Magic Relations and Critical Varieties of Feynman Integrals

Giulio Crisanti

Brookhaven, 27/05/26

Based on very soon to appear work with

H. Frellesvig, A. Pokraka, S. Smith

Introduction and Motivation (1/1)

Integration by Parts

Modern state of the art scattering amplitude computations require the evaluation of enormous amounts of Feynman integrals

Fortunately this number can be drastically reduced by performing an integral reduction

Introduction and Motivation (1/1)

Integration by Parts

Modern state of the art scattering amplitude computations require the evaluation of enormous amounts of Feynman integrals

Fortunately this number can be drastically reduced by performing an integral reduction

$\mathcal{O}(10^{4-5})$ integrals $\longrightarrow$ **Integration by Parts** $\longrightarrow$ $\mathcal{O}(10^{1-2})$ master integrals

[Chetyrkin, Tkachov, 1981]

[Laporta, 2001]

Introduction and Motivation (1/1)

Integration by Parts

Modern state of the art scattering amplitude computations require the evaluation of enormous amounts of Feynman integrals

Fortunately this number can be drastically reduced by performing an integral reduction

$$\mathcal{O}(10^{4-5}) \text{ integrals} \longrightarrow \boxed{\text{Integration by Parts}} \longrightarrow \mathcal{O}(10^{1-2}) \text{ master integrals}$$

[Chetyrkin, Tkachov, 1981]

[Laporta, 2001]

Generate relations with

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0$$

Solve (row reduce) the resulting linear system of equations

Introduction and Motivation (1/1)

Integration by Parts

Modern state of the art scattering amplitude computations require the evaluation of enormous amounts of Feynman integrals

Fortunately this number can be drastically reduced by performing an integral reduction

$\mathcal{O}(10^{4-5})$ integrals → Integration by Parts → $\mathcal{O}(10^{1-2})$ master integrals

[Chetyrkin, Tkachov, 1981]

[Laporta, 2001]

Generate relations with

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0$$

Solve (row reduce) the resulting linear system of equations

Can be very expensive! A lot of effort has gone to improving IBP algorithms

One such strategy: break problem down into smaller chunks — spanning cuts

[Larsen, Zhang, 2016]

IBPs and Spanning Cuts (1/1)

Role of contours in IBPs

Integration by parts identities are valid at the level of the integrand (form)

IBPs and Spanning Cuts (1/1)

Role of contours in IBPs

Integration by parts identities are valid at the level of the integrand (form)

Thus valid for any allowed choice of contour (eg: respects boundary assumptions)

Includes taking (residues) cuts!

IBPs and Spanning Cuts (1/1)

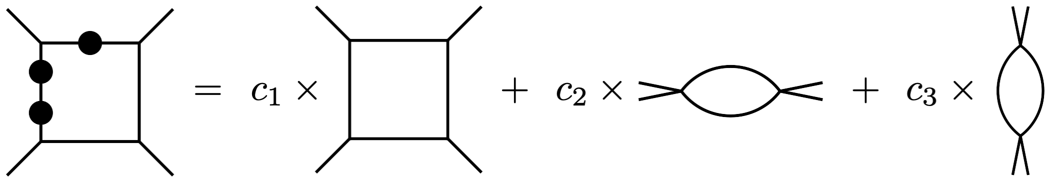
Role of contours in IBPs

Integration by parts identities are valid at the level of the integrand (form)

Thus valid for any allowed choice of contour (eg: respects boundary assumptions)

Includes taking (residues) cuts!

IBPs on cuts



(true for any valid contour)

IBPs and Spanning Cuts (1/1)

Role of contours in IBPs

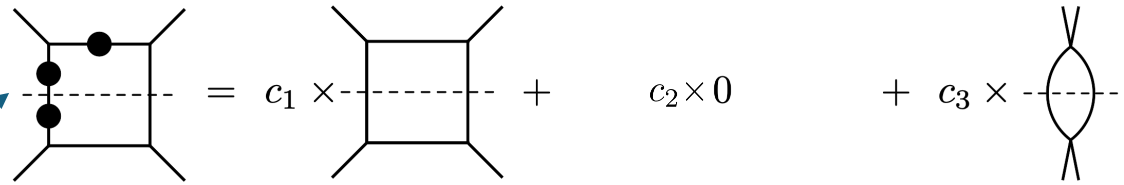
Integration by parts identities are valid at the level of the integrand (form)

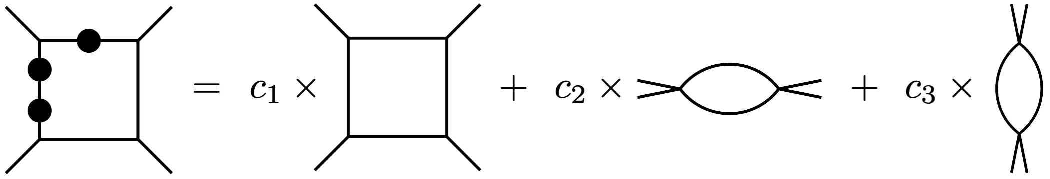
Thus valid for any allowed choice of contour (eg: respects boundary assumptions)

Includes taking (residues) cuts!

IBPs on cuts

(cut contour)





(true for any valid contour)

IBPs and Spanning Cuts (1/1)

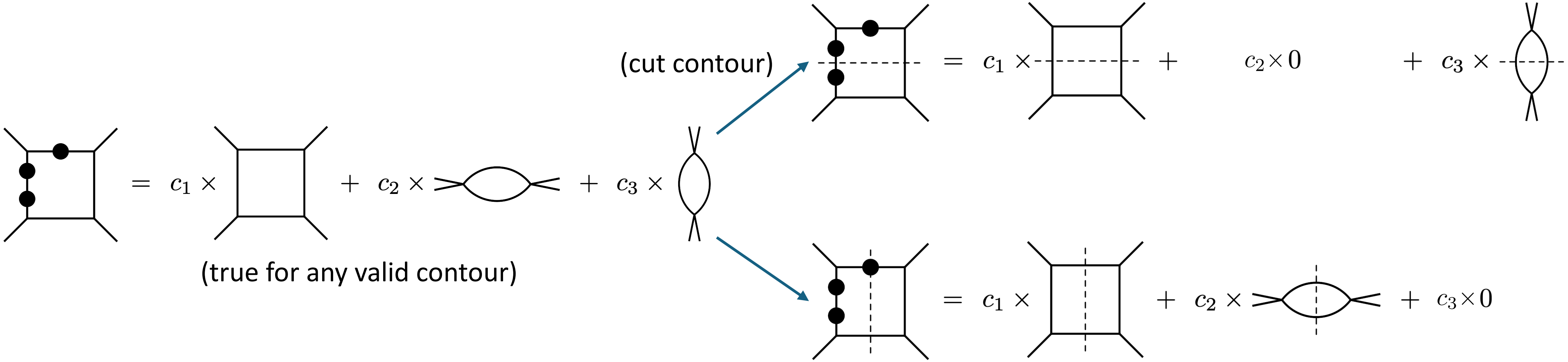
Role of contours in IBPs

Integration by parts identities are valid at the level of the integrand (form)

Thus valid for any allowed choice of contour (eg: respects boundary assumptions)

Includes taking (residues) cuts!

IBPs on cuts



Solving IBPs on a cut is easier!

IBPs and Spanning Cuts (1/1)

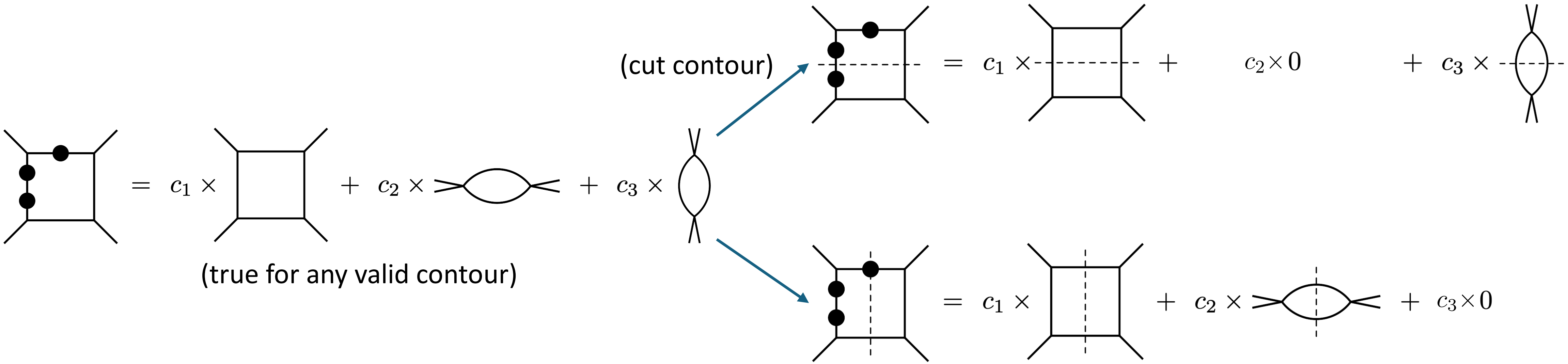
Role of contours in IBPs

Integration by parts identities are valid at the level of the integrand (form)

Thus valid for any allowed choice of contour (eg: respects boundary assumptions)

Includes taking (residues) cuts!

IBPs on cuts

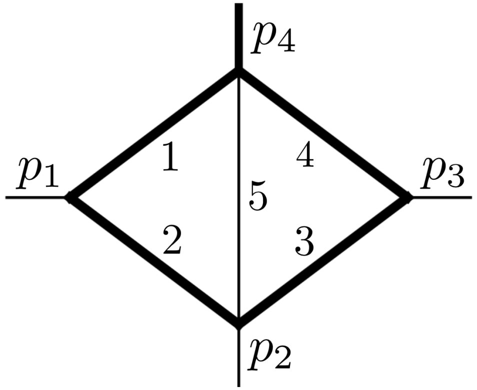


General Consensus: **Cuts & IBPs commute**

Solving IBPs on a cut is easier!

Magic Relations (1/3)

An unusual IBP identity

Consider 

$$P_1 = k_1^2 - m_t^2\,, \qquad P_2 = (k_1-p_1)^2 - m_t^2\,, \qquad P_3 = (k_2-p_1-p_2)^2 - m_t^2\,,$$
$$P_4 = (k_2+p_4)^2 - m_t^2\,, \qquad P_5 = (k_1-k_2)^2\,; \qquad P_6 = (k_1-p_1-p_2)^2 - m_t^2\,,$$
$$P_7 = k_2^2 - m_t^2\,, \qquad P_8 = (k_1+p_4)^2 - m_t^2\,, \qquad P_9 = (k_2-p_1)^2 - m_t^2\,,$$

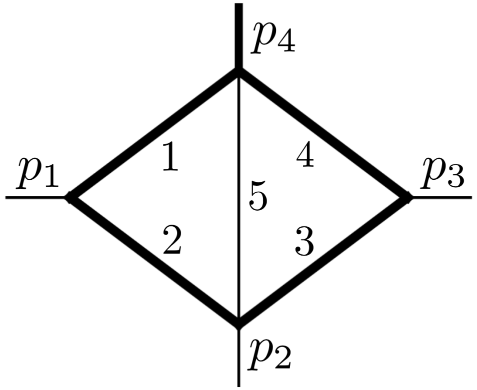
$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad p_4^2 = m_H^2\,,$$
$$(p_1+p_2)^2 = s\,, \qquad (p_2+p_3)^2 = t\,, \qquad (p_1+p_3)^2 = m_H^2-s-t$$

[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, 2016]

Magic Relations (1/3)

An unusual IBP identity

Consider



$$P_1 = k_1^2 - m_t^2\,, \qquad P_2 = (k_1-p_1)^2 - m_t^2\,, \qquad P_3 = (k_2-p_1-p_2)^2 - m_t^2\,,$$
$$P_4 = (k_2+p_4)^2 - m_t^2\,, \qquad P_5 = (k_1-k_2)^2\,; \qquad P_6 = (k_1-p_1-p_2)^2 - m_t^2\,,$$
$$P_7 = k_2^2 - m_t^2\,, \qquad P_8 = (k_1+p_4)^2 - m_t^2\,, \qquad P_9 = (k_2-p_1)^2 - m_t^2\,,$$

$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad p_4^2 = m_H^2\,,$$
$$(p_1+p_2)^2 = s\,, \qquad (p_2+p_3)^2 = t\,, \qquad (p_1+p_3)^2 = m_H^2-s-t$$

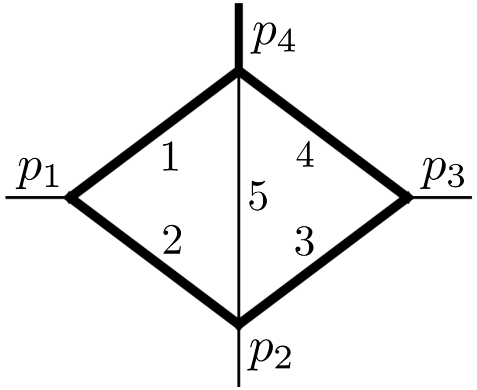
[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, 2016]

Take the IBP:

$$0 = \int \frac{d^dk_1 d^dk_2}{-\pi^d}\left(\frac{\partial}{\partial k_1^\mu}\frac{p_1^\mu(P_8-P_6)}{P_1P_2P_3P_4P_5} + \frac{\partial}{\partial k_2^\mu}\frac{p_3^\mu(P_7-P_9)}{P_1P_2P_3P_4P_5}\right)$$

Magic Relations (1/3)

An unusual IBP identity

Consider 

$$P_1 = k_1^2 - m_t^2\,, \qquad P_2 = (k_1 - p_1)^2 - m_t^2\,, \qquad P_3 = (k_2 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_4 = (k_2 + p_4)^2 - m_t^2\,, \qquad P_5 = (k_1 - k_2)^2\,; \qquad P_6 = (k_1 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_7 = k_2^2 - m_t^2\,, \qquad P_8 = (k_1 + p_4)^2 - m_t^2\,, \qquad P_9 = (k_2 - p_1)^2 - m_t^2\,,$$

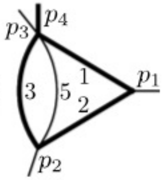
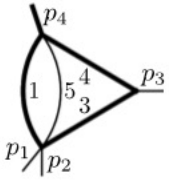
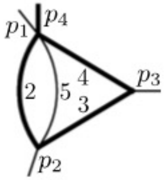
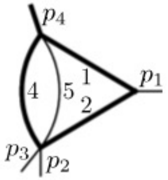
$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad p_4^2 = m_H^2\,,$$
$$(p_1 + p_2)^2 = s\,, \qquad (p_2 + p_3)^2 = t\,, \qquad (p_1 + p_3)^2 = m_H^2 - s - t$$

[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, 2016]

Take the IBP:

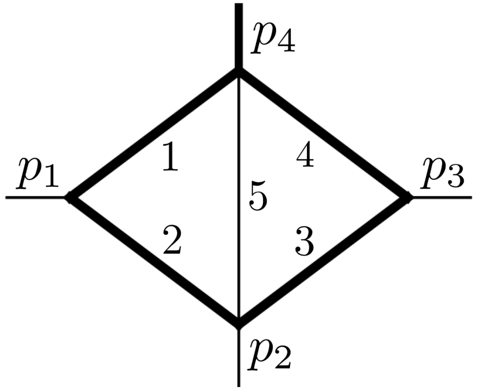
$$0 = \int \frac{d^d k_1 d^d k_2}{-\pi^d} \left(\frac{\partial}{\partial k_1^\mu} \frac{p_1^\mu (P_8 - P_6)}{P_1 P_2 P_3 P_4 P_5} + \frac{\partial}{\partial k_2^\mu} \frac{p_3^\mu (P_7 - P_9)}{P_1 P_2 P_3 P_4 P_5} \right)$$

$$s\, I_{11101;0000} + (m_H^2 - s) I_{10111;0000} = t\, I_{01111;0000} + (m_H^2 - t) I_{11011;0000}$$

$s \times$  $+ (m_H^2 - s) \times$  $= t \times$  $+ (m_H^2 - t) \times$ 

Magic Relations (1/3)

An unusual IBP identity

Consider 

$$P_1 = k_1^2 - m_t^2\,, \qquad P_2 = (k_1 - p_1)^2 - m_t^2\,, \qquad P_3 = (k_2 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_4 = (k_2 + p_4)^2 - m_t^2\,, \qquad P_5 = (k_1 - k_2)^2\,; \qquad P_6 = (k_1 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_7 = k_2^2 - m_t^2\,, \qquad P_8 = (k_1 + p_4)^2 - m_t^2\,, \qquad P_9 = (k_2 - p_1)^2 - m_t^2\,,$$

$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad p_4^2 = m_H^2\,,$$
$$(p_1+p_2)^2 = s\,, \qquad (p_2+p_3)^2 = t\,, \qquad (p_1+p_3)^2 = m_H^2 - s - t$$

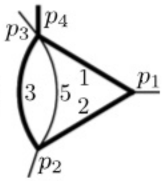
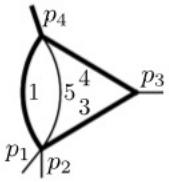
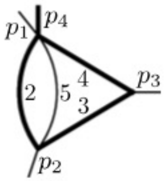
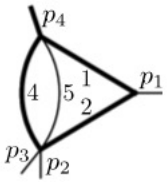
[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, 2016]

Take the IBP:

$$0 = \int \frac{d^d k_1 d^d k_2}{-\pi^d} \left(\frac{\partial}{\partial k_1^\mu} \frac{p_1^\mu (P_8 - P_6)}{P_1 P_2 P_3 P_4 P_5} + \frac{\partial}{\partial k_2^\mu} \frac{p_3^\mu (P_7 - P_9)}{P_1 P_2 P_3 P_4 P_5} \right)$$

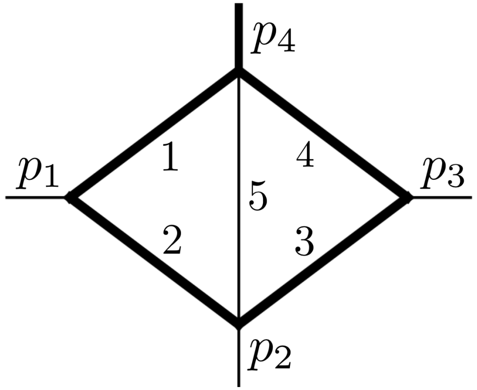
$$s\, I_{11101;0000} + (m_H^2 - s) I_{10111;0000} = t\, I_{01111;0000} + (m_H^2 - t) I_{11011;0000}$$

No top sector!

$s \times$  $+ (m_H^2 - s) \times$  $= t \times$  $+ (m_H^2 - t) \times$ 

Magic Relations (1/3)

An unusual IBP identity

Consider 

$$P_1 = k_1^2 - m_t^2\,, \qquad P_2 = (k_1 - p_1)^2 - m_t^2\,, \qquad P_3 = (k_2 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_4 = (k_2 + p_4)^2 - m_t^2\,, \qquad P_5 = (k_1 - k_2)^2\,; \qquad P_6 = (k_1 - p_1 - p_2)^2 - m_t^2\,,$$
$$P_7 = k_2^2 - m_t^2\,, \qquad P_8 = (k_1 + p_4)^2 - m_t^2\,, \qquad P_9 = (k_2 - p_1)^2 - m_t^2\,,$$

$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad p_4^2 = m_H^2\,,$$
$$(p_1 + p_2)^2 = s\,, \qquad (p_2 + p_3)^2 = t\,, \qquad (p_1 + p_3)^2 = m_H^2 - s - t$$

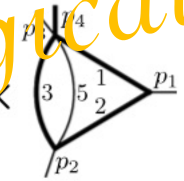
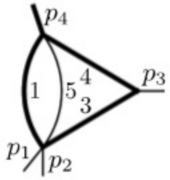
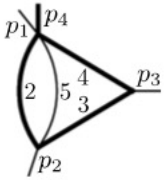
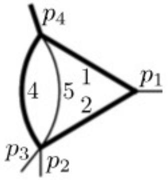
[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, 2016]

Take the IBP:
$$0 = \int \frac{d^d k_1 d^d k_2}{-\pi^d} \left(\frac{\partial}{\partial k_1^\mu} \frac{p_1^\mu (P_8 - P_6)}{P_1 P_2 P_3 P_4 P_5} + \frac{\partial}{\partial k_2^\mu} \frac{p_3^\mu (P_7 - P_9)}{P_1 P_2 P_3 P_4 P_5} \right)$$

$$s\, I_{11101;0000} + (m_H^2 - s) I_{10111;0000} = t\, I_{01111;0000} + (m_H^2 - t) I_{11011;0000}$$

No top sector!

Pretty Magical

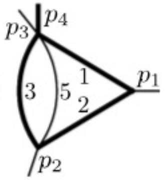
$s \times$  $+ (m_H^2 - s) \times$  $= t \times$  $+ (m_H^2 - t) \times$ 

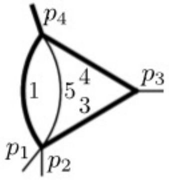
[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

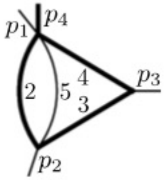
Magic Relations (2/3)

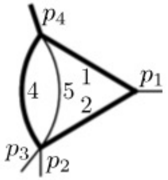
Magic Relations and Cuts

$$s\, I_{11101;0000} + (m_H^2 - s) I_{10111;0000} = t\, I_{01111;0000} + (m_H^2 - t) I_{11011;0000} = 0$$

$s \times$ 

$+ (m_H^2 - s) \times$ 

$= t \times$ 

$+ (m_H^2 - t) \times$ 

$= 0$

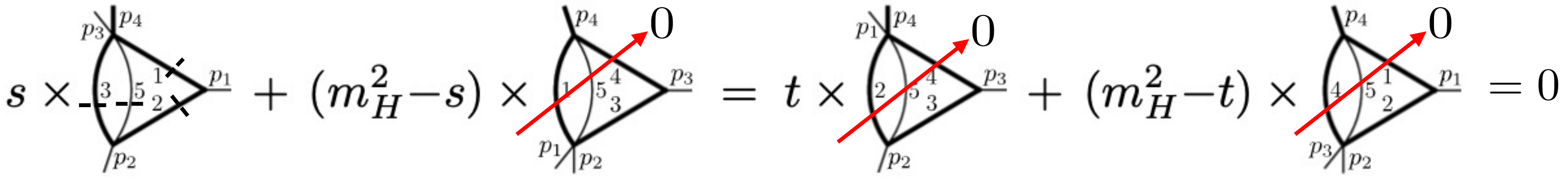
Magic Relations (2/3)

Magic Relations and Cuts

Cut propagators

$$\{1,2,3,5\}$$

$$s\, I_{\cancel{1}\cancel{1}\cancel{1}0\cancel{1};0000} + (m_H^2-s)\cancelto{0}{I_{10111;0000}} = t\, \cancelto{0}{I_{01111;0000}} + (m_H^2-t)\cancelto{0}{I_{11011;0000}} = 0$$

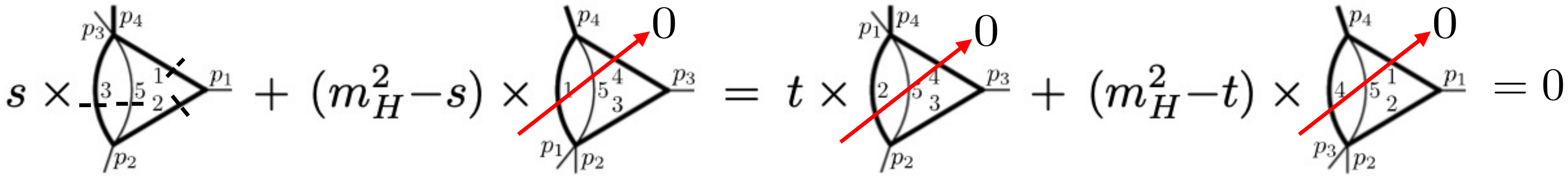


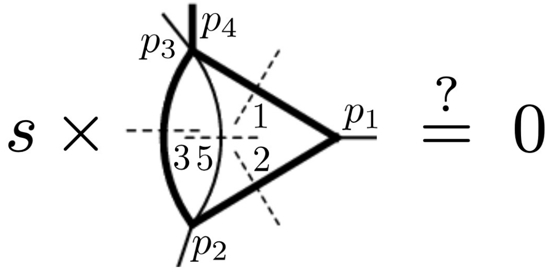
Magic Relations (2/3)

Magic Relations and Cuts

Cut propagators
$\{1, 2, 3, 5\}$

$$s\, I_{11101;0000} + (m_H^2 - s)\cancelto{0}{I_{10111;0000}} = t\, \cancelto{0}{I_{01111;0000}} + (m_H^2 - t)\cancelto{0}{I_{11011;0000}} = 0$$





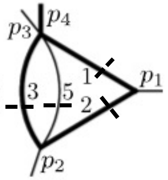
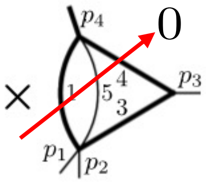
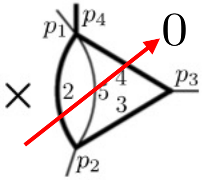
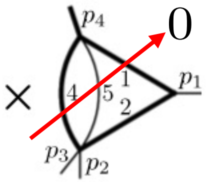
Magic Relations (2/3)

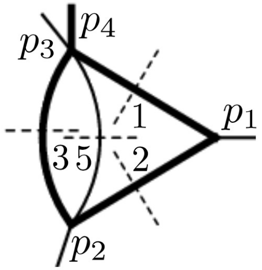
Magic Relations and Cuts

Cut propagators

$\{1, 2, 3, 5\}$

$$s\, I_{\cancel{11101};0000} + (m_H^2 - s)\cancelto{0}{I_{10111;0000}} = t\, \cancelto{0}{I_{01111;0000}} + (m_H^2 - t)\cancelto{0}{I_{11011;0000}} = 0$$

$s \times$  $+ (m_H^2 - s) \times$  $= t \times$  $+ (m_H^2 - t) \times$  $= 0$

$s \times$ 

$\stackrel{?}{=} 0$ **NO!**

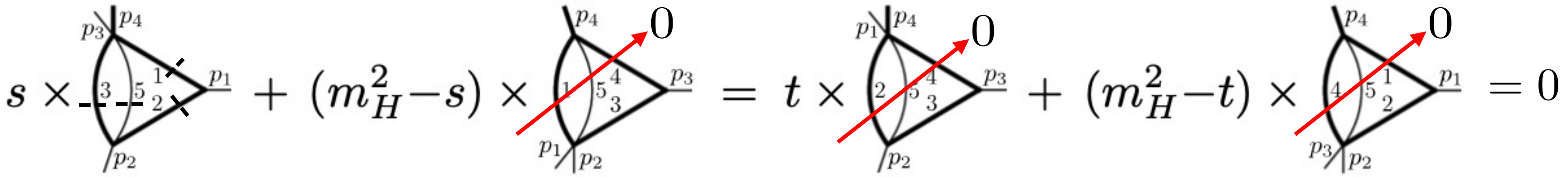
$$= \frac{(-1)^{3d/2}\,(m_t^2)^{d-3}\,\Gamma((4-d)/2)}{4\pi^2 s \Gamma(d/2)} \neq 0$$

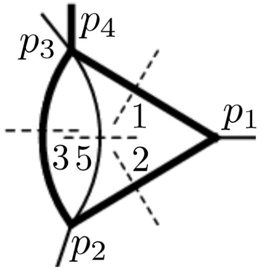
Magic Relations (2/3)

Magic Relations and Cuts

Cut propagators
$\{1,2,3,5\}$

$$s\, I_{\cancel{11101};0000} + (m_H^2 - s)\cancelto{0}{I_{10111;0000}} = t\, \cancelto{0}{I_{01111;0000}} + (m_H^2 - t)\cancelto{0}{I_{11011;0000}} = 0$$



$s \times$  $\stackrel{?}{=} 0$

NO!

$$= \frac{(-1)^{3d/2}\,(m_t^2)^{d-3}\,\Gamma((4-d)/2)}{4\pi^2 s\Gamma(d/2)} \neq 0$$

Cuts & IBPs do NOT commute in the presence of magic relations

↓

Magic relations break the notion of “sectors” in Feynman integrals

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

The generating sector cannot appear in the identity, only its subsectors can appear

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that: so not a symmetry relation for example

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

so not a symmetry relation for example

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Important questions to answer

Is there a systematic way to detect if a Feynman integral will have magic relations?

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

so not a symmetry relation for example

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Important questions to answer

Is there a systematic way to detect if a Feynman integral will have magic relations?

Is there a systematic algorithm to find such magic relations?

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that:

so not a symmetry relation for example

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Important questions to answer

Is there a systematic way to detect if a Feynman integral will have magic relations?

Is there a systematic algorithm to find such magic relations?

What is the exact mechanism that causes the magic IBP to no longer be valid on the cut?

Magic Relations (3/3)

A (somewhat) precise definition

A magic relation is an IBP identity such that: so not a symmetry relation for example

The generating sector cannot appear in the identity, only its subsectors can appear

Only the generating sector can produce this identity

It cannot be obtained by combining IBPs generated in the subsectors themselves

[Lee, Smirnov, 2010]
[Smirnov, Smirnov, 2013]
[Maierhöfer, Usovitsch, 2018]

Important questions to answer

Is there a systematic way to detect if a Feynman integral will have magic relations?

Following slides

Yes! (Conjectured — partial proof)

Is there a systematic algorithm to find such magic relations?

Yes! (Conjectured — partial proof)

What is the exact mechanism that causes the magic IBP to no longer be valid on the cut?

We have literally no idea please help us

Finding Magic Relations Systematically (1/2)

Computational Procedure

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij}, m) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij}, m) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle$$

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij},m) = \int_0^\infty G(x,s_{ij},m)^{-d/2}\,\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1},\ldots,\frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle \quad \dim(\mathcal{I}) := \text{dimension of solutions to the eq syst } \mathcal{I} = 0$$

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij}, m) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle$$ $\dim(\mathcal{I}) :=$ dimension of solutions to the eq syst $\mathcal{I} = 0$

Key observation

$$\dim(\mathcal{I}) = 0 \longleftrightarrow \text{no magic relations}$$

$$\dim(\mathcal{I}) \neq 0 \longleftrightarrow \text{magic relations}$$

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij}, m) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle \quad \dim(\mathcal{I}) := \text{dimension of solutions to the eq syst } \mathcal{I} = 0$$

Key observation

$$\dim(\mathcal{I}) = 0 \longleftrightarrow \text{no magic relations}$$

$$\dim(\mathcal{I}) \neq 0 \longleftrightarrow \text{magic relations}$$

Possibly additionally an even stronger version:

$$\dim(\mathcal{I}) = \#\text{ of magic relations}$$

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij},m) = \int_0^\infty G(x,s_{ij},m)^{-d/2}\,\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1},\ldots,\frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle \quad \dim(\mathcal{I}) := \text{dimension of solutions to the eq syst } \mathcal{I}=0$$

Key observation

$$\dim(\mathcal{I}) = 0 \longleftrightarrow \text{no magic relations}$$ ← Proof (under some minor technical assumptions)

$$\dim(\mathcal{I}) \neq 0 \longleftrightarrow \text{magic relations}$$ ← Conjecture — very reasonable based on why the proof above breaks down in this case

Possibly additionally an even stronger version:

$$\dim(\mathcal{I}) = \#\text{ of magic relations}$$ ← Need more data to be confident in conjecturing but it seems plausible

Finding Magic Relations Systematically (1/2)

Computational Procedure

Convert the Feynman integral to parametric representation:

Lee-Pomeransky: $$I(s_{ij}, m) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$ [Lee, Pomeransky, 2013]

Critical point ideal: $$\mathcal{I} := \left\langle \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}, 1 - x_0 G \right\rangle$$ $\dim(\mathcal{I}) :=$ dimension of solutions to the eq syst $\mathcal{I} = 0$

Key observation

$$\dim(\mathcal{I}) = 0 \longleftrightarrow \text{no magic relations}$$ ← Proof (under some minor technical assumptions)

$$\dim(\mathcal{I}) \neq 0 \longleftrightarrow \text{magic relations}$$ ← Conjecture — very reasonable based on why the proof above breaks down in this case

Possibly additionally an even stronger version:

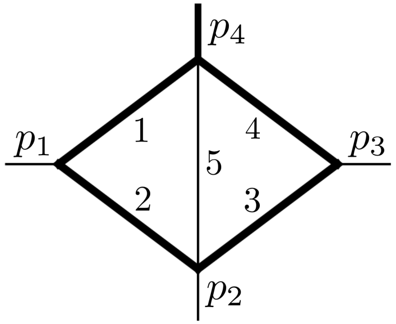
$$\dim(\mathcal{I}) = \# \text{ of magic relations}$$ ← Need more data to be confident in conjecturing but it seems plausible

$\dim(\mathcal{I})$ is in practice easy to compute — Mathematica code that implements this check

Finding Magic Relations Systematically (2/2)

Super quick example

Taking the example from the previous slides



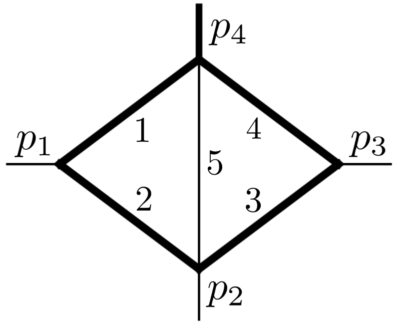

```
gpol = (x1 + x2) (x3 + x4) (1 + mt^2 (x1 + x2 + x3 + x4)) +
    (mt^2 x1^2 + x2 + x3 + x4 - t x2 x4 + mt^2 (x2 + x3 + x4)^2 + x1 (1 - s x3 - mh^2 x4 + 2 mt^2 (x2 + x3 + x4)))
     x5;
ideal = D[gpol, {{x1, x2, x3, x4, x5}}] ~Join~ {1 - x0 * gpol};
Solve[ideal == 0, {x0, x1, x2, x3, x4, x5}];
```

Solve: Equations may not give solutions for all "solve" variables.

Finding Magic Relations Systematically (2/2)

Super quick example

Taking the example from the previous slides




```
gpol = (x1 + x2) (x3 + x4) (1 + mt^2 (x1 + x2 + x3 + x4)) +
    (mt^2 x1^2 + x2 + x3 + x4 - t x2 x4 + mt^2 (x2 + x3 + x4)^2 + x1 (1 - s x3 - mh^2 x4 + 2 mt^2 (x2 + x3 + x4)))
     x5;
ideal = D[gpol, {{x1, x2, x3, x4, x5}}] ~Join~ {1 - x0 * gpol};
Solve[ideal == 0, {x0, x1, x2, x3, x4, x5}];
```

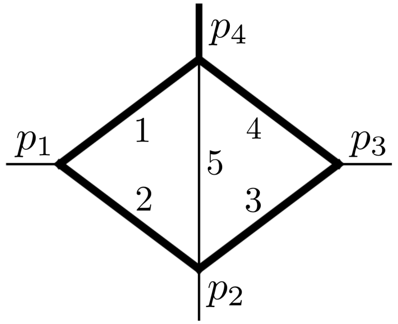
Solve: Equations may not give solutions for all "solve" variables.

$\dim(\mathcal{I}) \neq 0$

Finding Magic Relations Systematically (2/2)

Super quick example

Taking the example from the previous slides




```
gpol = (x1 + x2) (x3 + x4) (1 + mt^2 (x1 + x2 + x3 + x4)) +
    (mt^2 x1^2 + x2 + x3 + x4 - t x2 x4 + mt^2 (x2 + x3 + x4)^2 + x1 (1 - s x3 - mh^2 x4 + 2 mt^2 (x2 + x3 + x4)))
     x5;
ideal = D[gpol, {{x1, x2, x3, x4, x5}}]~Join~{1 - x0 * gpol};
Solve[ideal == 0, {x0, x1, x2, x3, x4, x5}];
```

Solve: Equations may not give solutions for all "solve" variables.

$\dim(\mathcal{I}) \neq 0$

This is definitely not the best way to do this!

Ideal dimensions can be computed more efficiently — provide a Mathematica implementation

```
MagicQ[gpol, {x1, x2, x3, x4, x5}]

True
```

Also very doable with Singular/Macaulay2/msolve and many more

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

$$\int_{\Gamma}\left(\frac{d}{2}\phi_0+\sum_{k=1}^{n}\partial_k\phi_k\right)G^{-\frac{d}{2}}\bigwedge_{i=1}^{n}\mathrm{d}x_i=-\sum_{k=1}^{n}\int_{\partial\Gamma_k}\left(\phi_k G^{-\frac{d}{2}}\right)_{x_k=0}\mathrm{d}\widehat{x}_k\,.$$

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

$$\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^n \partial_k \phi_k \right) G^{-\frac{d}{2}} \bigwedge_{i=1}^n \mathrm{d}x_i = -\sum_{k=1}^n \int_{\partial\Gamma_k} \left(\phi_k G^{-\frac{d}{2}} \right)_{x_k=0} \mathrm{d}\widehat{x}_k \, .$$

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

$$\underbrace{\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^{n} \partial_k \phi_k \right) G^{-\frac{d}{2}} \bigwedge_{i=1}^{n} \mathrm{d}x_i}_{\text{Top Sector}} = \underbrace{- \sum_{k=1}^{n} \int_{\partial\Gamma_k} \left(\phi_k G^{-\frac{d}{2}} \right)_{x_k=0} \mathrm{d}\widehat{x}_k}_{\text{Subsectors}} \, .$$

Intuition for connection (1/2)

IBPs in Parametric Representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

Consider IBPs in Lee-Pomeransky representation

$$\underbrace{\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^{n} \partial_k \phi_k\right) G^{-\frac{d}{2}} \bigwedge_{i=1}^{n} \mathrm{d}x_i}_{\text{Top Sector}} = \underbrace{-\sum_{k=1}^{n} \int_{\partial\Gamma_k} \left(\phi_k G^{-\frac{d}{2}}\right)_{x_k=0} \mathrm{d}\widehat{x}_k}_{\text{Subsectors}}\,.$$

Want the top sector contribution to vanish:

“divergence free syzygy”

$$\phi_0 = \sum_{k=1}^{n} \partial_k \phi_k = 0$$

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

$$\underbrace{\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^n \partial_k \phi_k \right) G^{-\frac{d}{2}} \bigwedge_{i=1}^n \mathrm{d}x_i}_{\text{Top Sector}} = \underbrace{- \sum_{k=1}^n \int_{\partial \Gamma_k} \left(\phi_k G^{-\frac{d}{2}} \right)_{x_k = 0} \mathrm{d}\widehat{x}_k}_{\text{Subsectors}} .$$

Want the top sector contribution to vanish:

"divergence free syzygy"

$$\phi_0 = \sum_{k=1}^n \partial_k \phi_k = 0$$

Necessary but not sufficient: Magic relations cannot be obtained by combining IBPs generated in the subsectors themselves

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

$$\underbrace{\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^{n} \partial_k \phi_k\right) G^{-\frac{d}{2}} \bigwedge_{i=1}^{n} \mathrm{d}x_i}_{\text{Top Sector}} = \underbrace{-\sum_{k=1}^{n} \int_{\partial\Gamma_k} \left(\phi_k G^{-\frac{d}{2}}\right)_{x_k=0} \mathrm{d}\widehat{x}_k}_{\text{Subsectors}} \, .$$

Want the top sector contribution to vanish:

"divergence free syzygy"

$$\phi_0 = \sum_{k=1}^{n} \partial_k \phi_k = 0$$

Necessary but not sufficient: Magic relations cannot be obtained by combining IBPs generated in the subsectors themselves

$\dim(\mathcal{I}) = 0$ places strong constraints on the allowed syzygies

only "trivial" syzygies allowed: $\phi_i = \partial_j \mathcal{G} \, , \quad \phi_j = -\partial_i \mathcal{G} \, , \quad \phi_k = 0 \;\; \text{for} \;\; k \neq i, j \, .$

Intuition for connection (1/2)

IBPs in Parametric Representation

Consider IBPs in Lee-Pomeransky representation

Syzygies: $\phi_0 \mathcal{G} + \sum_k \phi_k \partial_k \mathcal{G} = 0$

$$\underbrace{\int_\Gamma \left(\frac{d}{2}\phi_0 + \sum_{k=1}^n \partial_k \phi_k \right) G^{-\frac{d}{2}} \bigwedge_{i=1}^n \mathrm{d}x_i}_{\text{Top Sector}} = \underbrace{- \sum_{k=1}^n \int_{\partial\Gamma_k} \left(\phi_k G^{-\frac{d}{2}} \right)_{x_k=0} \mathrm{d}\widehat{x}_k}_{\text{Subsectors}} \,.$$

Want the top sector contribution to vanish:

"divergence free syzygy"

$$\phi_0 = \sum_{k=1}^n \partial_k \phi_k = 0$$

Necessary but not sufficient: Magic relations cannot be obtained by combining IBPs generated in the subsectors themselves

$\dim(\mathcal{I}) = 0$ places strong constraints on the allowed syzygies

only "trivial" syzygies allowed: $\phi_i = \partial_j \mathcal{G}\,, \quad \phi_j = -\partial_i \mathcal{G}\,, \quad \phi_k = 0 \text{ for } k \neq i, j\,.$

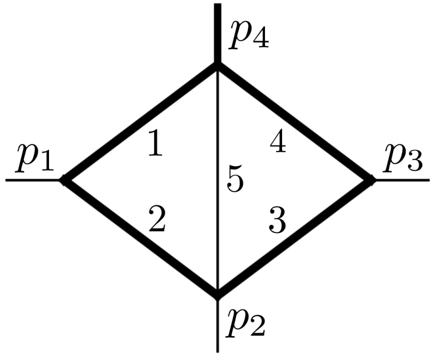
plugging this into the IBP identity above one can show that these IBPs are already generated by the subsectors themselves $\longrightarrow$ not a magic relation

Modulo technical details (see paper)

Intuition for connection (2/2)

Building magic relations explicitly

If $\dim(\mathcal{I}) \neq 0$ then a broader class of syzygies can exist — can generate magic relations



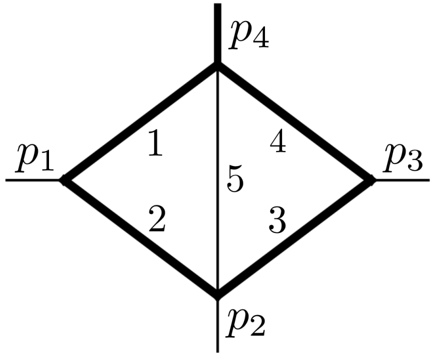
One "non-trivial" divergence free syzygy:

$$\vec{\phi} = \Big(0\,,\ \ (s-m_H^2)x_1 - tx_2\,,\ \ (m_H^2-s)x_1 + tx_2\,,\ \ -(m_H^2 - t)x_4 - sx_3\,,\ \ sx_3 + (m_H^2-t)x_4\,,\ \ 0\Big)$$

Intuition for connection (2/2)

Building magic relations explicitly

If $\dim(\mathcal{I}) \neq 0$ then a broader class of syzygies can exist — can generate magic relations



One "non-trivial" divergence free syzygy:

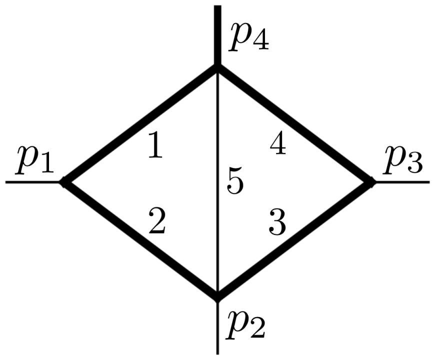
$$\vec{\phi} = \Big(0\,,\ \ (s-m_H^2)x_1 - tx_2\,,\ \ (m_H^2-s)x_1 + tx_2\,,\ \ -(m_H^2 - t)x_4 - sx_3\,,\ \ sx_3 + (m_H^2-t)x_4\,,\ \ 0\Big)$$

$$-tJ_{0,2,1,1,1} + (m_H^2-s)J_{2,0,1,1,1} - (m_H^2-t)J_{1,1,0,2,1} + sJ_{1,1,2,0,1} = 0$$

Intuition for connection (2/2)

Building magic relations explicitly

If $\dim(\mathcal{I}) \neq 0$ then a broader class of syzygies can exist — can generate magic relations



One "non-trivial" divergence free syzygy:

$$\vec{\phi} = \Big(0\,,\;\; (s - m_H^2)x_1 - t x_2\,,\;\; (m_H^2 - s)x_1 + t x_2\,,\;\; -(m_H^2 - t)x_4 - s x_3\,,\;\; s x_3 + (m_H^2 - t)x_4\,,\;\; 0\Big)$$

$$-tJ_{0,2,1,1,1} + (m_H^2 - s)J_{2,0,1,1,1} - (m_H^2 - t)J_{1,1,0,2,1} + sJ_{1,1,2,0,1} = 0$$

Implemented this procedure in a Mathematica function:

```
FindMagicRelations[gpol, {x1, x2, x3, x4, x5}]
{-t j[0, 2, 1, 1, 1] + (-mh^2 + t) j[1, 1, 0, 2, 1] + s j[1, 1, 2, 0, 1] + (mh^2 - s) j[2, 0, 1, 1, 1]}
```

Conclusions and Outlook

In this work we have made some progress into understanding the workings of magic relations.

Shown that the presence of magic relations is deeply connected to the dimensionality of critical point varieties generated by Feynman integrals.

From a computational standpoint this provides an efficient test to see if magic relations can exist. Implemented in proof-of-concept Mathematica functions.

Definition of magic relations (presented here) is designed to be incompatible with IBPs on a cut. It is however still very unclear what assumptions about IBPs are broken when taking cuts in the presence of magic relations.

Is there a way to “fix” the notion of a cut in the presence of magic relations? Is there a more general definition of what a sector should be in such cases?

Can the conjectures presented in this work be fully proven or if not disproved?

Thank you for listening!