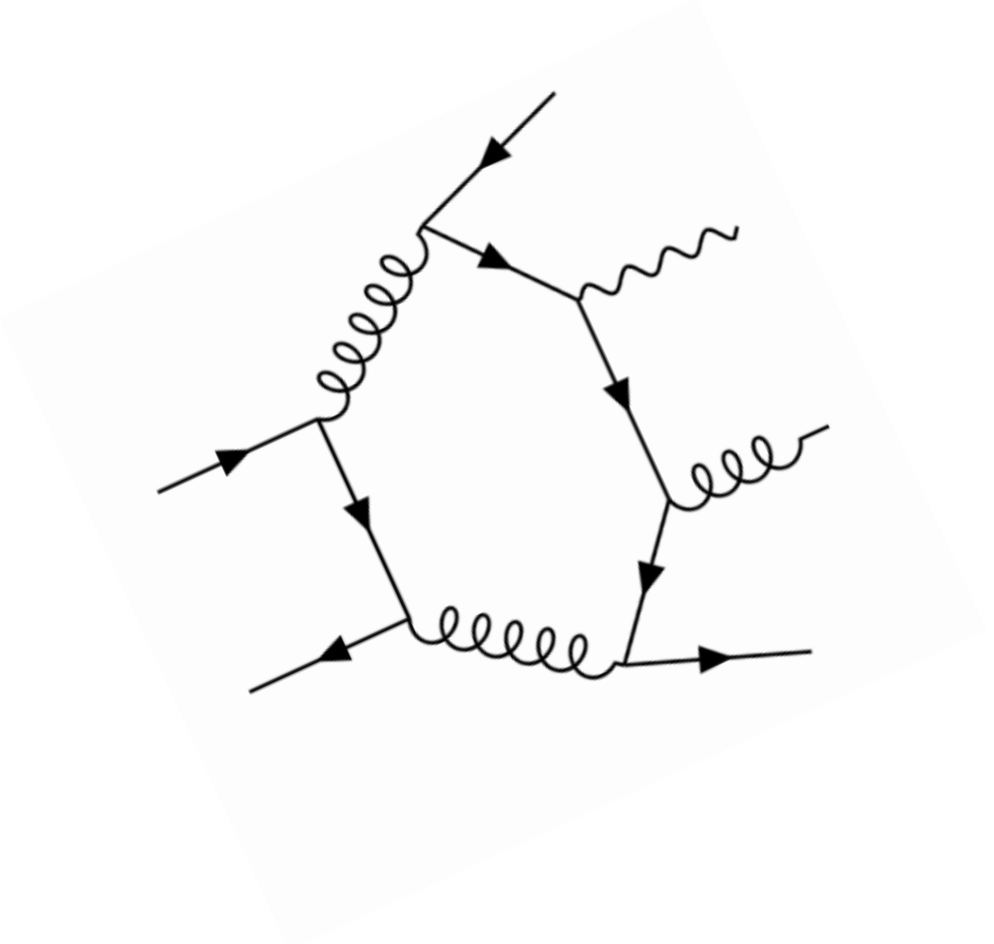
[Henn]





$$I \sim \int [\prod_{l=1}^{L} d^d k_l] \frac{N(\{k_i \cdot p_j\})}{D_1\, D_2\, \cdots\, D_n}$$

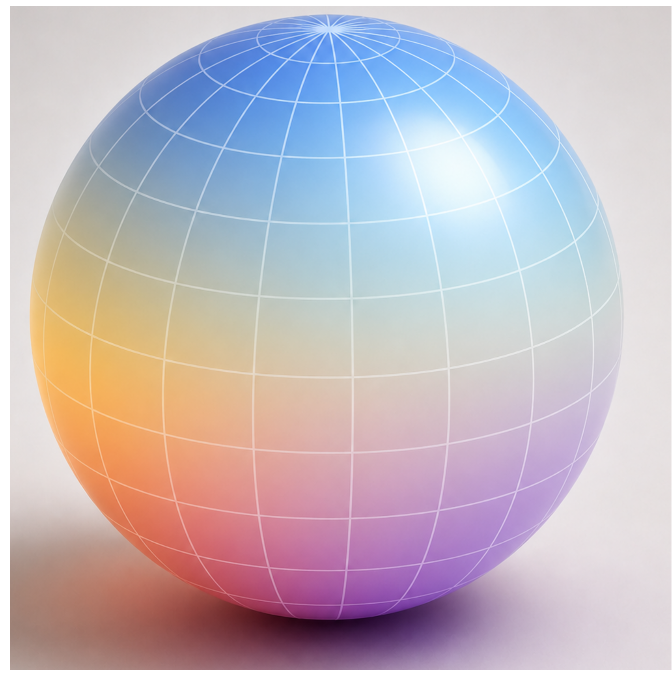


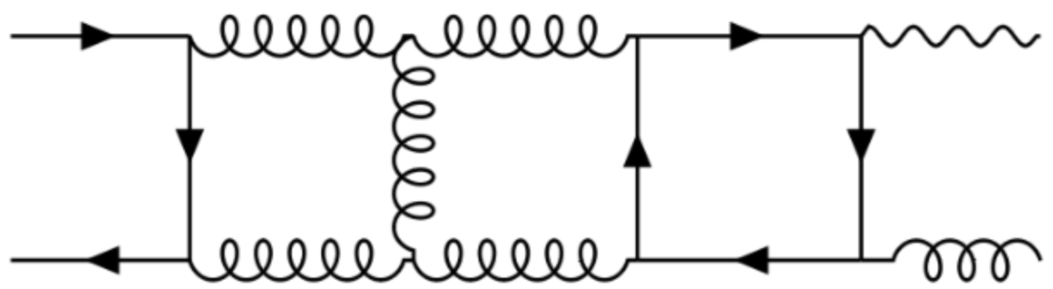


Propose an *integrand level* characterisation of canonical integrals on general geometries

FEYNMAN INTEGRALS AND GEOMETRIES

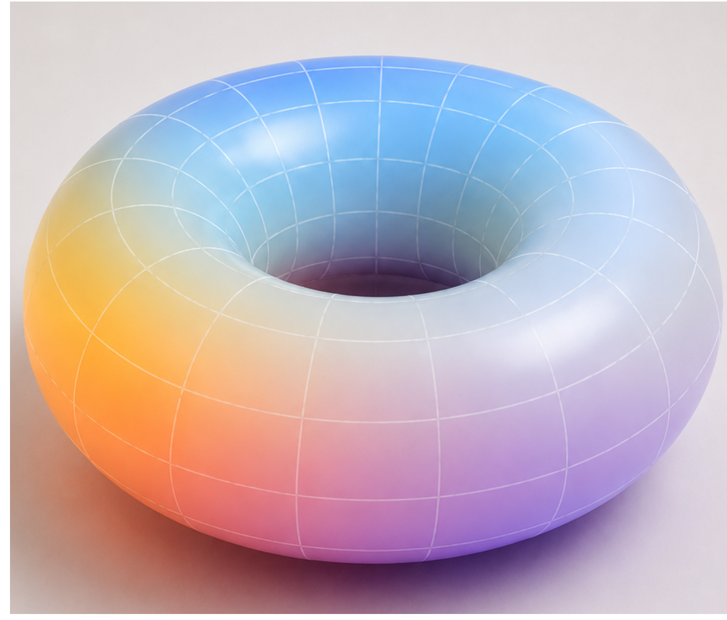
Riemann Sphere (aka Polylogarithms)

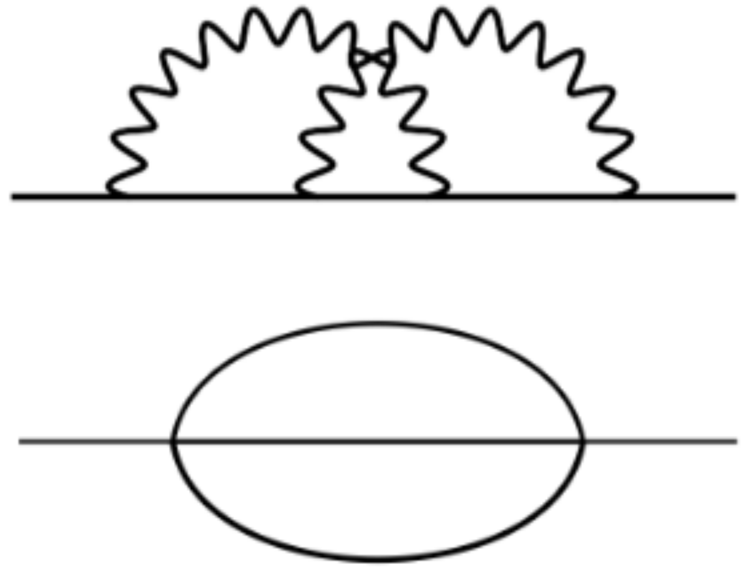




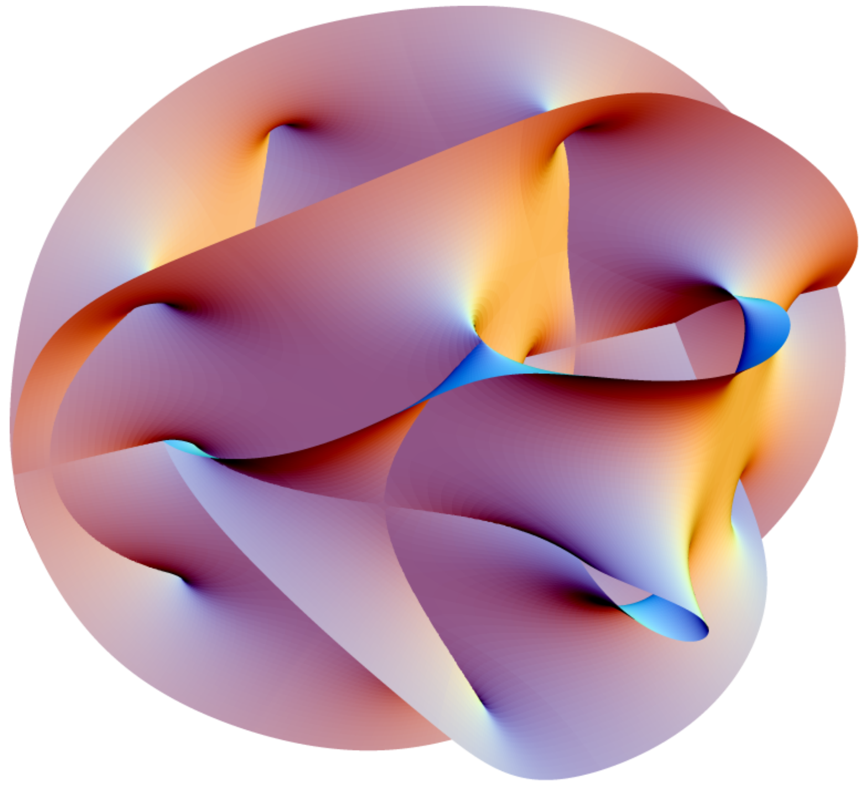
Integrands with ***unit leading singularity*** [Henn]

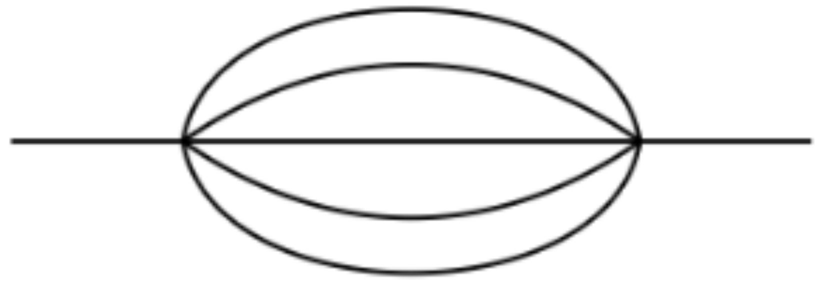
Elliptic Curves





Calabi -Yau





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(talks by Martin, 2 x Christoph)

Multiple approaches put forward to deal with more complicated geometries

New interpretation of integrands with unit leading singularities for general geometries

[Görges, Nega, Tancredi, Wagner]

[$\epsilon-$collaboration, Weinzierl et al.]

[Duhr, Maggio, Nega,Sauer, Tancredi]

START EASY: POLYLOGARITHMS

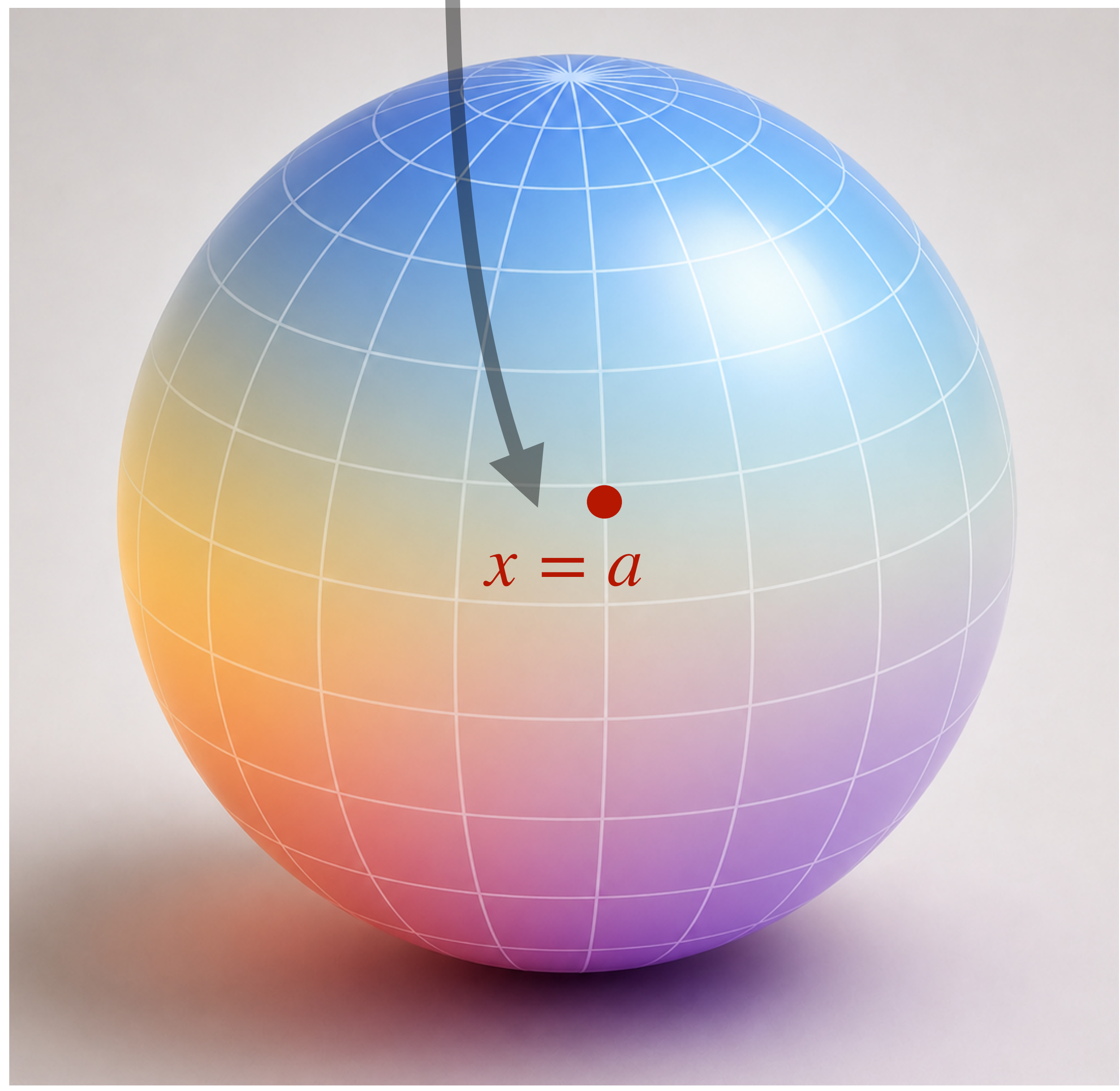
POLYLOGARITHMS

Look for integrals whose integrand can be written, ***for*** $d \sim d_0$, *in terms of* $d\log$ ***integration forms with unit leading singularity***

In practice, we want … $I \sim \mathcal{N} \int d\log(f_1(\vec{x})) \wedge \cdots \wedge d\log(f_n(\vec{x}))$

The "leading singularity" $\mathcal{N}$ is ***iterated residue around the single poles*** of the integrand

$$\log(1 - x/a) = \int_0^x \frac{dt}{t-a}$$



$$d\vec{I}(s_{ij}) = \epsilon\, M(s_{ij}) \cdot \vec{I} \qquad M(s_{ij}) = \sum_i A_i \, d\log(\alpha_i)$$

Singularities of scattering amplitudes in gauge theories are ***single poles and logarithms***

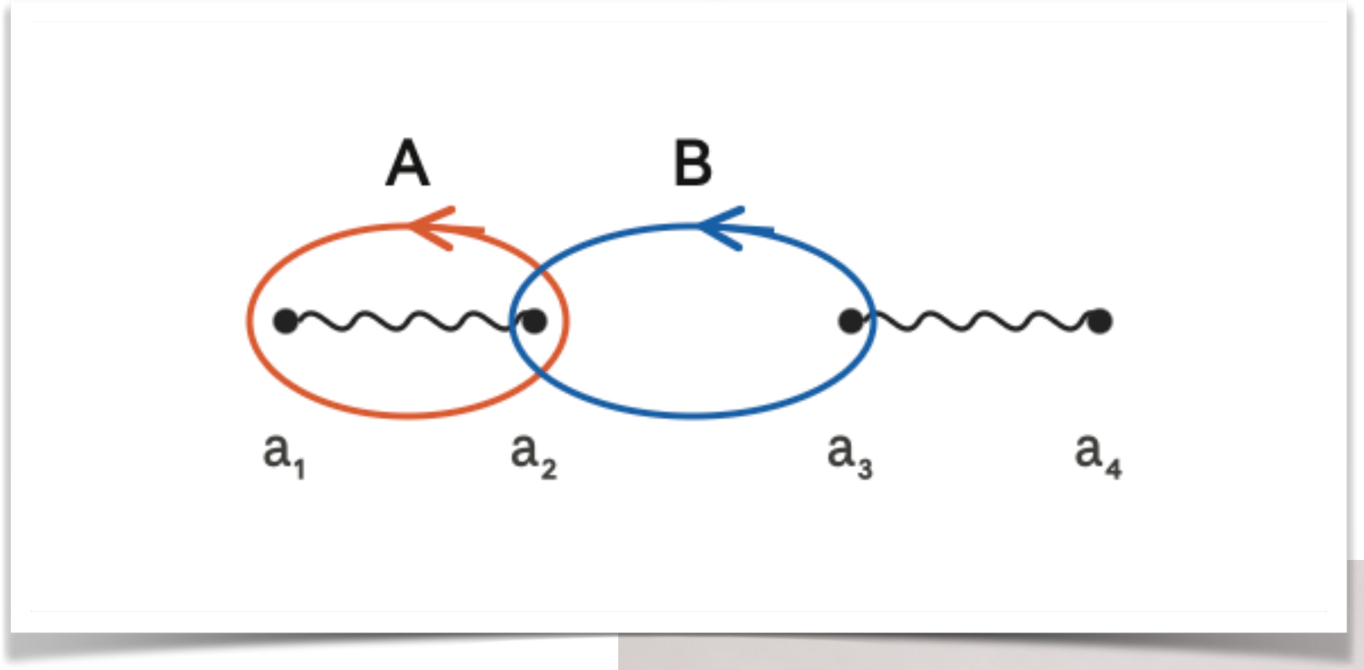
In the polylogarithmic world, this is ***globally*** manifest

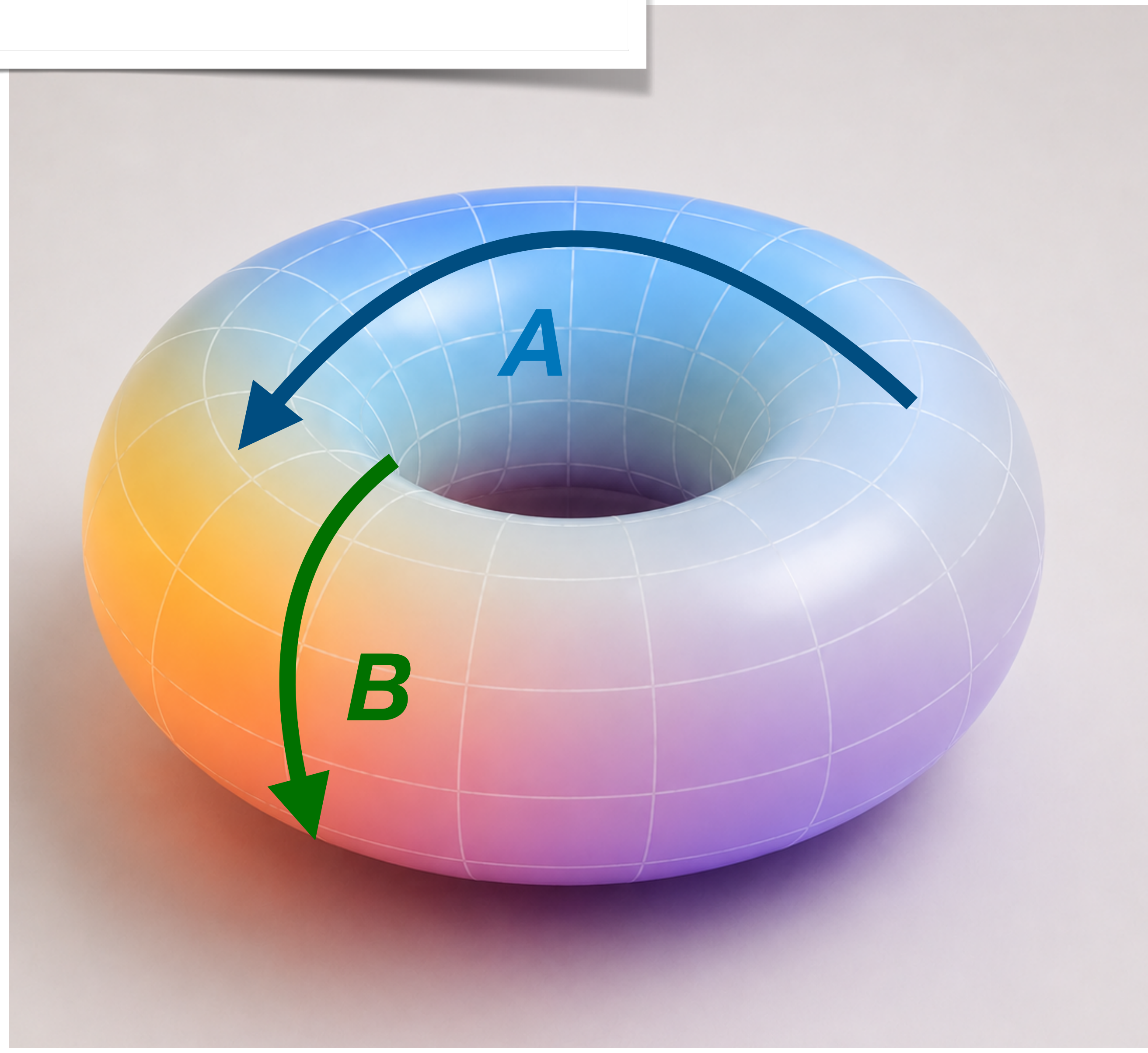
BEYOND POLYLOGARITHMS

BEYOND POLYLOGS: Integration forms and contours

$$I \sim \int \frac{dx}{\sqrt{P_4(x)}}$$

→ This forces you out of the Riemann sphere





$$P_4(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

Beyond the Riemann sphere, ***richer variety of integration forms and integration cycles***

First kind (no poles)	**Second kind** (higher poles)	**Third kind** (single poles)
$\frac{dx}{\sqrt{P_4(x)}}$	$\partial_x \frac{dx}{\sqrt{P_4(x)}}$	$\frac{x\,dx}{\sqrt{P_4(x)}}$

Cannot reduce all forms to third kind using IBPs →

What forms do we choose?

How do extend the concept of leading sing.?

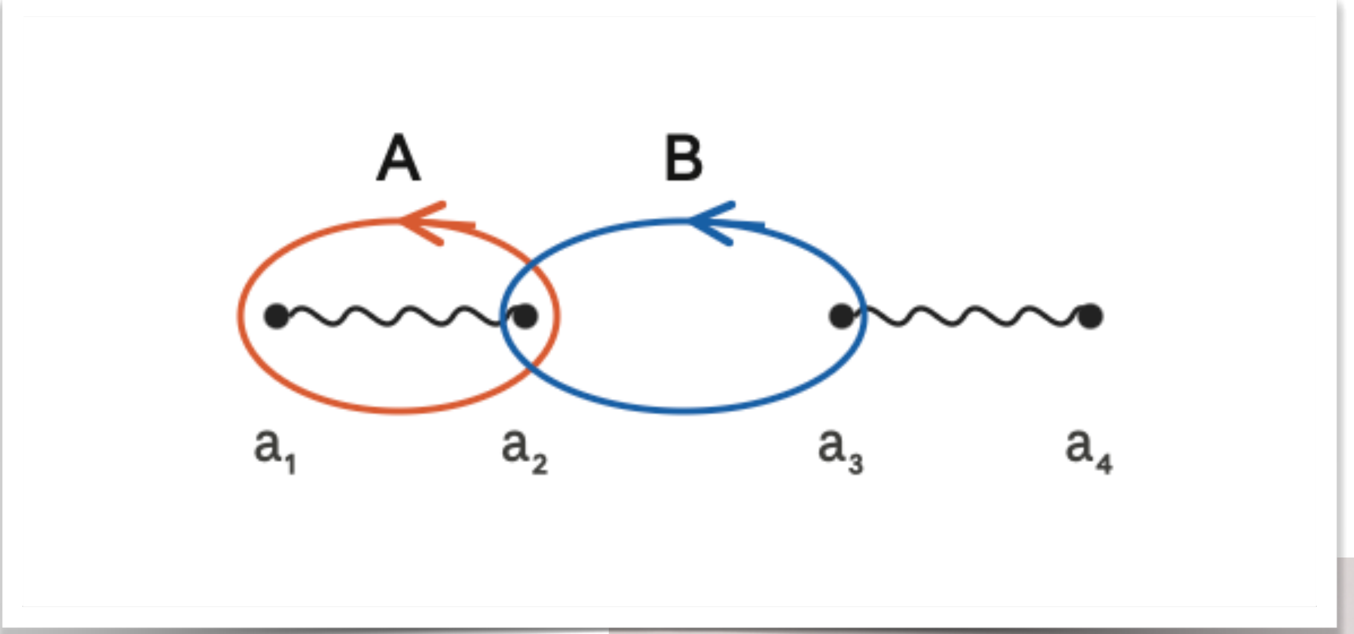
ELLIPTICS: A toy example

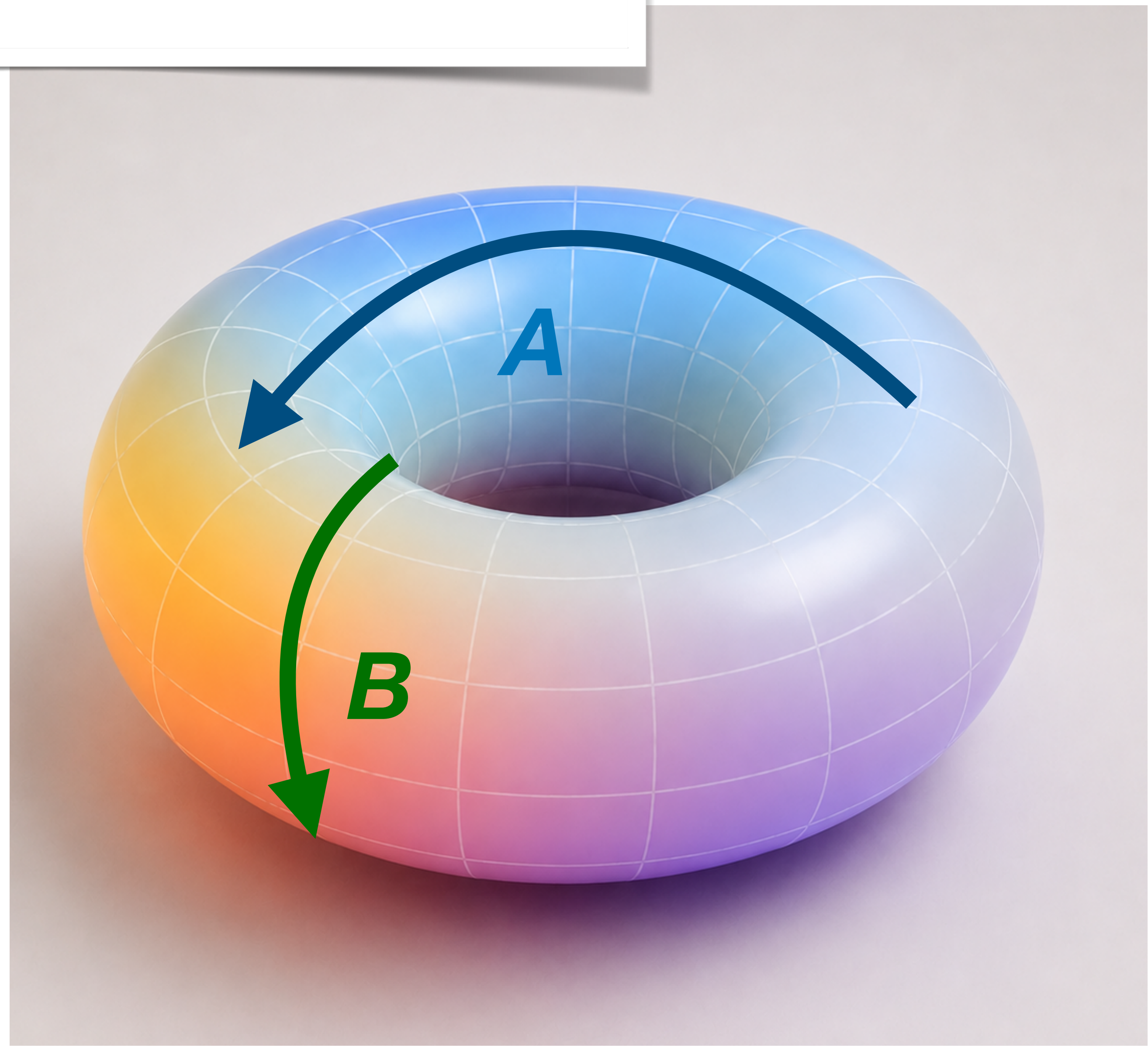
$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x^n}{\sqrt{x(1-x)(1-ax)}} \left(x(1-x)(1-a\,x)\right)^{\epsilon}, \; 0 < a < 1 \,, n \in \mathbb{Z}$$

Periods of the elliptic curve

$$\varpi_0 \sim \oint_A \frac{dx}{\sqrt{P_4(x)}}$$ A-cycle

$$\varpi_1 \sim \oint_B \frac{dx}{\sqrt{P_4(x)}}$$ B-cycle





ELLIPTICS: A toy example

$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x^n}{\sqrt{x(1-x)(1-ax)}} \left(x(1-x)(1-a\,x)\right)^{\epsilon} \,, 0 < a < 1 \,, n \in \mathbb{Z}$$

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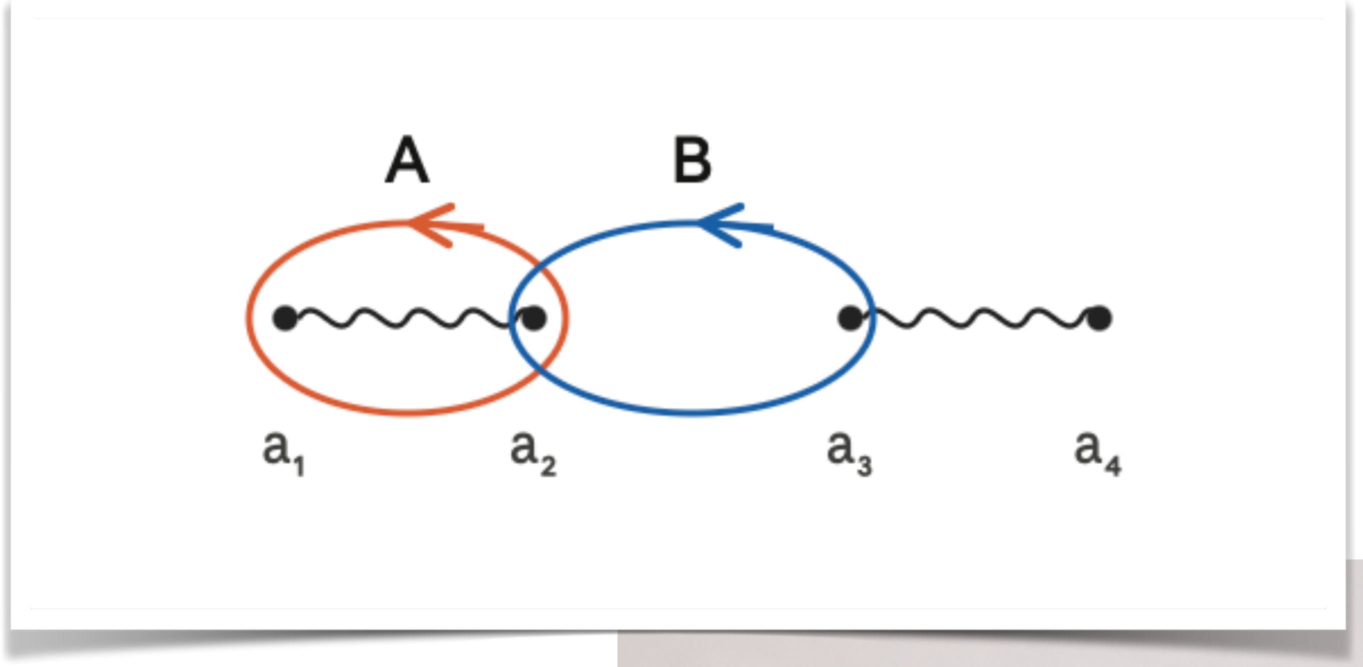
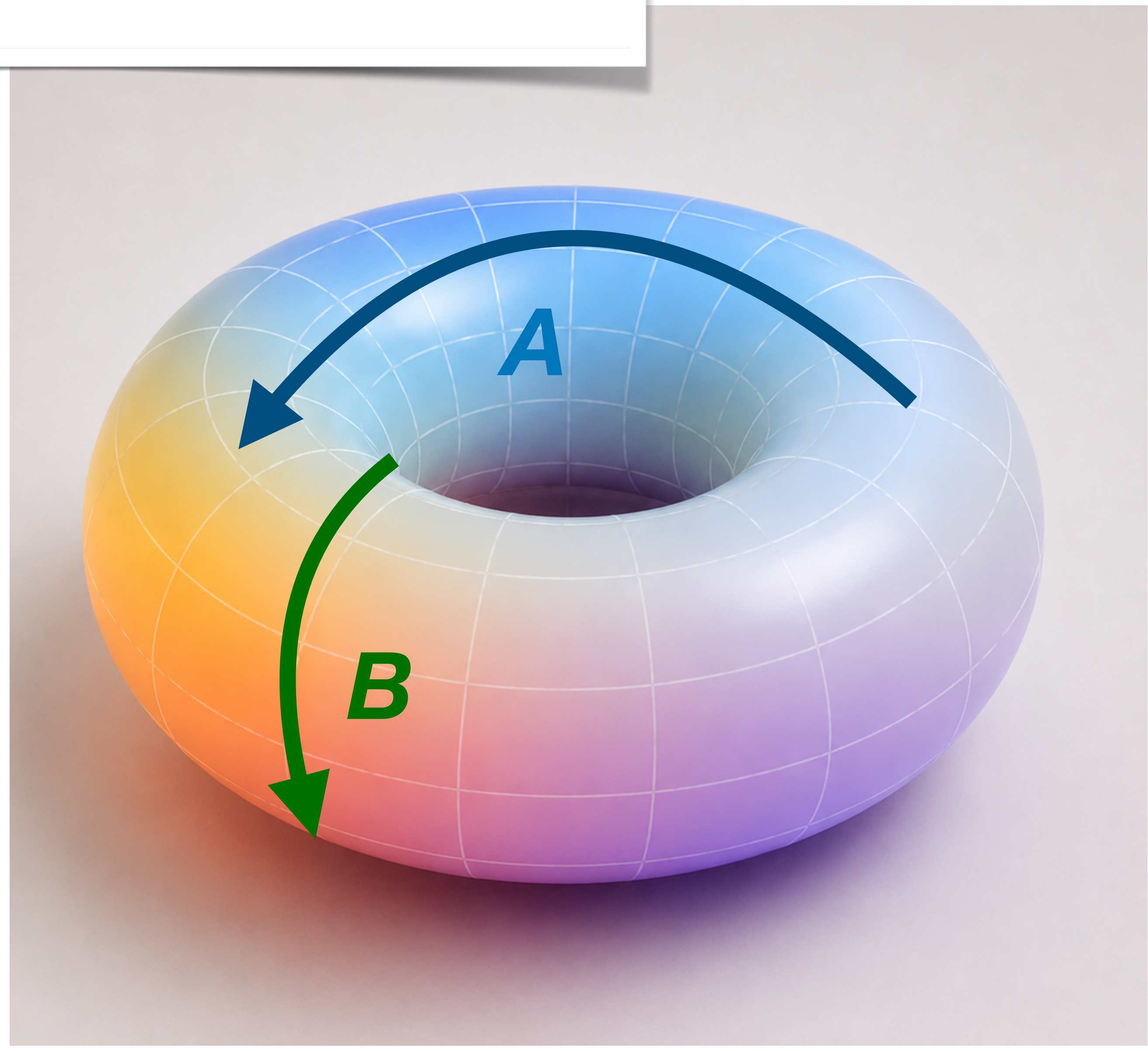




Leading singularities are all functions obtained upon evaluating the integrand along all independent contours

What forms do we pick?

ELLIPTICS: A toy example

$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x^n}{\sqrt{x(1-x)(1-ax)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon} \,,\, 0 < a < 1 \,,\, n \in \mathbb{Z}$$

- Work out IBPs: 2 master integrals

$$I_0^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{1}{\sqrt{x(1-x)(1-a\,x)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon}$$

At $\epsilon = 0$, no poles (first kind), logarithmic behaviour for $\epsilon \neq 0$

Fixing the leading singularity at $\epsilon = 0$ guarantees a canonical differential equation

This is a good candidate

$$\varpi_0 \sim I_0^A(a)$$

$$J_1(a,\epsilon) = \frac{I_0(a,\epsilon)}{\varpi_0} \longrightarrow \begin{cases} LS[J_1]_A \sim 1 + \mathcal{O}(\epsilon) \\ LS[J_1]_B \sim \tau + \mathcal{O}(\epsilon) \,,\, \tau = \dfrac{\varpi_1}{\varpi_0} \sim \log(a) \end{cases}$$

ELLIPTICS: A toy example

$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x^n}{\sqrt{x(1-x)(1-ax)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon} \;, 0 < a < 1 \,, n \in \mathbb{Z}$$

- Work out IBPs: 2 master integrals

$$I_1^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x}{\sqrt{x(1-x)(1-a\,x)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon}$$

At $\epsilon = 0$, higher poles (second kind), bad behaviour $\epsilon \neq 0$!!

This is NOT a good candidate, but cannot avoid it …

ELLIPTICS: A toy example

$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x^n}{\sqrt{x(1-x)(1-ax)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon} \,,\, 0 < a < 1 \,,\, n \in \mathbb{Z}$$

- Work out IBPs: 2 master integrals

$$I_1^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}} \int_C \mathrm{d}x \, \frac{x}{\sqrt{x(1-x)(1-a\,x)}} \big(x(1-x)(1-a\,x)\big)^{\epsilon}$$

At $\epsilon = 0$, higher poles (second kind), bad behaviour $\epsilon \neq 0$!!

This is NOT a good candidate, but cannot avoid it …

Minimise pollution at higher order in ϵ: take the ***derivative of the first master*** and ***compensate the weight drop***

$$\tilde{J}_2 = \frac{1}{\epsilon}\partial_a\left(\frac{I_0}{\varpi_0}\right) \qquad \begin{cases} LS[\tilde{J}_2]_A \sim 0 + \mathcal{O}(\epsilon^0) \\ LS[\tilde{J}_2]_B \sim \frac{1}{\epsilon}\partial_a \tau + \mathcal{O}(\epsilon^0)\,, \quad \tau = \frac{\varpi_1}{\varpi_0} \sim \log(a) \end{cases} \longrightarrow \quad J_2 = \frac{1}{\epsilon}\frac{\varpi_0^2}{\Delta}\partial_a J_1$$

Is this enough? No, one order more …

ELLIPTICS: A toy example

$$I_n^C(a,\epsilon) = \frac{1}{2\sqrt{\pi}}\int_C \mathrm{d}x\, \frac{x^n}{\sqrt{x(1-x)(1-ax)}}\left(x(1-x)(1-a\,x)\right)^{\epsilon}, 0<a<1\,, n\in\mathbb{Z}$$

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \frac{I_0}{\varpi_0} \\ \frac{1}{\epsilon}\frac{\varpi_0^2}{\Delta}\partial_a J_1 \end{pmatrix}$$

$$\frac{\partial}{\partial a}\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \left[\epsilon\begin{pmatrix} 0 & -\frac{1}{a(1-a)\varpi_0^2} \\ \varpi_0^2 & -\frac{2}{1-a} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2\varpi_0'\varpi_0 & 0 \end{pmatrix}\right]\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

This is a total derivative (it is not always a total derivative!!!)

From the differential equation:

$$\partial_a(LS[J_2]_A) = 2\varpi_0'\varpi_0 + \mathcal{O}(\epsilon^1)$$

$$\partial_a(LS[J_2]_B) = 2\tau\varpi_0'\varpi_0 + \frac{2}{1-a} + \mathcal{O}(\epsilon^1)$$

$$\rightarrow \quad K_2 = \frac{\varpi_0^2}{\Delta(a)}\left(\frac{1}{\epsilon}\partial_a J_1\right) - \varpi_0^2 J_1$$

Taking the derivative ensures that **we need to do this only once** (one weight drop)

ELLIPTICS: A toy example

$$\begin{pmatrix} J_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} \dfrac{I_0}{\varpi_0} \\ \dfrac{\varpi_0^2}{\Delta(a)}\left(\frac{1}{\epsilon}\partial_a J_1\right) - \varpi_0^2 J_1 \end{pmatrix}$$

$$\frac{\partial}{\partial a}\begin{pmatrix} J_1 \\ K_2 \end{pmatrix} = \varepsilon \begin{pmatrix} -\dfrac{1}{a(1-a)} & -\dfrac{1}{a(1-a)\varpi_0^2} \\ -\dfrac{(1-a+a^2)\varpi_0^2}{a(1-a)} & -\dfrac{1}{a(1-a)} \end{pmatrix} \begin{pmatrix} J_1 \\ K_2 \end{pmatrix}$$

Comments:

- Locally only single poles in the differential equation

!

- To canonicalise we had to fix leading singularities
  one order higher in ϵ

ELLIPTICS: leading singularities without differential equations

Let's go back

$$u^\epsilon = (x(1-x)(1-ax))^\epsilon$$

$$\frac{1}{\epsilon}\partial_a J_1 \propto \frac{\Delta(a)}{\varpi_0}\int_C dx \left(\frac{Z_3(x)}{x} + \frac{Z_3(x)}{x-1} + \frac{Z_3(x)}{x-1/a} - 6\widetilde{\Phi}_3(x)\right) u^\epsilon + \Delta(a)\, J_1$$

Need to remove this!

Combination of pure eMPLs kernels with leading sing $\frac{\Delta(a)}{\varpi_0^2}$

[Broedel, Duhr, Dulat, Tancredi]

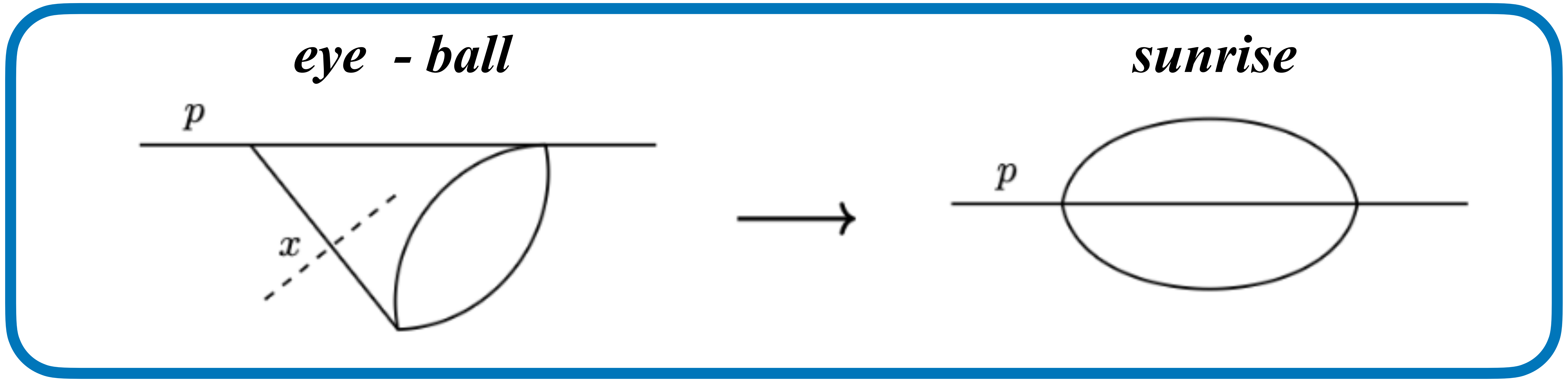
Rescale by $\left(\frac{\Delta(a)}{\varpi_0^2}\right)^{-1}$ and subtract $\varpi_0^2 J_1$

$$K_2 = \frac{\varpi_0^2}{\Delta(a)}\left(\frac{1}{\epsilon}\partial_a J_1\right) - \varpi_0^2 J_1$$

Exactly as from diff. Eq.!

G FUNCTIONS

G FUNCTIONS



$z = m^2/p^2$

$$I_{n_1 \cdots n_4} = \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{x_1^{n_1} x_2^{n_2} x_3^{n_3} x^{n_4}}$$

On the maximal cut, $I_{1,1,1,1}$ is a string of $d\log \rightarrow$ leading singularity as iterated residue

$$LS[I_{1,1,1,1}] = \frac{z}{\sqrt{1-4z}}$$

$$I = \frac{\sqrt{1-4z}}{z} I_{1,1,1,1}$$

The correct canonical candidate needs a correction from the sunrise subsector

$$I = I - \left(G(z) + \frac{5\varpi_0}{3\sqrt{1-4z}} \right) \frac{J_1}{\varpi_0}$$

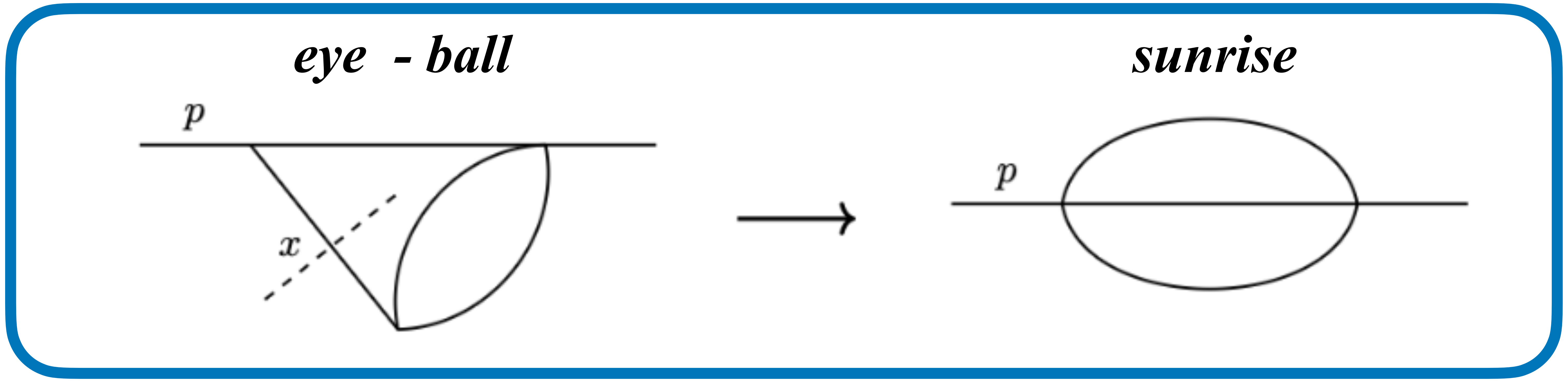
$$G'(z) = \frac{2(1+z)\varpi_0}{3z(1-4z)^{3/2}}$$

New independent function!

J_1 is the candidate for the sunrise associated to the form of the first kind

Cancels a leading singularity related to the coupling to the sunrise

G FUNCTIONS

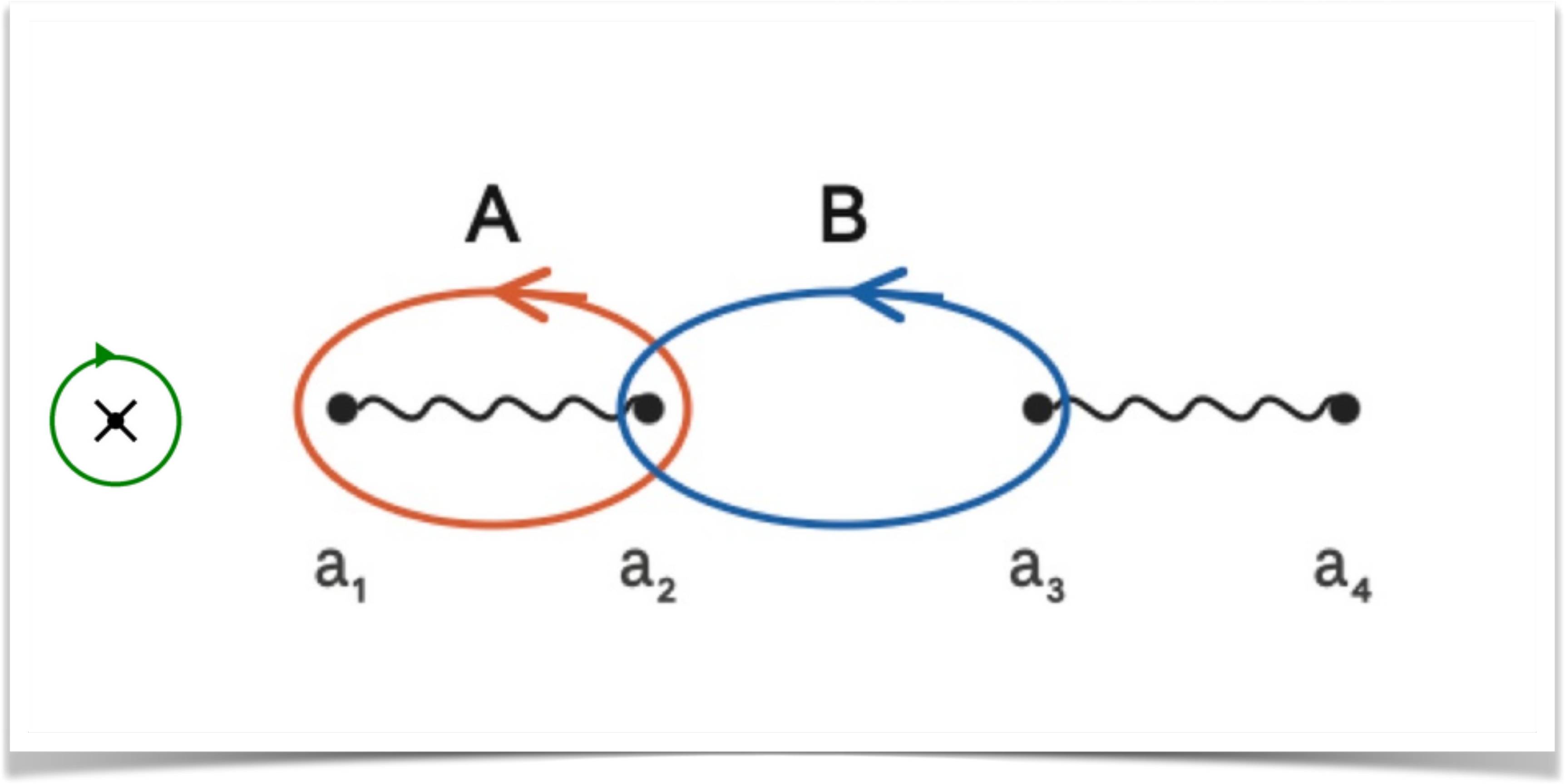


Use Baikov representation and integrate over x_1, x_2, x_3

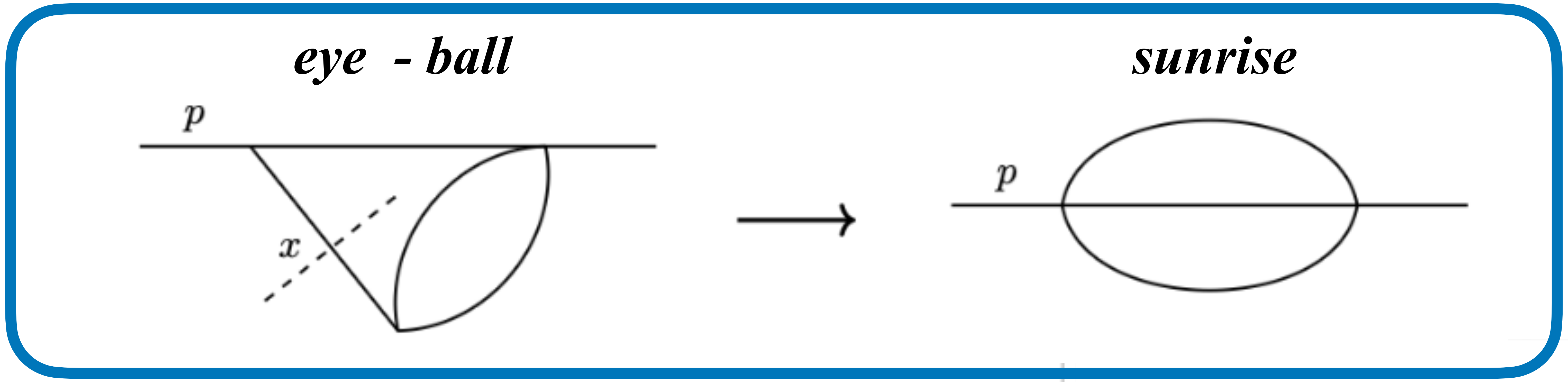
$$I_{1,1,1,1} \sim z \int \mathrm{d}x \, \underbrace{\frac{1}{x} \frac{1}{\sqrt{P_4(x;z)}}}$$

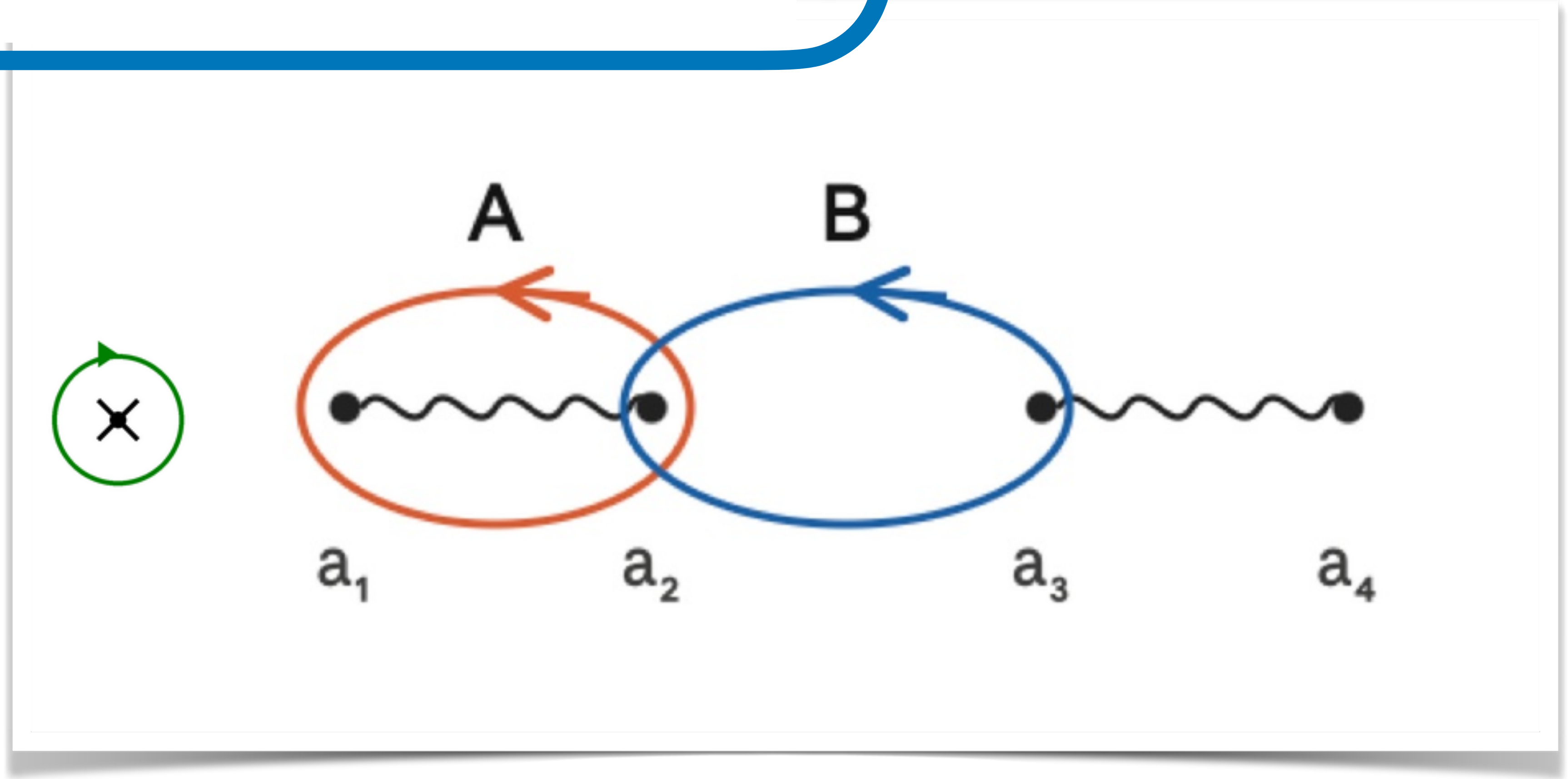
→ Ellitpic curve of the 2 loop equal mass sunrise

→ Form of the ***third kind*** on the geometry of the sunrise



G FUNCTIONS





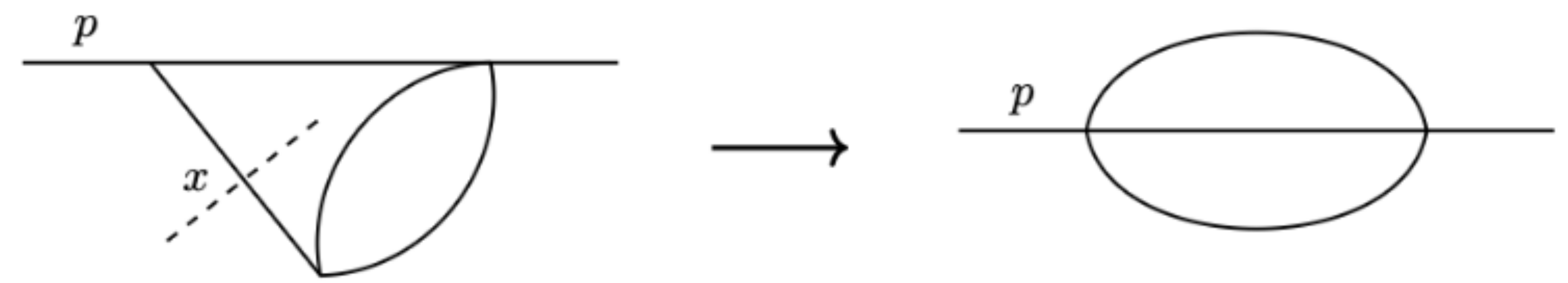
$$I_{1,1,1,1} \sim z \int \mathrm{d}x \, \frac{1}{x} \frac{1}{\sqrt{P_4(x;z)}}$$

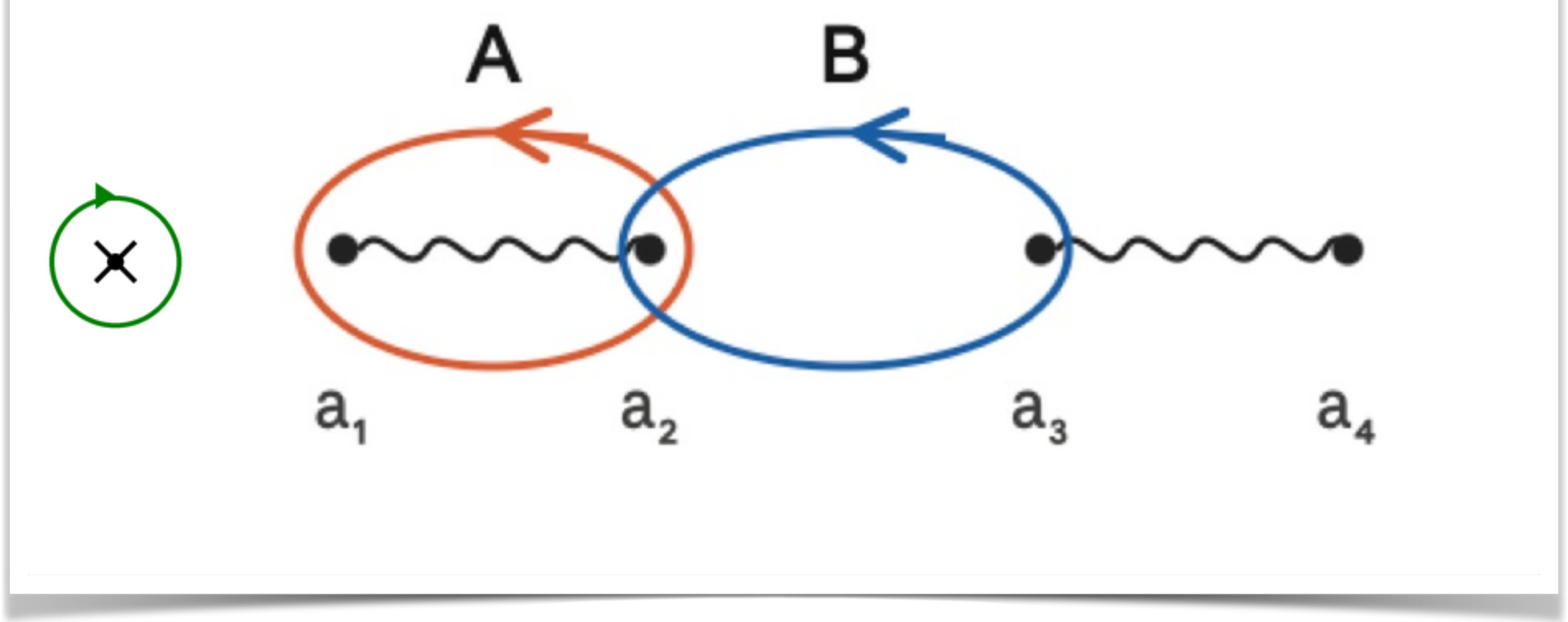
We want to normalize all leading singularities to constants:

- Taking the residue at $x \sim 0 \longrightarrow$ projects on the maximal cut $\quad LS[\tilde{I}_1] = \dfrac{z}{\sqrt{1-4z}}$

- If the integral over the remaining cycles does not vanish, we need to subtract a contribution proportional to the sunrise

G FUNCTIONS





Integrate over the A-cycle $\tilde{G} \sim z\left(z + 7z^2 + 45z^3 + \mathcal{O}(z^4)\right)$

Which satisfies the ***exact*** differential equation

$$(1-2z)\,\tilde{G}(z) - (1-4z)z\,\partial_z \tilde{G}(z) = -\frac{2}{3}\varpi_0 + \frac{5}{3}z\,\varpi_0'$$

$$G(z) \equiv \frac{\sqrt{1-4z}}{z}\tilde{G}(z) - \frac{5}{3\sqrt{1-4z}}\varpi_0(z) \qquad G'(z) = \frac{2(1+z)\varpi_0}{3z(1-4z)^{3/2}}$$

Same G function of the canonical basis

$$I = I_1 - \left(G(z) + \frac{5\varpi_0}{3\sqrt{1-4z}}\right)\frac{J_1}{\varpi_0}$$

$$\begin{cases} LS[I]_{x=0} = 1 + \mathcal{O}(\epsilon) \\ LS[I]_A = 0 + \mathcal{O}(\epsilon) \end{cases}$$

CONCLUSIONS

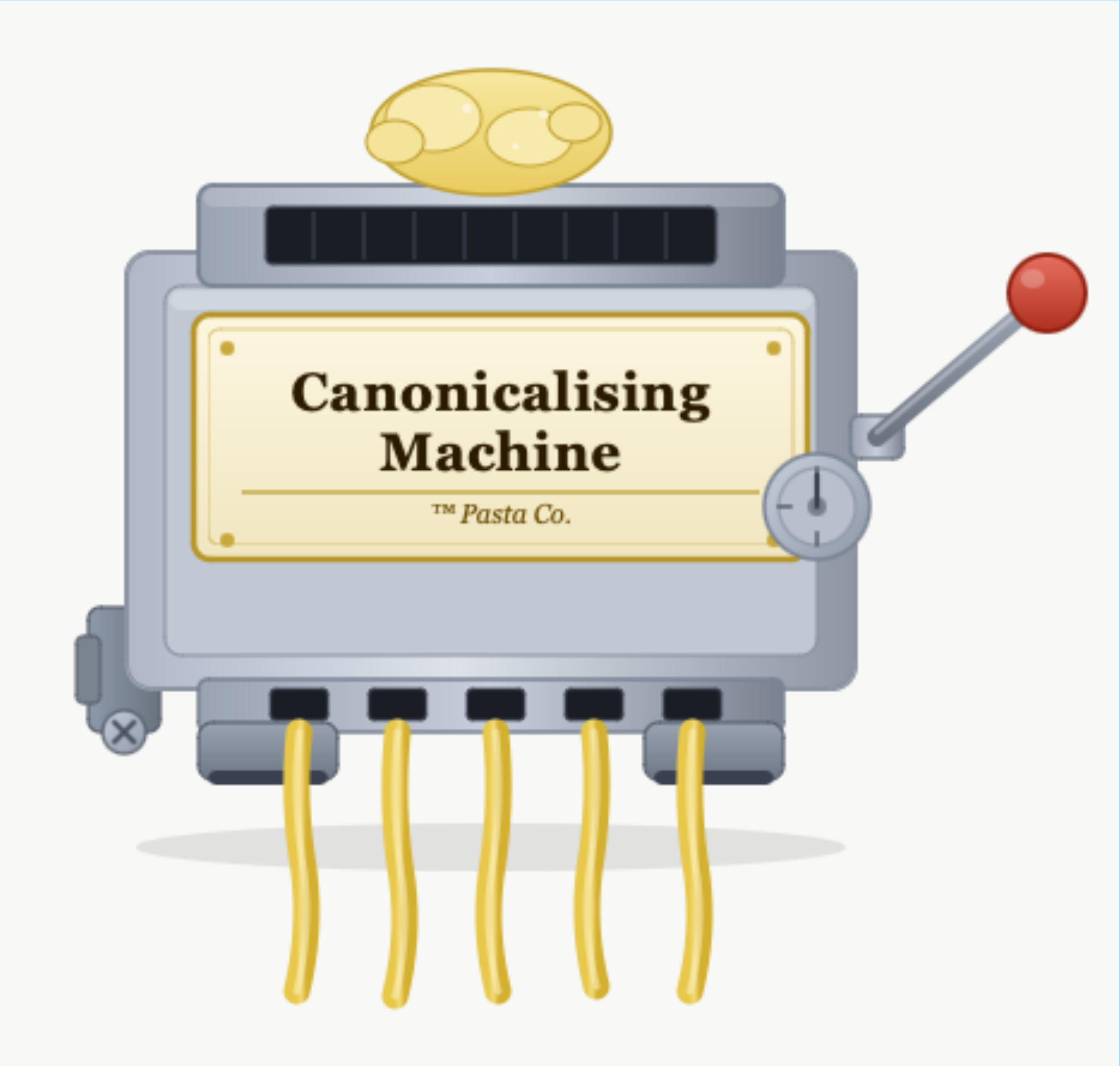
1. It is possible to generalise the concept of ***integral with unit leading singularity*** beyond polylogarithms

2. Choice of a starting basis has general recipe which ***does not depend on the particular geometry*** and only requires to find the correct integration form

3. Analysis at exactly $\epsilon = 0$ not enough!

4. Forms of the the ***third kind*** imply the appearance of new independent G functions (also happen on the maximal cut!)

$$d\vec{I}(s_{ij}) = M(\epsilon, s_{ij}) \cdot \vec{I}$$



$$d\vec{I}(s_{ij}) = \epsilon\, M(s_{ij}) \cdot \vec{I}$$

THANK YOU !

BACKUP

ELLIPTICS: a toy example, the Wronksian method

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} \frac{1}{\varpi_0} & 0 \\ -\frac{\varpi_0}{\Delta(a)} & \frac{\varpi_0}{\Delta(a)} \end{pmatrix} \begin{pmatrix} I_1 \\ \partial_a I_1 \end{pmatrix}$$

$$W(a) = \begin{pmatrix} \varpi_0 & \varpi_1 \\ \varpi_0' & \varpi_1' \end{pmatrix} = \underbrace{\begin{pmatrix} \varpi_0 & 0 \\ \varpi_0' & \frac{\Delta(a)}{\varpi_0} \end{pmatrix}}_{W_{ss}} \underbrace{\begin{pmatrix} 1 & \frac{\varpi_1}{\varpi_0} \\ 0 & 1 \end{pmatrix}}_{W_u}$$

First step of Wronskian method

[Görges, Nega, Tancredi, Wagner ’23]

[Duhr, Maggio, Nega, Tancredi, Wagner ’25]

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix} \cdot W_{ss}^{-1} \cdot \begin{pmatrix} I_1 \\ \partial_a I_1 \end{pmatrix}$$

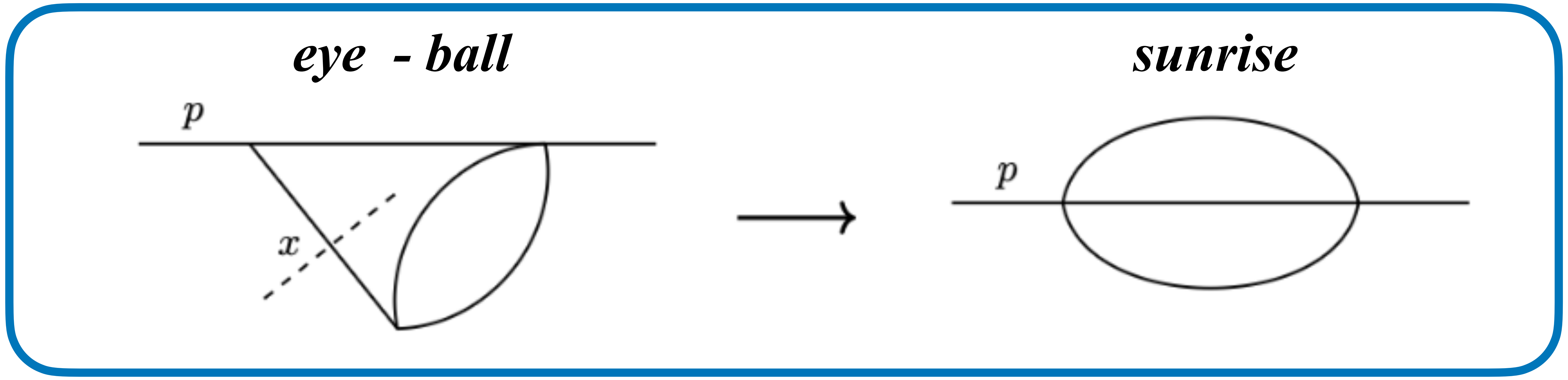
Rotate by inverse semi-simple part of Wronskian

Followed by a ***clean-up step***

$$K_2 = \frac{\varpi_0^2}{\Delta(a)} \left(\frac{1}{\epsilon} \partial_a J_1 \right) - \varpi_0^2 J_1$$

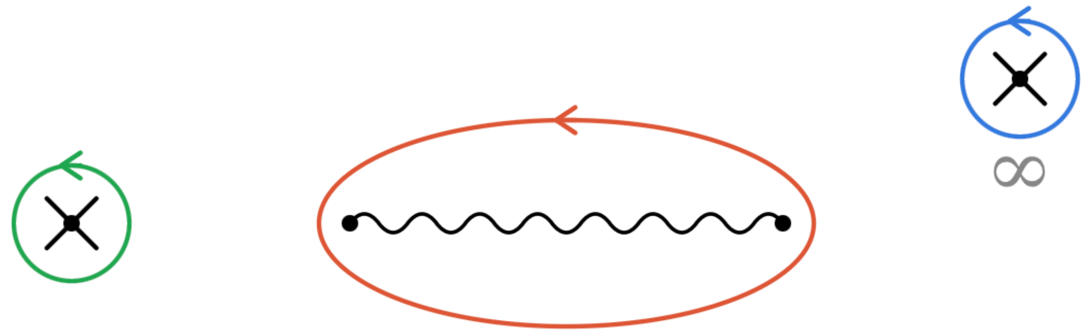
We now understand it as ***fixing leading singularity one order higher in ϵ!***

G FUNCTIONS



To evaluate over the A-cycle, expand the integrand in powers of z

$$\tilde{G} \sim \int dX \frac{1 + (1+X)z + (1 + 4X + X^2)z^2 + (1 + 9X + 9X^2 + X^3)z^3 + \mathcal{O}(z^4)}{(1-X)\sqrt{X(4-X)}}$$



$$x = X - 1$$

Expand the integrand in powers of z → collapse the quartic root into a quadratic one

Theres is an extra pole at infinity: upon taking the residue $\tilde{G} \sim z\left(z + 7z^2 + 45z^3 + \mathcal{O}(z^4)\right)$

By computing enough order in the expansion we find the exact differential equation

$$(1-2z)\,\tilde{G}(z) - (1-4z)z\,\partial_z \tilde{G}(z) = -\frac{2}{3}\varpi_0 + \frac{5}{3}z\,\varpi_0'$$

A K_3 example

$$I^C_{n,m}(a,\epsilon) = \int_C \mathrm{d}x\mathrm{d}y\,\frac{x^n y^m}{\sqrt{x(1-x)y(1-y)(1-a\,xy)}}(x(1-x)y(1-y)(1-a\,xy))^\epsilon$$

K_3 - manifold

$$\frac{\partial}{\partial a}\begin{pmatrix} I_{0,0} \\ I'_{0,0} \\ I''_{0,0} \end{pmatrix} = \left(A_0 + \epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3\right)\begin{pmatrix} I_{0,0} \\ I'_{0,0} \\ I''_{0,0} \end{pmatrix}$$

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{8a^2(1-a)} & \frac{4-13a}{4a^2(a-1)} & \frac{6-9a}{2a(a-1)} \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4a^2(1-a)} & \frac{2(2-a)}{a^2(a-1)} & \frac{4-a}{a(a-1)} \end{pmatrix}$$

etc...

A K_3 example

$$I^C_{n,m}(a,\epsilon) = \int_C \mathrm{d}x\mathrm{d}y \, \frac{x^n y^m}{\sqrt{x(1-x)y(1-y)(1-a\,xy)}}(x(1-x)y(1-y)(1-a\,xy))^\epsilon$$

K_3 - manifold

After semi-simple splitting

$$J_1 = \frac{1}{\varpi_0} I_{0,0}$$

1st candidates is a form of the first kind normalised by its leading sing.

$$J_2 = \frac{1}{\epsilon}\frac{\varpi_0^2}{\Delta_{01}}\partial_a J_1 = \frac{1}{\epsilon}\left(-\frac{\varpi_0'}{\Delta_{01}} I_{0,0} + \frac{\varpi_0}{\Delta_{01}} I'_{0,0}\right)$$

2nd candidate is a form of second kind, corrected by its leading sing.

$$J_3 = \frac{1}{\epsilon}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a J_2 = \frac{1}{\epsilon^2}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a\left(\frac{\varpi_0^2}{\Delta_{01}}\partial_a J_1\right)$$

3rd candidate is a form of second kind, corrected by its leading sing.

A K_3 example

$$J_1 = \frac{1}{\varpi_0} I_{0,0}$$

1st candidates is a form of the first kind normalised by its leading sing.

$$J_2 = \frac{1}{\epsilon}\frac{\varpi_0^2}{\Delta_{01}}\partial_a J_1 = \frac{1}{\epsilon}\left(-\frac{\varpi_0'}{\Delta_{01}}I_{0,0} + \frac{\varpi_0}{\Delta_{01}}I_{0,0}'\right)$$

2nd candidate is a form of second kind, corrected by its leading sing.

$$J_3 = \frac{1}{\epsilon}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a J_2 = \frac{1}{\epsilon^2}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a\left(\frac{\varpi_0^2}{\Delta_{01}}\partial_a J_1\right)$$

3rd candidate is a form of second kind, corrected by its leading sing.

$$\frac{\partial}{\partial a}\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix} = \left[\frac{1}{\epsilon}B_{-1} + B_0 + \epsilon B_1\right]\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix}$$

Need to fix second and third candidates: they have weight drops, semi - simple splitting is fixing only the **leading - leading singularities**

A K_3 example

$$J_3 = \frac{1}{\epsilon}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a J_2 = \frac{1}{\epsilon^2}\left[\partial_a\left(\frac{\Delta_{02}}{\Delta_{01}}\right)\right]^{-1}\partial_a\left(\frac{\varpi_0^2}{\Delta_{01}}\partial_a J_1\right)$$

$$L_3 = J_3 - \frac{1}{\epsilon}\left(-\frac{1}{6}a\,(4-a)\,\varpi_0\varpi_0' - \frac{a\,(2+a)\,\varpi_0^2}{12(1-a)} + G_1(a)\right)J_1 + \cdots$$

$$G_1(a) = \frac{1}{8}\int \mathrm{d}a\,\frac{(1+a)\varpi_0^2}{(1-a)^2}$$

$$K_2 = J_2 - \left(-\frac{4-a}{3\sqrt{1-x}}\varpi_0 + 2\,G_2(a)\right)J_1$$

$$G_2(a) = \int \mathrm{d}a\,\frac{1}{a\sqrt{1-a}}\,\frac{G_1}{\varpi_0}$$

New independent functions! They end up in the differential equation

A K_3 example

$$\frac{\partial}{\partial a}\begin{pmatrix} J_1 \\ K_2 \\ L_3 \end{pmatrix} = \epsilon\, M \begin{pmatrix} J_1 \\ K_2 \\ L_3 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{\sqrt{1-a}G_2}{(1-a)a\varpi_0} - \frac{4-a}{3(1-a)a} & \frac{1}{\sqrt{1-a}a\varpi_0} & 0 \\ \frac{\left(4+a+4a^2\right)\varpi_0}{6(1-a)^{3/2}a} - \frac{3G_2^2}{2\sqrt{1-a}a\varpi_0} & -\frac{2G_2}{\sqrt{1-a}a\varpi_0} - \frac{4-a}{3(1-a)a} & \frac{2}{\sqrt{1-a}a\varpi_0} \\ \frac{\left(4+a+4a^2\right)G_2\varpi_0}{6(1-a)^{3/2}a} + \frac{\left(16-75a+21a^2-16a^3\right)\varpi_0^2}{54(1-a)^2a} - \frac{G_2^3}{2\sqrt{1-a}a\varpi_0} & \frac{\left(4+a+4a^2\right)\varpi_0}{12(1-a)^{3/2}a} - \frac{3G_2^2}{4\sqrt{1-a}a\varpi_0} & -\frac{\sqrt{1-a}G_2}{(a-1)a\varpi_0} - \frac{4-a}{3(1-a)a} \end{pmatrix}$$