Lattice Chiral Quark revisited

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May 1, 2026

Outline

Chiral symmetry in the continuum

Consider the Euclidean Dirac operator in a gauge background:

$$\not{D} = \gamma_\mu(\partial_\mu + iA_\mu)$$

For massless fermions,

$$S_F = \int d^4x \, \bar{\psi} \not{D} \psi.$$

Since

$$\{\gamma_5, \not{D}\} = 0,$$

the massless action is invariant under

$$\psi \to e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}.$$

This is the basic continuum chiral symmetry.

Spectrum of the continuum Dirac operator

In Euclidean space,

$$\not{D}^\dagger = -\not{D}$$

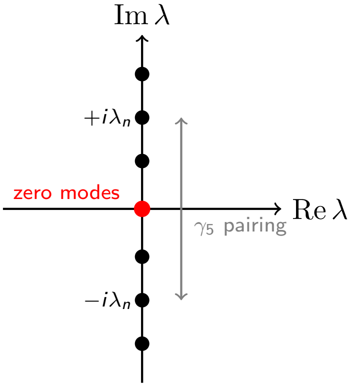
for a smooth gauge background. Therefore its eigenvalues are purely imaginary:

$$\not{D}\psi_n = i\lambda_n\psi_n, \qquad \lambda_n \in \mathbb{R}.$$

Chiral symmetry implies pairing:

$$\not{D}(\gamma_5\psi_n) = -i\lambda_n(\gamma_5\psi_n).$$

Non-zero modes are chirality zero: $\chi = \langle\psi|\gamma_5|\psi\rangle = \psi_n^\dagger\gamma_5\psi_n = 0$ for the paired orthogonal eigenvectors.



finite volume: discrete spectrum

Spectrum of the continuum Dirac operator (free)

Free Dirac operator, Plane-wave eigenvectors:

$$D = \gamma_\mu \partial_\mu, \psi_p(x) = u(p)\, e^{ip\cdot x}$$

Eigenvalue equation:

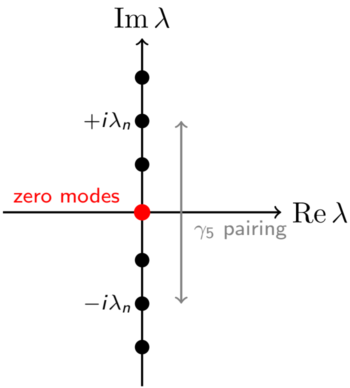
$$D\psi_p = i\gamma_\mu p_\mu\, \psi_p, (D)^2\psi_p = -p^2\psi_p$$

Eigenvalues:

$$\boxed{\lambda = \pm i|p|}$$

Each p gives a pair:

$$+i|p|, \quad -i|p|$$



finite volume: discrete spectrum

Hermitian Dirac operator (free, massless)

Define

$$H = \gamma_5 D, \qquad D = \gamma_\mu \partial_\mu .$$

Plane-wave:

$$\psi_p(x) = u(p)\, e^{ip\cdot x}.$$

Then

$$H\psi_p = \gamma_5 (i\gamma_\mu p_\mu)\psi_p = i\gamma_5 \not{p}\, \psi_p .$$

Square:

$$H^2 = D^\dagger D = -D^2 = p^2 .$$

Eigenvalues:

$$\boxed{H\psi_p = \pm|p|\,\psi_p}$$

- real spectrum
- symmetric under $\lambda \leftrightarrow -\lambda$

Hermitian Dirac operator (free, massive)

Dirac operator:

$$D = \gamma_\mu \partial_\mu + m.$$

Hermitian operator:

$$H = \gamma_5 D = \gamma_5 (\not{\partial} + m).$$

On plane waves:

$$H\psi_p = (i\gamma_5 \not{p} + m\gamma_5)\psi_p.$$

Square:

$$H^2 = (\not{p} + m)^\dagger (\not{p} + m) = p^2 + m^2.$$

Eigenvalues:

$$\boxed{H\psi_p = \pm\sqrt{p^2 + m^2}\,\psi_p}$$

- mass opens a gap
- still symmetric under $\lambda \leftrightarrow -\lambda$

Zero modes and chirality

For zero modes,

$$\not{D}\psi_0 = 0,$$

the paired-mode argument no longer applies.
Zero modes can be chosen to have definite chirality since $\{\not{D}, \gamma_5\}\psi_0 = [\not{D}, \gamma_5]\psi_0 = 0$:

$$\gamma_5\psi_0 = \pm\psi_0.$$

Define

$n_+ = \#$ right-handed zero modes, $\quad n_- = \#$ left-handed zero modes.

The index of the Dirac operator is

$$\mathrm{index}(\not{D}) = n_+ - n_-.$$

Chirality in the continuum (free)

Chirality operator:

$$\gamma_5^2 = 1, \qquad P_\pm = \frac{1 \pm \gamma_5}{2}.$$

Define chirality of a mode:

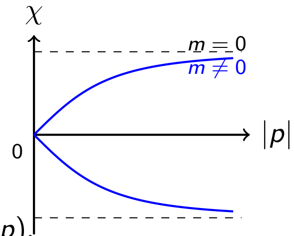
$$\chi(p) = \langle \psi_p | \gamma_5 | \psi_p \rangle.$$

Massless Dirac:

$$D = \not\partial, \qquad \{D, \gamma_5\} = 0.$$

For plane waves:

$$\psi_p(x) = u(p) e^{ip\cdot x}, \qquad \gamma_5 u_\pm(p) = \pm u_\pm(p).$$



Axial Ward identity and PCAC

For a massive Dirac fermion,

$$S_F = \int d^4x\, \bar{\psi}(\not{D} + m)\psi.$$

The axial current is

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5 T^a\psi.$$

Classically,

$$\partial_\mu A_\mu^a = 2m\, P^a, \qquad P^a = \bar{\psi}\gamma_5 T^a\psi.$$

This is the PCAC relation:

$$\partial_\mu A_\mu^a \simeq 2mP^a.$$

Chiral symmetry is exact only in the massless limit.

Axial Ward identity and pion mass

PCAC relation:

$$\partial_\mu A^a_\mu = 2m\, P^a, \qquad P^a = \bar{\psi}\gamma_5 T^a \psi.$$

Consider matrix elements with a one-pion state:

$$\langle 0|A^a_\mu|\pi^b(p)\rangle = i f_\pi p_\mu \delta^{ab},$$

$$\langle 0|P^a|\pi^b(p)\rangle = G_\pi \delta^{ab}.$$

Taking the divergence:

$$\langle 0|\partial_\mu A^a_\mu|\pi\rangle = f_\pi m_\pi^2.$$

Using PCAC:

$$f_\pi m_\pi^2 = 2m\, G_\pi.$$

$$\boxed{m_\pi^2 \propto m} \quad (m \to 0)$$

GMOR relation

Using chiral symmetry and current algebra,

$$f_\pi^2 m_\pi^2 = -2m\langle\bar{\psi}\psi\rangle + O(m^2).$$

Thus:

- pion is a pseudo-Goldstone boson
- small quark mass $\Rightarrow$ small m_π

$$m_\pi^2 \sim m \quad \text{(leading order)}$$

Axial anomaly

For the flavor-singlet axial current,

$$A^0_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi,$$

the quantum theory gives

$$\partial_\mu A^0_\mu = 2m\,\bar{\psi}\gamma_5\psi + \frac{g^2 N_f}{16\pi^2}\mathrm{tr}\,F_{\mu\nu}\widetilde{F}_{\mu\nu}.$$

The second term is the axial anomaly.

Thus the classical $U(1)_A$ symmetry is broken by quantum effects.

Fujikawa method

Under a chiral rotation

$$\psi \to e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5},$$

the classical massless action is invariant.
However, the fermion path-integral measure is not:

$$D\bar{\psi}D\psi \to D\bar{\psi}D\psi \, \exp\left[2i\alpha \, \mathrm{Tr}\,\gamma_5\right].$$

With gauge-invariant regularization,

$$\mathrm{Tr}\,\gamma_5 = \frac{g^2}{32\pi^2}\int d^4x \, \mathrm{tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}.$$

The anomaly comes from the Jacobian of the fermion measure.

Instanton number and topology

The topological charge is

$$Q = \frac{g^2}{32\pi^2} \int d^4x \, \mathrm{tr} \, F_{\mu\nu} \widetilde{F}_{\mu\nu}.$$

For smooth gauge fields with appropriate boundary conditions,

$$Q \in \mathbb{Z}.$$

It counts the winding number of the gauge field.

Instantons are classical gauge-field configurations with

$$Q = \pm 1.$$

Atiyah–Singer index theorem

The index of the Dirac operator is tied to gauge-field topology:

$$\mathrm{index}(\not{D}) = n_+ - n_- = Q.$$

Thus:

$$\text{chiral zero modes} \quad \Longleftrightarrow \quad \text{topological charge.}$$

This relation is one of the central reasons chiral symmetry is so important in QCD.

- topology of gauge bundle
- (more generally) topology of spacetime + bundle

Dirac operator "measures" topology

Strong CP problem

The QCD action may contain the topological term

$$S_\theta = i\theta Q = i\theta \frac{g^2}{32\pi^2} \int d^4x \, \mathrm{tr} \, F_{\mu\nu} \widetilde{F}_{\mu\nu}.$$

A chiral rotation changes the fermion mass phase and shifts θ.
The physical CP-violating parameter is

$$\bar{\theta} = \theta + \arg\det M_q.$$

Experimentally,

$$|\bar{\theta}| \ll 1.$$

Why is this parameter so small? This is the strong CP problem.

η' mass and the $U(1)_A$ problem

Classically, massless QCD has

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A.$$

If $U(1)_A$ were a true symmetry, spontaneous breaking would imply an extra light Goldstone boson.
But the η' is heavy:

$$m_{\eta'} \gg m_\pi.$$

Resolution:

$$U(1)_A$$

is broken by the anomaly and by topological gauge fluctuations.

Witten–Veneziano relation

In large-N_c Yang–Mills theory, the η' mass is related to the topological susceptibility:

$$m_{\eta'}^2 \simeq \frac{2N_f}{f_\pi^2}\,\chi_t.$$

where

$$\chi_t = \frac{\langle Q^2 \rangle}{V}.$$

Thus topology gives mass to the η'.

This connects:

$$\text{anomaly} \quad \Longleftrightarrow \quad \text{topology} \quad \Longleftrightarrow \quad \eta' \text{ mass.}$$

Chiral symmetry on the lattice

Naively, one wants a lattice Dirac operator satisfying

$$\{D, \gamma_5\} = 0.$$

This would imply exact chiral symmetry:

$$\psi \to e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}.$$

However, on the lattice this conflicts with locality and absence of doublers.

This is the Nielsen–Ninomiya theorem.

Naive fermion and doublers

The naive lattice Dirac operator is

$$D_{\mathrm{naive}}(p) = i \sum_{\mu} \gamma_\mu \sin p_\mu .$$

It satisfies

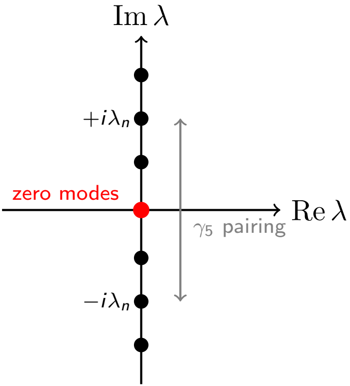
$$\{D_{\mathrm{naive}}, \gamma_5\} = 0.$$

But it has zeros whenever

$$\sin p_\mu = 0 \quad \Rightarrow \quad p_\mu = 0 \text{ or } \pi .$$

In d dimensions, the unwanted doublers/tastes appears:

$$\#\text{zeros} = 2^d .$$



finite volume: discrete spectrum

Nielsen–Ninomiya theorem

One cannot have a lattice Dirac operator satisfying all of:

- locality
- translational invariance
- correct continuum behavior
- exact chiral symmetry

$$\{D, \gamma_5\} = 0$$

- no fermion doublers

Thus lattice chiral symmetry requires a compromise.

Wilson fermions choose:

remove doublers $\Longleftrightarrow$ break chiral symmetry explicitly.

Wilson Dirac operator (difference form)

Define gauge-covariant forward/backward differences:

$$\nabla_\mu \psi(x) = U_\mu(x)\psi(x+\hat{\mu}) - \psi(x),$$

$$\nabla_\mu^* \psi(x) = \psi(x) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}).$$

Wilson Dirac operator:

$$D_W = m_0 + \frac{1}{2}\sum_\mu \left[\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - r\,\nabla_\mu^*\nabla_\mu\right].$$

- first term: naive Dirac operator
- second term: Wilson term (lifts doublers)

Wilson fermion

Wilson adds a momentum-dependent mass term:

$$D_W(p) = i\sum_\mu \gamma_\mu \sin p_\mu + m_0 + r\sum_\mu (1-\cos p_\mu).$$

The Wilson term vanishes near $p=0$:

$$\sum_\mu (1-\cos p_\mu) \simeq \frac{p^2}{2}.$$

But near doubler momenta it gives masses of order cutoff:

$$p_\mu = \pi \quad \Rightarrow \quad 1-\cos p_\mu = 2.$$

Therefore doublers become heavy and decouple.

Cost:

$$\{D_W, \gamma_5\} \neq 0.$$

Spectrum of free Wilson fermion

For free Wilson fermions,

$$D_W(p) = i \sum_\mu \gamma_\mu \sin p_\mu + M(p),$$

where

$$M(p) = m_0 + r \sum_\mu (1 - \cos p_\mu).$$

The eigenvalues are

$$\lambda_\pm(p) = M(p) \pm i \sqrt{\sum_\mu \sin^2 p_\mu}.$$

Thus the spectrum is shifted into the complex plane by the Wilson mass term.

$$\mathrm{Re}\,\lambda = M(p).$$

Wilson spectrum in the complex plane

Naive fermion:

$$\lambda(p) = \pm i \sqrt{\sum_{\mu} \sin^2 p_\mu}$$

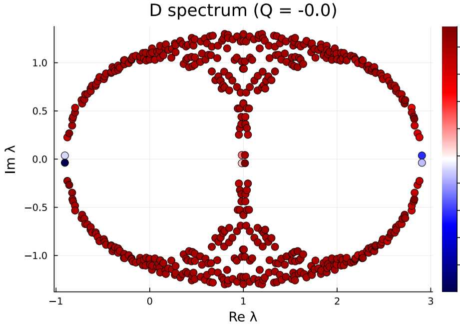
lies on the imaginary axis.

Wilson fermion:

$$\lambda(p) = M(p) \pm i |\sin p|$$

moves the doublers to large positive real mass.

This removes unwanted light species.



doubler branches acquire Wilson masses

γ_5-Hermiticity

Wilson fermions do not anticommute with γ_5:

$$\{D_W, \gamma_5\} \neq 0.$$

But they satisfy

$$D_W^\dagger = \gamma_5 D_W \gamma_5.$$

This is called γ_5-Hermiticity.

Consequences:

$$\lambda \in \mathrm{spec}(D_W) \quad \Rightarrow \quad \lambda^* \in \mathrm{spec}(D_W).$$

So the Wilson spectrum is symmetric under complex conjugation.

Define the Hermitian Wilson operator:

$$H_W = \gamma_5 D_W.$$

Then:

$$H_W^\dagger = H_W.$$

Chirality of Wilson eigenvectors

Wilson operator satisfies

$$D_W^\dagger = \gamma_5 D_W \gamma_5.$$

Let

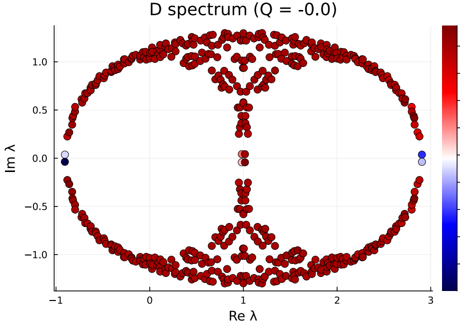
$$D_W \psi = \lambda \psi.$$

Then

$$D_W^\dagger (\gamma_5 \psi) = \lambda^* (\gamma_5 \psi).$$

Thus:

- ψ and $\gamma_5 \psi$ are related eigenvectors
- eigenvalues appear as $\lambda \leftrightarrow \lambda^*$



doubler branches acquire Wilson masses

Chirality expectation value S. Durr, J.H. Weber 2203.15699

Define chirality of a mode:

$$\chi = \langle\psi|\gamma_5|\psi\rangle.$$

Since

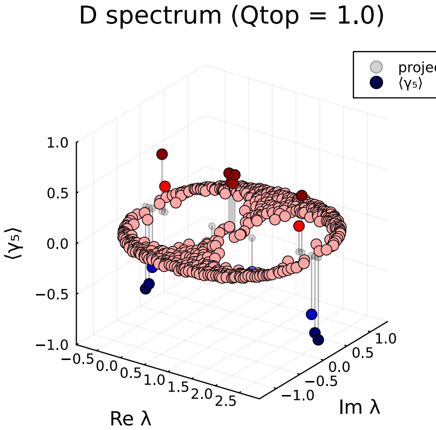
$$\langle\psi|\gamma_5 D_W|\psi\rangle = \lambda\langle\psi|\gamma_5|\psi\rangle$$

$$= \langle\psi|D_W^\dagger\gamma_5|\psi\rangle = \lambda^*\langle\psi|\gamma_5|\psi\rangle$$

$$(\lambda - \lambda^*)\langle\psi|\gamma_5|\psi\rangle = 0$$

for Wilson fermions:

- complex modes $\Rightarrow \chi = 0$
- real modes: $\chi \neq 0$ possible but without constraint $\chi = \pm 1$ (D is not normal matrix) and $|\chi| < 1$.



Ginsparg–Wilson relation

Naive chiral symmetry on the lattice,

$$\{D, \gamma_5\} = 0,$$

is incompatible with locality and no doublers.

Ginsparg–Wilson relation modifies it to

$$\boxed{\gamma_5 D + D\gamma_5 = D\gamma_5 D}$$

This allows exact lattice chiral symmetry while avoiding doublers.

Continuum limit:

$$D \sim aD_{\text{cont}} \quad \Rightarrow \quad D\gamma_5 D = O(a^2),$$

so

$$\{D_{\text{cont}}, \gamma_5\} = 0$$

is recovered.

Lattice chiral symmetry

For a Ginsparg–Wilson operator,

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D.$$

The action

$$S = \bar{\psi} D \psi$$

is invariant under Lüscher's lattice chiral transformation:

$$\delta\psi = \gamma_5(1 - D)\psi, \qquad \delta\bar{\psi} = \bar{\psi}\gamma_5.$$

The symmetry is exact at finite lattice spacing.

But it is realized in a lattice-modified way:

$$\gamma_5 \quad \longrightarrow \quad \hat{\gamma}_5 = \gamma_5(1 - D).$$

Overlap operator

Define the Hermitian Wilson kernel

$$H_W = \gamma_5 D_W(-M_0),$$

with a negative Wilson mass parameter.

Then define

$$\epsilon(H_W) = \mathrm{sign}(H_W) = \frac{H_W}{\sqrt{H_W^2}}.$$

The massless overlap Dirac operator is

$$\boxed{D_{\mathrm{ov}} = 1 + \gamma_5\, \epsilon(H_W)}$$

or

$$D_{\mathrm{ov}} = 1 + V, \qquad V = \gamma_5 \epsilon(H_W).$$

Here V is unitary:

$$V^\dagger V = 1.$$

Why overlap satisfies Ginsparg–Wilson

Let

$$D = 1 + V, \qquad V = \gamma_5 \epsilon(H_W).$$

Using

$$V^\dagger = V^{-1}, \qquad \gamma_5 V \gamma_5 = V^\dagger,$$

one finds

$$D^\dagger = \gamma_5 D \gamma_5.$$

Moreover,

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D.$$

Thus the overlap operator is an explicit solution of the Ginsparg–Wilson relation.

Overlap spectrum

Because

$$D_{\rm ov} = 1 + V, \qquad V^\dagger V = 1,$$

the eigenvalues of V lie on the unit circle:

$$\lambda(V) = e^{i\theta}.$$

Therefore the eigenvalues of $D_{\rm ov}$ lie on the circle

$$\boxed{\lambda(D_{\rm ov}) = 1 + e^{i\theta}}$$

So the overlap spectrum lies on a circle centered at 1 with radius 1.

Special points:

$$\lambda(D) = 0 \quad \Longleftrightarrow \quad V = -1,$$

$$\lambda(D) = 2 \quad \Longleftrightarrow \quad V = +1.$$

Overlap spectrum in the complex plane

$$D_{\rm ov} = 1 + V, \qquad V = e^{i\theta}.$$

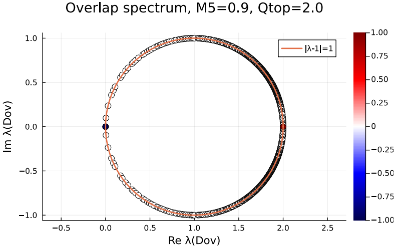
$$\lambda(D_{\rm ov}) = 1 + e^{i\theta}.$$

The circle passes through:

$$0, \qquad 2.$$

Zero modes of $D_{\rm ov}$ correspond to

$$V = -1.$$



Index theorem with overlap fermions

For overlap fermions, the lattice index is

$$\boxed{\mathrm{index}(D_{\mathrm{ov}}) = \frac{1}{2}\mathrm{Tr}\,\gamma_5(1 - D_{\mathrm{ov}})}$$

Using

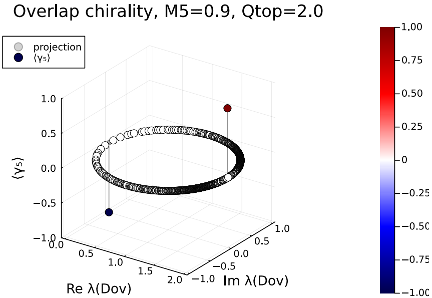
$$D_{\mathrm{ov}} = 1 + \gamma_5 \epsilon(H_W),$$

this becomes

$$\mathrm{index}(D_{\mathrm{ov}}) = -\frac{1}{2}\mathrm{Tr}\,\epsilon(H_W).$$

Equivalently,

$$\mathrm{index} = n_+ - n_-.$$



Topological charge from overlap

The overlap definition of topological charge is

$$Q_{\mathrm{ov}} = \frac{1}{2}\mathrm{Tr}\,\gamma_5(1 - D_{\mathrm{ov}}).$$

This is integer-valued for admissible gauge fields.

It gives a fermionic definition of topology:

$$Q_{\mathrm{ov}} = \mathrm{index}(D_{\mathrm{ov}}).$$

Continuum limit:

$$Q_{\mathrm{ov}} \longrightarrow \frac{g^2}{32\pi^2}\int d^4x\,\mathrm{tr}\,F_{\mu\nu}\widetilde{F}_{\mu\nu}.$$

Sign function and numerical cost

The central object is

$$\epsilon(H_W) = \mathrm{sign}(H_W).$$

For an eigenvector

$$H_W \phi_n = \lambda_n \phi_n,$$
$$\epsilon(H_W)\phi_n = \mathrm{sign}(\lambda_n)\phi_n.$$

Numerically, this is expensive because it requires approximating

$$\frac{H_W}{\sqrt{H_W^2}}.$$

Common approximations:

- rational approximation
- Zolotarev approximation
- Chebyshev polynomial approximation
- low-mode deflation

Angles between two subspaces C. Jordan (1875), P.R. Halmos (1969)

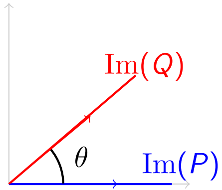
Wikipedia "Angles between flats"

Let P and Q be orthogonal projections:

$$P^2 = P, \quad Q^2 = Q, \quad P^\dagger = P, \quad Q^\dagger = Q$$

- P: chirality projector
- Q: spectral projector

Goal: quantify the misalignment between subspaces



Principal angles (1st definition)

There exist orthonormal bases $\{u_i\}$ in $\mathrm{Im}(P)$ and $\{v_i\}$ in $\mathrm{Im}(Q)$ such that

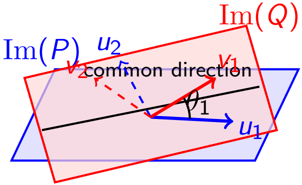
$$\langle u_i, v_j \rangle = \delta_{ij} \cos \theta_i .$$

The angles

$$0 \leq \theta_i \leq \frac{\pi}{2}$$

are called principal angles.

- $\theta_i = 0$: common direction
- $\theta_i = \pi/2$: orthogonal direction



Halmos decomposition

There exists a basis where P and Q take block form:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

(each block corresponds to one angle θ)
This reduces the problem to 2×2 geometry.

Operator formulation (2nd definition)

Consider the operator

$$C = PQP \quad \text{on } \mathrm{Im}(P)$$

Then:

$$Cu_i = \cos^2\theta_i \, u_i$$

Equivalently, singular values of PQ:

$$\sigma_i(PQ) = \cos\theta_i$$

Thus:

$$\text{angles} \longleftrightarrow \text{spectrum of } PQP$$

Principal angles: (3rd definition)

Let P_+ and Q_+ define two subspaces

$$\mathcal{P} = \mathrm{Im}(P_+), \qquad \mathcal{Q} = \mathrm{Im}(Q_+).$$

The first principal angle is defined by

$$\cos\theta_1 = \max_{\substack{u\in\mathcal{P},\, v\in\mathcal{Q} \\ \|u\|=\|v\|=1}} |\langle u, v\rangle|.$$

Subsequent angles are obtained with orthogonality constraints:

$$\cos\theta_k = \max_{\substack{u\in\mathcal{P},\, v\in\mathcal{Q} \\ \|u\|=\|v\|=1 \\ u\perp u_1,\dots,u_{k-1} \\ v\perp v_1,\dots,v_{k-1}}} |\langle u, v\rangle|.$$

Equivalently:

$$P_+ Q_+ P_+ \, u_k = \cos^2\theta_k \, u_k.$$

Numerically:

$$\boxed{\text{large-angle search} \iff \text{minimize } \langle u|Q_+P_+Q_+|u\rangle \quad \text{or} \quad \text{minimize } \langle u|P_+Q_+P_+|u\rangle}$$

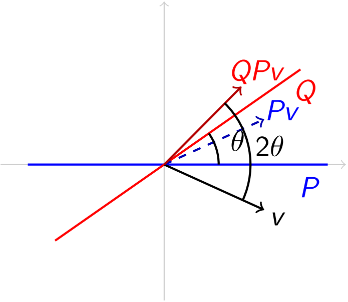
Geometry: reflections $\Rightarrow$ rotation

$$V = \gamma_5 \epsilon(H_W) = (2P_+ - 1)(2Q_+ - 1)$$

- γ_5 : reflection about $\mathrm{Im}(P_+)$
- $\epsilon(H_W)$: reflection about $\mathrm{Im}(Q_+)$

- reflection $\times$ reflection $=$ rotation
- rotation angle $= 2\theta$

$$V \sim e^{\pm i\,2\theta}$$



Polar part from SVD of D_W

Take the SVD:

$$D_W = U\,\Sigma\,W^\dagger, \qquad \Sigma = \mathrm{diag}(\sigma_i) \geq 0.$$

Then

$$D_W^\dagger D_W = W\,\Sigma^2\,W^\dagger, \qquad (D_W^\dagger D_W)^{-1/2} = W\,\Sigma^{-1}\,W^\dagger \quad (\sigma_i > 0).$$

Hence the unitary (polar) part is

$$\boxed{V \;=\; D_W (D_W^\dagger D_W)^{-1/2} \;=\; U\,W^\dagger}$$

Equivalently,

$$V = \sum_i u_i\, w_i^\dagger, \quad D_W = \sum_i \sigma_i\, u_i\, w_i^\dagger,$$

so V is obtained by replacing $\sigma_i \to 1$.

Spectral interpretation

Eigenvalues of V:

$$\lambda_i(V) = e^{\pm i\,2\theta_i}$$

- $\theta_i \approx \frac{\pi}{2}$
  - $\lambda_i \approx -1$
  - topological / near-zero modes
  - Useful for:
    - deflation strategies
    - understanding overlap spectrum
    - relation to domain-wall fermions
- $\theta_i \approx 0$
  - $\lambda_i \approx +1$
  - UV modes

Spectrum: H_W vs overlap

- H_W: real spectrum
- $V = \gamma_5 \,\mathrm{sign}(H_W)$: unit circle

Mapping:

$$\lambda(H_W) > 0 \Rightarrow V \sim +1, \qquad \lambda(H_W) < 0 \Rightarrow V \sim -1.$$

$$\boxed{\lambda(H_W) \approx 0 \iff V \approx -1 \iff \lambda(D_{\mathrm{ov}}) \approx 0}$$

Low modes of overlap correspond to near-zero modes of H_W.

Geometry: principal angles

Define projectors:

$$P_+ = \frac{1+\gamma_5}{2}, \qquad Q_+ = \frac{1+\mathrm{sign}(H_W)}{2}.$$

Principal angles θ between subspaces:

$$V \sim e^{\pm i\,2\theta}.$$

$$\boxed{\theta \approx \frac{\pi}{2} \iff V \approx -1}$$

Thus:

- small-angle modes $\Rightarrow$ UV / trivial
- $\theta \approx \pi/2 \Rightarrow$ topological / low modes

Search strategy in Q_+ space

Goal:

$$V \approx -1 \iff \theta \approx \frac{\pi}{2}.$$

Idea:

- restrict search to Q_+ subspace
- search in Krylov space of H, which preserves Q_+ subspace
- look for vectors with large principal angle

Advantages:

- dimension reduction
- suppress modes with $\theta \approx 0$
- enhances physically relevant modes

Occasional Q_+ projection:

- stabilize search
- correct drift

Polynomial filtering without repeated sign

Use polynomial filter:

$$\boxed{f(H) = \left(1 - \frac{H^2}{\lambda_{\text{max}}^2}\right)^n \left(1 + \frac{H}{\lambda_{\text{max}}}\right)^s}$$

Design:

- $\left(1 - \frac{H^2}{\lambda_{\text{max}}^2}\right)^n$ suppresses large $|H|$
- $\left(1 + \frac{H}{\lambda_{\text{max}}}\right)^s$ enhances positive modes

Effect:

$$\boxed{\text{enhance } \lambda(H) \approx 0^+, \quad \text{suppress } \lambda < 0 \text{ and large } |\lambda|}$$

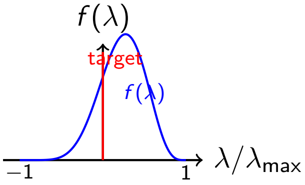
Spectral shaping by polynomial filter

Filter:

$$f(\lambda) = \left(1 - \frac{\lambda^2}{\lambda_{\text{max}}^2}\right)^n \left(1 + \frac{\lambda}{\lambda_{\text{max}}}\right)^s$$

Key features:

- suppress $|\lambda| \sim \lambda_{\text{max}}$
- suppress $\lambda < 0$
- enhance $\lambda \approx 0^+$



$\boxed{\text{focus spectral weight near } \lambda \approx 0^+}$

Commutation with sign and Q_+ projection

Since $f(H)$ is a function of H:

$$\boxed{[f(H), \mathrm{sign}(H)] = 0}$$

Define

$$Q_+ = \frac{1 + \mathrm{sign}(H)}{2}.$$

Then

$$f(H) Q_+ = Q_+ f(H),$$

so projection and filtering commute.

- apply Q_+ once (using $\mathrm{sign}(H)$)
- then iterate using only $f(H)$

no need to repeatedly apply expensive $\mathrm{sign}(H)$

Algorithm

- a (random) initial vector r
- project once:

$$v_0 \longrightarrow Q_+ r$$

- iterate:

$$v_0 \longrightarrow v_1 = f(H)v, \quad v_2 = f(H)^2 v_0, \dots$$

- search max principal angles $\theta \sim \pi/2$ and associated vectors in $\mathcal{K}_n = \{v_0, v_1, \cdots, v_{n-1}\}$ using $\langle v_n|\gamma_5|v_m\rangle$.

Because $[f(H), \mathrm{sign}(H)] = 0$:

$$Q_+ f(H)^k v = f(H)^k Q_+ v$$

Thus:

- Q_+ projection is stable
- re-application needed only when numerical errors accumulate

$\boxed{\text{efficient low-mode search in } Q_+ \text{ space}}$

Summary: geometry $\leftrightarrow$ numerics

$$\text{low modes} \iff V \approx -1 \iff \theta \approx \frac{\pi}{2}$$

- overlap = polar decomposition of Wilson operator
- principal angles encode spectral information
- search = finding large-angle directions

$$\boxed{\text{low-mode search} \sim \text{principal angle maximization}}$$

Krylov subspace and spectral weights

Krylov subspace:

$$\mathcal{K}_n(H_W, v) = \mathrm{span}\{v, H_W v, H_W^2 v, \ldots, H_W^{n-1} v\}.$$

Expand initial vector in eigenbasis:

$$v = \sum_i c_i \phi_i, \qquad H_W \phi_i = \lambda_i \phi_i.$$

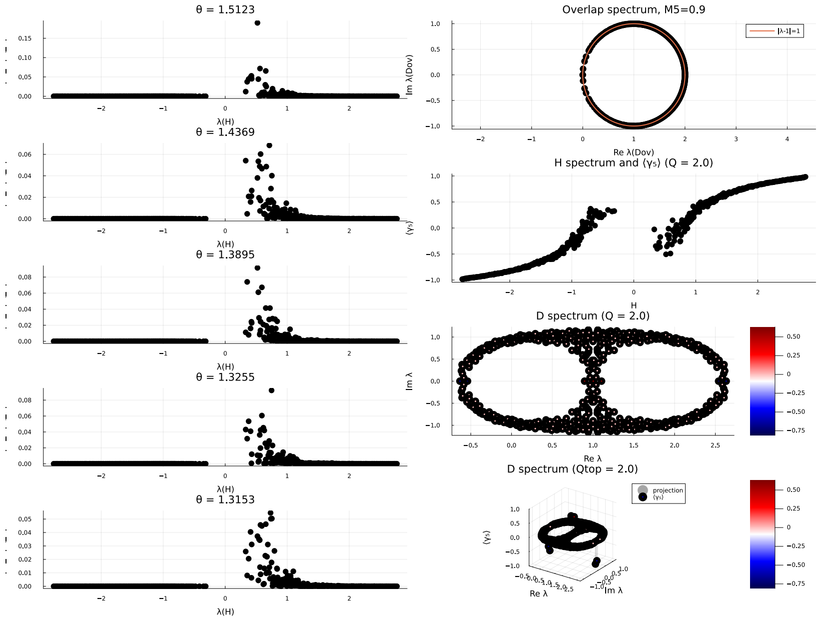
After applying polynomial filter $\phi(H_W)$:

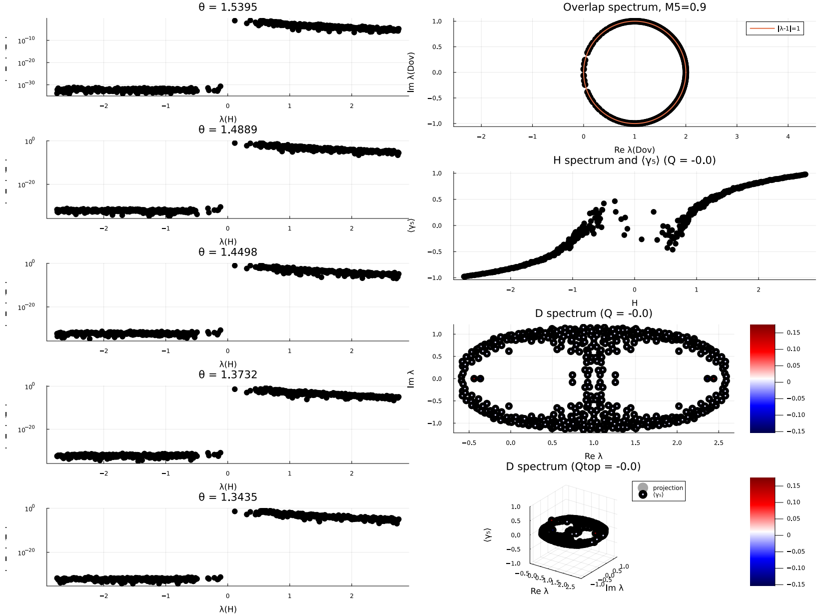
$$\phi(H_W) v = \sum_i c_i \, \phi(\lambda_i) \, \phi_i.$$

$$\boxed{\text{weight } c_i \longrightarrow c_i \, \phi(\lambda_i)}$$

Goal:

- enhance $\lambda_i \approx 0^+$
- suppress $\lambda_i < 0$ and large $|\lambda_i|$


θ = 1.5123
θ = 1.4369
θ = 1.3895
θ = 1.3255
θ = 1.3153
λ(H)
Overlap spectrum, M5=0.9
|λ-1|=1
Re λ(Dov)
Im λ(Dov)
H spectrum and ⟨γ5⟩ (Q = 2.0)
H
⟨γ5⟩
D spectrum (Q = 2.0)
Re λ
Im λ
D spectrum (Qtop = 2.0)
projection
⟨γ5⟩


θ = 1.5395
λ(H)
θ = 1.4889
λ(H)
θ = 1.4498
λ(H)
θ = 1.3732
λ(H)
θ = 1.3435
λ(H)
Overlap spectrum, M5=0.9
|λ-1|=1
Im λ(Dov)
Re λ(Dov)
H spectrum and ⟨γ5⟩ (Q = -0.0)
⟨γ5⟩
H
D spectrum (Q = -0.0)
Im λ
Re λ
D spectrum (Qtop = -0.0)
projection
⟨γ5⟩
Re λ
Im λ

Summary

- Chiral symmetry
- Chiral symmetry of Lattice Quarks
- It took 3 days for this feasibility study of overlap fermions in 2D U(1) using ChatGPT

Chirality and γ_5

Define

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \qquad \gamma_5^2 = 1.$$

Chiral projectors:

$$P_\pm = \frac{1 \pm \gamma_5}{2}.$$

Decomposition:

$$\psi = \psi_L + \psi_R, \qquad \psi_{L,R} = P_\mp \psi.$$

- ψ_L: left-handed
- ψ_R: right-handed

Doubler transformation of Wilson operator

Define

$$\eta(x) = (-1)^{\sum_\mu x_\mu}, \qquad \eta(x \pm \hat{\mu}) = -\eta(x).$$

Gauge-covariant differences:

$$\nabla_\mu = T_\mu^+ - 1, \qquad \nabla_\mu^* = 1 - T_\mu^-,$$

where

$$T_\mu^+ \psi(x) = U_\mu(x)\psi(x+\hat{\mu}), \qquad T_\mu^- \psi(x) = U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}).$$

Since

$$\eta T_\mu^\pm \eta = -T_\mu^\pm,$$

we get, for the kinematic term,

$$\eta(\nabla_\mu + \nabla_\mu^*)\eta = -(\nabla_\mu + \nabla_\mu^*),$$

but, for the Wilson term,

$$\eta(\nabla_\mu^* \nabla_\mu)\eta = -\left(T_\mu^+ + T_\mu^- + 2\right).$$

Using

$$\nabla_\mu^* \nabla_\mu = T_\mu^+ + T_\mu^- - 2,$$

Doubler transformation Wilson operator

the Wilson term transforms as

$$\eta \left(-\frac{r}{2} \sum_{\mu} \nabla_{\mu}^{*} \nabla_{\mu} \right) \eta = 2rd - \left(-\frac{r}{2} \sum_{\mu} \nabla_{\mu}^{*} \nabla_{\mu} \right).$$

Therefore

$$D_W = m_0 + \frac{1}{2} \sum_{\mu} \gamma_{\mu} (\nabla_{\mu} + \nabla_{\mu}^{*}) - \frac{r}{2} \sum_{\mu} \nabla_{\mu}^{*} \nabla_{\mu}$$

satisfies

$$\boxed{\eta D_W \eta = 2(rd + m_0) - D_W.}$$

Thus

$$D_W \psi = \lambda \psi \quad \Rightarrow \quad D_W(\eta\psi) = \left[2(rd + m_0) - \lambda \right] (\eta\psi).$$

In $d = 4$, $r = 1$:

$$\lambda \longleftrightarrow 8 + 2m_0 - \lambda.$$

Doubler chirality

Near a doubler corner

$$p_\mu = \pi n_\mu + k_\mu, \qquad n_\mu \in \{0, 1\},$$

we have

$$\sin p_\mu \simeq (-1)^{n_\mu} k_\mu.$$

Thus the effective gamma matrices are

$$\gamma_\mu^{(n)} = (-1)^{n_\mu} \gamma_\mu.$$

The effective chirality matrix becomes

$$\gamma_5^{(n)} = \gamma_1^{(n)} \gamma_2^{(n)} \gamma_3^{(n)} \gamma_4^{(n)} = (-1)^{\sum_\mu n_\mu} \gamma_5.$$

So doublers carry alternating chirality.

Staggered fermions

Staggered fermions reduce the number of doublers by spin diagonalization.
Naive fermion in four dimensions:

16 species.

Staggered fermion:

4 tastes.

The staggered Dirac operator has remnant chiral symmetry:

$$\{D_{\mathrm{stag}}, \epsilon(x)\} = 0, \qquad \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}.$$

This protects the mass from additive renormalization, but taste symmetry is broken at finite lattice spacing.

Staggered spectrum

Because

$$D_{\text{stag}}^{\dagger} = -D_{\text{stag}}$$

for massless staggered fermions, its eigenvalues are purely imaginary:

$$D_{\text{stag}}\chi_n = i\lambda_n\chi_n, \qquad \lambda_n \in \mathbb{R}.$$

The remnant chiral symmetry implies pairing:

$$i\lambda_n \quad \leftrightarrow \quad -i\lambda_n.$$

At finite lattice spacing:

taste multiplets are split.

In the continuum limit:

$a \to 0,$ taste symmetry is restored.

Chirality expectation value

Define chirality of a mode:

$$\chi = \langle \psi | \gamma_5 | \psi \rangle.$$

Since

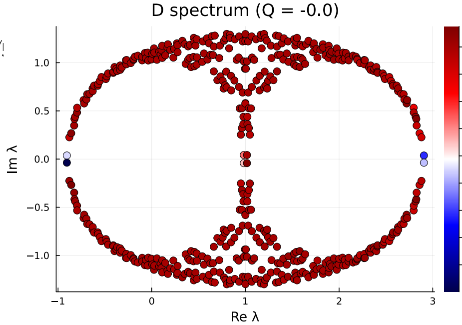
$$\langle \psi | \gamma_5 D_W | \psi \rangle = \lambda \langle \psi | \gamma_5 | \psi \rangle = \langle \psi | D_W^\dagger \gamma_5$$

that is

$$(\lambda - \lambda^*) \langle \psi | \gamma_5 | \psi \rangle = 0$$

for Wilson fermions:

- complex modes $\Rightarrow \chi = 0$ (similar to the continuum)
- real modes: $\chi \neq 0$ possible



doubler branches acquire Wilson masses