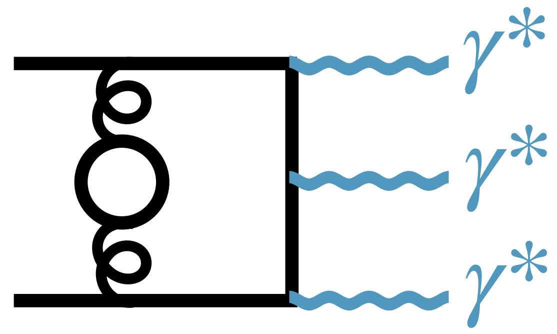
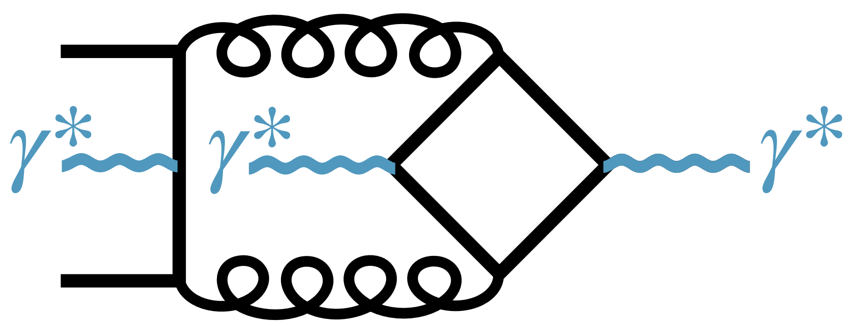
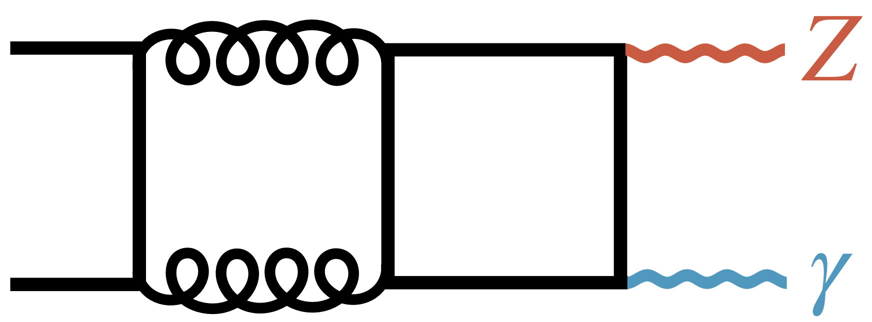
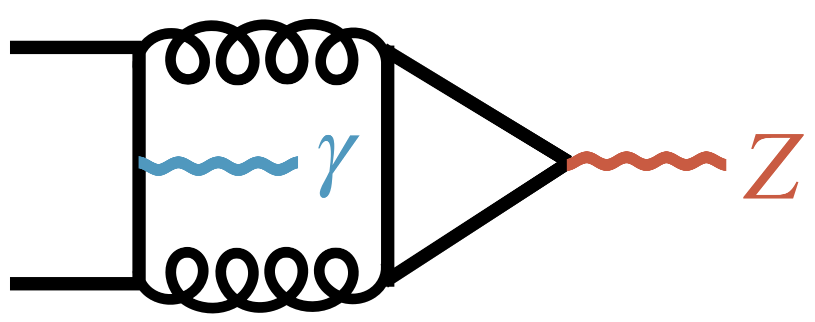
Off-shell triphoton production via quark loops from numerical integration

On 2407.18051, 2510.18801, 2603.03169, 2603.03171 in collaboration with Matilde Vicini







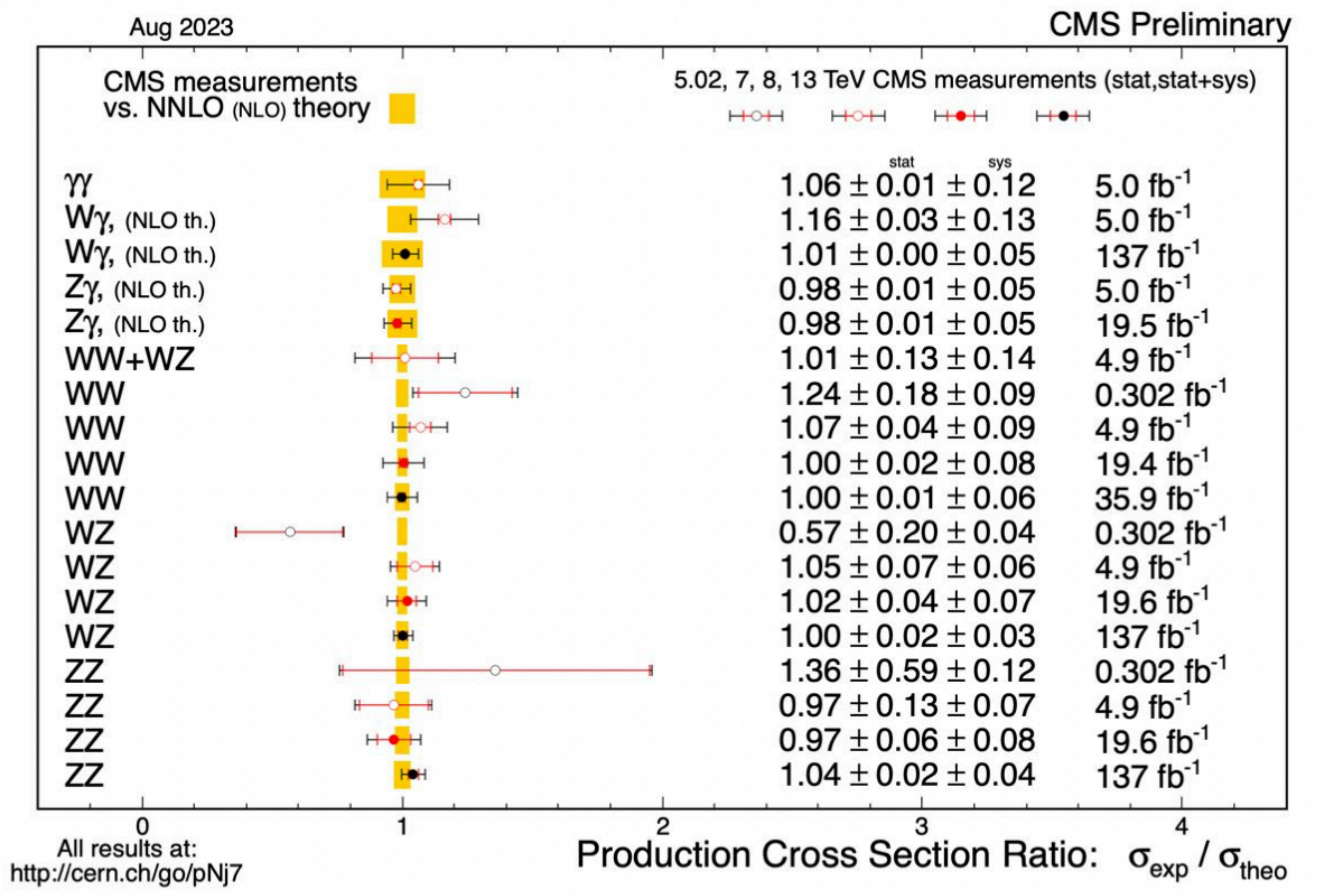


Dario Kermanschah

LoopFest XXIV, Brookhaven National Laboratory, 28 May 2026

Multi-boson production at the LHC?

- Diboson production measured to few percent uncertainty



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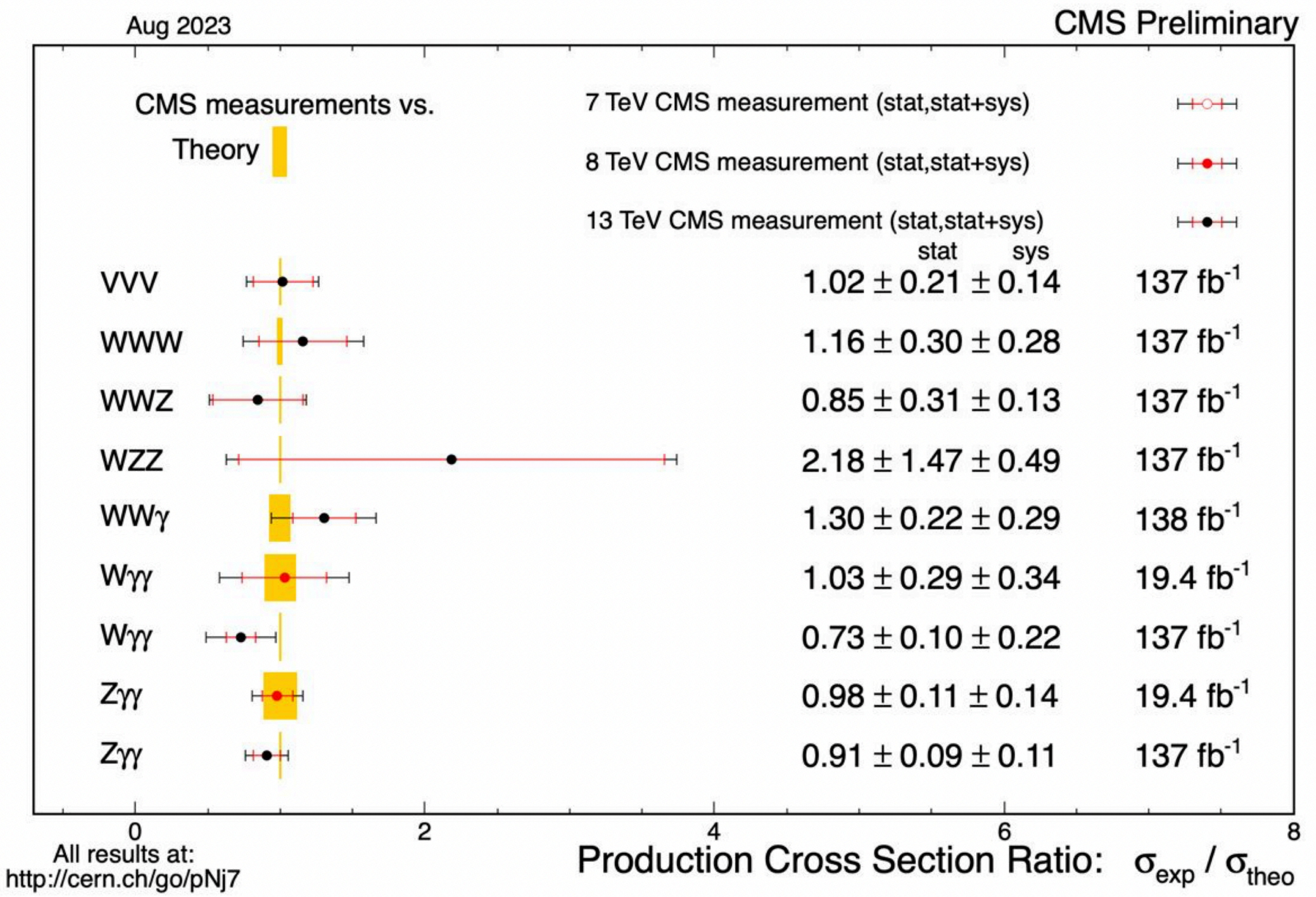
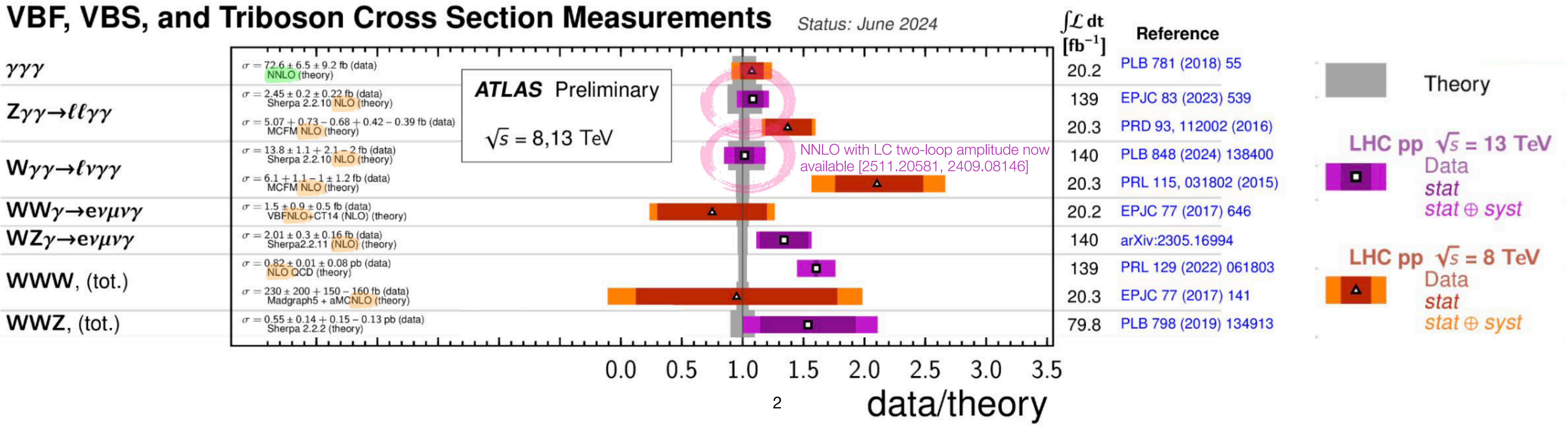
- New observations of rare triboson processes
- No $2 \to 3$ NNLO calculations with multiple massive bosons
- Bottleneck is availability of two-loop amplitudes

  Double-real and real-virtual known [MATRIX, q_T-subtraction, OpenLoops]





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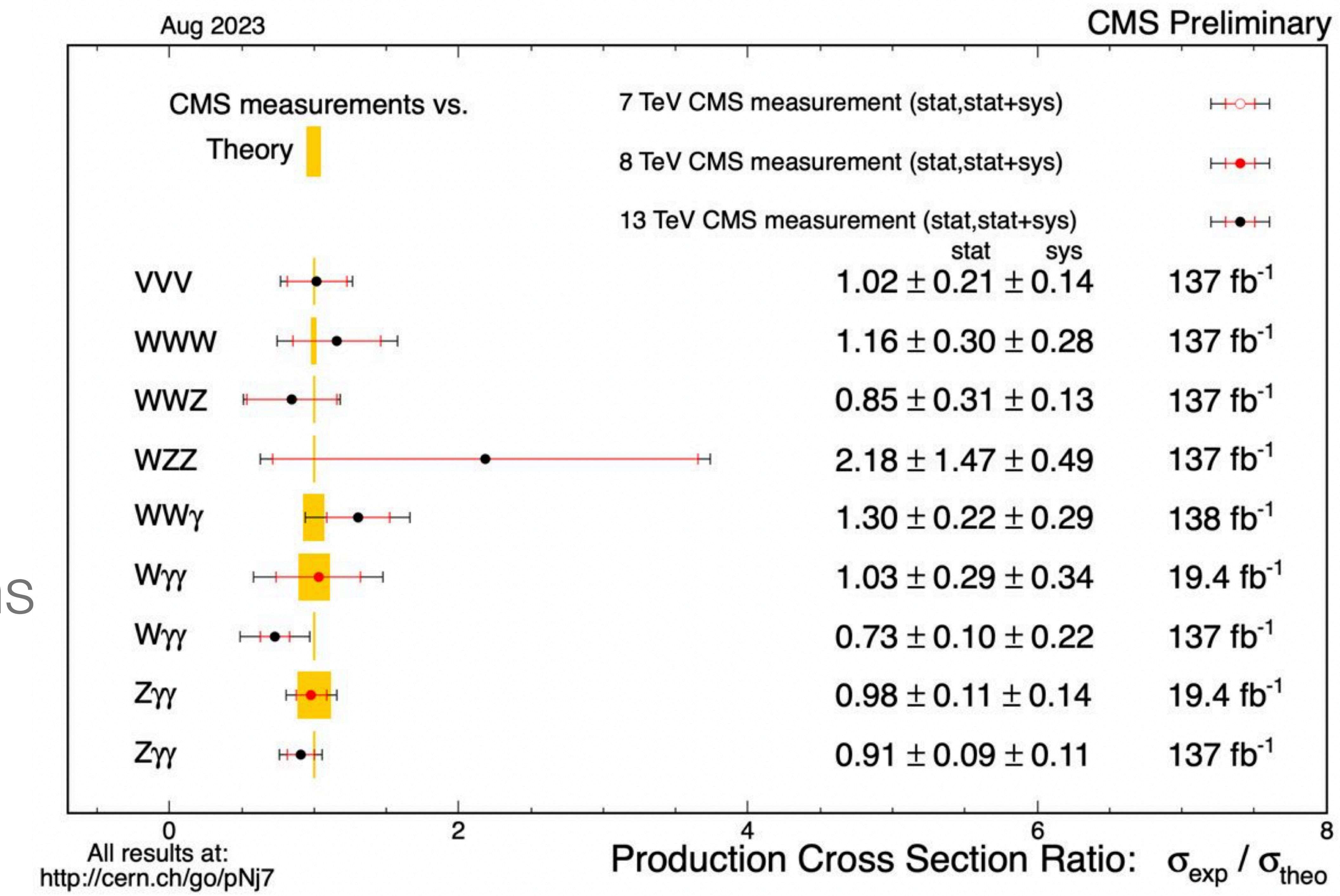
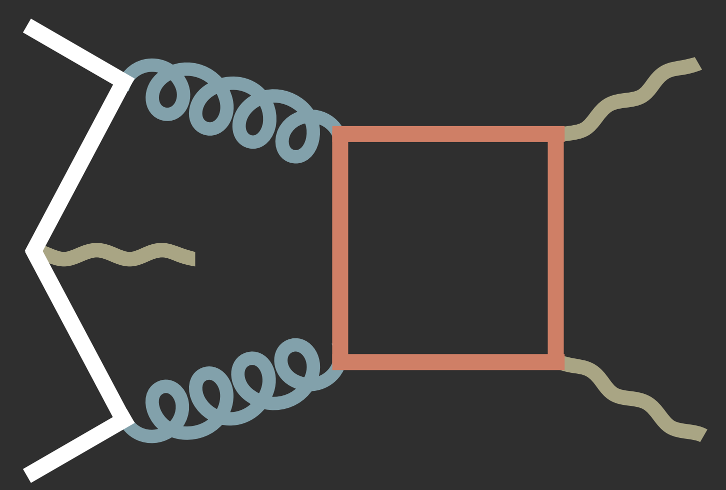


Break new ground in multi-scale multi-loop calculations!

- Two-loop 5-point integrals: planar, non-planar, light- & heavy-quark loops, IR, UV and threshold singularities, …
- Intractable IBP reduction & unknown Master Integrals
- Breakthroughs in analytic methods needed… Resort to numerical methods!



Our approach for the two-loop virtual corrections:
Local subtraction & direct numerical integration

finite remainder: $R^{(2)} = M^{(2)} \underbrace{-\frac{2\beta_0}{\epsilon} M^{(1)}}_{\text{UV renorm.}} \underbrace{- \mathbf{Z}^{(1)} M^{(1)} - \mathbf{Z}^{(2)} M^{(0)}}_{\text{Catani IR poles}}$

Anastasiou, Haindl, Karlen, Sahoo
Sterman, Venkata, Vicini, Yang, Zeng
[2603.12862, 2601.22936, 2509.07805, 2403.13712, 2212.12162, 2008.12293, 1812.03753]

$C_{\text{IR\&UV}}$ from local factorisation

hard scattering amplitude

$$M^{(2)}_{\text{hard}} = M^{(2)} - C_{\text{IR\&UV}}$$

- finite in $D = 4$ dimensions, no dim reg. ($\gamma^5 \ldots$)
- integrate numerically with Monte Carlo methods
- directly in momentum space
- no IBPs, no Master integrals, no sector decomposition

renormalisation & factorisation scheme change

$$+ C_{\text{IR\&UV}} - \frac{2\beta_0}{\epsilon} M^{(1)} - \mathbf{Z}^{(1)} M^{(1)} - \mathbf{Z}^{(2)} M^{(0)}$$

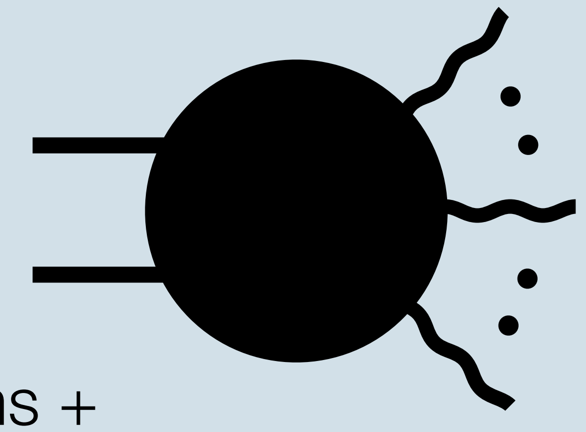
calculate analytically in $D = 4 - 2\epsilon$ dimensions

$$= c_1 M^{(1)}_{\text{hard}} + c_0 M^{(0)}$$

interfere with tree & integrate over phase space

to get the virtual corrections: $\int \mathrm{d}\Pi \sum_{\text{hel.}} |M|^2$

Local singularities of finite loop integrals

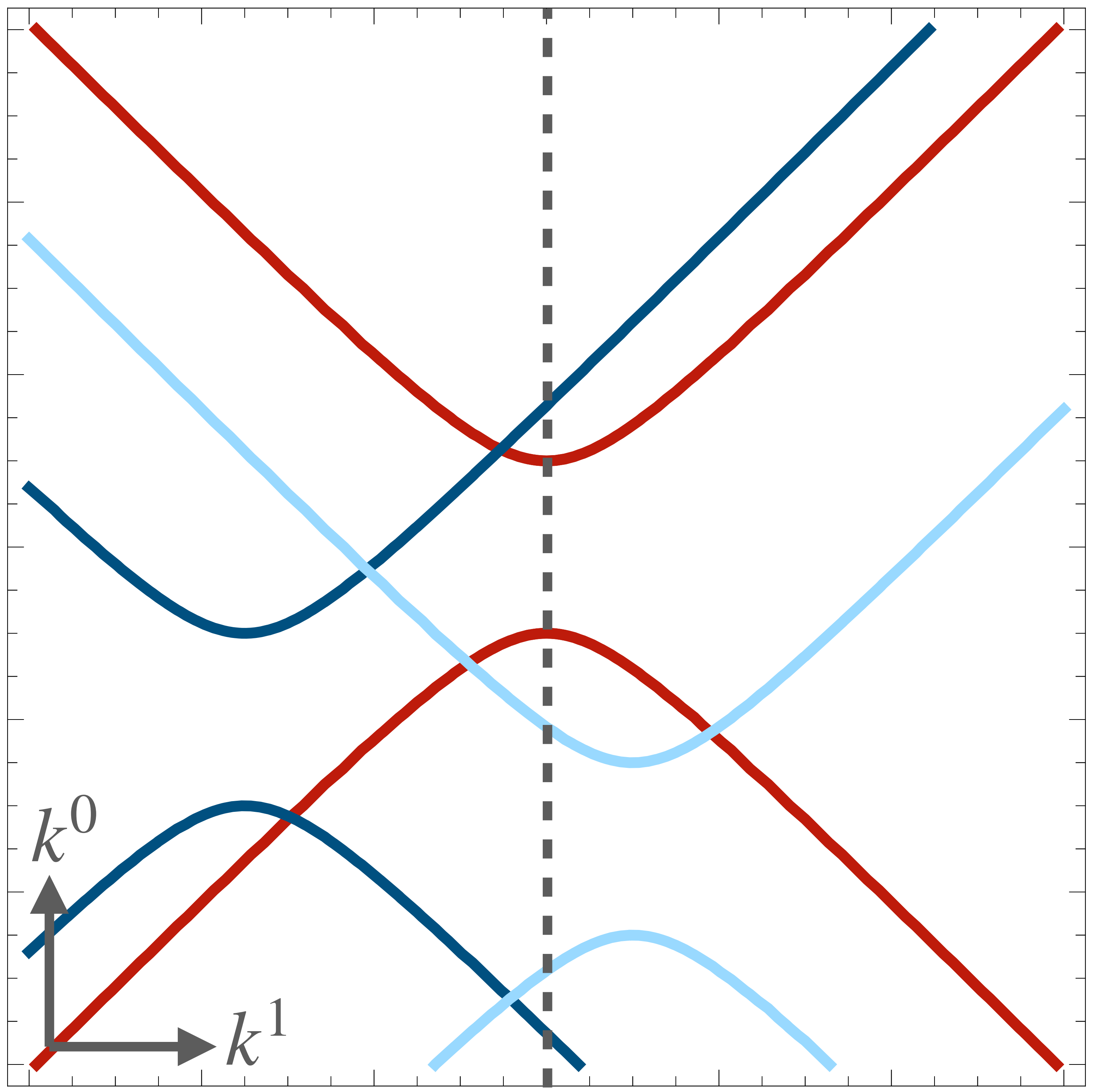
$$M_{\text{hard}} = \sum$$ 

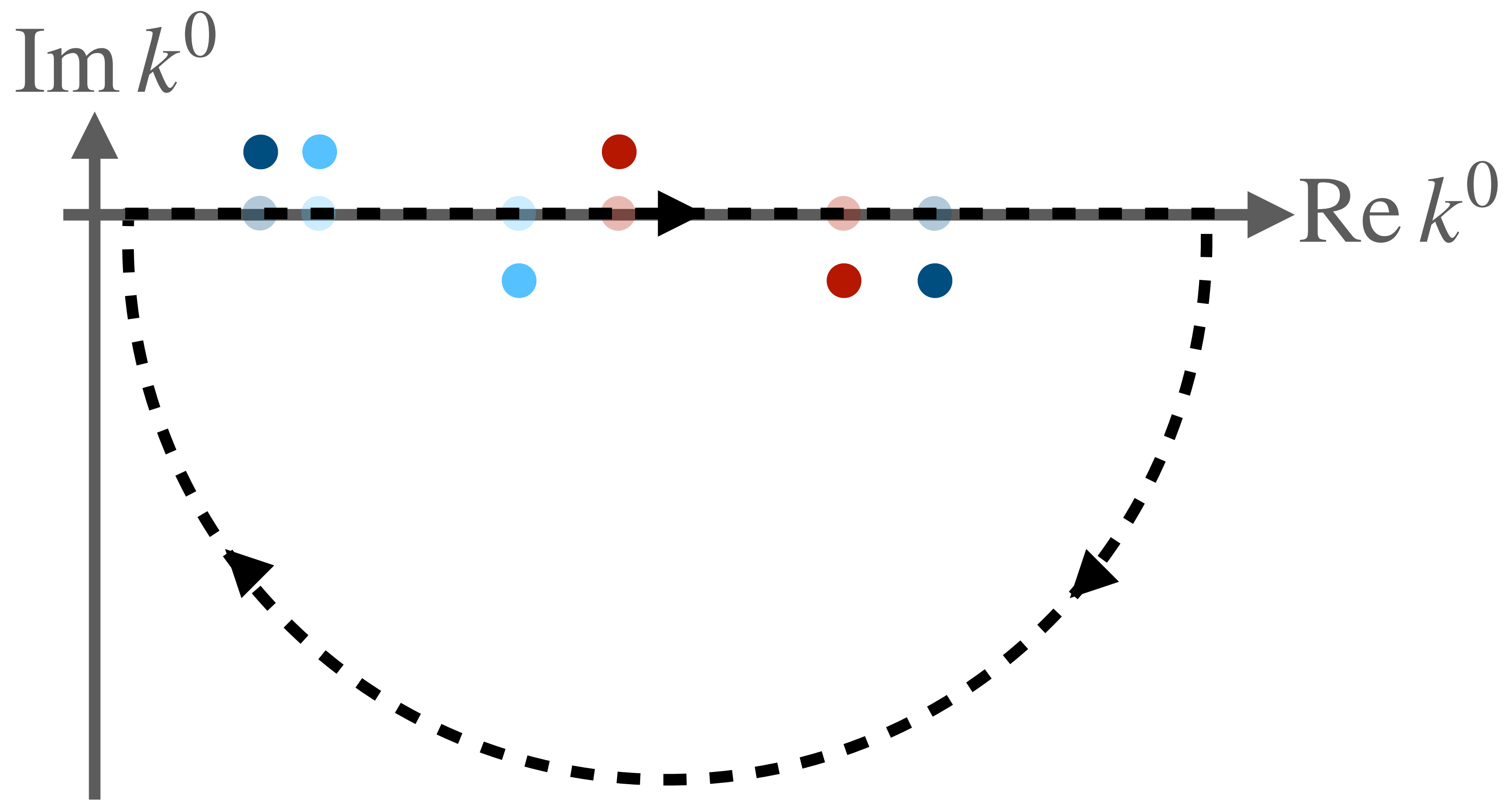
Feyn. diagrams +
local IR & UV CTs

$$= \lim_{\epsilon \to 0} \int [\mathrm{d}^4 k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$$

- ↯ poles in the integration domain
- ✔ causal prescription
- ⚠️ implement causal prescription for numerical integration

  → analytic integration over k^0





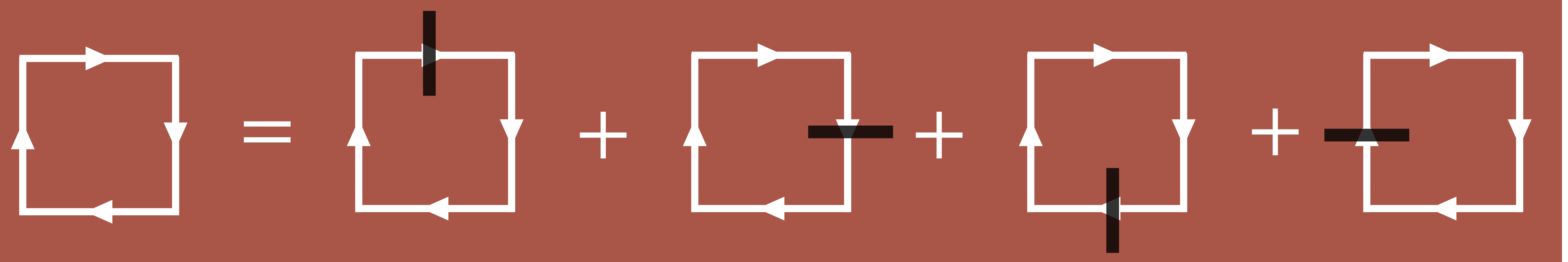
Loop integrals are rational functions in the energy component of the loop momentum

→ integrate using the residue theorem: from D to $D-1$ integration dimensions per loop

Catani, Rodrigo et al. [0804.3170], ETHZ [1906.06138], Mainz [1902.02135], Valencia [2001.03564, 2010.12971]

Loop-Tree Duality

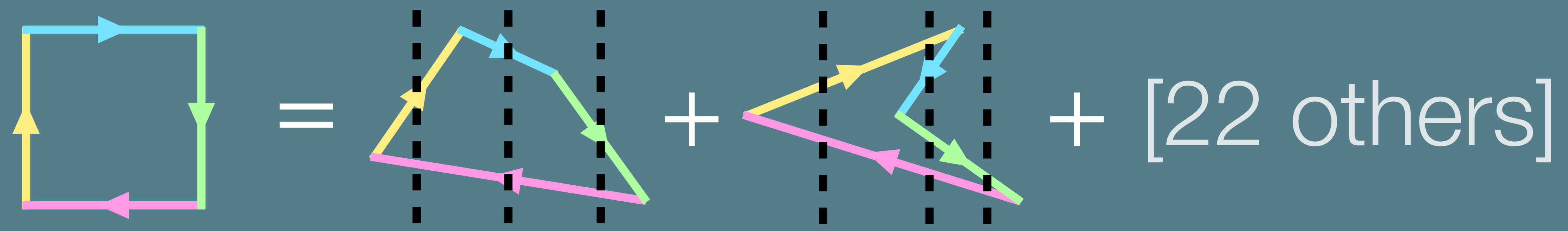
compact expression but problematic spurious singularities and derivatives for raised propagators



cf. Collins, Soper, Sterman ’85, …

Time-Ordered Perturbation Theory

still some spurious singularities and more terms

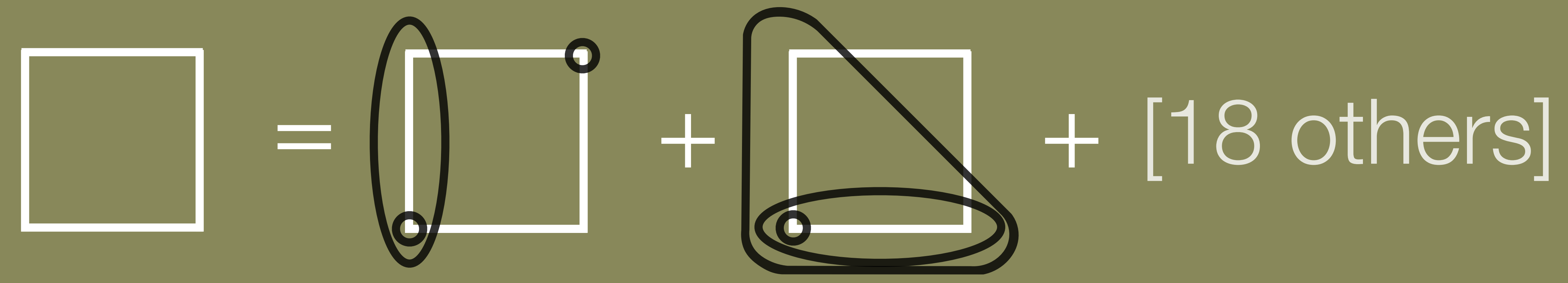


Valencia [2006.11217, 2112.09028, 2103.09237, 2102.05062]
ETHZ [2009.05509], Mainz [2208.01060 Stony Brook [2309.13023]

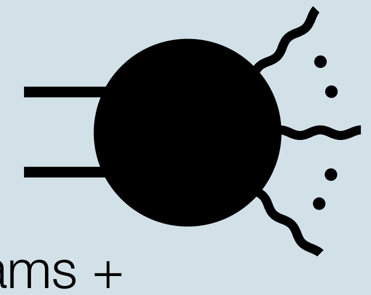
Causal representations

CFF: Capatti [2211.09653]

no spurious singularities



Threshold singularities

$M_{\text{hard}} = \sum$  $= \lim_{\epsilon \to 0} \int [\mathrm{d}^4 k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$

Feyn. diagrams +
local IR & UV CTs

$$= \lim_{\epsilon \to 0} \int [\mathrm{d}^3 \vec{k}] \sum \cdots \frac{\cdots}{E_1 + E_2 - p^0 - \mathrm{i}\epsilon} \cdots$$

$$E_i = \sqrt{\vec{q}_i^{\,2} + m_i^2}$$

- ↯ poles in the integration domain
- ✔ causal prescription
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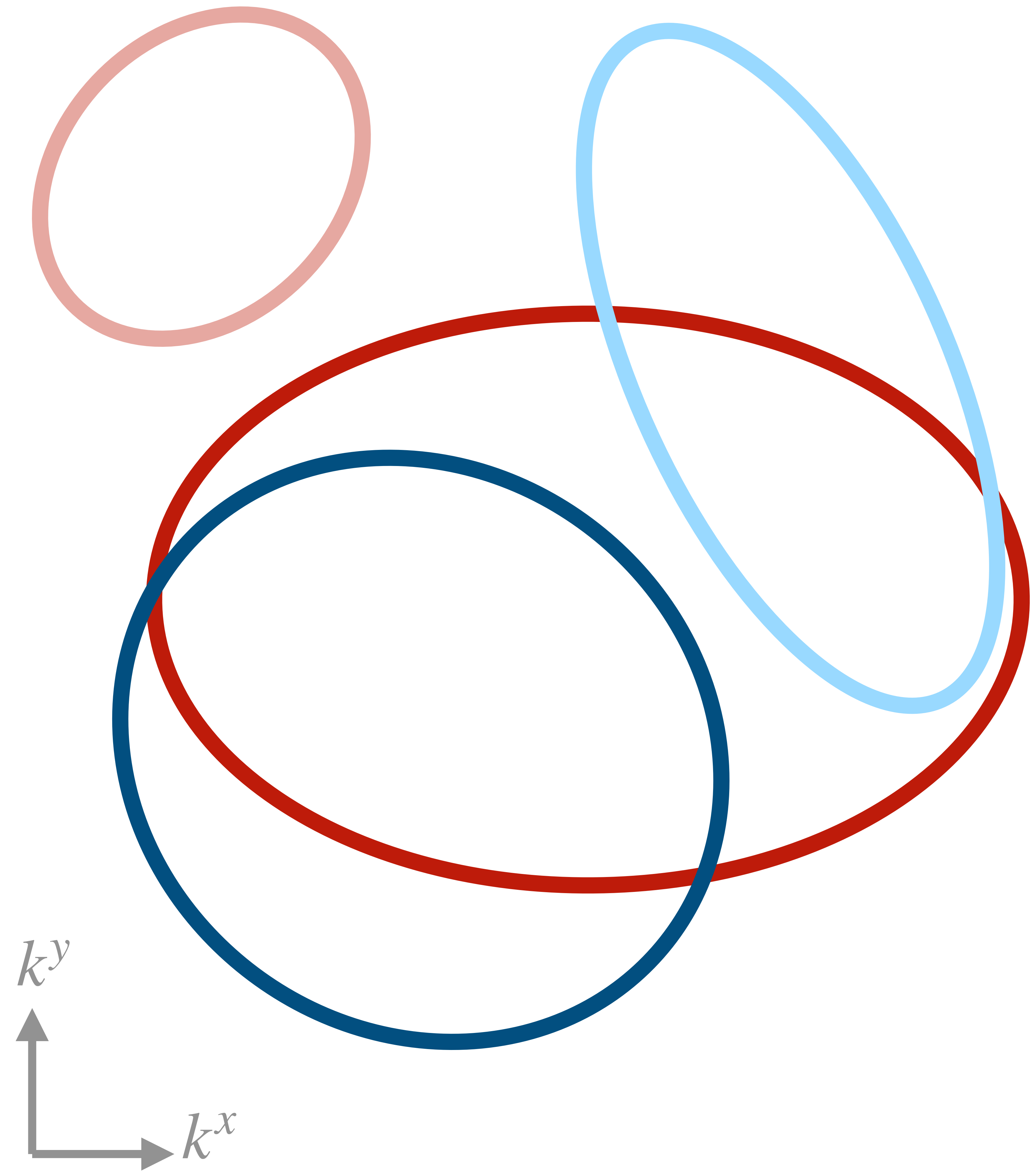
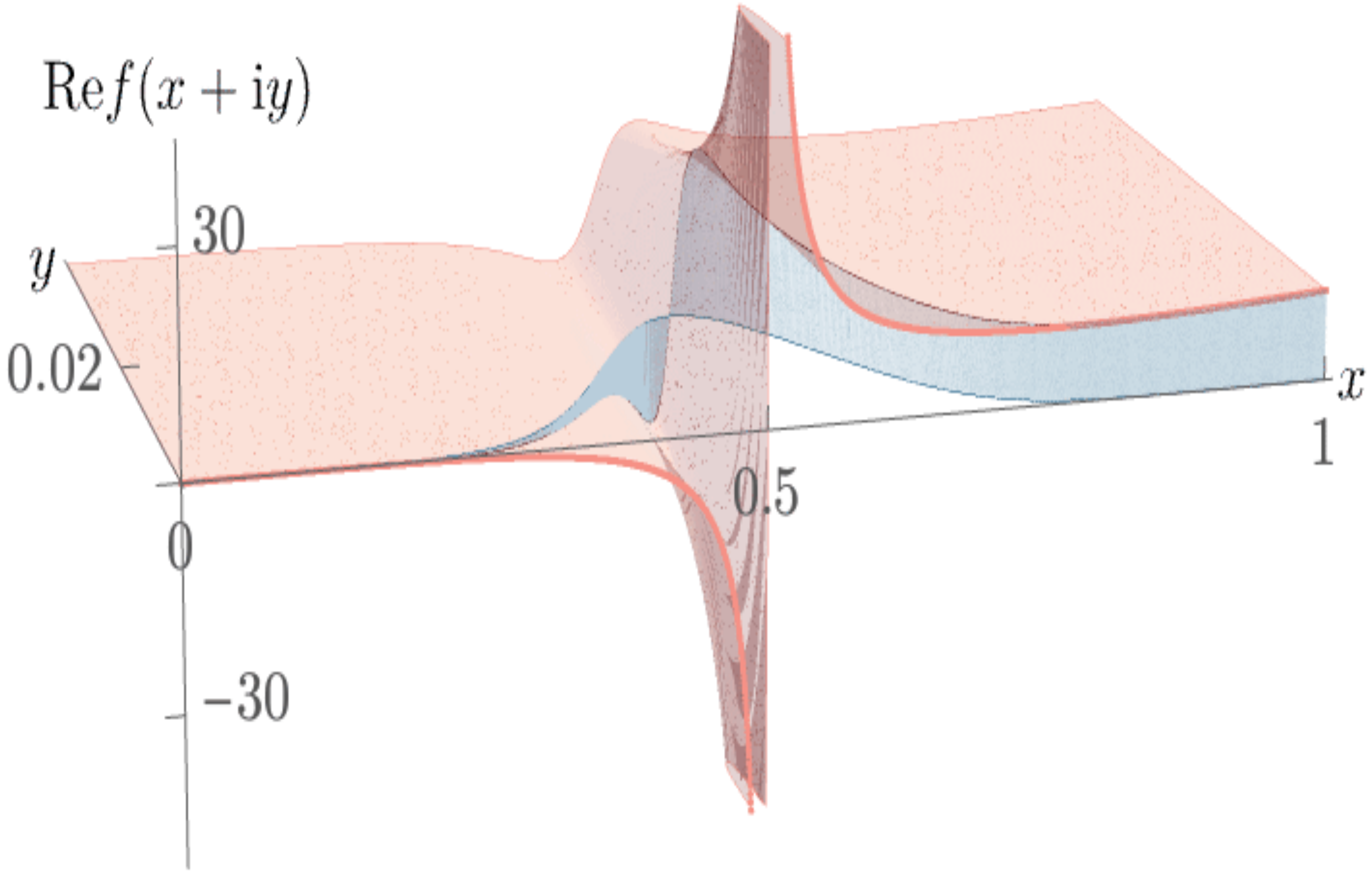
→ Same problems? Yes but fewer integration dimensions & fewer integrand singularities in compact region!



contour deformation

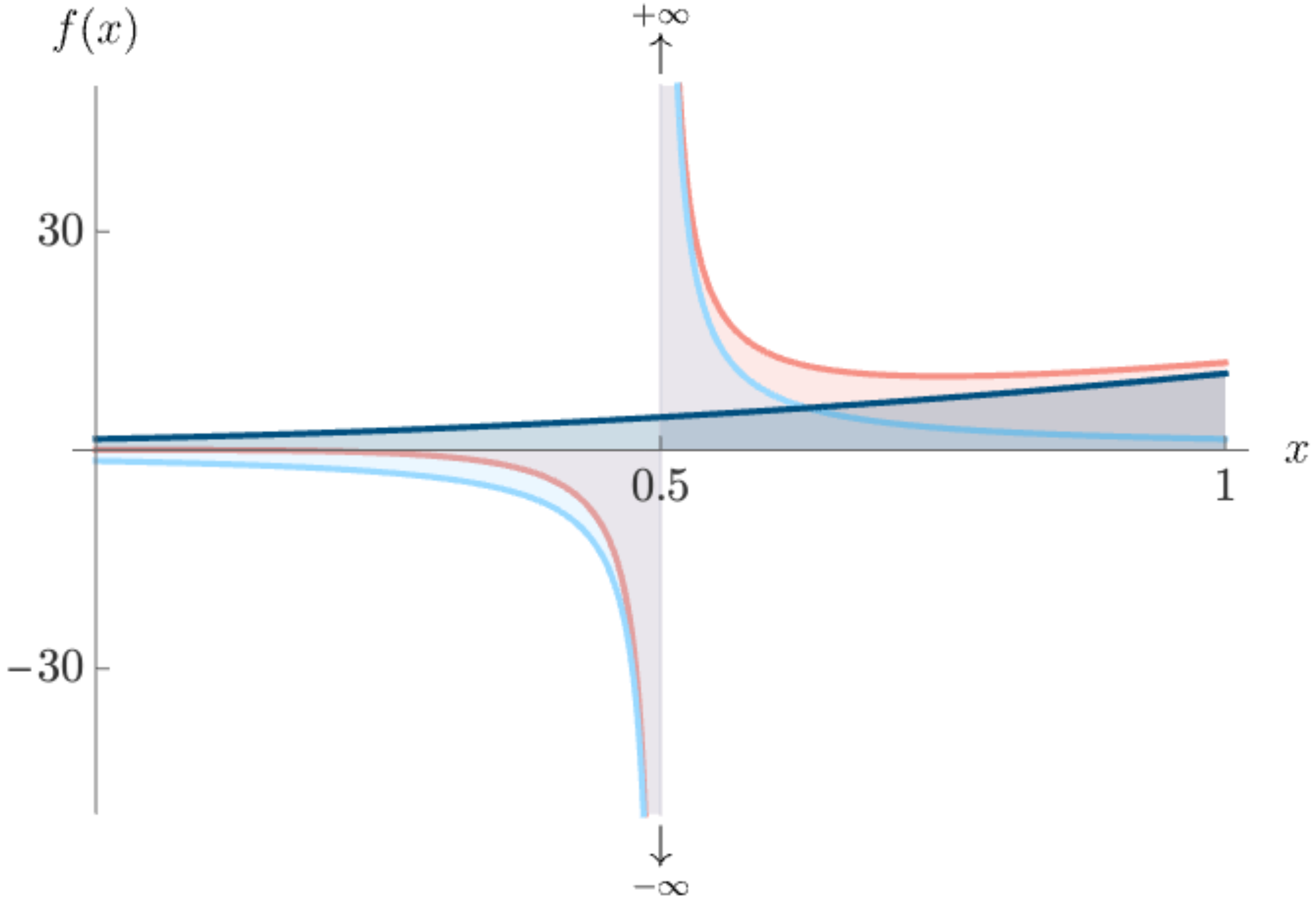
$$\mathbb{R} \to \mathbb{C}$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} \mathrm{Re} f(z(x_i)) j_z(x_i) = 4.9948$$

subtraction

$$\frac{1}{x - x_0 + \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x - x_0} - \mathrm{i}\pi\delta(x_0)$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} \left(f(x_i) - f_{\mathrm{ct}}(x_i)\right) = 5.0008$$

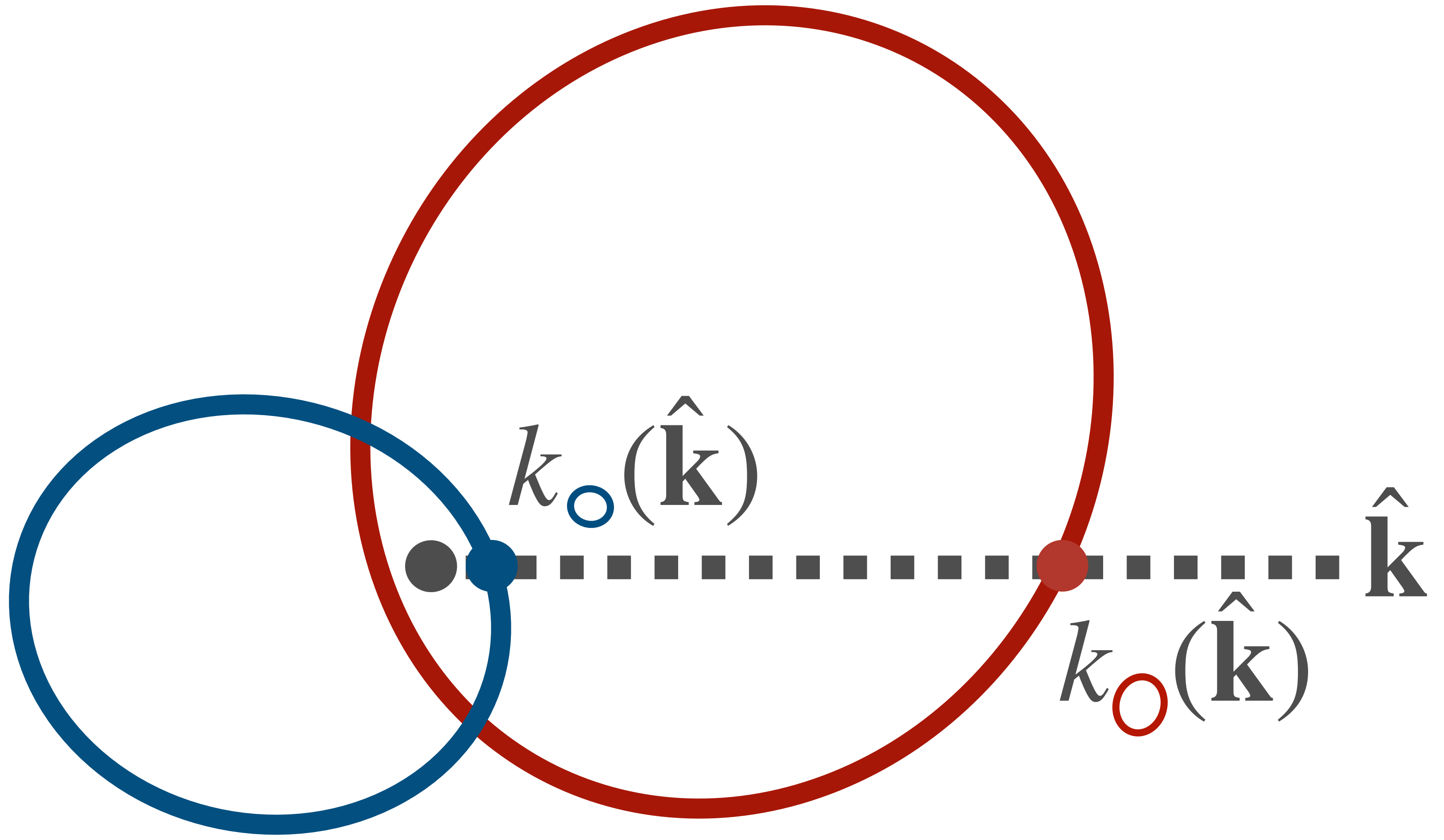
DK: [2110.06869], DK, Matilde Vicini [2407.21511]
Kilian, Kleinschmidt [0912.3495]

Subtraction of threshold singularities

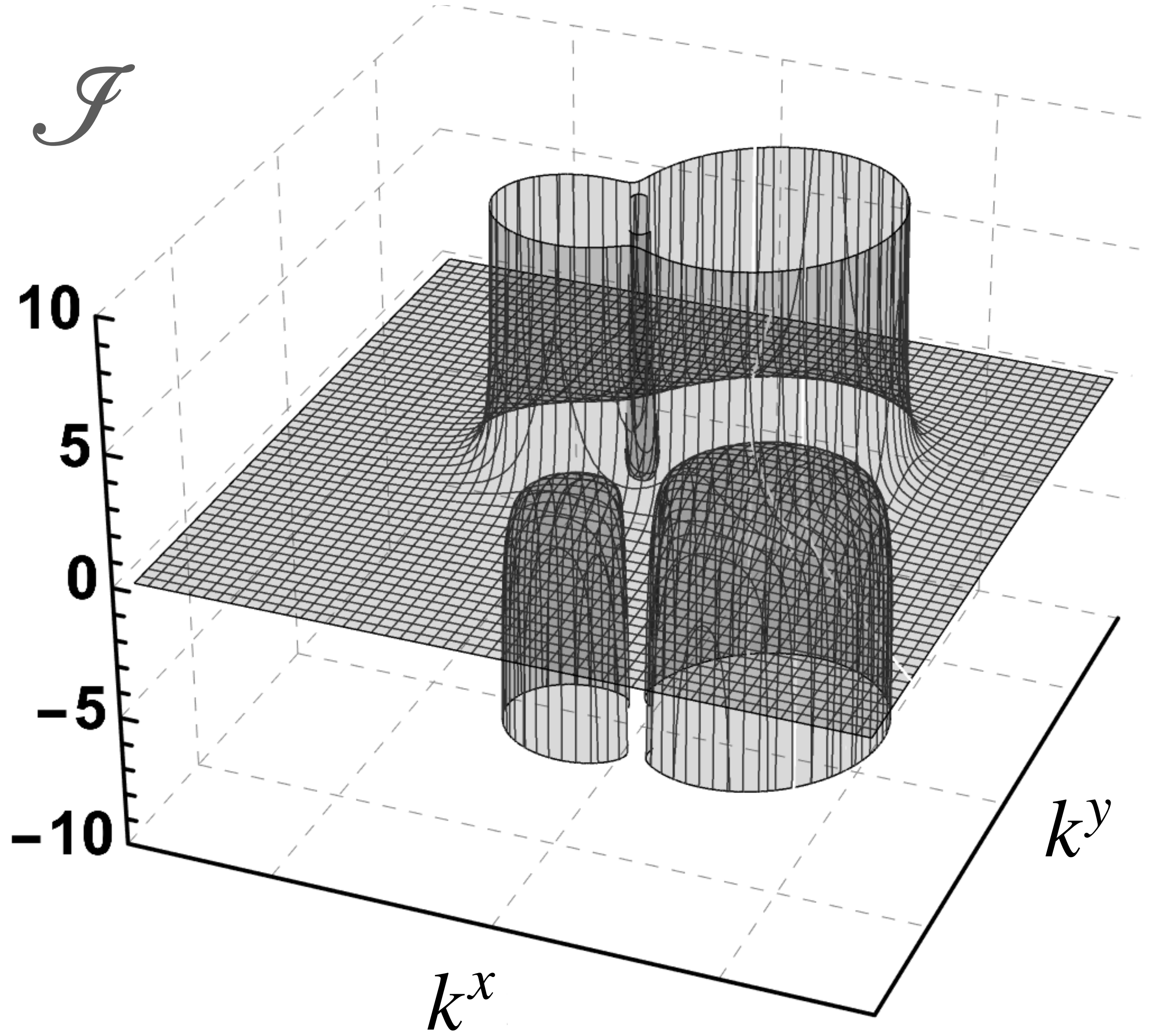
around a threshold the integrand behaves as

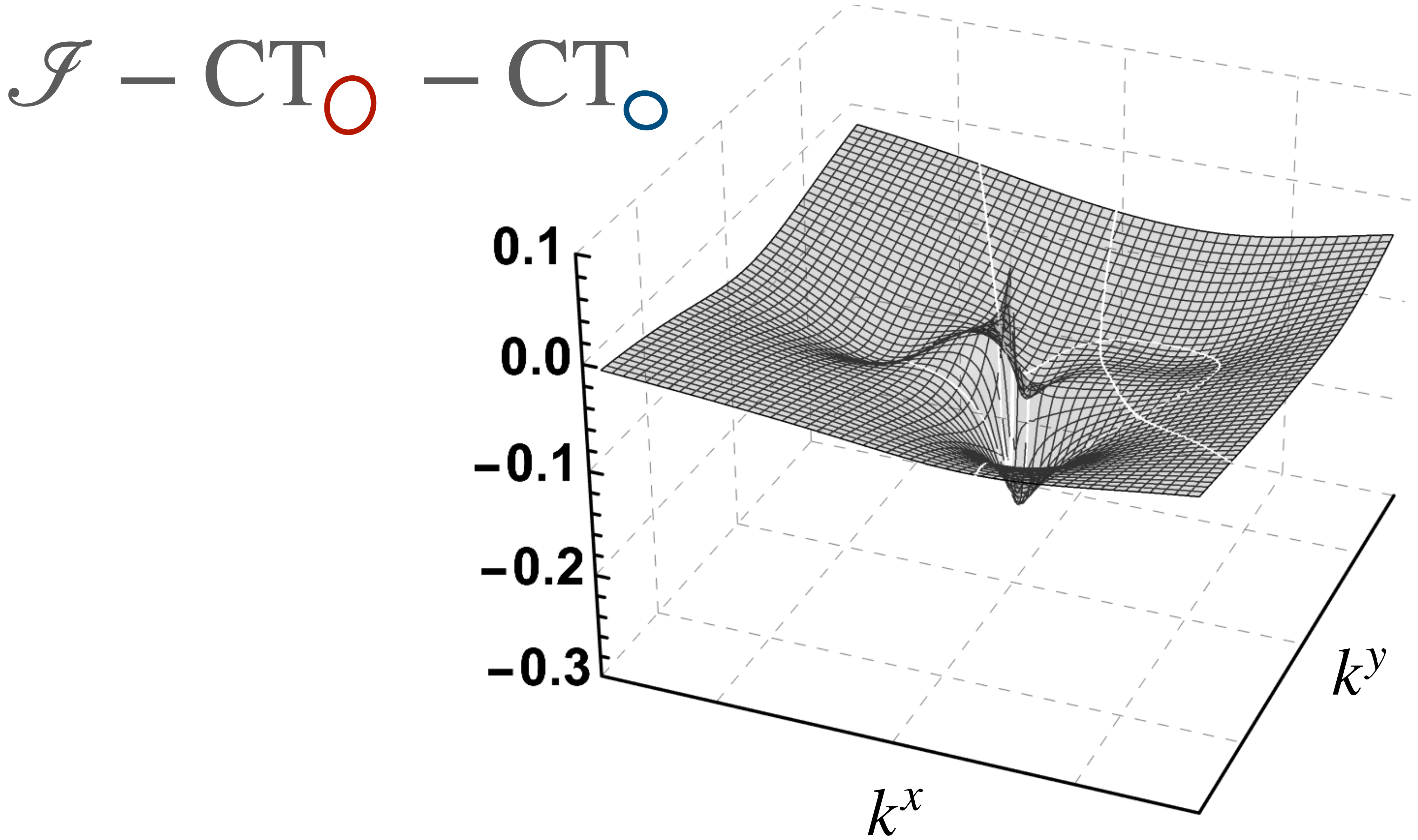
$$\mathcal{I} \sim \frac{\mathrm{Res}_i\,\mathcal{I}}{|\mathbf{k}| - k_i(\hat{\mathbf{k}}) - \mathrm{i}\epsilon} \to \mathrm{CT}_i \text{ threshold counterterm}$$

$$\mathrm{Re}\, I = \int \mathrm{d}^{3n}\mathbf{k} \left(\mathcal{I} - \sum_i \mathrm{CT}_i \right)$$ dispersive part



well suited for numerical integration!





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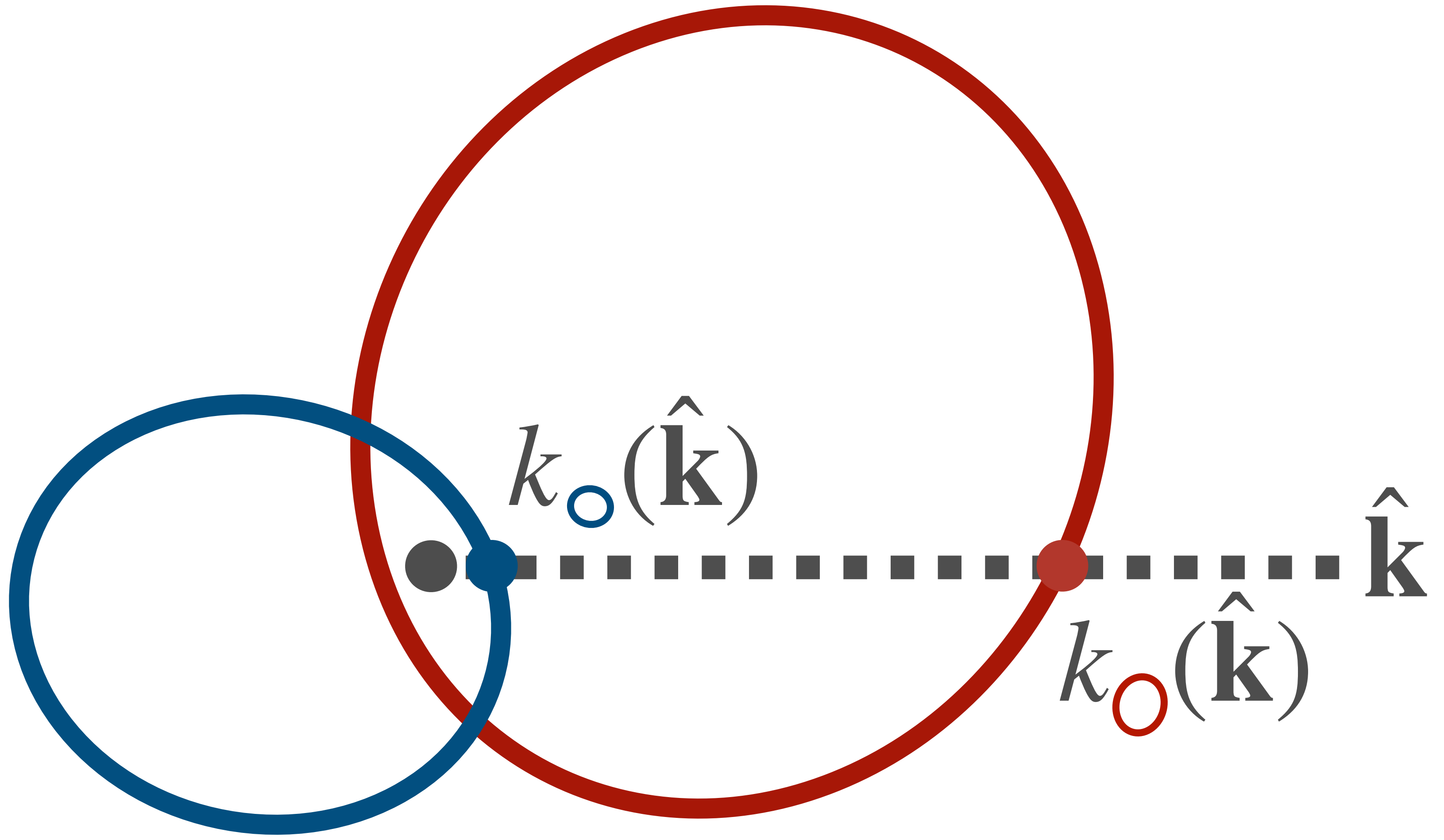

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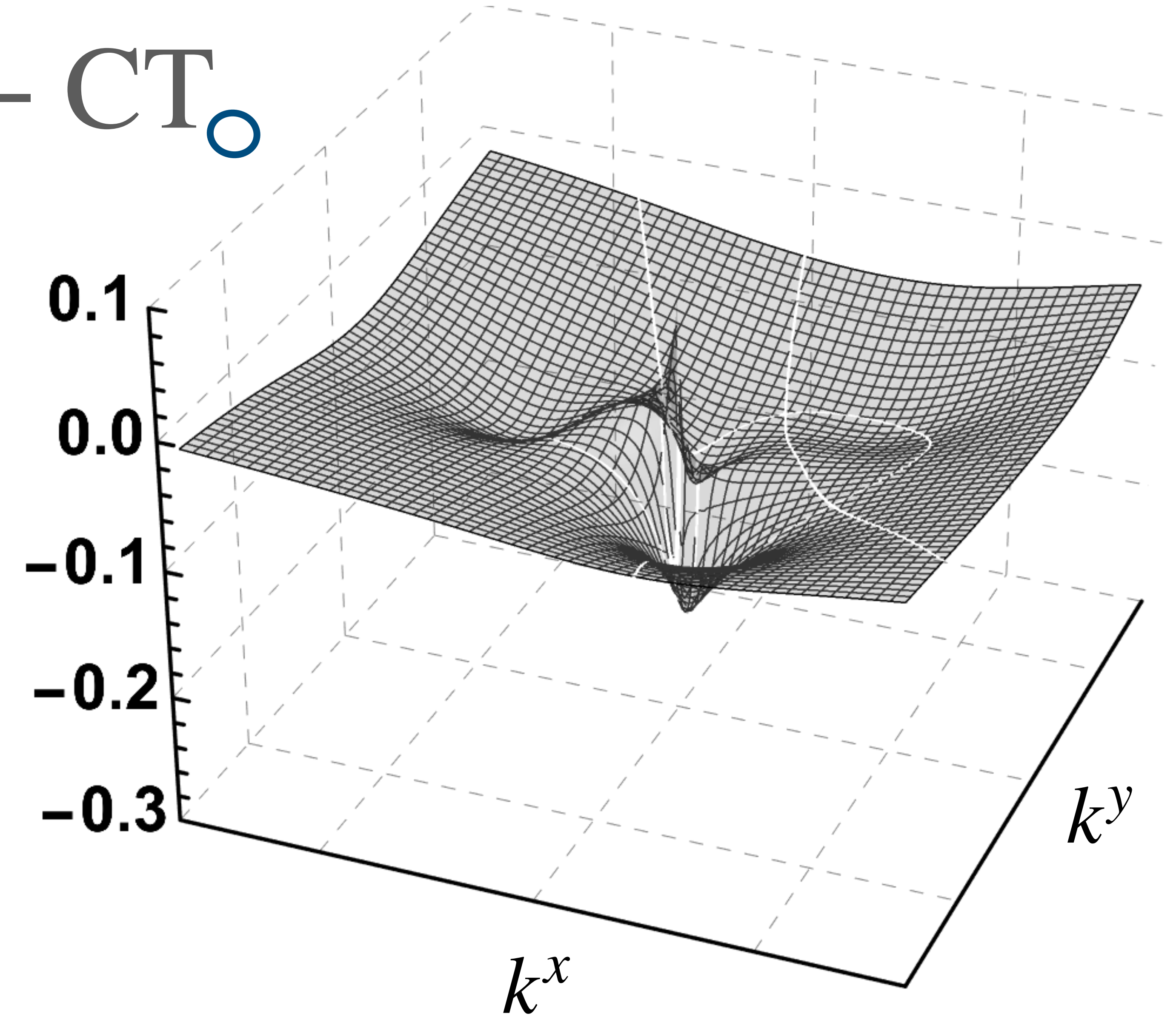
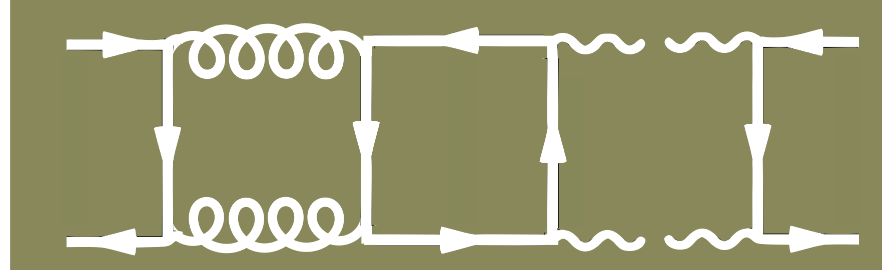
$\mathscr{I} - \mathrm{CT}_{\circ} - \mathrm{CT}_{\circ}$



We will only need the dispersive part!

for parity symmetric processes or observables

$$\int \mathrm{d}\Pi\, \mathrm{d}^3\vec{k}\, \mathrm{d}^3\vec{l} \sum_{\text{hel.}} 2\,\mathrm{Re}\left[\quad + \cdots \right]$$



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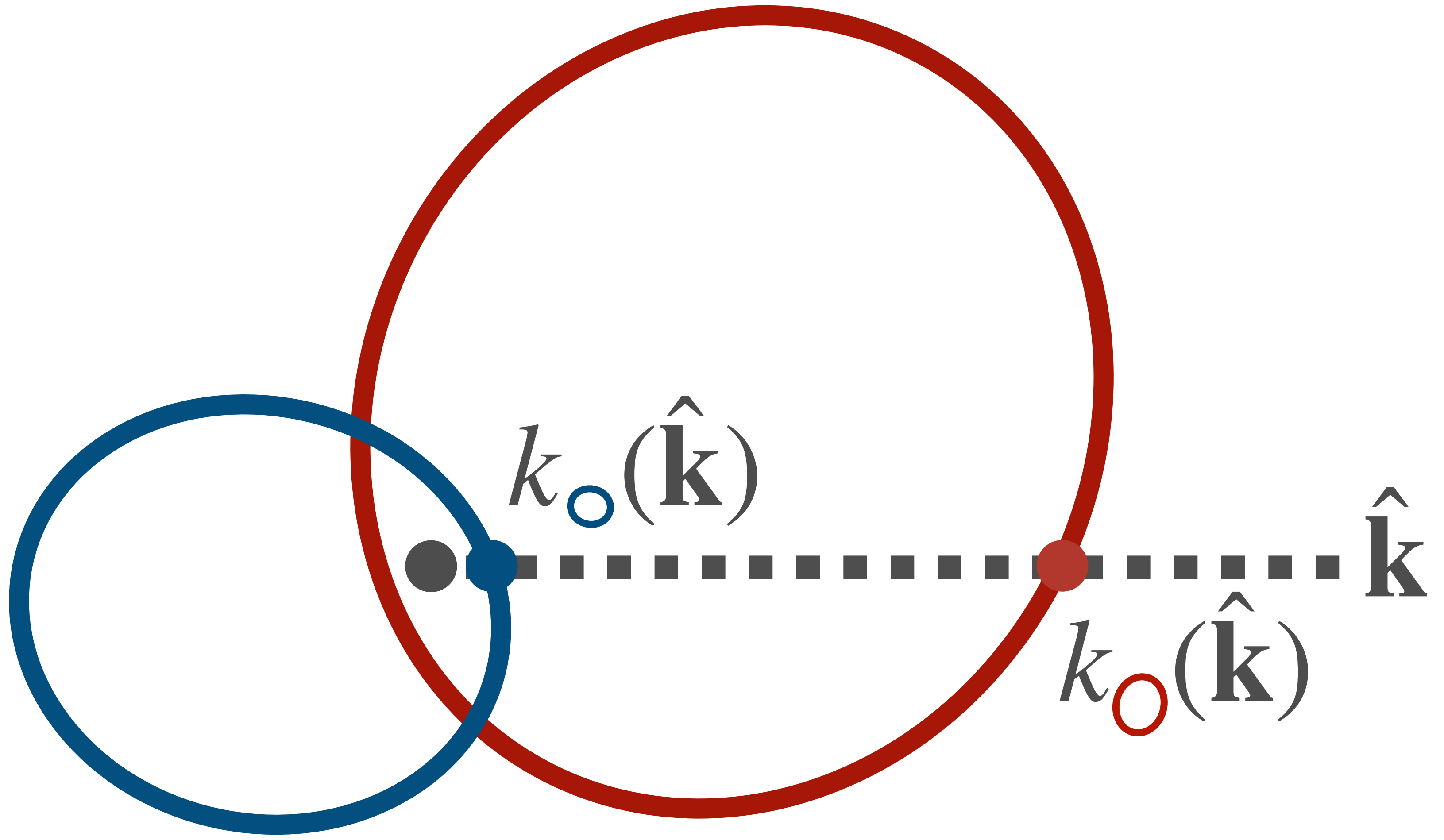

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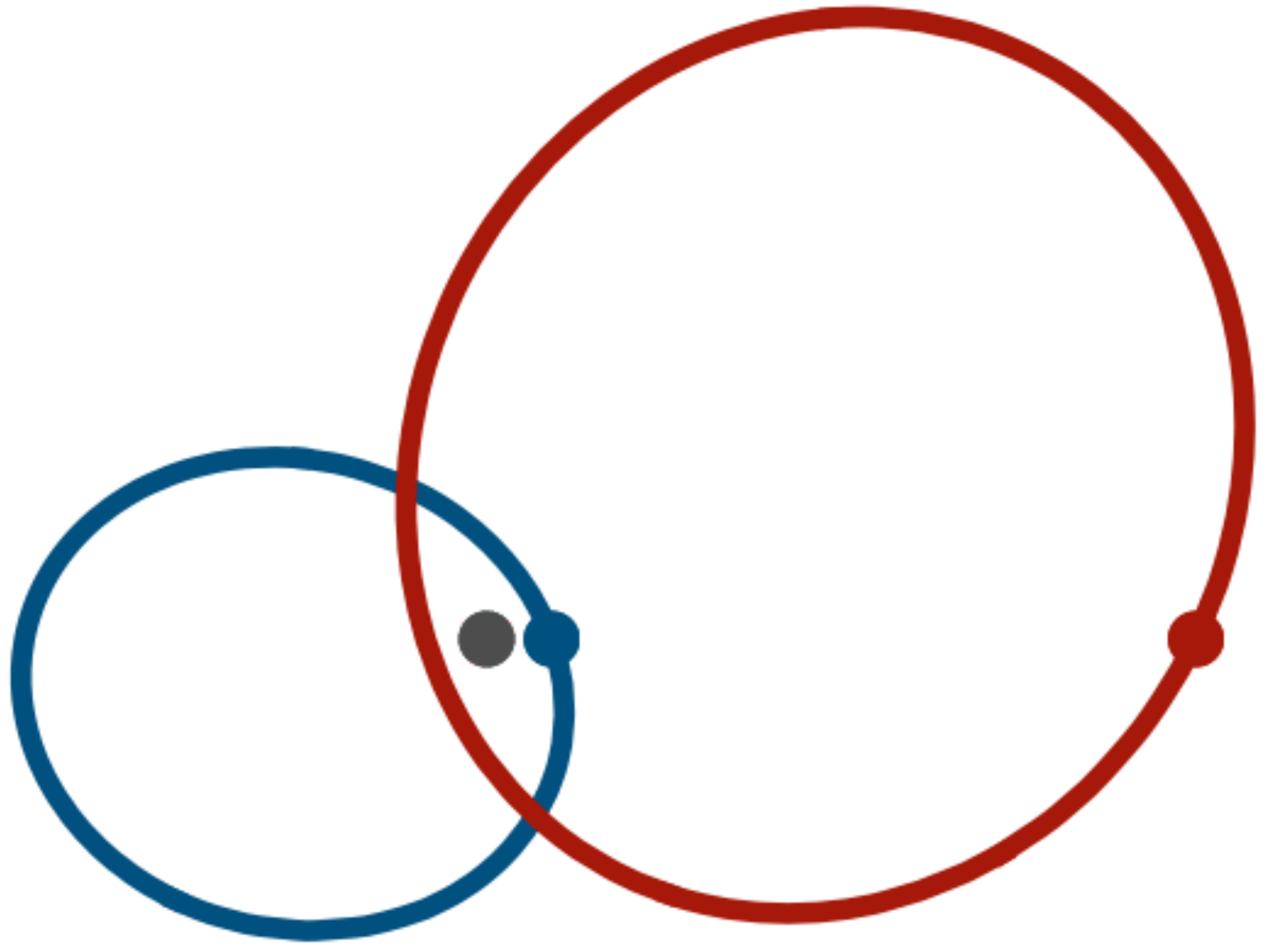
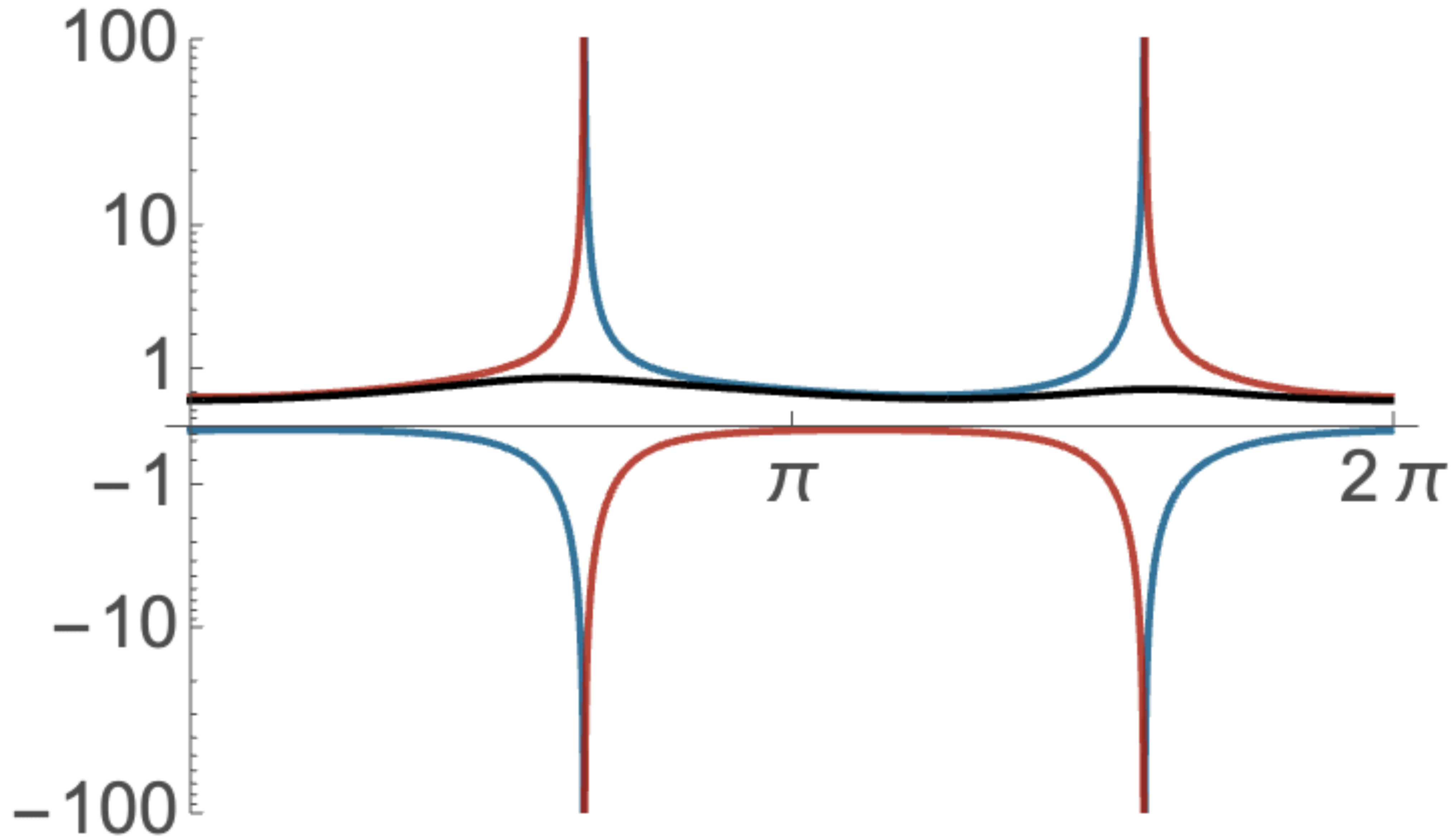
$$\mathrm{Im}\, I = \pi \int \mathrm{d}^{3n-1}\hat{\mathbf{k}} \sum_i \mathrm{Res}_i\, \mathcal{I}$$ absorptive part



parameterisation aligns singularities

$\mathrm{Res}_{\circ}\, \mathcal{I} + \mathrm{Res}_{\circ}\, \mathcal{I}$



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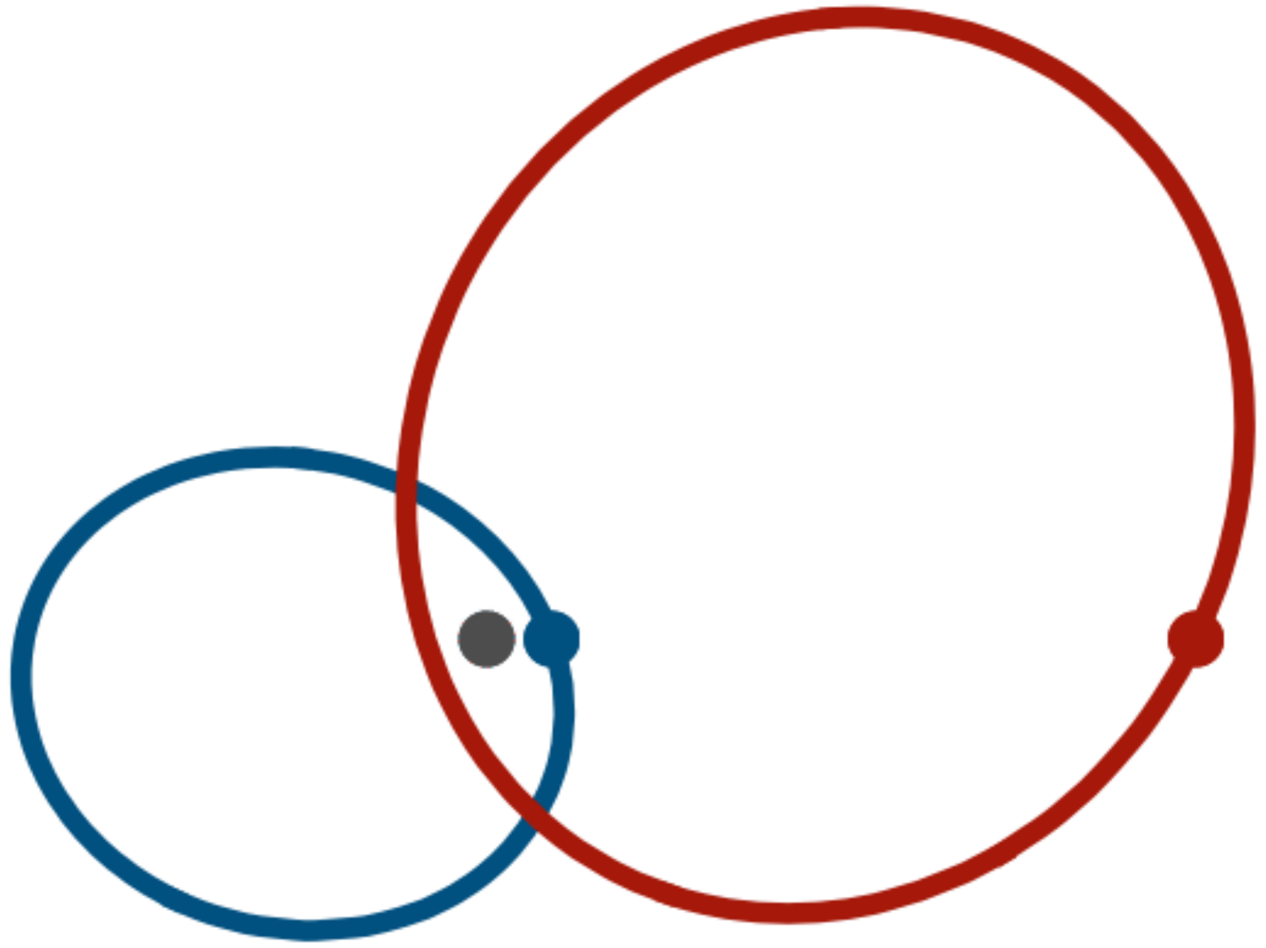

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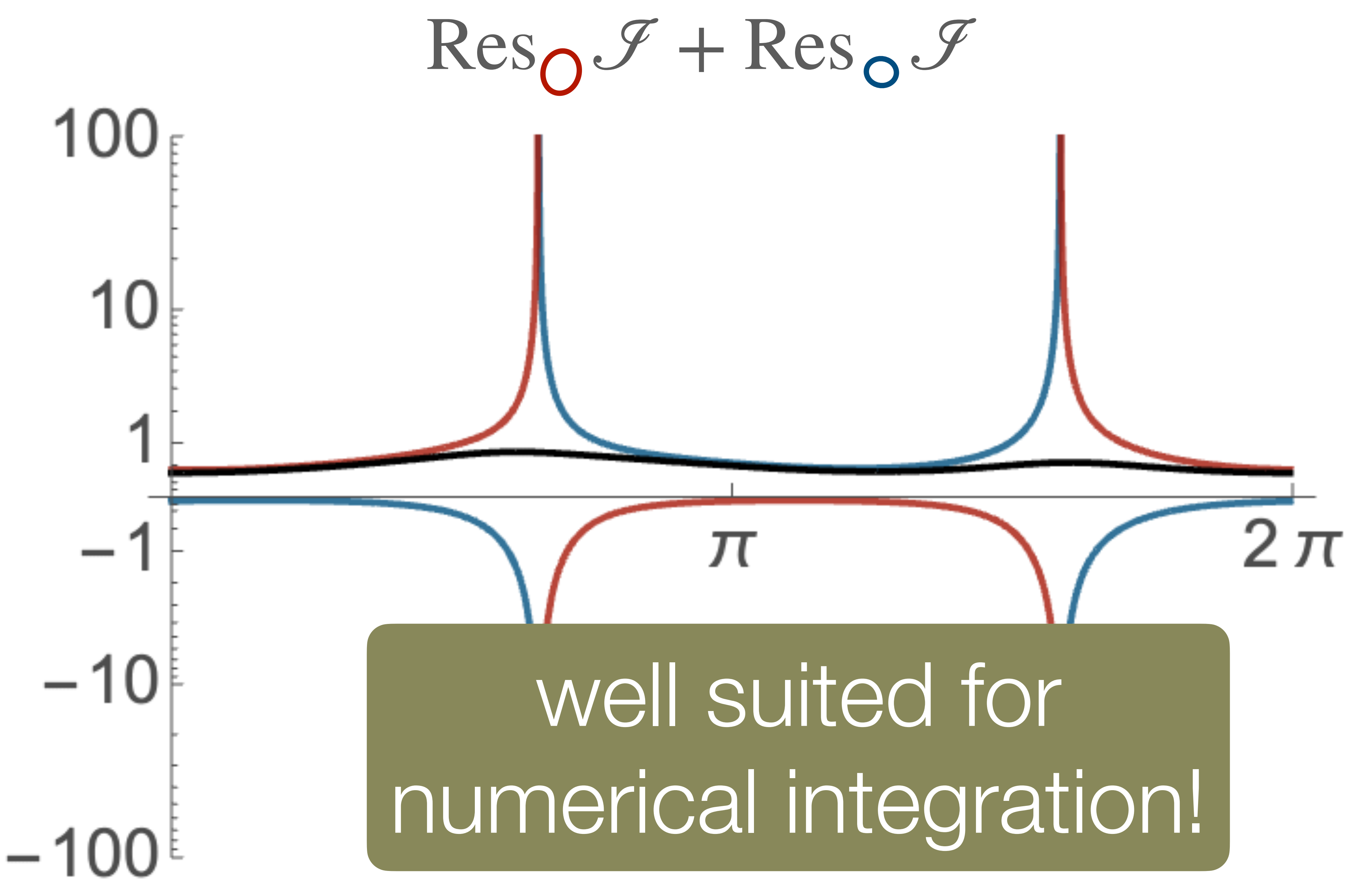

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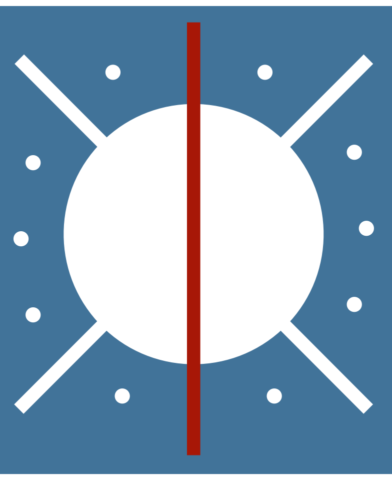
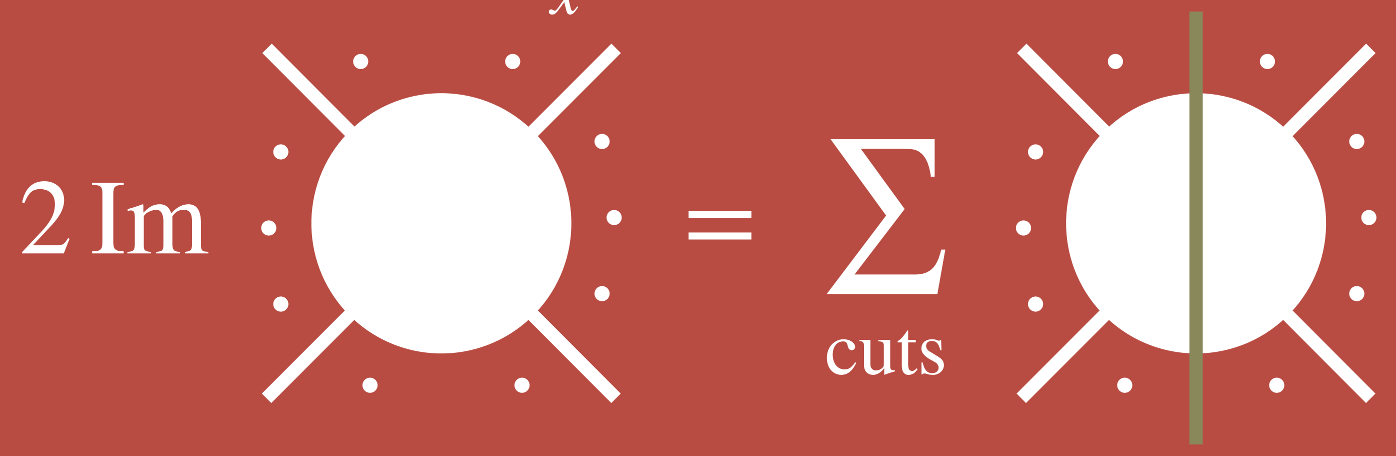


phase space integral

$$\mathrm{Im}\, I = \pi \int \mathrm{d}^{3n-1}\hat{\mathbf{k}} \sum_i \mathrm{Res}_i\, \mathcal{I}$$ absorptive part

Optical theorem

$$2\, \mathrm{Im}\, A(i \to f) = \sum_x \int \mathrm{d}\Pi_x\, A(i \to x)\, A^*(f \to x)$$

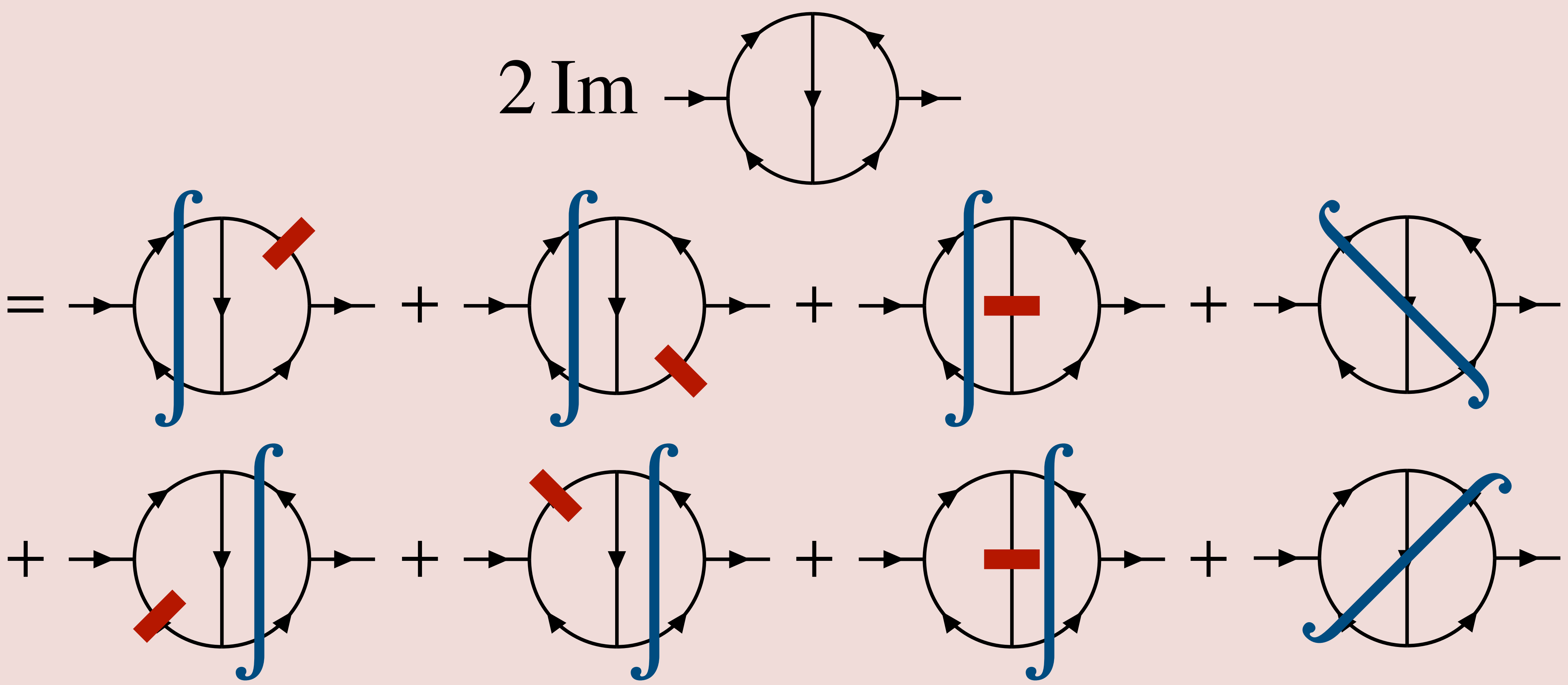


locally finite!*

*with the right threshold parametrisation

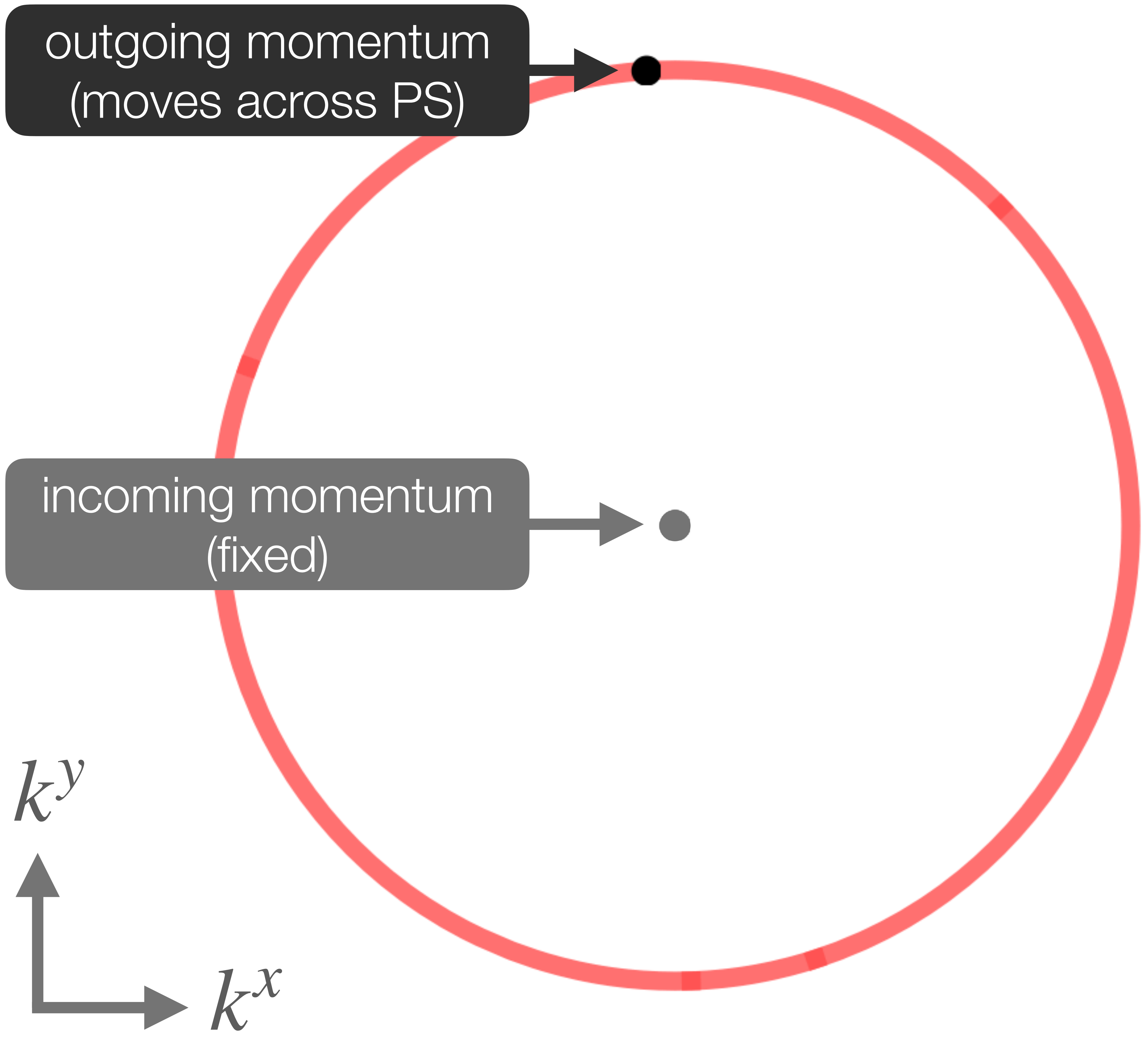
aligns singularities incl. FSR IR between real and virtual

analogous to: [Soper: hep-ph/9804454, hep-ph/9910292],
Local Unitarity [Capatti, Hirschi, Pelloni, Ruijl: 2010.01068, 2203.11038]

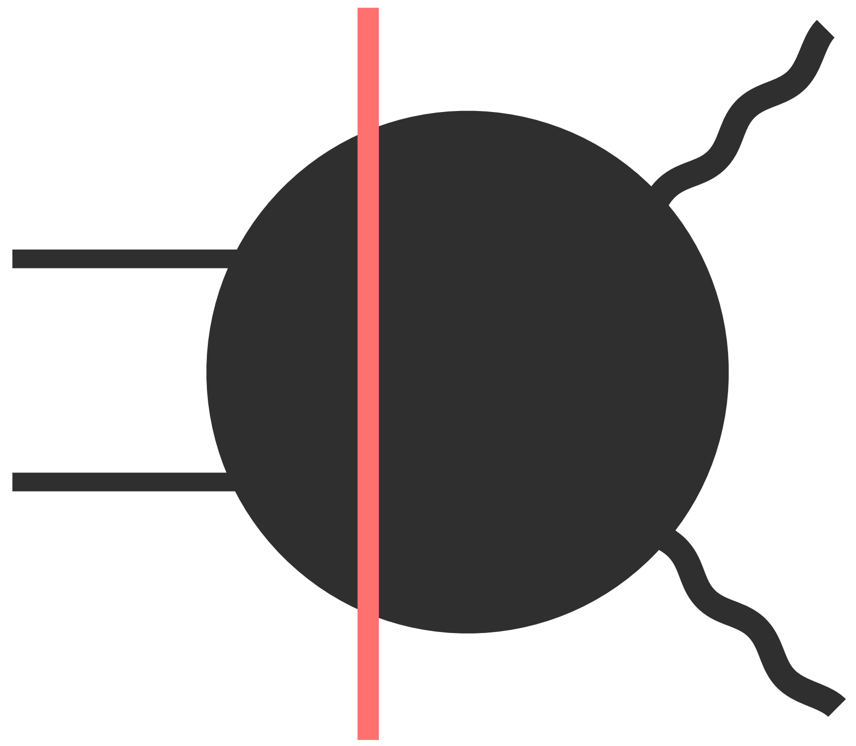


Threshold singularities DK, Matilde Vicini [2407.21511]
one-loop & two-loop Nf amplitude

$$q\bar{q} \to \gamma\gamma$$



corresponding Cutkosky cuts



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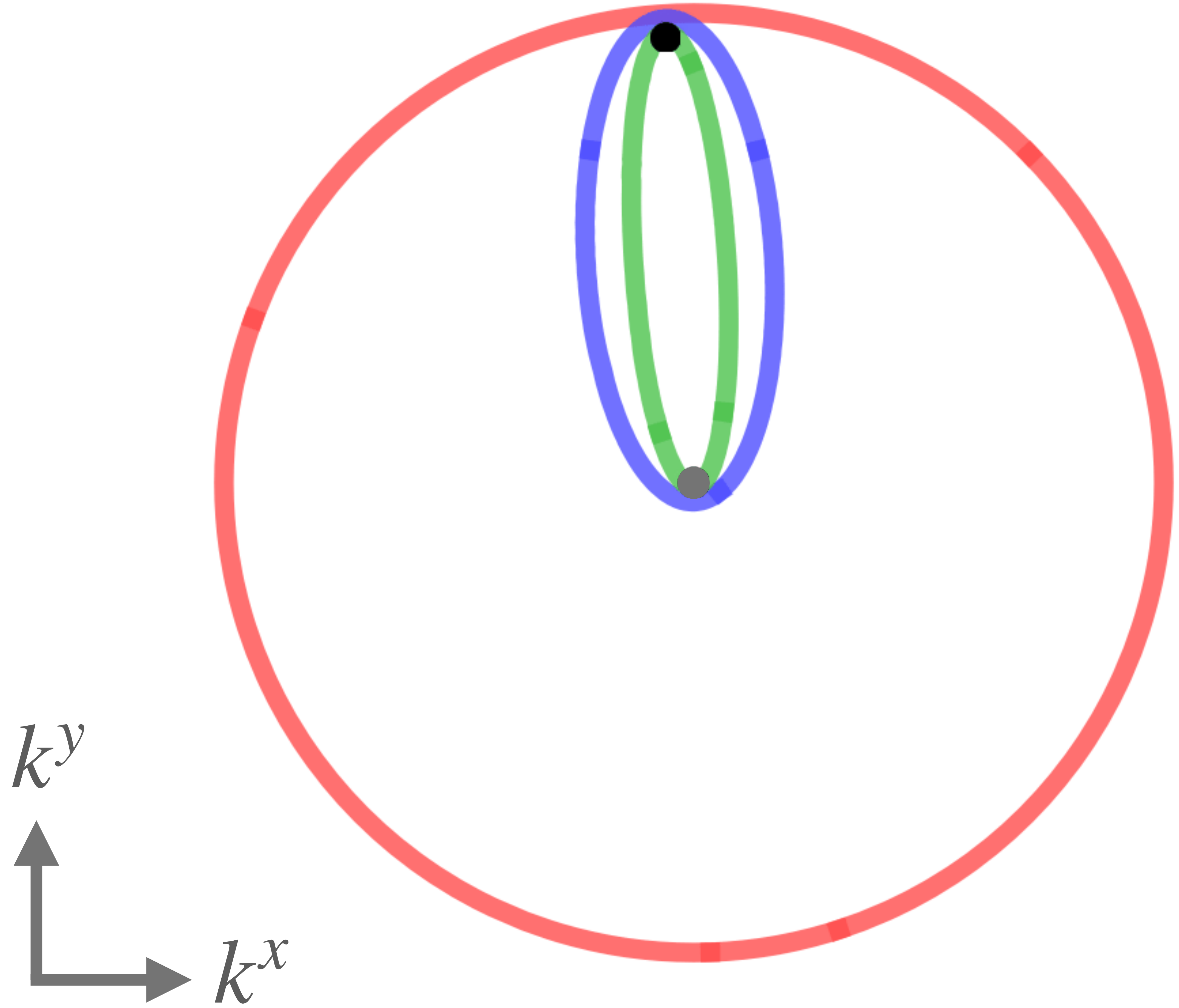

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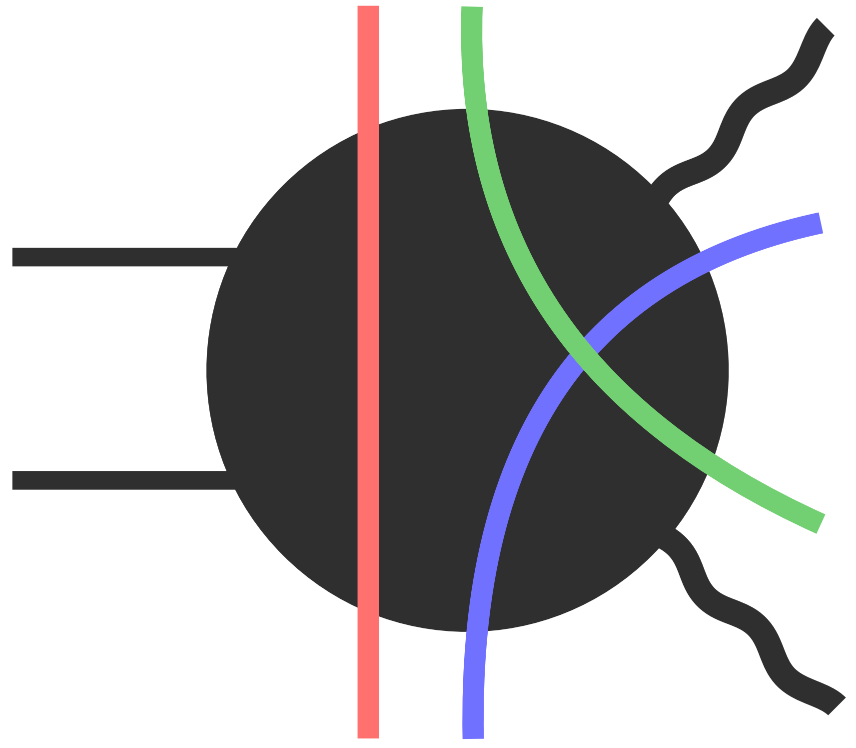

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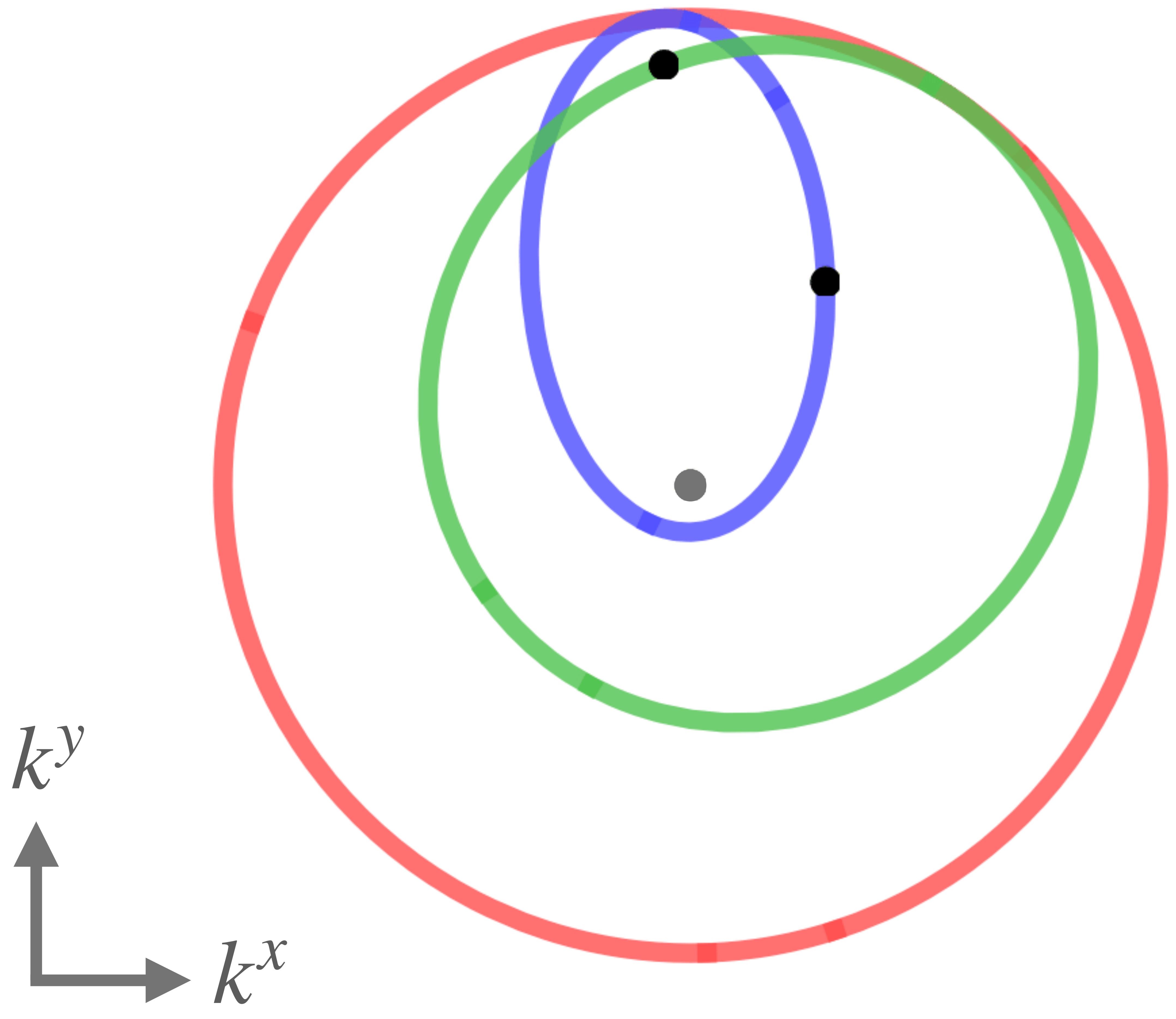

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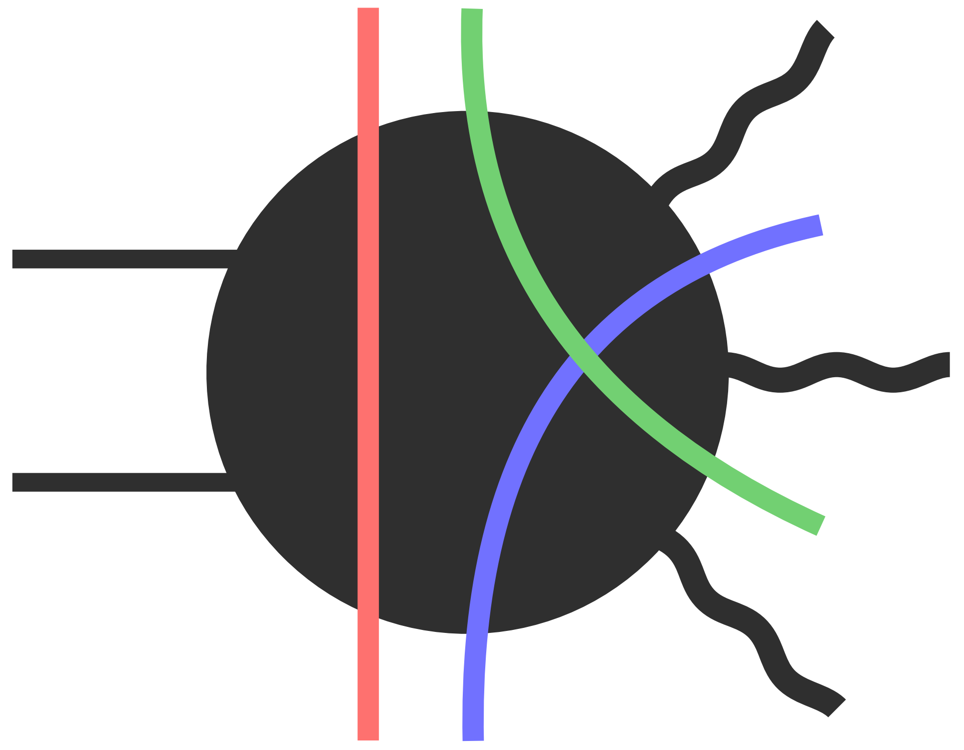

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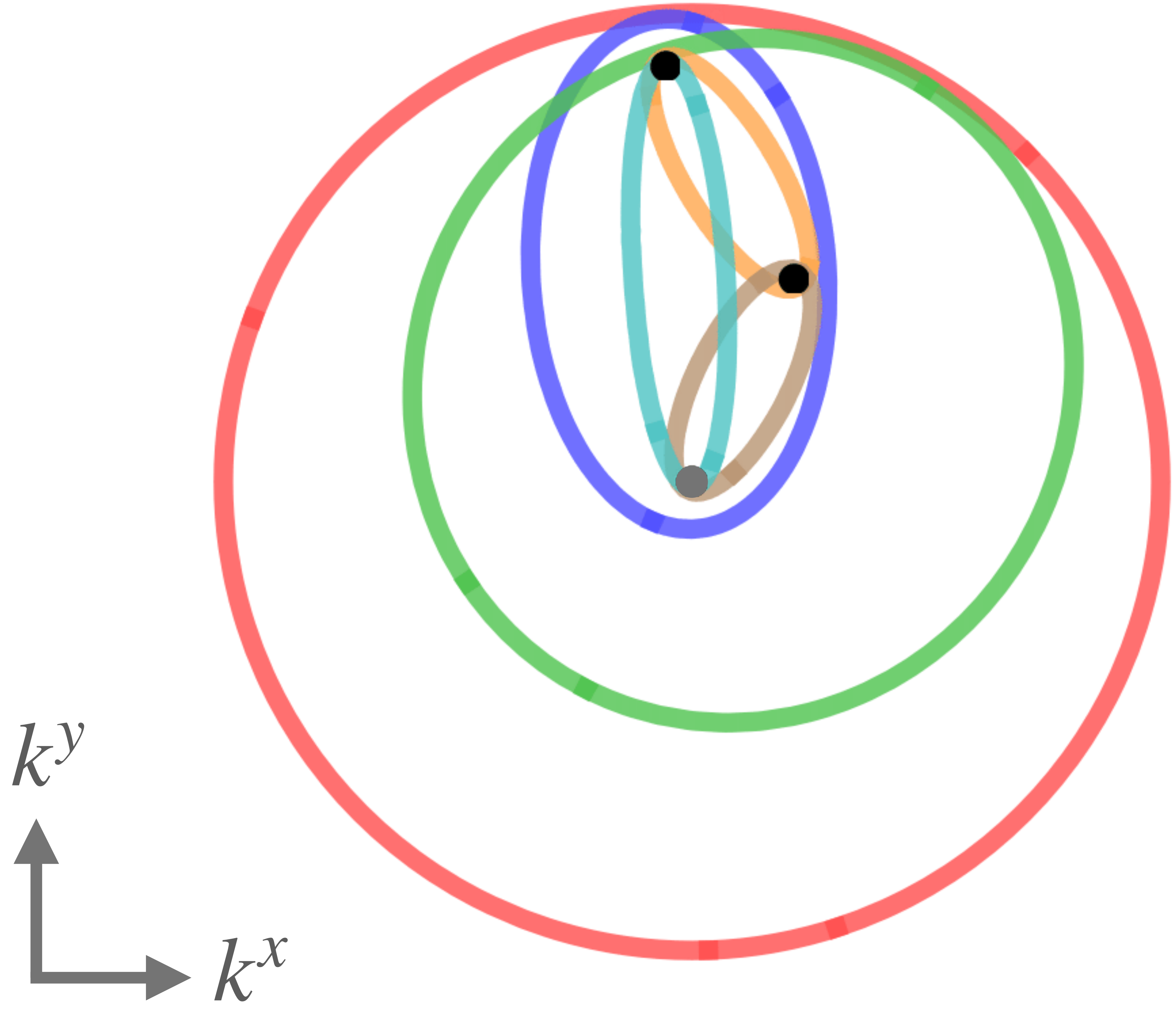

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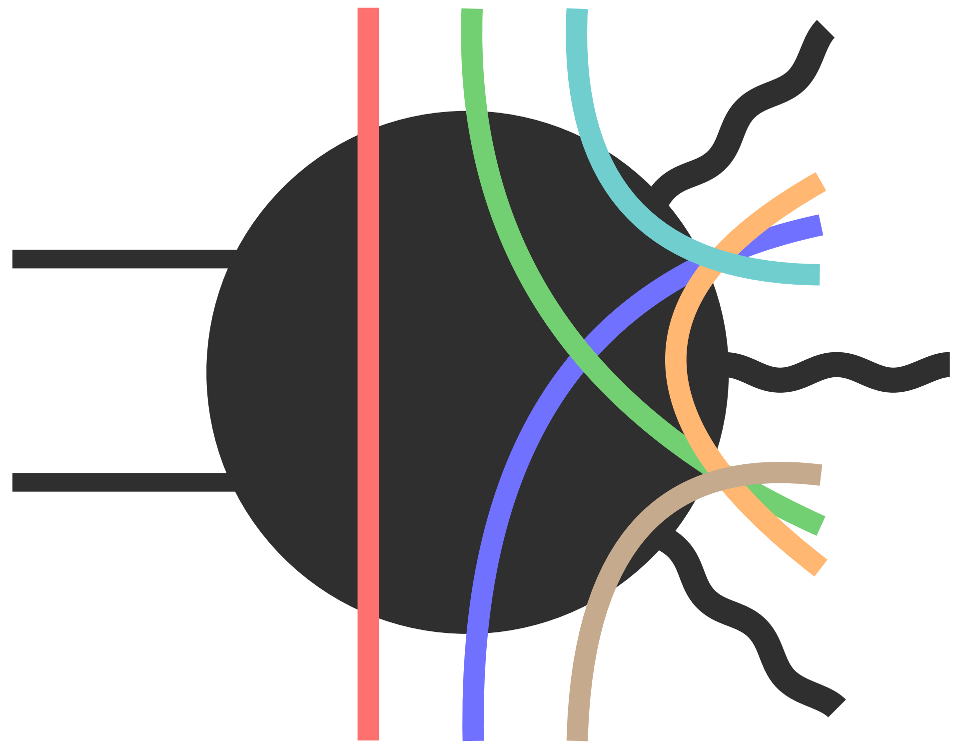

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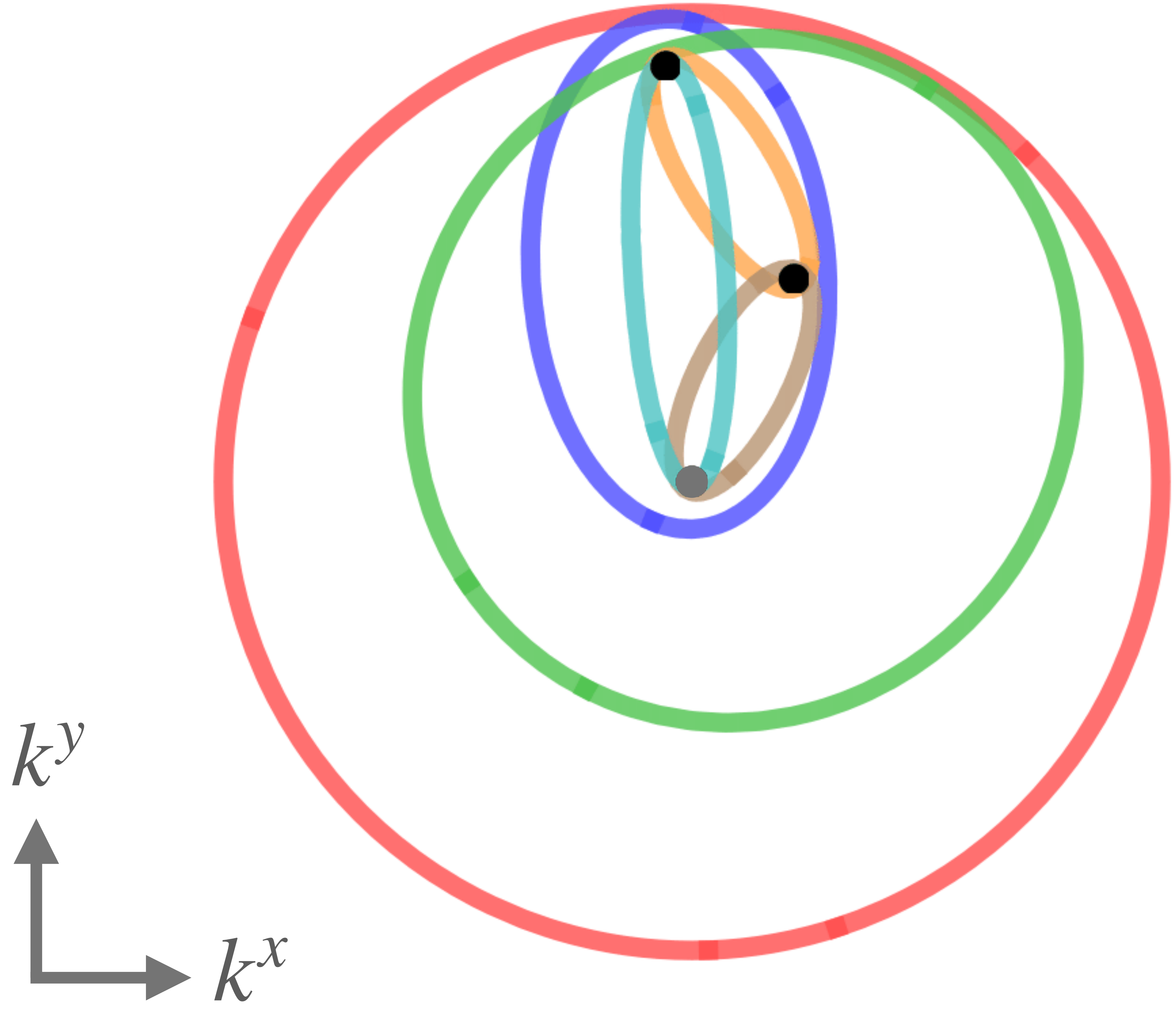

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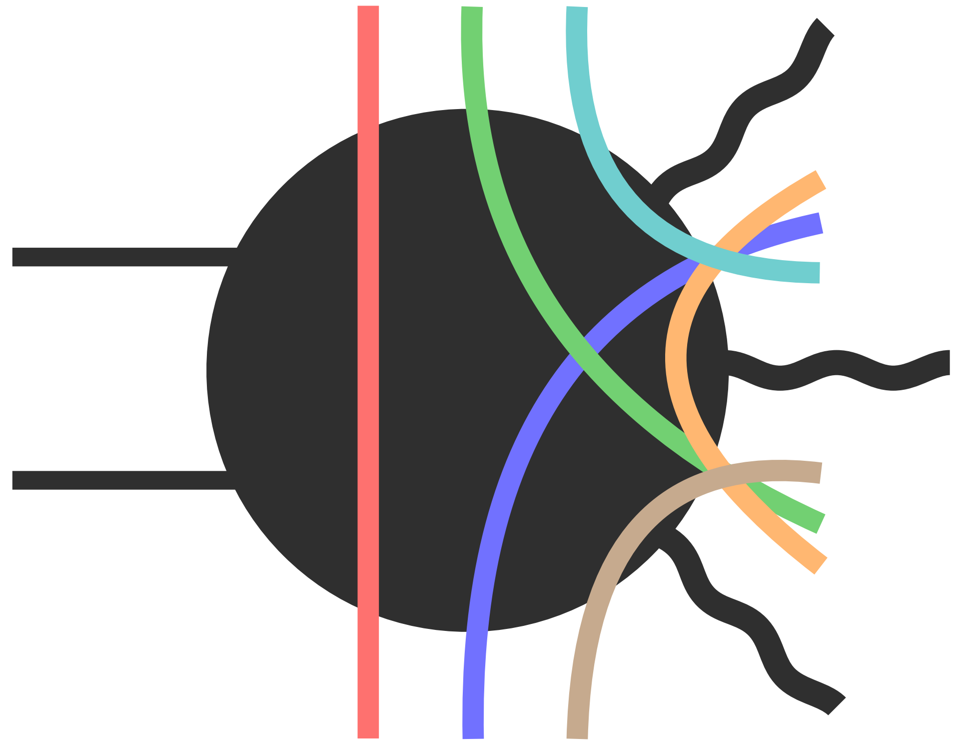
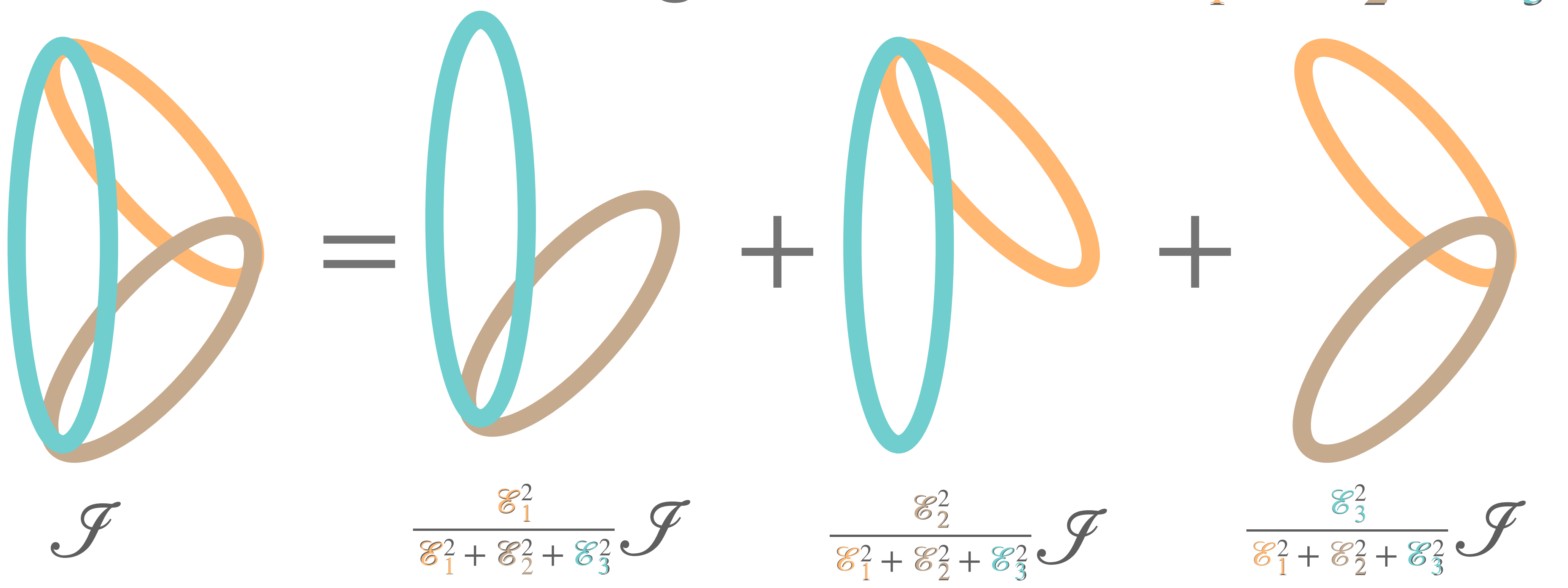


overlapping thresholds
multi-channelling

$$1 = \frac{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2}{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2}$$

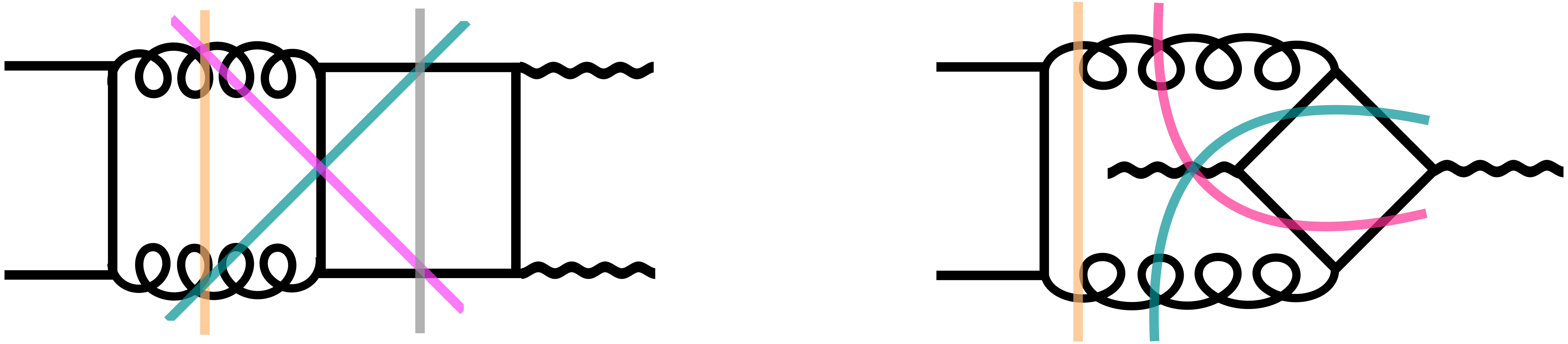


Threshold singularities

DK, Matilde Vicini [2510.18801]

quark-loop mediated two-loop amplitude

$q\bar{q} \to \gamma\gamma$

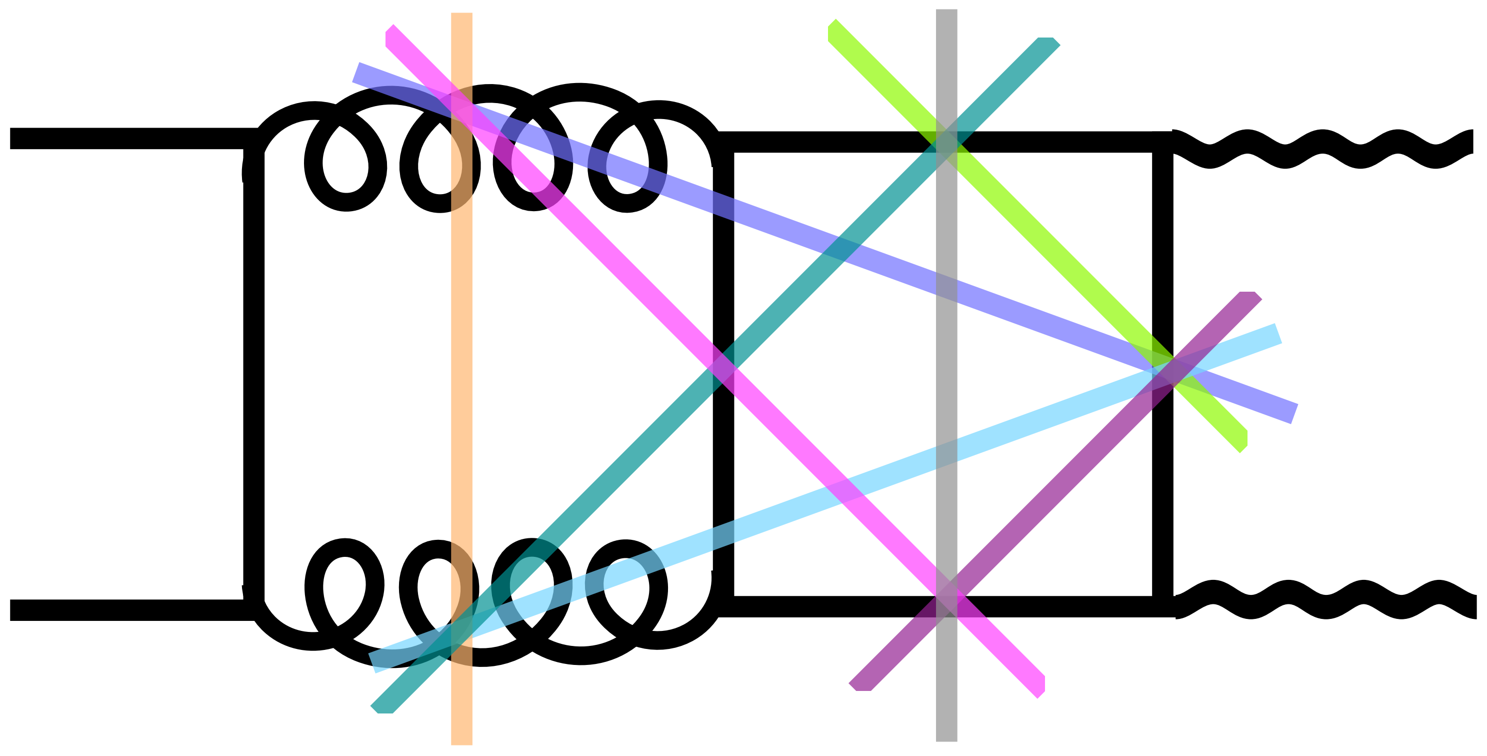


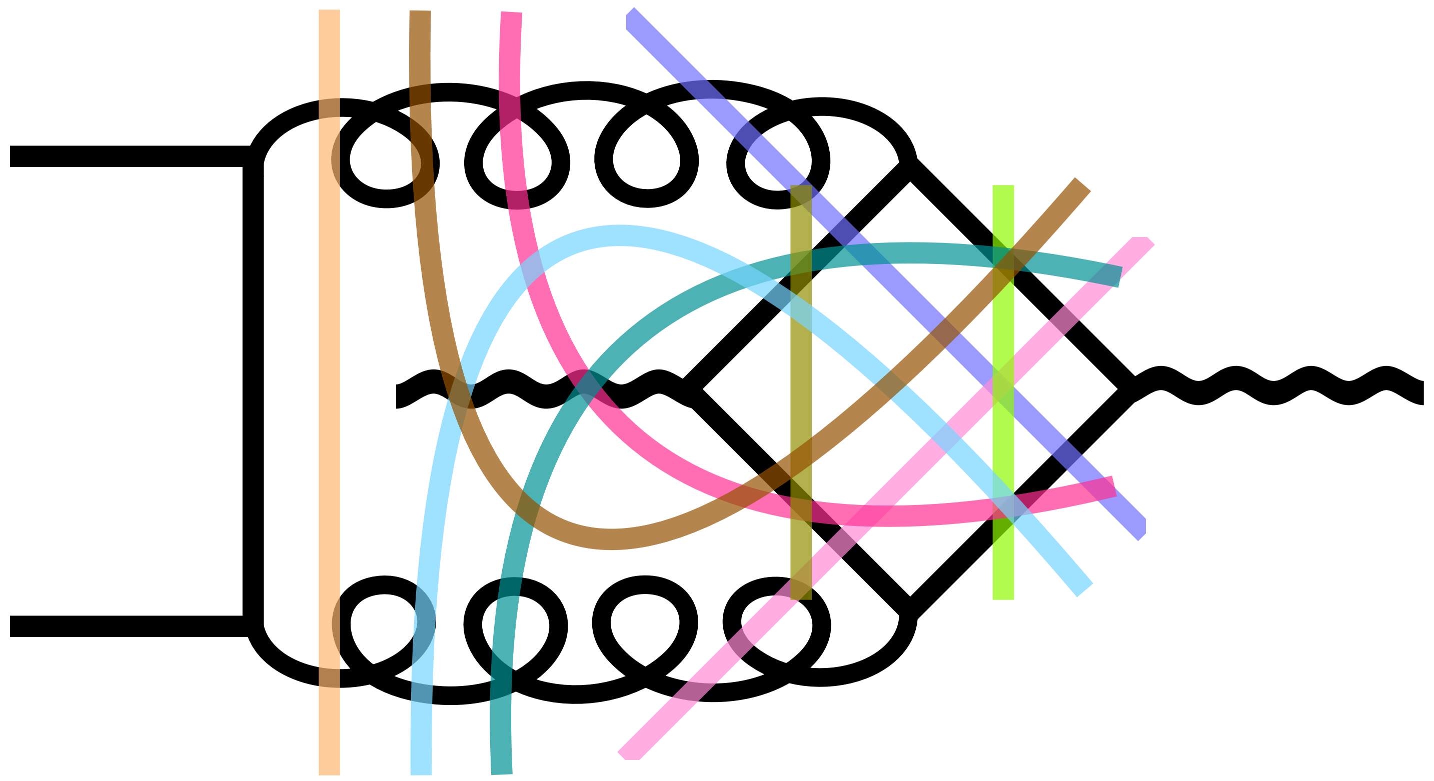
$$\text{all} \quad \text{threshold channels} \quad \frac{\text{all except } t}{\text{all except } t}$$

Threshold singularities
DK, Matilde Vicini [2510.18801]

quark-loop mediated two-loop amplitude

$q\bar{q} \to \gamma^*\gamma^*$





all

threshold channels

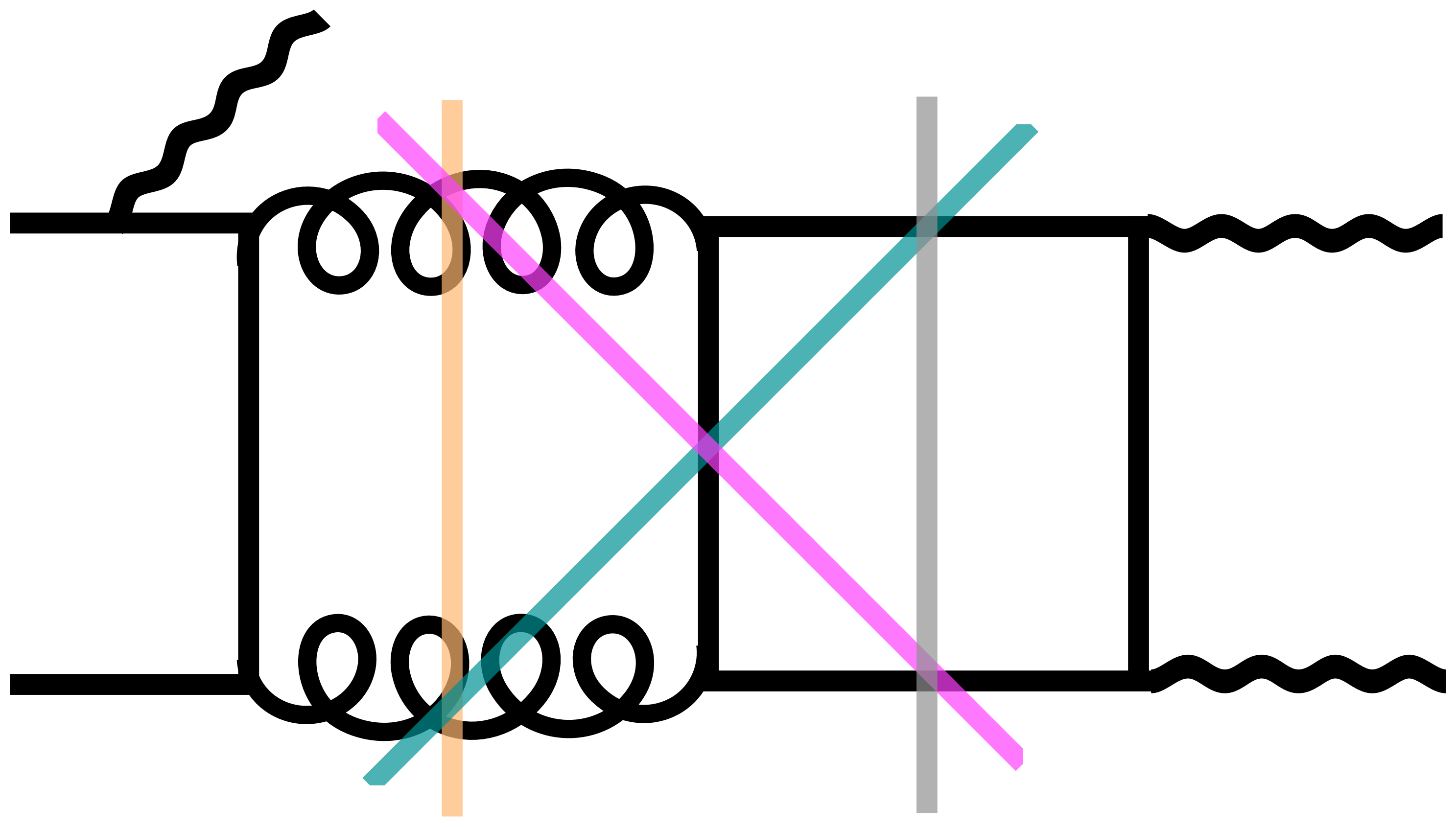
all except t

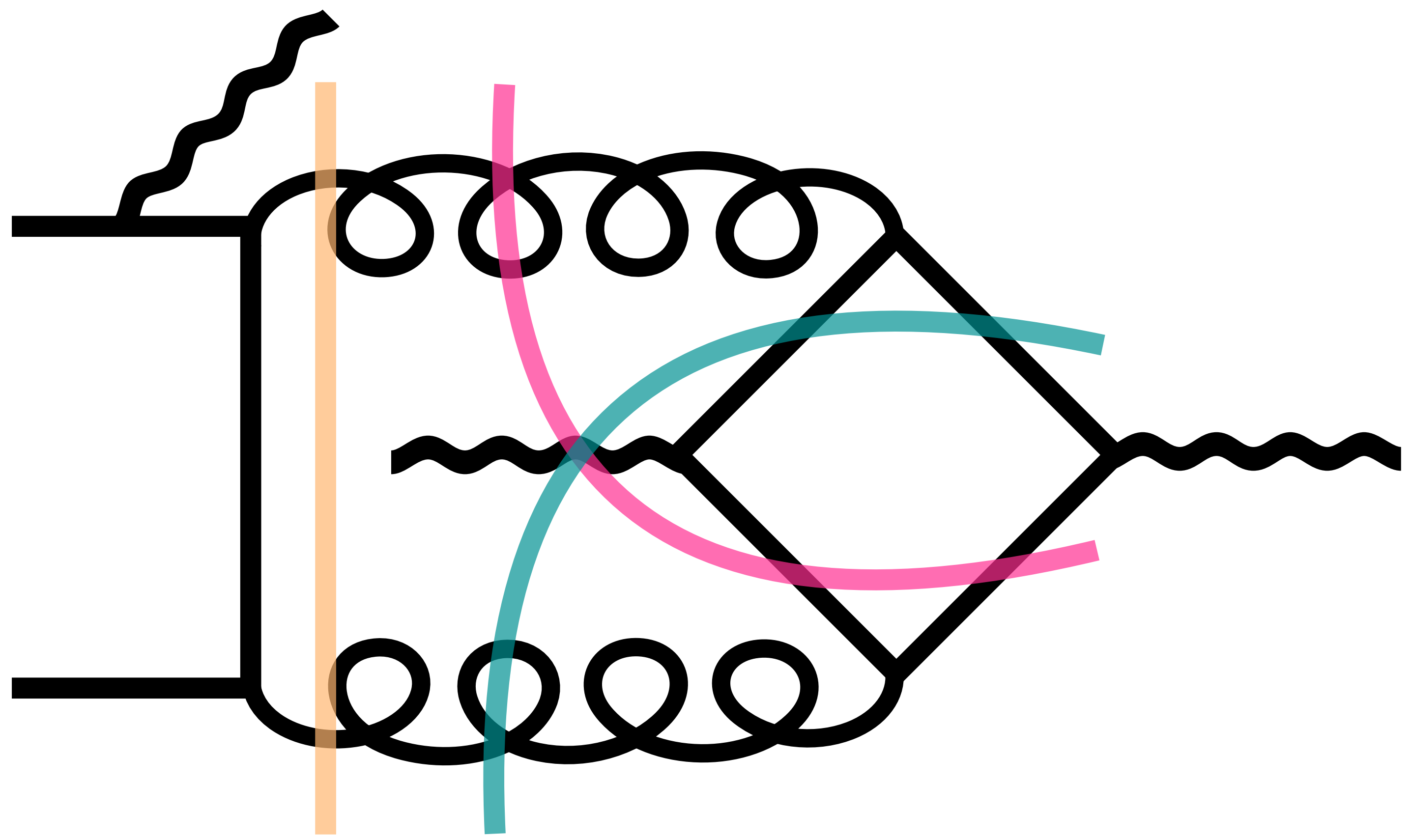
all except t

Threshold singularities
quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$



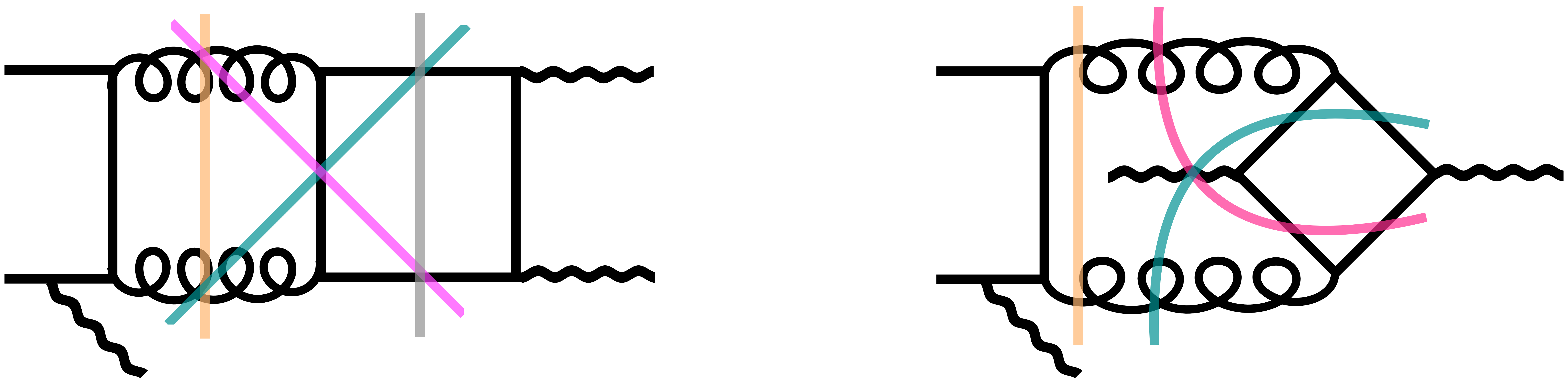


all	threshold channels	$\dfrac{\text{all except } t}{\text{all except } t}$

Threshold singularities
quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$



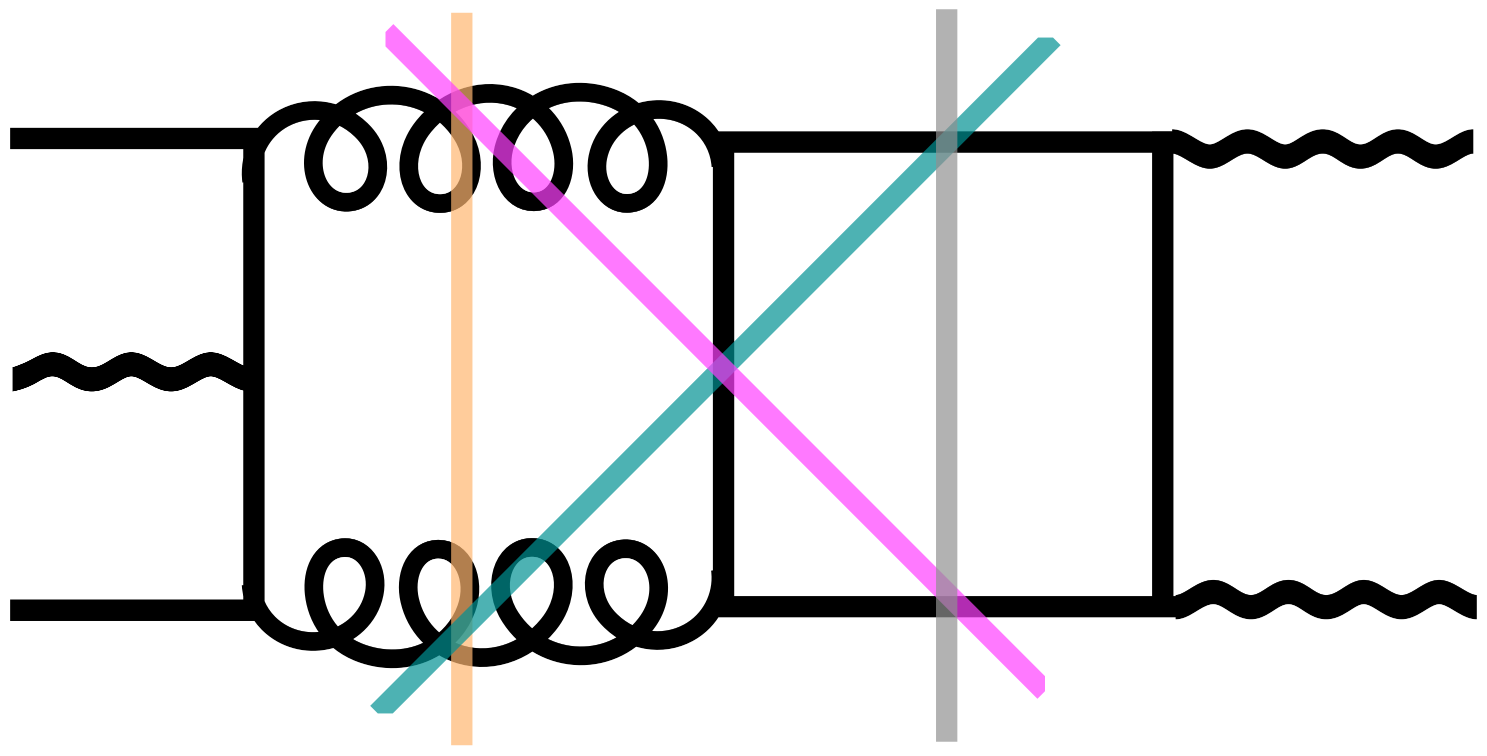
all	threshold channels	$\dfrac{\text{all except } t}{\text{all except } t}$

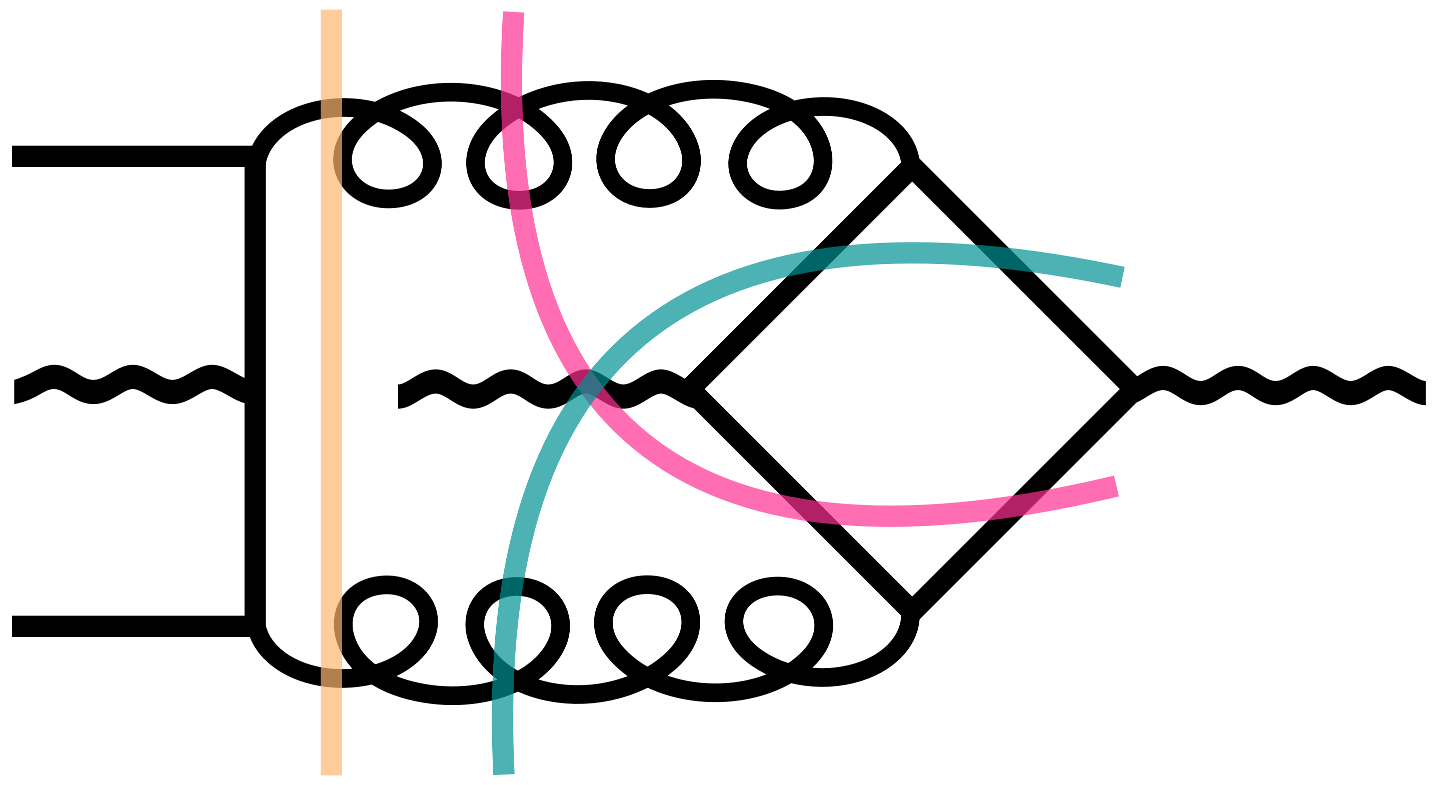
Threshold singularities

DK, Matilde Vicini [2510.18801]

quark-loop mediated two-loop amplitude

$q\bar{q} \to \gamma\gamma\gamma$





all

threshold channels

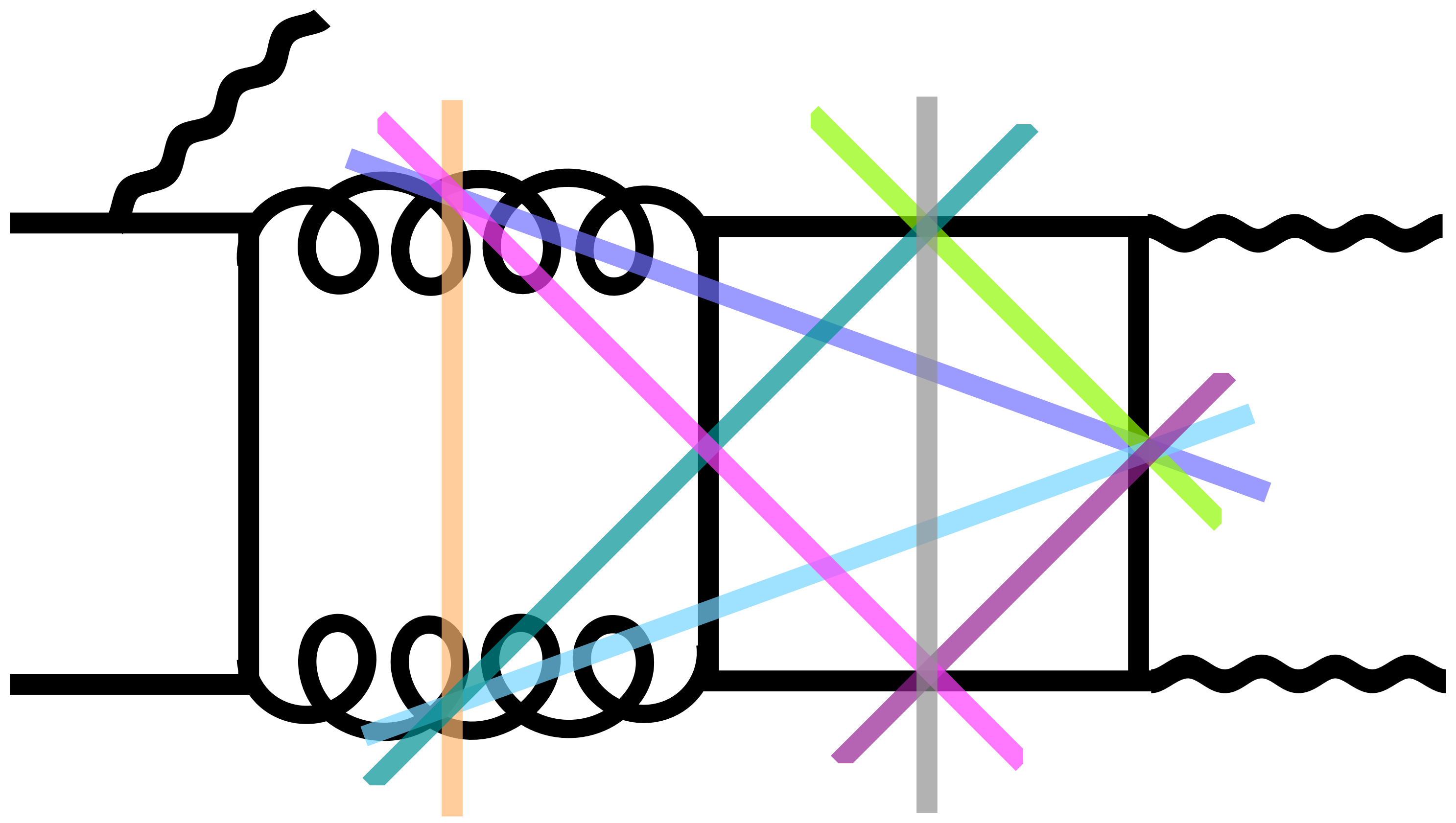
all except t

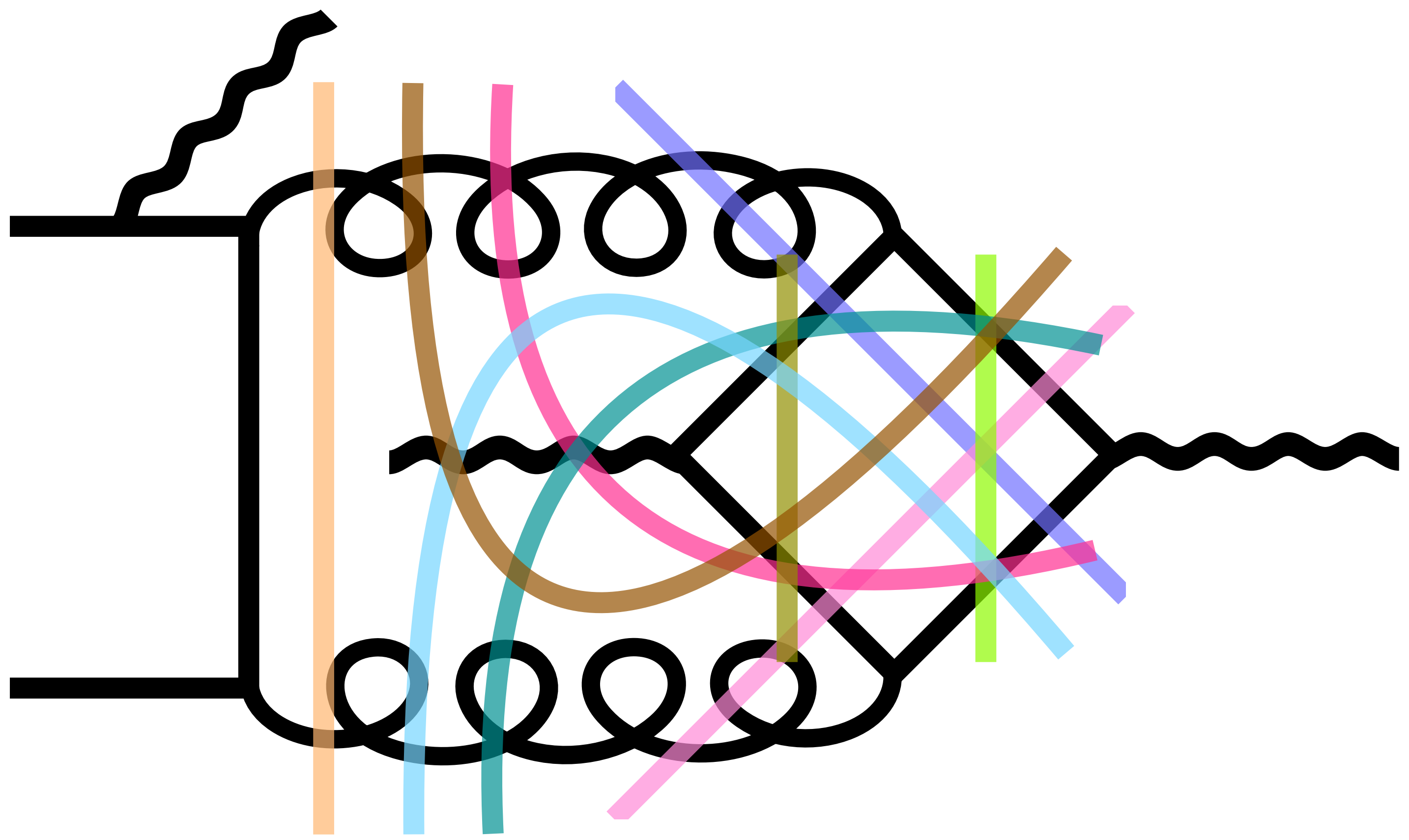
all except t

Threshold singularities
quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma^*\gamma^*\gamma^*$





$$\frac{\text{all}}{\text{threshold channels}} \quad \frac{\text{all except } t}{\text{all except } t}$$

all

threshold channels

all except t

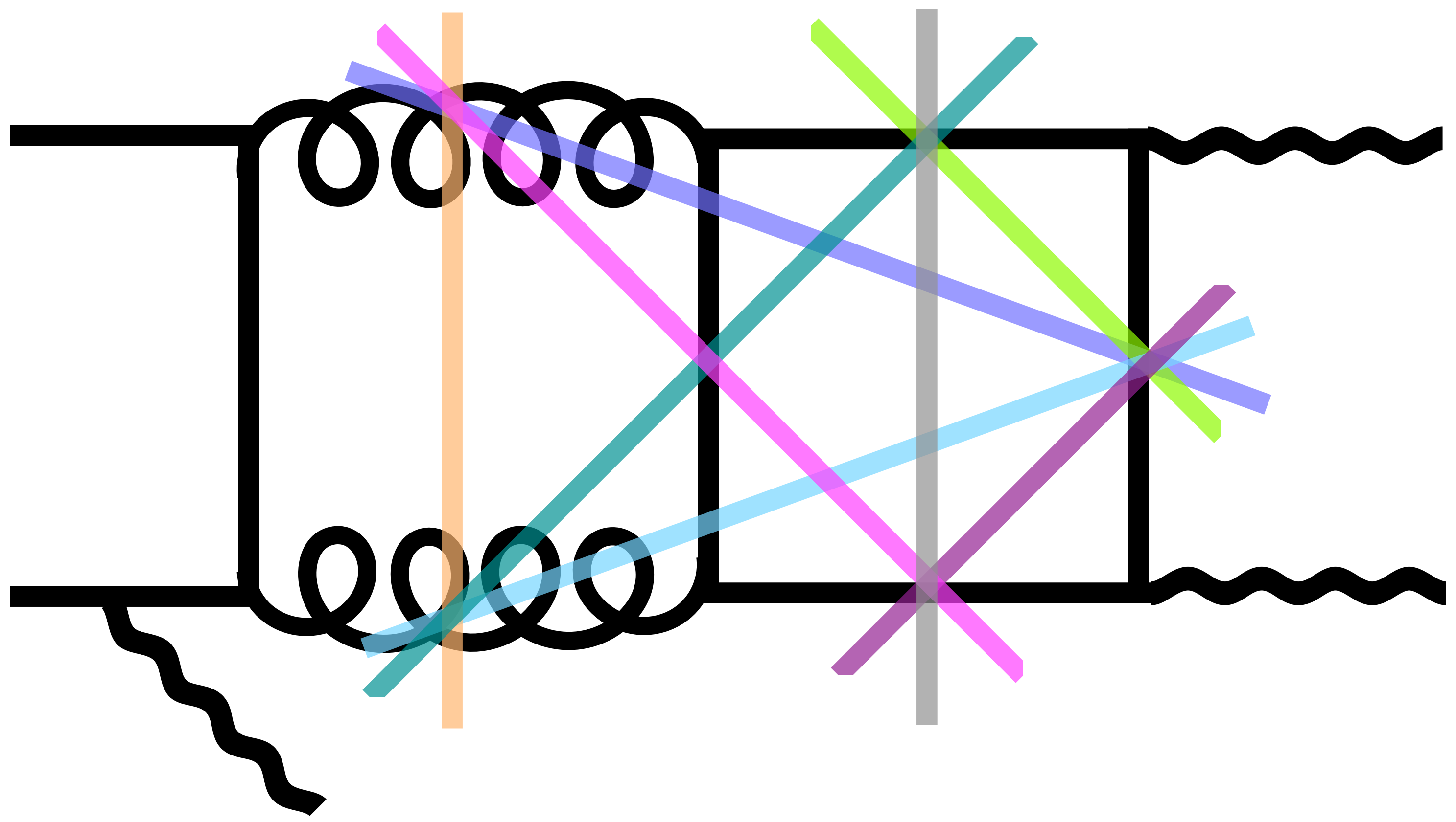
all except t

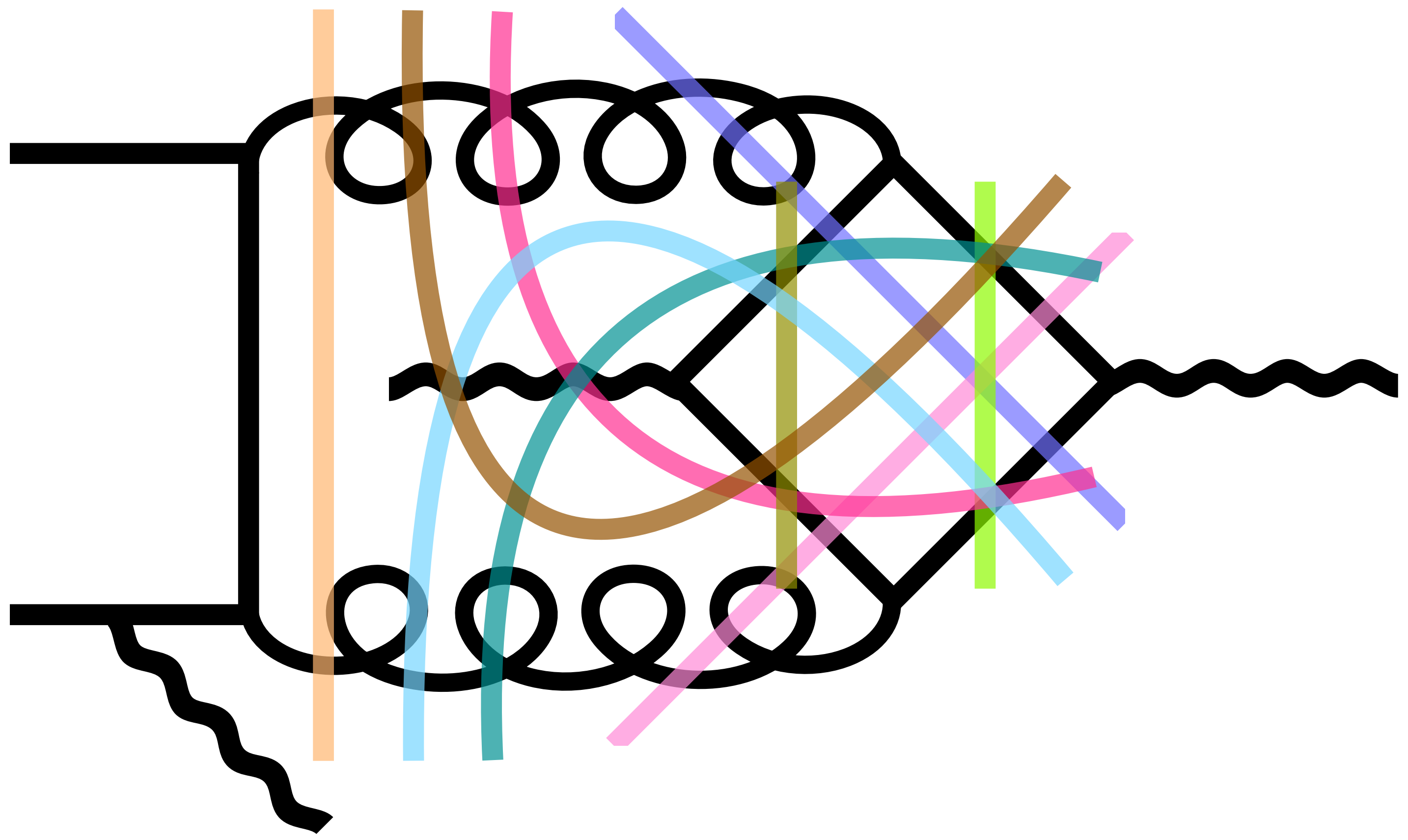
Threshold singularities

DK, Matilde Vicini [2510.18801]

quark-loop mediated two-loop amplitude

$q\bar{q} \to \gamma^*\gamma^*\gamma^*$





all

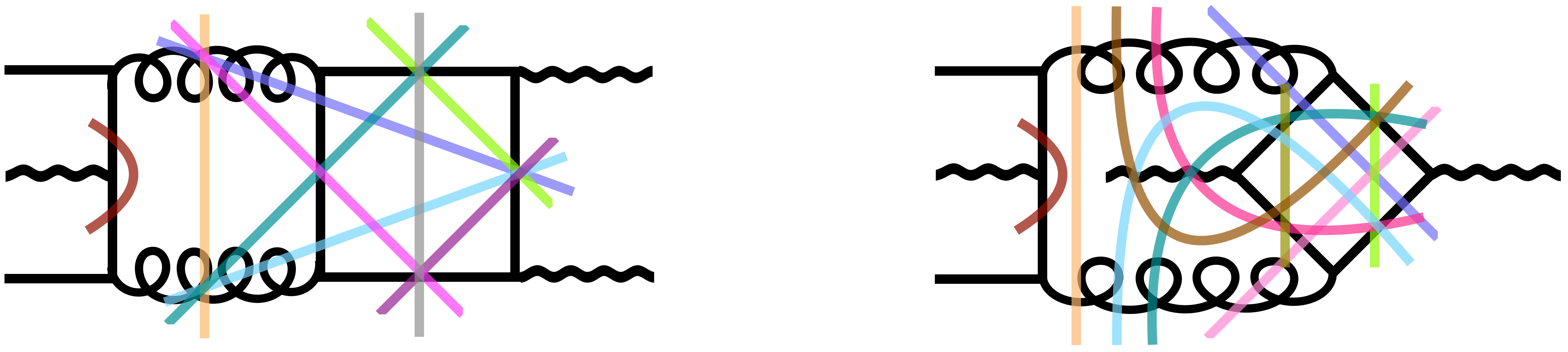
threshold channels

$$\frac{\text{all except } t}{\text{all except } t}$$

Threshold singularities quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma^*\gamma^*\gamma^*$



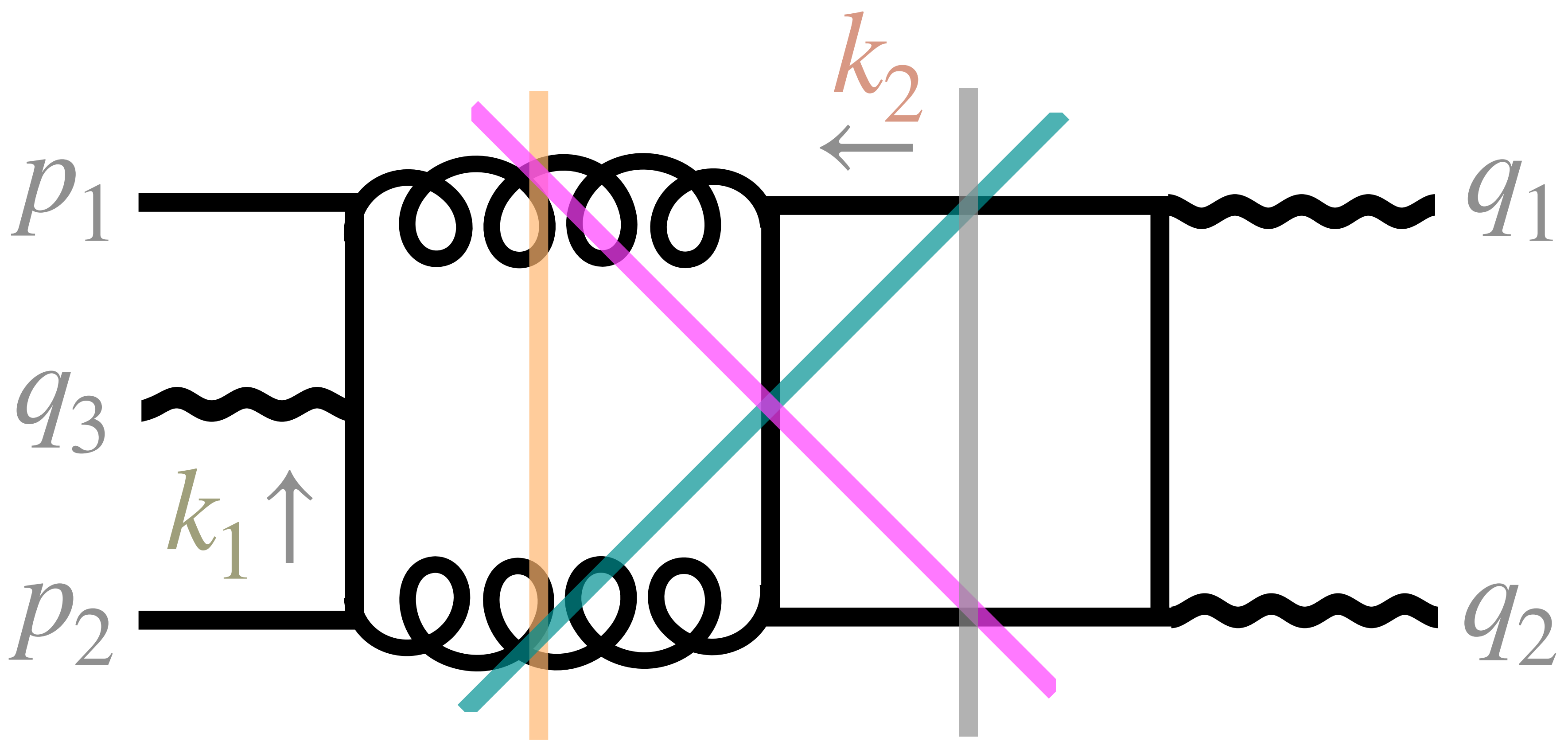
all except t / t, t, t, t	threshold channels	all except t, t	t, t
		all except t, t	t, t

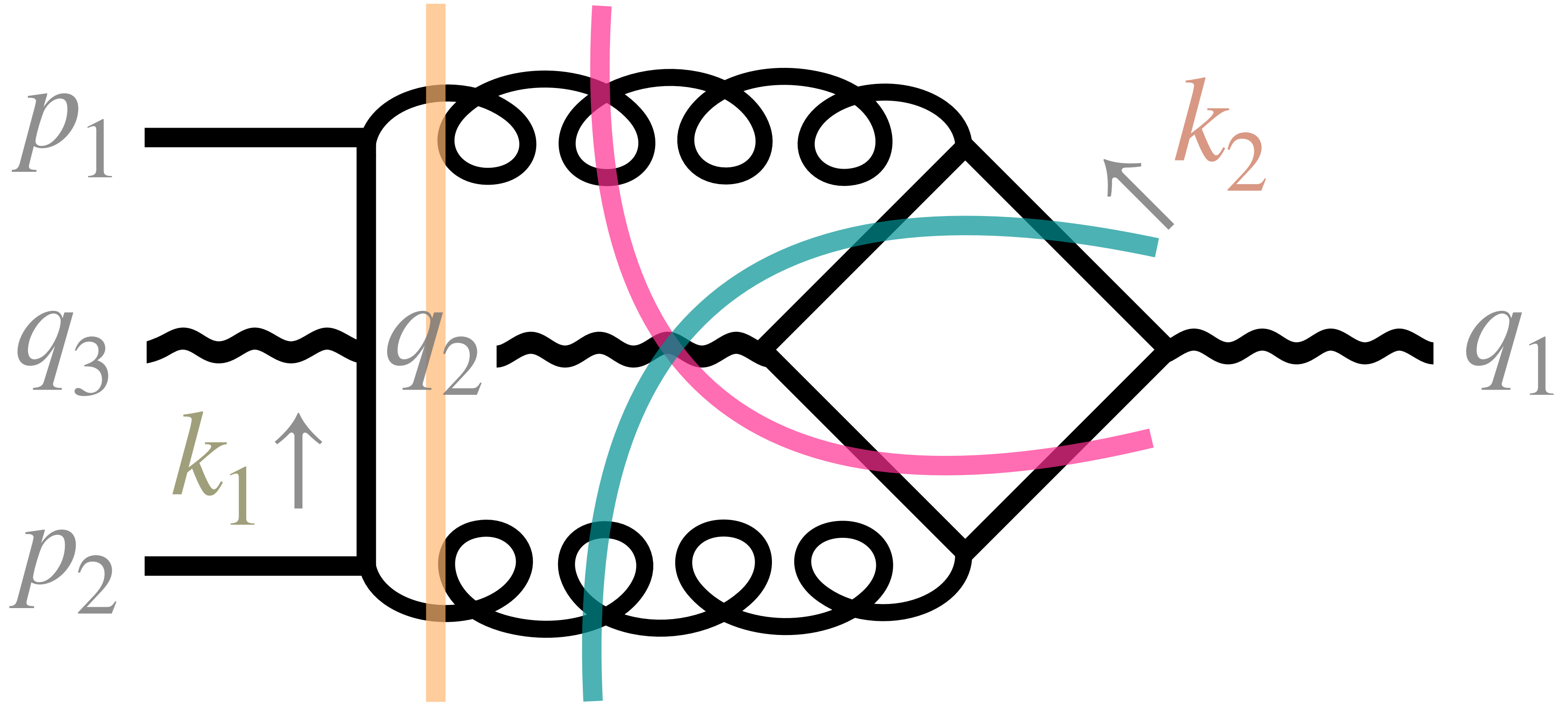
Threshold singularities

quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$

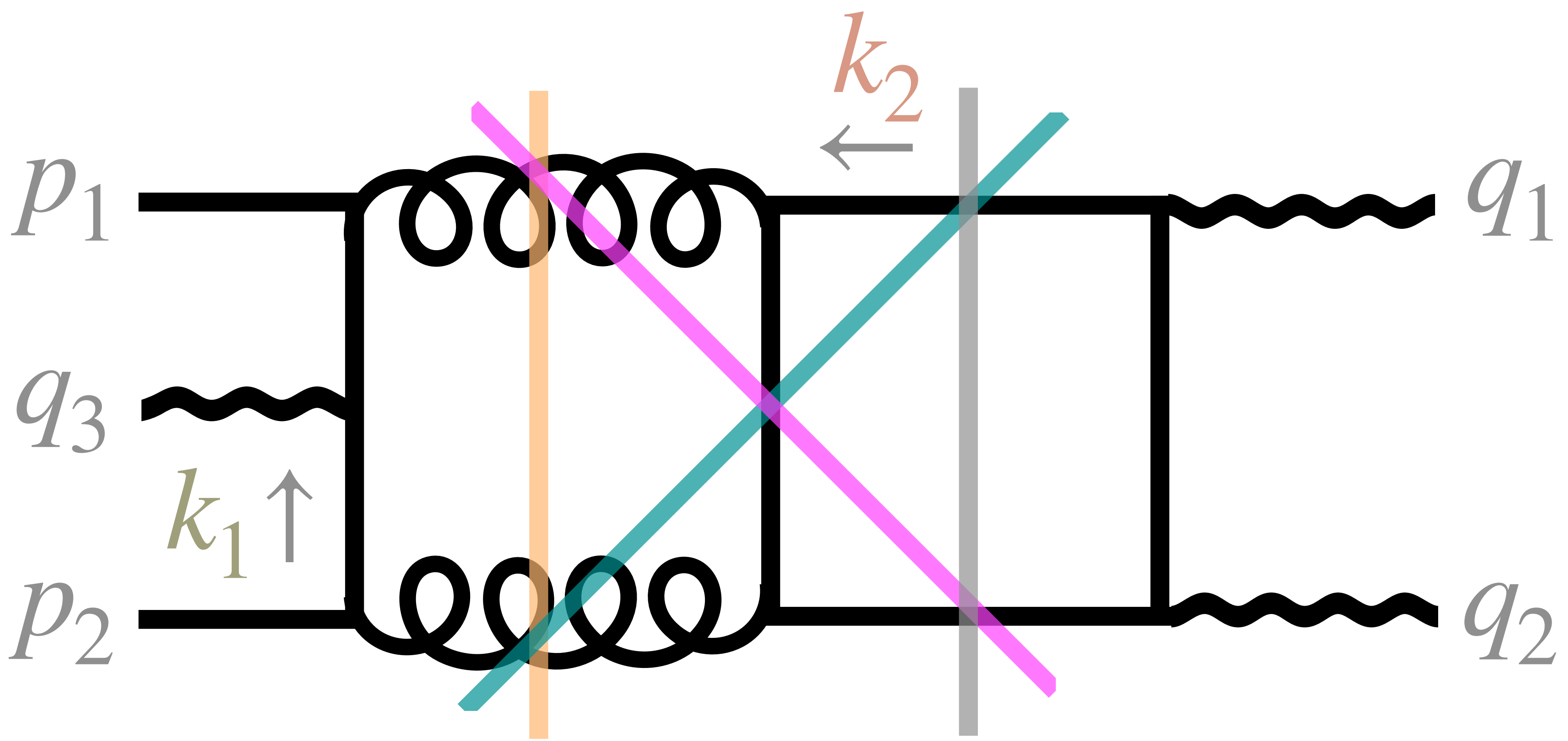


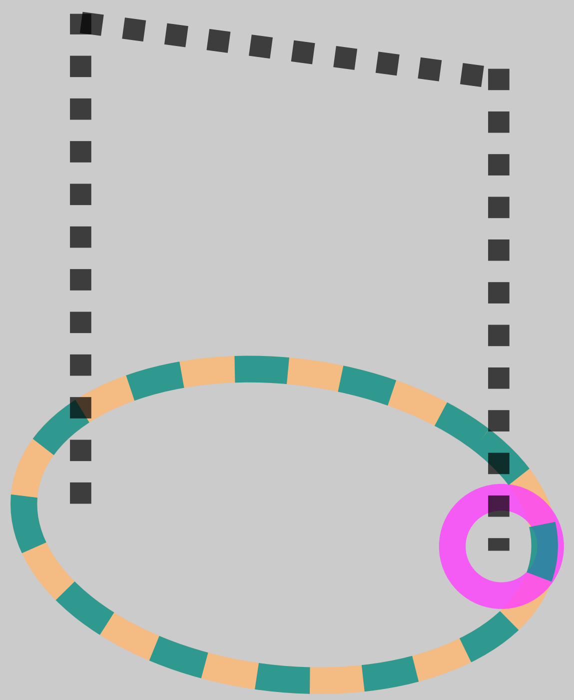
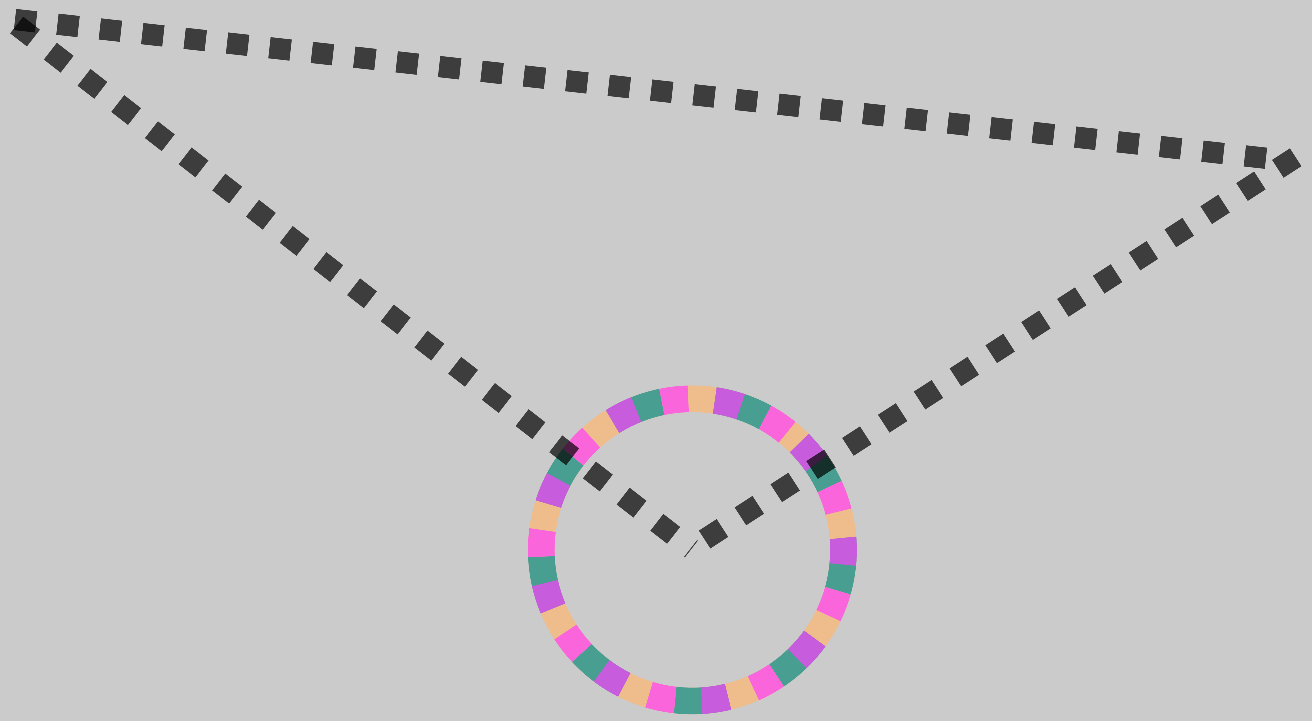


Threshold singularities
quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$



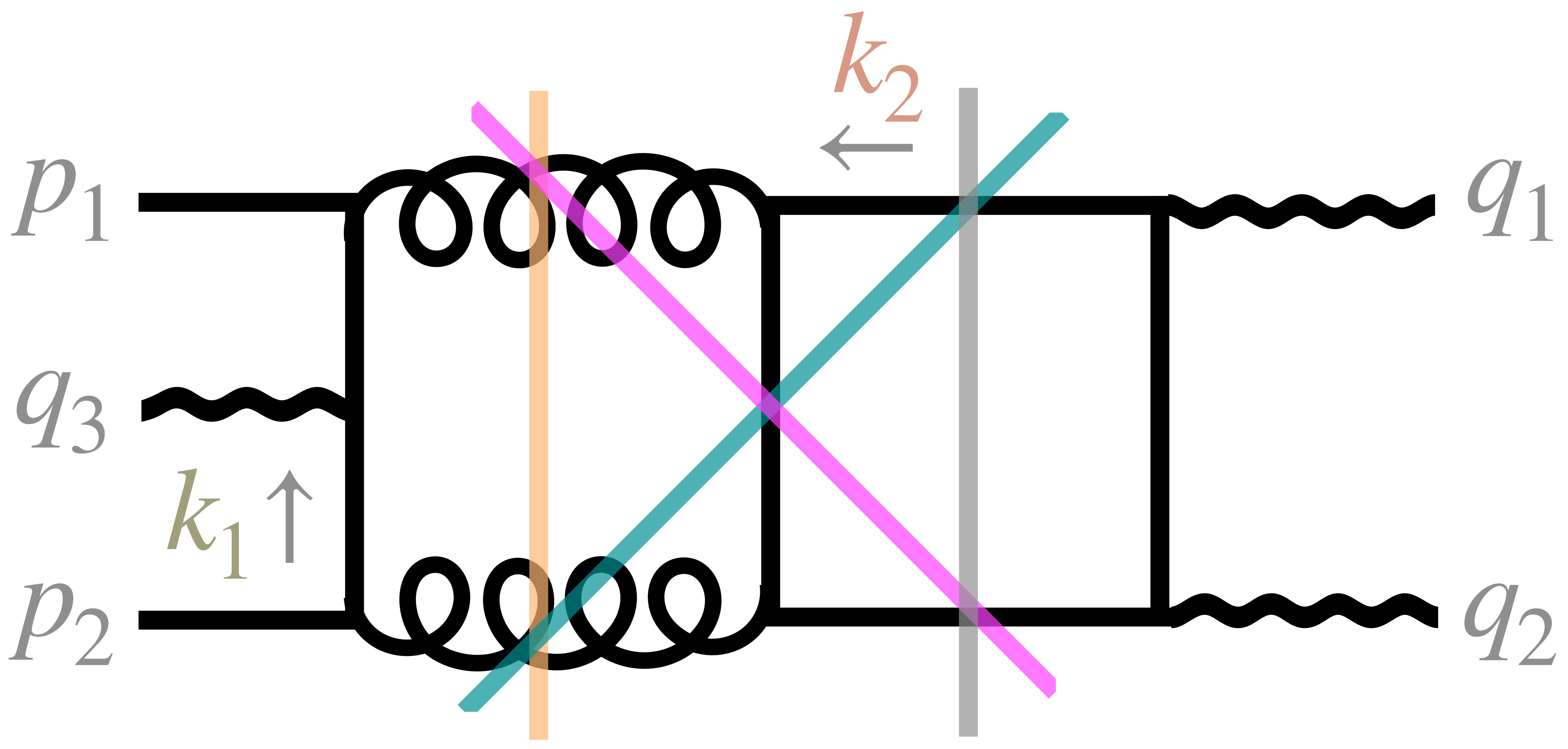
boost to simplify singularity structure	$p_1 + p_2$ at rest	$q_1 + q_2$ at rest
$\vec{k}_1$ subspace, $\vec{k}_2$ fixed		

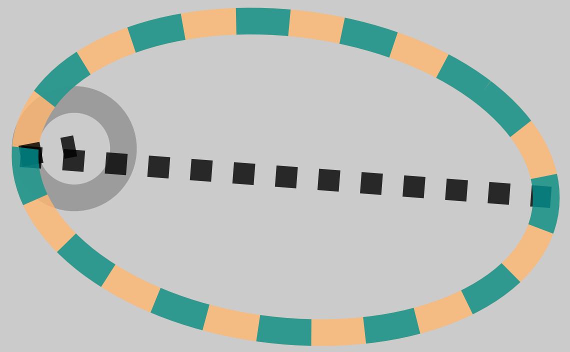
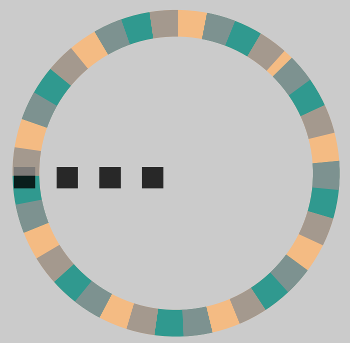
Threshold singularities

DK, Matilde Vicini [2510.18801]

quark-loop mediated two-loop amplitude

$q\bar{q} \to \gamma\gamma\gamma$



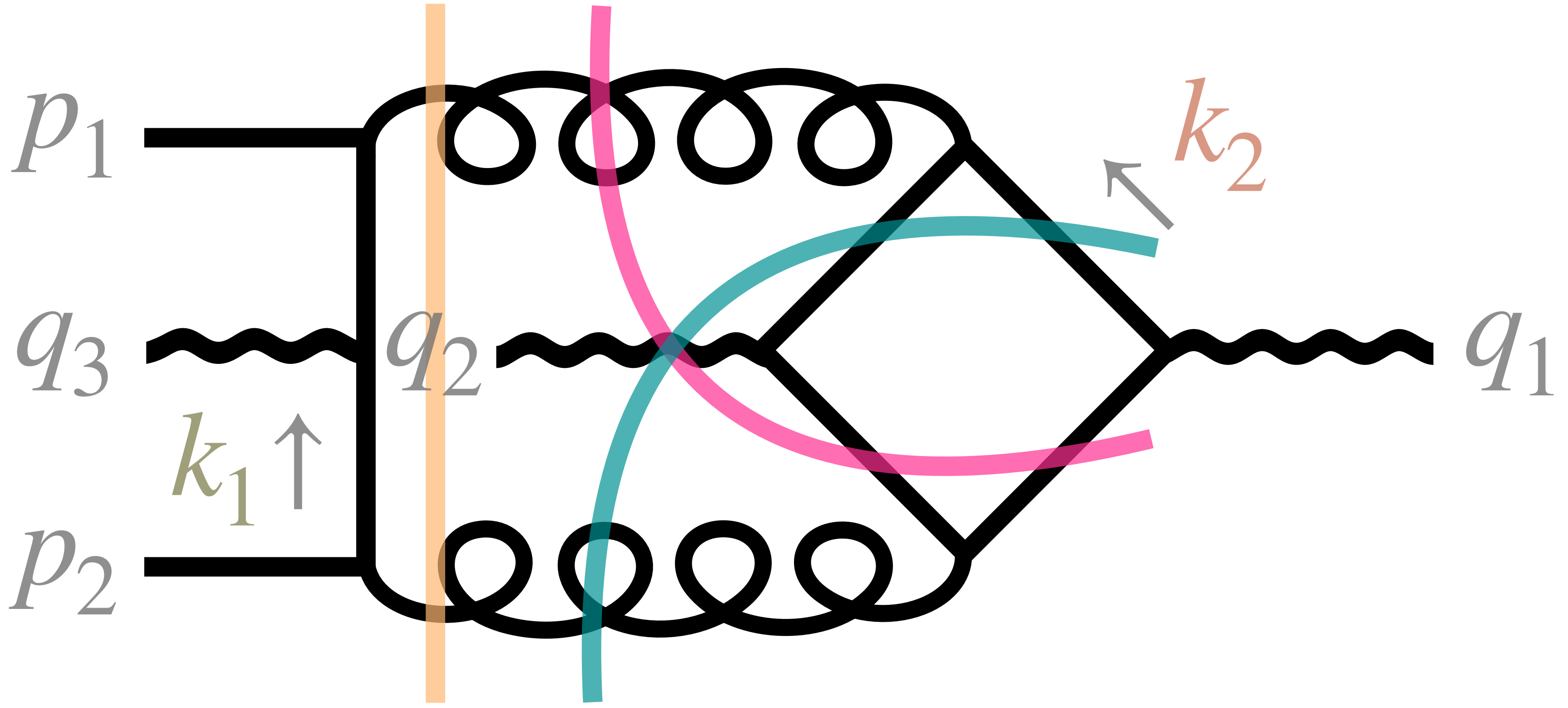
boost to simplify singularity structure	$p_1 + p_2$ at rest	$q_1 + q_2$ at rest
$\vec{k}_2$ subspace, $\vec{k}_1$ fixed		

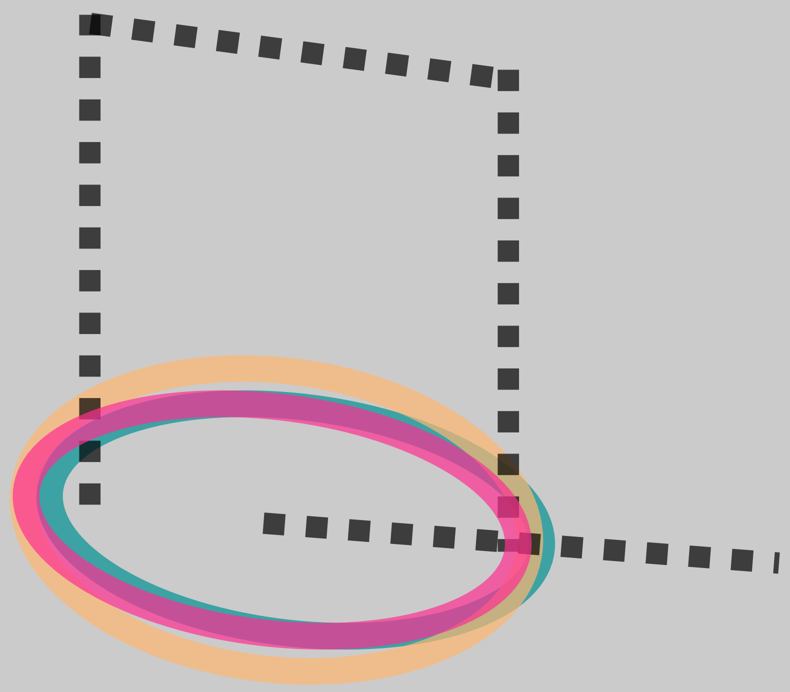
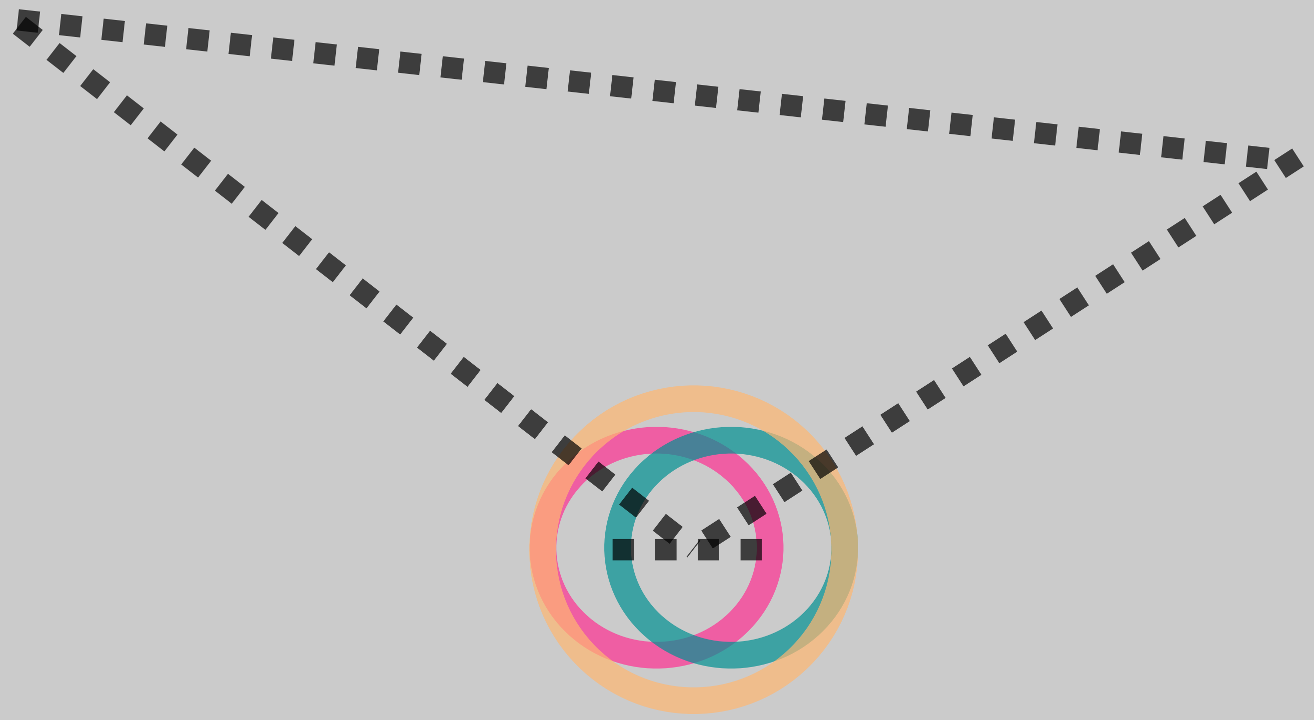
Threshold singularities

quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$

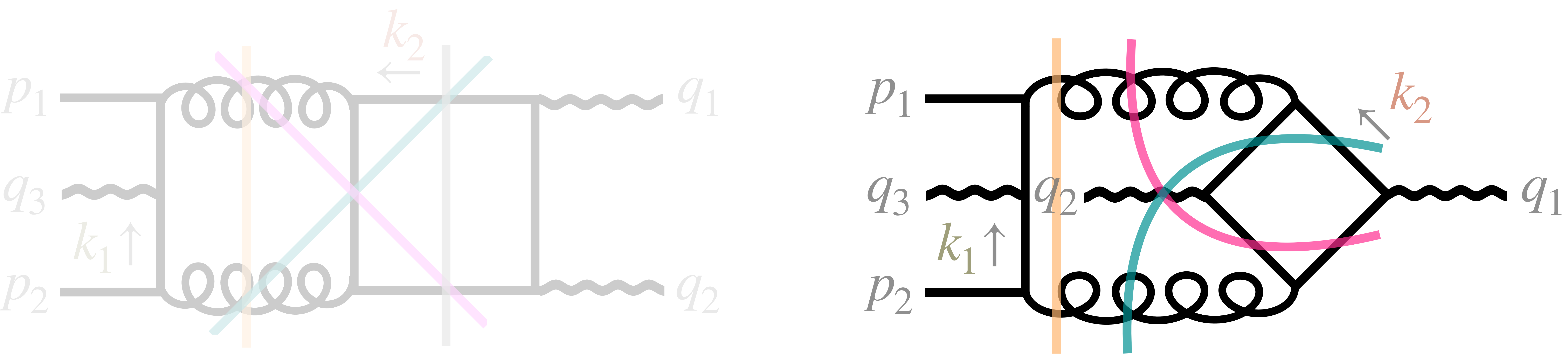


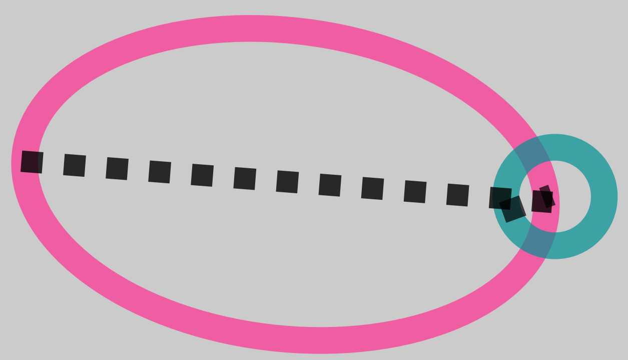
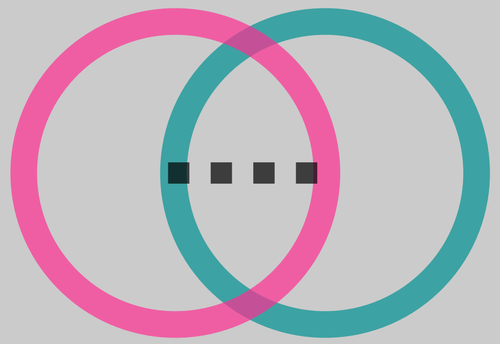
boost to simplify singularity structure	$p_1 + p_2$ at rest	$q_1 + q_2$ at rest
$\vec{k}_1$ subspace, $\vec{k}_2$ fixed		

Threshold singularities
quark-loop mediated two-loop amplitude

DK, Matilde Vicini [2510.18801]

$q\bar{q} \to \gamma\gamma\gamma$



boost to simplify singularity structure	$p_1 + p_2$ at rest	$q_1 + q_2$ at rest
$\vec{k}_2$ subspace, $\vec{k}_1$ fixed		

Integrable singularities

Kleiss, Denner, Dittmaier, Roth, Wackeroth, Soper, Weinzierl, …

Multi-channel Monte Carlo and importance sampling

Integral $\int \mathrm{d}\mathbf{k}\, f(\mathbf{k})$ → On the hypercube $\int \mathrm{d}\mathbf{x}\, \frac{f(\phi_c(\mathbf{x}))}{\rho_c(\phi_c(\mathbf{x}))}$ → Monte Carlo $\sum_i \frac{f(\phi_c(\mathbf{x}_i))}{\rho_c(\phi_c(\mathbf{x}_i))}$

ρ_c inv. Jacobian det. of ϕ_c

Integrable singularities

Kleiss, Denner, Dittmaier, Roth, Wackeroth, Soper, Weinzierl, …

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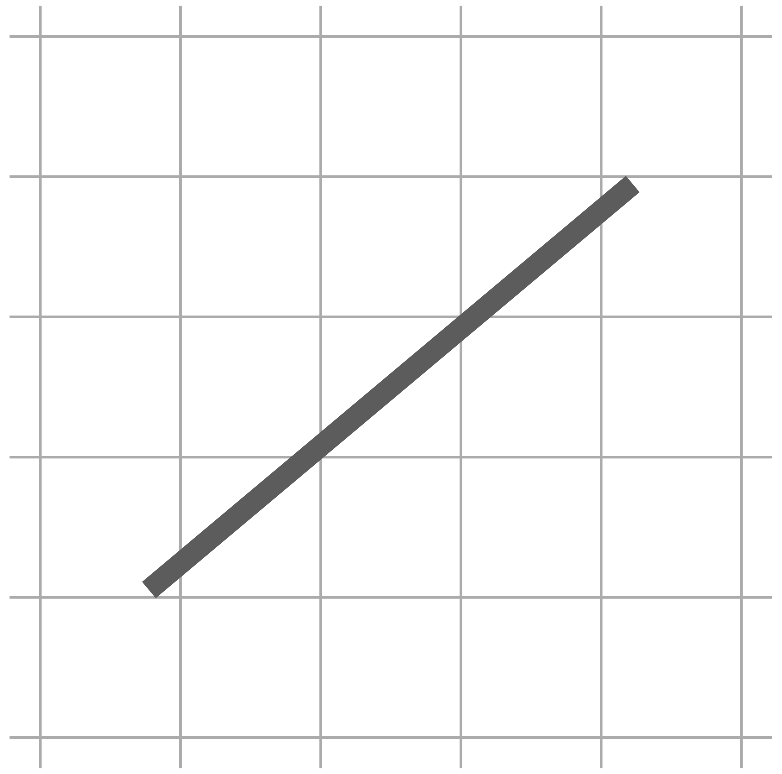

⚠️ Vegas grid can't adapt well if enhancements not aligned with integration variables

Integrable singularities

Kleiss, Denner, Dittmaier, Roth, Wackeroth, Soper, Weinzierl, …

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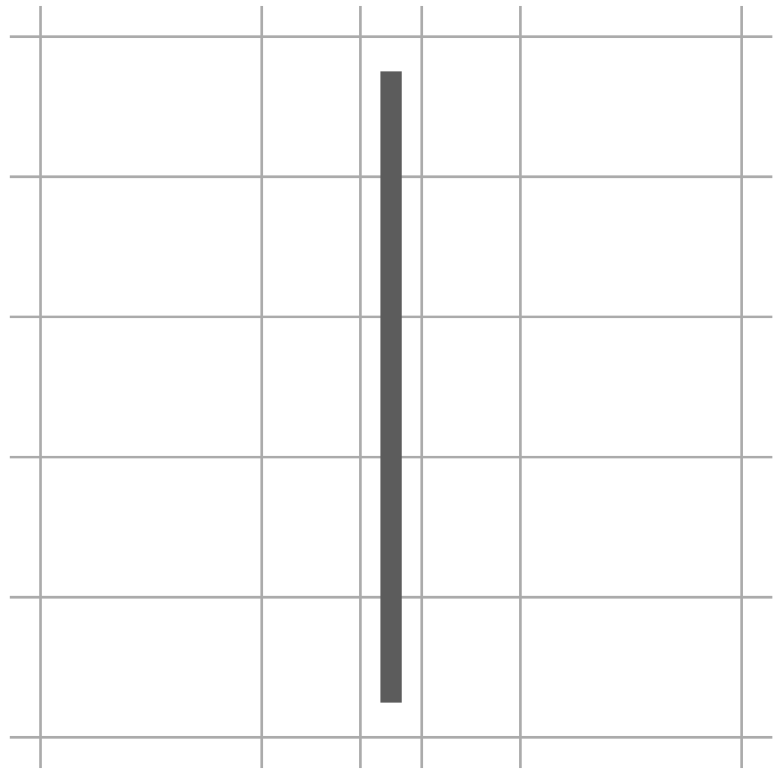

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Integral $\int \mathrm{d}\mathbf{k}\, f(\mathbf{k})$ → On the hypercube $\int \mathrm{d}\mathbf{x}\, \frac{f(\phi_c(\mathbf{x}))}{\rho_c(\phi_c(\mathbf{x}))}$ → Monte Carlo $\sum_i \frac{f(\phi_c(\mathbf{x}_i))}{\rho_c(\phi_c(\mathbf{x}_i))}$

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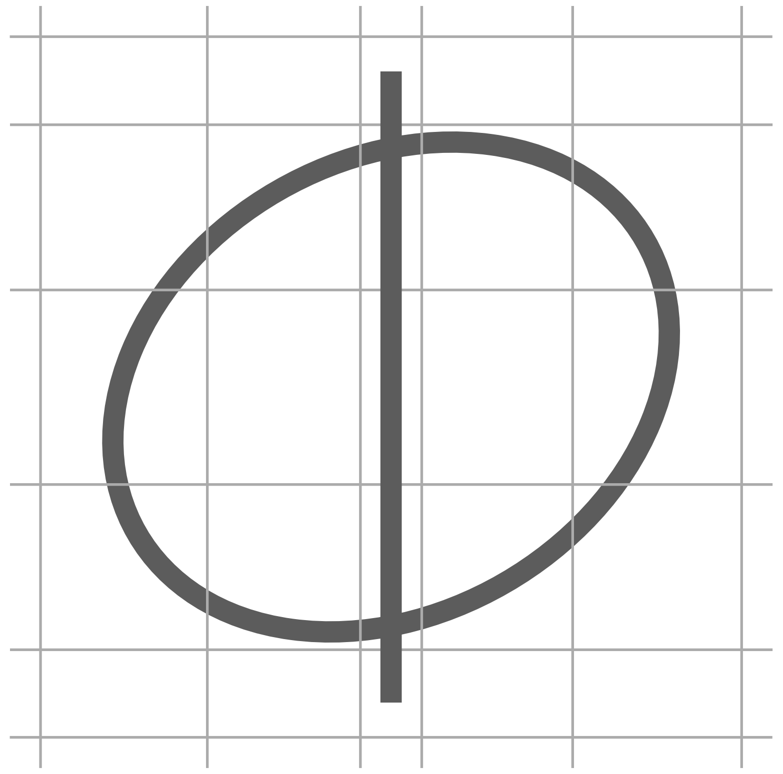

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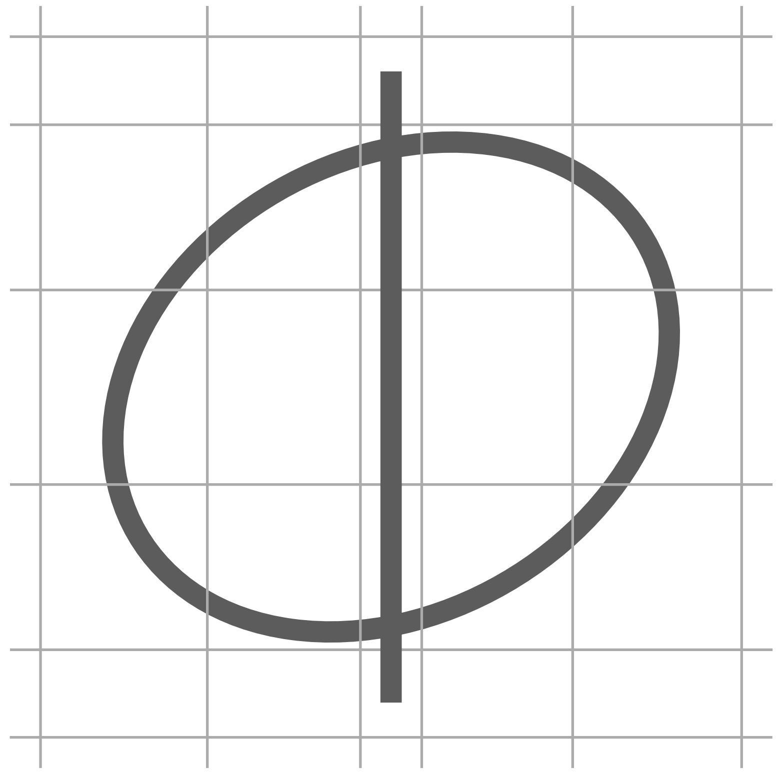
Integrable singularities

Kleiss, Denner, Dittmaier, Roth, Wackeroth, Soper, Weinzierl, …

Multi-channel Monte Carlo and importance sampling



Integral $\int \mathrm{d}\mathbf{k}\, f(\mathbf{k})$ → On the hypercube $\int \mathrm{d}\mathbf{x}\, \frac{f(\phi_c(\mathbf{x}))}{\rho_c(\phi_c(\mathbf{x}))}$, ρ_c inv. Jacobian det. of ϕ_c → Monte Carlo $\sum_i \frac{f(\phi_c(\mathbf{x}_i))}{\rho_c(\phi_c(\mathbf{x}_i))}$

↓

⚠️ Vegas grid can't adapt well if enhancements not aligned with integration variables

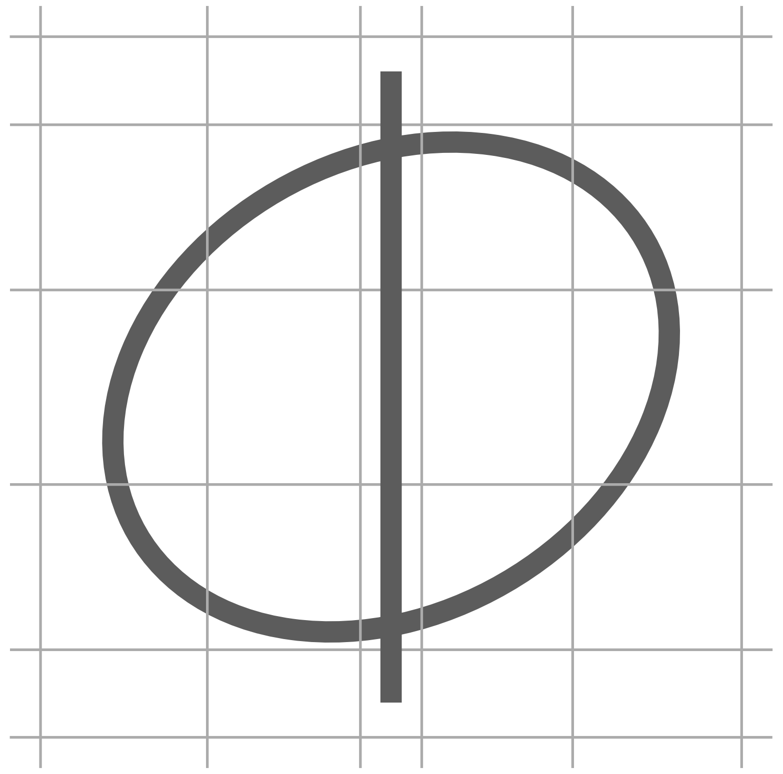
Combine multiple parameterisations $\int \mathrm{d}\mathbf{x} \sum_c \frac{f(\phi_c(\mathbf{x}))}{\rho(\phi_c(\mathbf{x}))}$, sampling density $\rho(\mathbf{k}) = \sum_c \rho_c(\mathbf{k})$ → Multi-channel Monte Carlo $\sum_i \frac{f(\phi_{c_i}(\mathbf{x}_i))}{\rho(\phi_{c_i}(\mathbf{x}_i))}$

⚠️ adds integration dimension ✔ flattens multiple peaks ✔ more accurate estimate & grid adaptation ✔ separate grids for channels

Integrable singularities

Kleiss, Denner, Dittmaier, Roth, Wackeroth, Soper, Weinzierl, …

Multi-channel Monte Carlo and importance sampling



Integral $\int \mathrm{d}\mathbf{k}\, f(\mathbf{k})$ → On the hypercube $\int \mathrm{d}\mathbf{x}\, \frac{f(\phi_c(\mathbf{x}))}{\rho_c(\phi_c(\mathbf{x}))}$ → Monte Carlo $\sum_i \frac{f(\phi_c(\mathbf{x}_i))}{\rho_c(\phi_c(\mathbf{x}_i))}$

ρ_c inv. Jacobian det. of ϕ_c

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sampling density $\rho(\mathbf{k}) = \sum_c \rho_c(\mathbf{k})$

⚠️ adds integration dimension ✔ flattens multiple peaks ✔ more accurate estimate & grid adaptation ✔ separate grids for channels

Soft $\sim \frac{1}{|\vec{q}\,|}$

from massless particles

spherical coordinates + $r \sim \frac{x}{1-x}$

improvements comparable to tropical sampling of Borinsky, Fraaije [2504.09613]

Euclidean (2, 3, 4 loops)		physical (2 & 3 loops)	
naive	adaptive	naive	adaptive
$10^2 - 10^4$	$10 - 10^3$	$10 - 10^2$	1

Collinear $\sim \frac{1}{\sqrt{|\vec{q}\,| + |\vec{q}+\vec{p}\,| - |\vec{p}\,|}}$

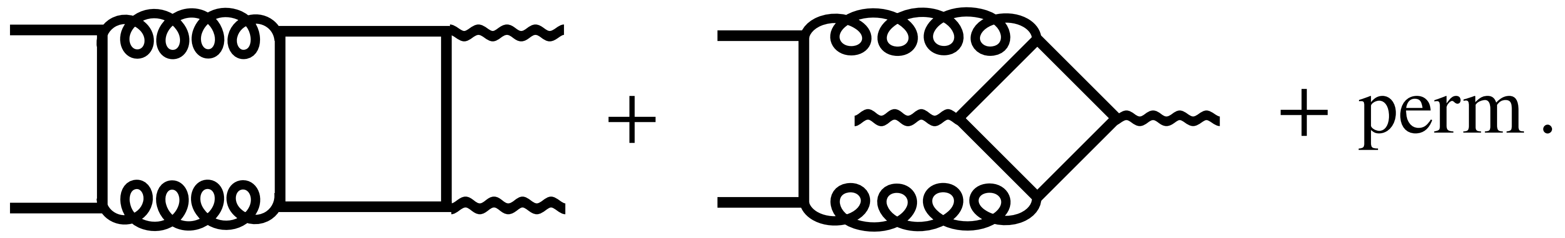
from splittings of massless particles

elliptic coordinates

physical (1 & 2 loops)	
naive	adaptive
10^2	2

How many more samples single-channel MC needs to match multi-channel MC error (extrapolated after 10^9 samples).

Two-loop corrections via quark loops



DK, Matilde Vicini [2510.18801]

Double-virtual corrections (PS integrated, incl. PDFs)

Process	Part	N_p [10^8]	Exp.	Result	Δ [%]
$q\bar{q}\to\gamma\gamma$, $m=0$	$\hat{\sigma}_{\mathrm{P}}^{(2,\tilde{N}_f),R}$	1000.8	10^{-4}	-5.4142 ± 0.0062	0.115
	$\hat{\sigma}_{\mathrm{NP}}^{(2,\tilde{N}_f),R}$	1000.8	10^{-4}	3.9034 ± 0.0030	0.076
	$R_{\mathrm{UV}}\hat{\sigma}^{(2,\tilde{N}_f)}$	1.4	10^{-4}	-1.0357 ± 0.0005	0.048
	$\boldsymbol{\sigma}^{(2,\tilde{N}_f)}$		10^{-3}	-1.4886 ± 0.0026	0.173
	benchmark	0.1	10^{-3}	-1.4905 ± 0.0004	0.025
$q\bar{q}\to\gamma^*\gamma^*$, $m=0$	$\hat{\sigma}_{\mathrm{P}}^{(2,\tilde{N}_f),R}$	352.0	10^{-4}	-1.0802 ± 0.0004	0.035
	$\hat{\sigma}_{\mathrm{NP}}^{(2,\tilde{N}_f),R}$	529.3	10^{-4}	2.8959 ± 0.0014	0.048
	$R_{\mathrm{UV}}\hat{\sigma}^{(2,\tilde{N}_f)}$	1.1	10^{-5}	-3.5849 ± 0.0017	0.047
	$\boldsymbol{\sigma}^{(2,\tilde{N}_f)}$		10^{-4}	1.1126 ± 0.0032	0.287
	benchmark	0.1	10^{-4}	1.1187 ± 0.0005	0.043
$q\bar{q}\to\gamma\gamma\gamma$, $m=0$	$\hat{\sigma}_{\mathrm{P}}^{(2,\tilde{N}_f),R}$	1808.6	10^{-6}	-4.0711 ± 0.1084	2.662
	$\hat{\sigma}_{\mathrm{NP}}^{(2,\tilde{N}_f),R}$	1570.3	10^{-6}	0.7717 ± 0.3021	39.142
	$R_{\mathrm{UV}}\hat{\sigma}^{(2,\tilde{N}_f)}$	46.1	10^{-7}	-6.9927 ± 0.0035	0.049
	$\boldsymbol{\sigma}^{(2,\tilde{N}_f)}$		10^{-5}	-4.6321 ± 0.2231	4.816
	benchmark	0.02	10^{-5}	-4.2678 ± 0.0084	0.197

Integrand evaluations times $\mathcal{O}$(0.1-10 ms)
Monte Carlo samples $\mathcal{O}(10^8\text{-}10^{10})$
Hours to days on a single node on ETH's Euler cluster

Squared matrix elements at RAMBO PS points

Analytic references from: Anastasiou, Glover, Tejeda-Yeomans [0201274] Gehrmann, von Manteuffel, Tancredi [1503.04812], Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov [2305.17056] Becchetti, Coro, Nega, Tancredi, Wagner [2502.00118]

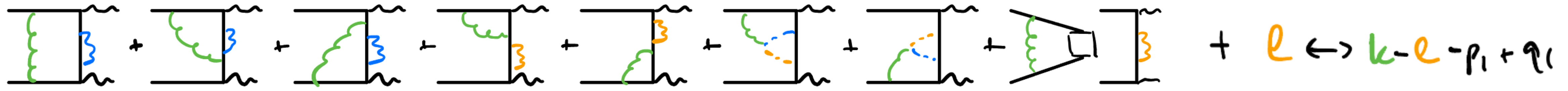
	Process	PSP	Exp.	Reference	Result	Δ [σ]	Δ [%]
	$q\bar{q}\to\gamma\gamma$, $m=0$	1	10^{+2}	-7.2127	-7.2574 ± 0.0458	0.976	0.631
		2	10^{+2}	-6.7813	-6.7649 ± 0.0421	0.391	0.622
		3	10^{+2}	-8.4533	-8.5154 ± 0.0558	1.113	0.655
	$q\bar{q}\to\gamma^*\gamma^*$, $m=0$	1	10^{+2}	-4.6281	-4.5791 ± 0.0347	1.411	0.758
		2	10^{+2}	-4.5407	-4.5137 ± 0.0352	0.768	0.780
		3	10^{+2}	-4.3357	-4.2968 ± 0.0339	1.149	0.788
	$q\bar{q}\to\gamma^*\gamma_2^*$, $m=0$	1	10^{+2}	-5.2774	-5.2742 ± 0.0087	0.363	0.164
		2	10^{+2}	-5.1052	-5.0972 ± 0.0087	0.913	0.170
		3	10^{+2}	-5.3523	-5.3433 ± 0.0110	0.821	0.206
	$q\bar{q}\to\gamma\gamma\gamma$, $m=0$	1	10^{-1}	-3.3918	-3.4066 ± 0.0159	0.929	0.467
		2	10^{+0}	-1.4313	-1.4335 ± 0.0101	0.217	0.707
		3	10^{-1}	-3.3987	-3.3930 ± 0.0167	0.341	0.493
off-shell triphoton light-quark loops	$q\bar{q}\to\gamma^*\gamma^*\gamma^*$, $m=0$	1	10^{-1}		-2.1869 ± 0.0209		0.956
		2	10^{-1}		-4.8302 ± 0.0481		0.996
		3	10^{-1}		-2.1826 ± 0.0208		0.951
	$q\bar{q}\to\gamma\gamma$, $m>0$	1	10^{+0}	4.0175	4.0709 ± 0.0358	1.491	0.879
		2	10^{+1}	-4.9939	-4.9853 ± 0.0098	0.883	0.196
		3	10^{+2}	-2.8525	-2.8432 ± 0.0075	1.236	0.264
on-shell diphoton heavy-quark loops	$q\bar{q}\to\gamma^*\gamma^*$, $m>0$	1	10^{+0}		4.0862 ± 0.0392		0.959
		2	10^{+1}		-5.3944 ± 0.0185		0.343
		3	10^{+2}		-2.9477 ± 0.0118		0.401
on-shell triphoton heavy-quark loops	$q\bar{q}\to\gamma\gamma\gamma$, $m>0$	1	10^{-2}		-5.5712 ± 0.0538		0.966
		2	10^{-1}		-3.6549 ± 0.0312		0.854
		3	10^{-2}		-3.5927 ± 0.0334		0.931

Preview of other two-loop contributions to $q\,\bar{q} \to \gamma\,\gamma$ at a random PS point

local CTs known: Anastasiou, Haindl, Karlen, Sahoo Sterman, Venkata, Vicini, Yang, Zeng [2509.07805, 2403.13712, 2212.12162, 2008.12293, 1812.03753]

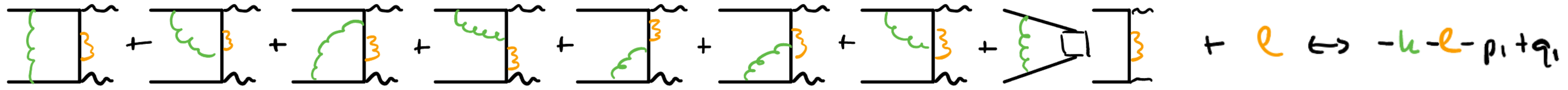
new features: triple-gluon vertex, initial- and final-state loop polarisations, transverse momentum symmetrisations, Tensor reduction through symmetrisation, …

LC SE



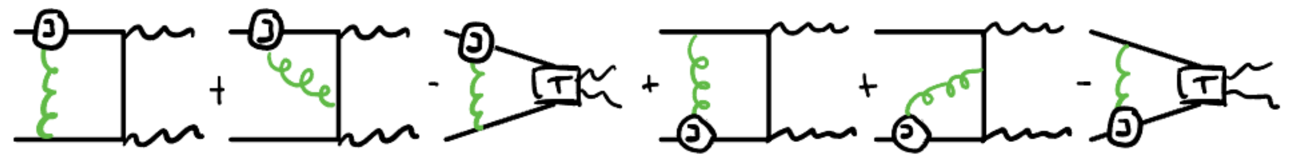
```
Iteration 133: 1010800000 evaluations so far
[1] -9.97817e-4 +- 7.83206e-7   chisq 101.26324 (132 df)
t/s: 0.006573 ms, runtime: 00:01:39:46
```

SLC SE



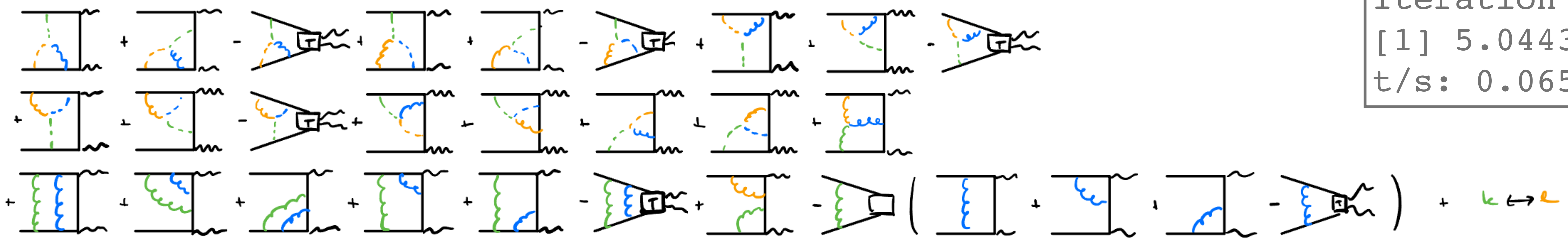
```
Iteration 133: 1010800000 evaluations so far
[1] -7.86300e-4 +- 4.65777e-7   chisq 141.01765 (132 df)
t/s: 0.007719 ms, runtime: 00:02:02:38
```

ISLP



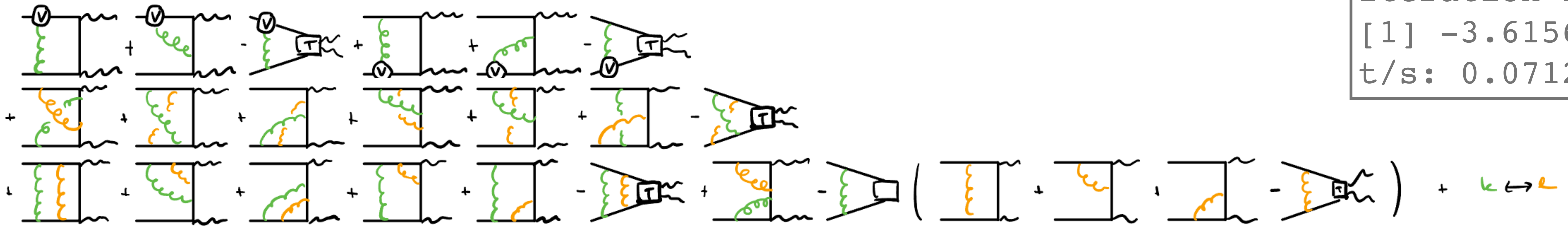
```
Iteration 133: 1010800000 evaluations so far
[1] 2.98793e-3 +- 1.57650e-6   chisq 118.49312 (132 df)
t/s: 0.013734 ms, runtime: 00:03:52:31
```

LC



```
Iteration 122: 860100000 evaluations so far
[1] 5.04434e-3 +- 6.85540e-6   chisq 129.57814 (121 df)
t/s: 0.065379 ms, runtime: 00:15:26:47
```

SLC



```
Iteration 212: 2448600000 evaluations so far
[1] -3.61560e-3 +- 6.02141e-5   chisq 204.41112 (211 df)
t/s: 0.071209 ms, runtime: 01:23:58:47
```

→ ongoing work with Matilde Vicini

Summary & Outlook

☑ State-of-the-art two-loop calculations using numerical integration over loop & phase space are feasible!

Local IR factorisation & UV renormalisation	Analytic loop energy integration LTD, CFF, TOPT, …	Threshold subtraction	Importance sampling

flexible and robust framework suited for automation

C++ • FORM • MATHEMATICA • PYTHON • QGRAF • RUST • SYMBOLICA • SPENSO • VEGAS

☑ Demonstrated for di- and triboson production processes (on- and off-shell) via quark loops (light or heavy)

☐ Apply techniques to computation of full two-loop amplitudes: work in progress!

☐ A lot more to explore…

- ☐ Multi-channel loop & phase-space integration, importance sampling, …
- ☐ Efficient integrand constructions, recursions, unitarity, …
- ☐ Combine with real radiation, processes with coloured final state, …