Multiplicity distributions of produced gluons in DIS: evolution equations and homotopy solutions

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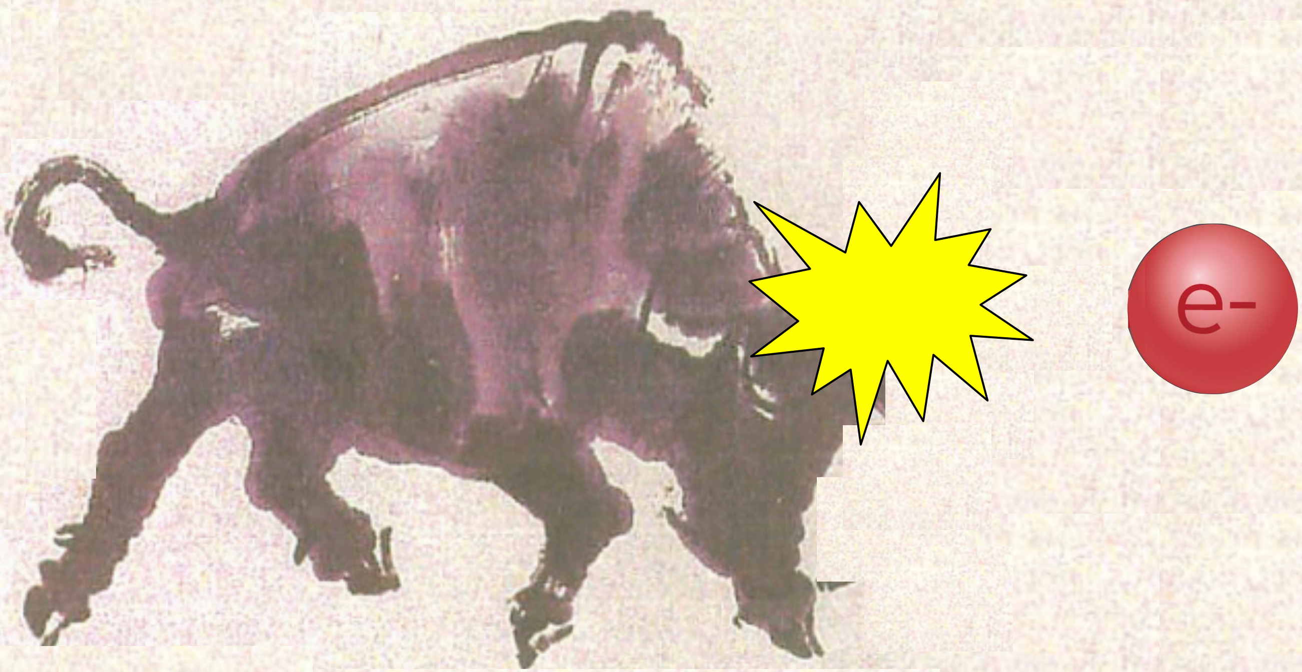


RBRC Seminar, May 7th, BNL, Upton, NY, USA

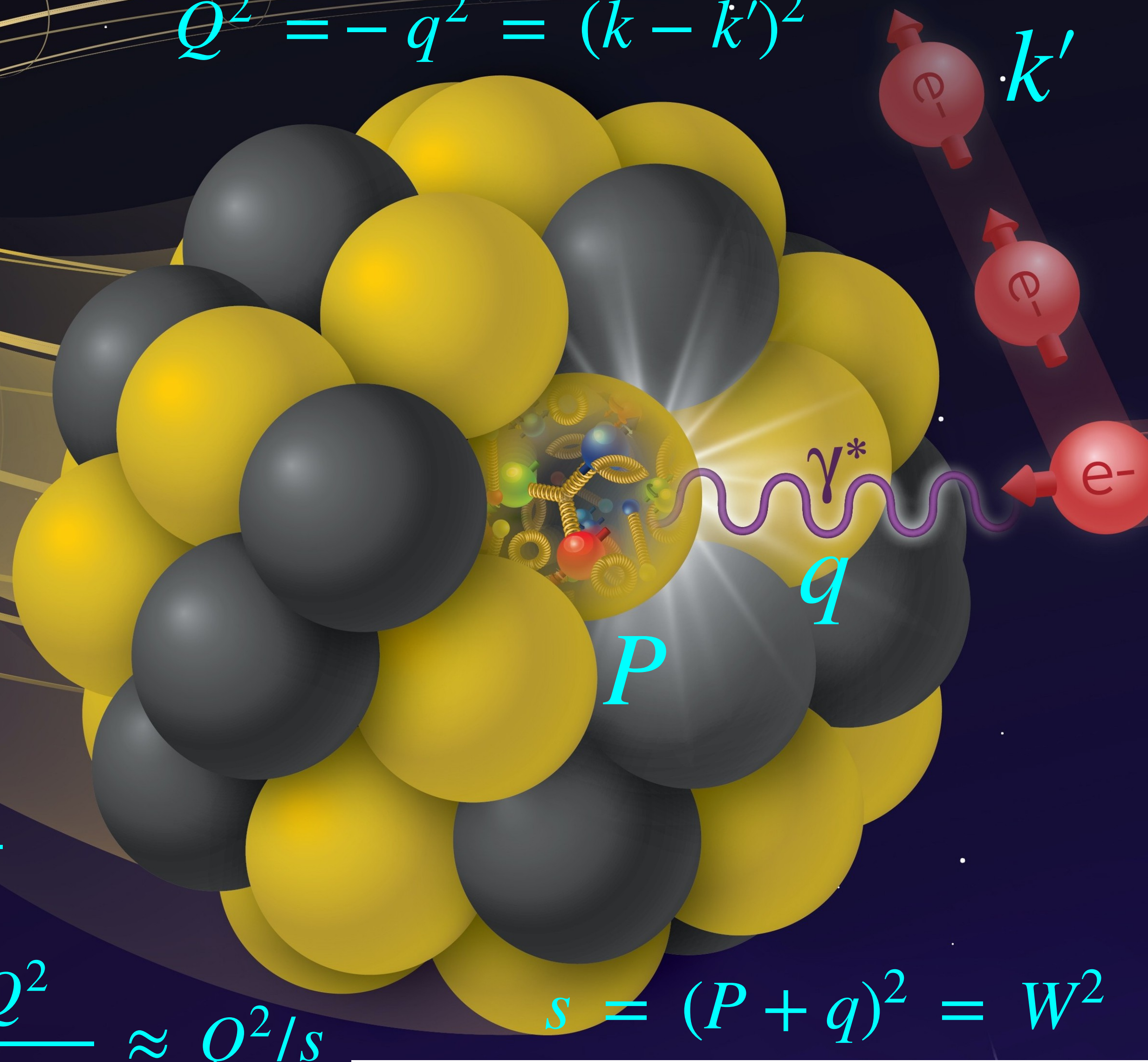
Outline

- **Brief summary of our theoretical achievements in high energy DIS.**
- **Multiplicity distribution of the produced gluons in high energy DIS:**
  - **The linear but with complicated kernel and non-homogeneous evolution equations for the cross sections of n-cut BFKL Pomeron production.**
  - **Approximate analytical solution to the equation (homotopy and large n solution)**
  - **Calculation of the multiplicity distribution and related quantities**
  - **Future work**

Based on arXiv:2603.21775 [hep-ph]

Deep inelastic scattering

$$Q^2 = -q^2 = (k - k')^2$$

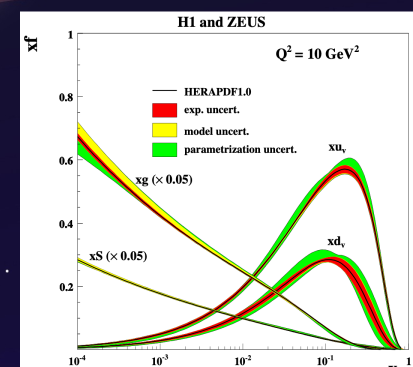


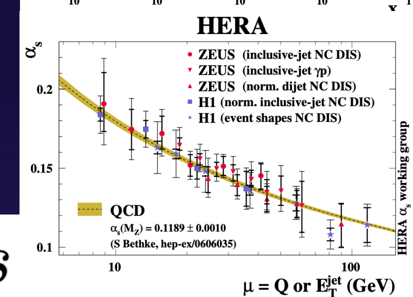
$$y = \frac{Q^2}{sx}$$

$$x = \frac{Q^2}{2P \cdot q} \approx Q^2/s$$

$$s = (P+q)^2 = W^2$$

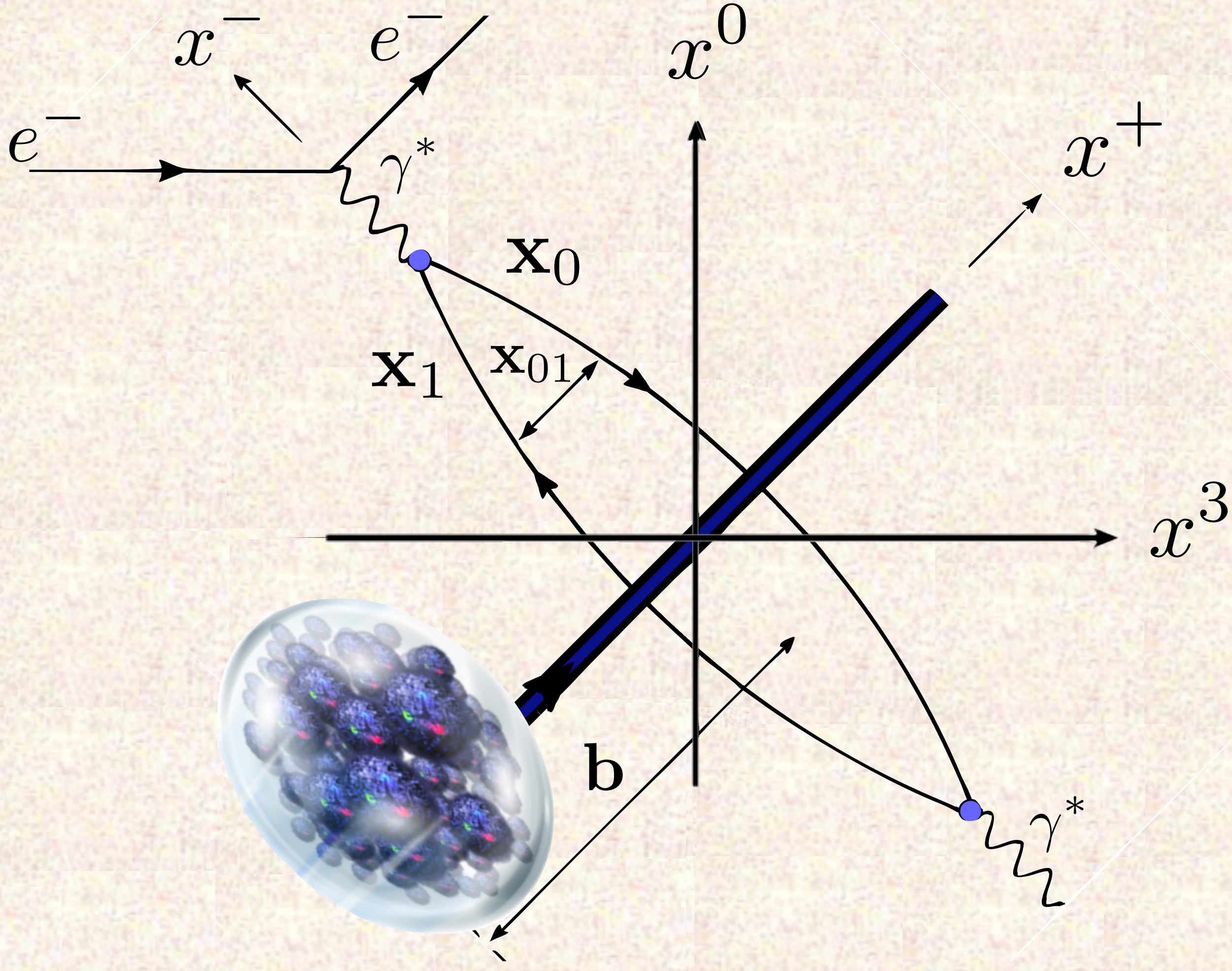
Small$-x$ $\Leftrightarrow$ High energy s





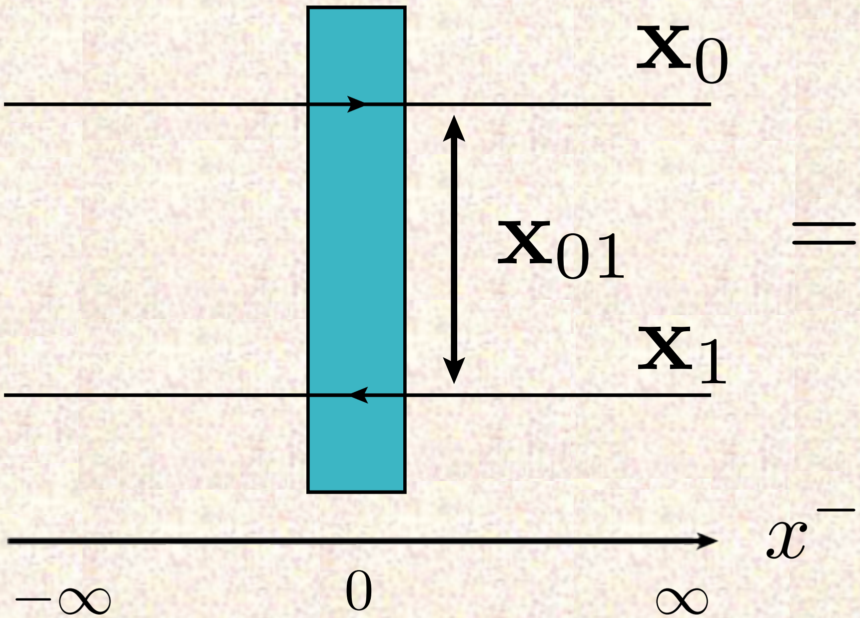
Dipole picture of DIS

- We consider DIS for $Q^2 \geq 1$ GeV^2 and in the region of small x.
- For x<0.01, the quark loop diagram dominates



The scattering amplitude

- For the calculation of the $q\bar{q}$ propagator through the nucleus, we make use of the quasi-classical approximation (the standard GGM/MV model):



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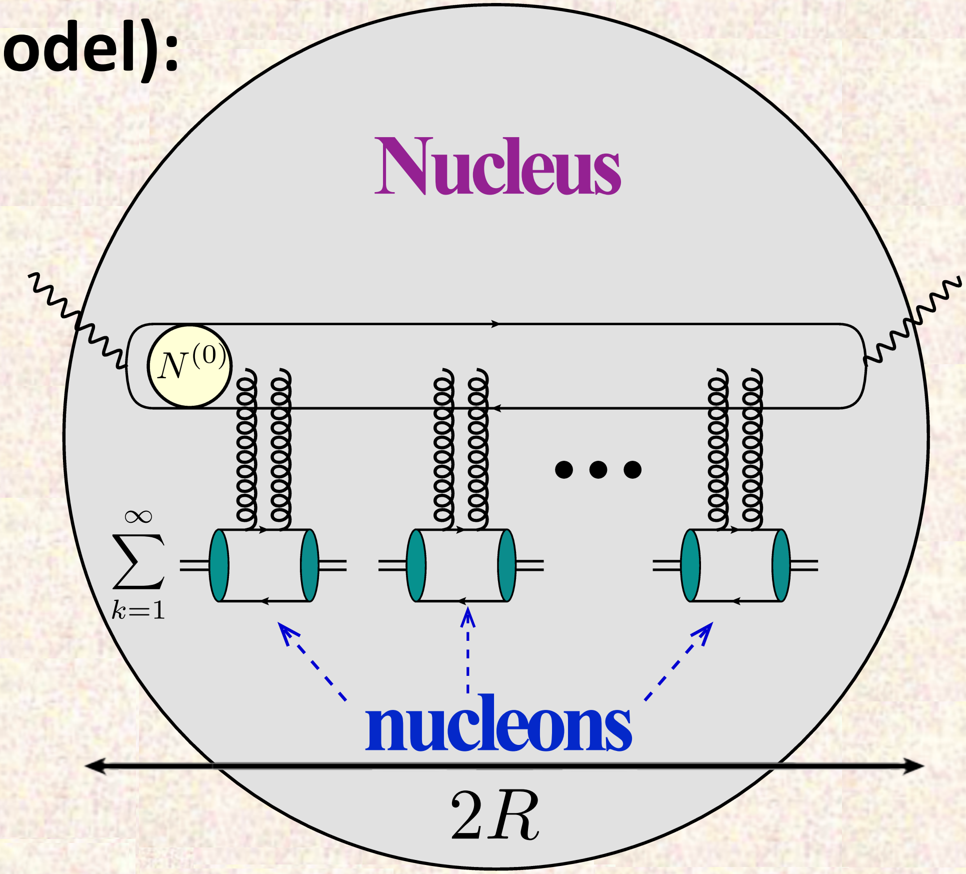
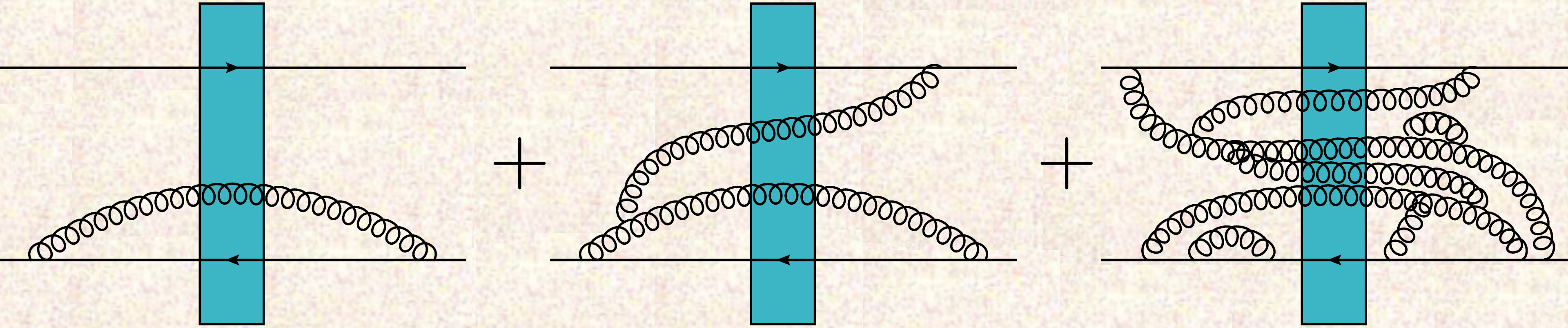


$$N\left(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} \left(\frac{x_{01}^2\, Q_s^2(Y_A, \mathbf{b})}{4} \right)^k = 1 - \exp\left(-\frac{1}{4}\, x_{01}^2\, Q_s^2(Y_A, \mathbf{b}) \right)$$

where $Q_s(b)$ is the initial saturation scale.

The scattering amplitude

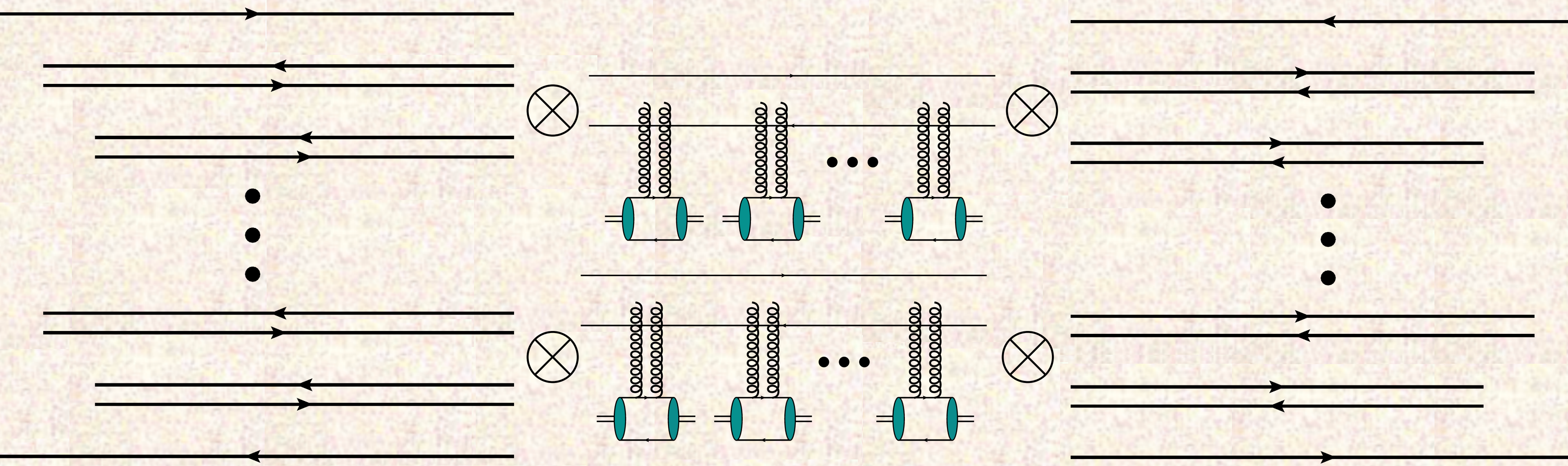
- To include the quantum evolution corrections to the quasi-classical scattering amplitude, we consider only long-lived gluons emissions and virtual gluons (as gluons emitted inside the shockwave are suppressed):



- The gluon cascade includes all possible emissions, both from the quark and the antiquark. Further, gluons emit gluons (recombination), etc.
- We can resume the cascade but only can do it at leading-log (and next-to-leading log), we cannot sum it exactly.

The scattering amplitude

- To further simplify it, we employ the 't Hooft's large-N_c limit, where only the planar diagrams survive and our gluon cascade turns into a cascade of color dipoles:



- Resuming this cascade, we arrive at the following evolution equation for the dipole S-matrix:

$$\frac{\partial}{\partial Y} S\left(Y, \mathbf{x}_{01}, \mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\,\pi} \int d^2 x_2 \, \frac{x_{01}^2}{x_{12}^2 \, x_{02}^2} \left[S\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) \, S\left(Y; \mathbf{x}_{02}, \mathbf{b}\right) - S\left(Y; \mathbf{x}_{01}, \mathbf{b}\right) \right]$$

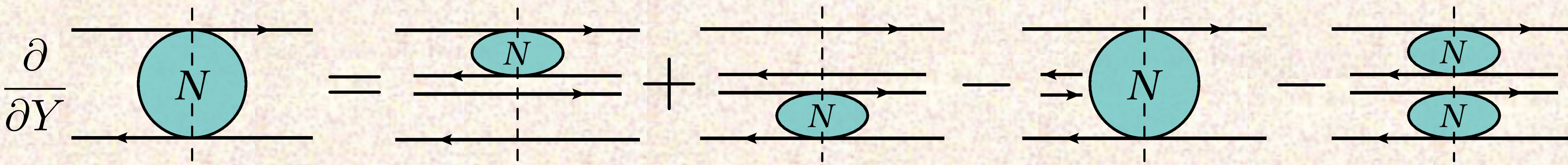
where $Y = \ln 1/x \sim \ln s\,,\ \bar{\alpha}_S = \alpha_S \, N_c/\pi$

The scattering amplitude

- **For N=1-S, it can be re-written in the more conventional form as (Balitsky and Kovchegov (BK), 1999)**

$$\frac{\partial}{\partial Y} N\left(Y, \mathbf{x}_{01}, \mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\,\pi} \int d^2 x_2 \, \frac{x_{01}^2}{x_{12}^2 \, x_{02}^2} \left[N\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) + N\left(Y; \mathbf{x}_{02}, \mathbf{b}\right)\right.$$

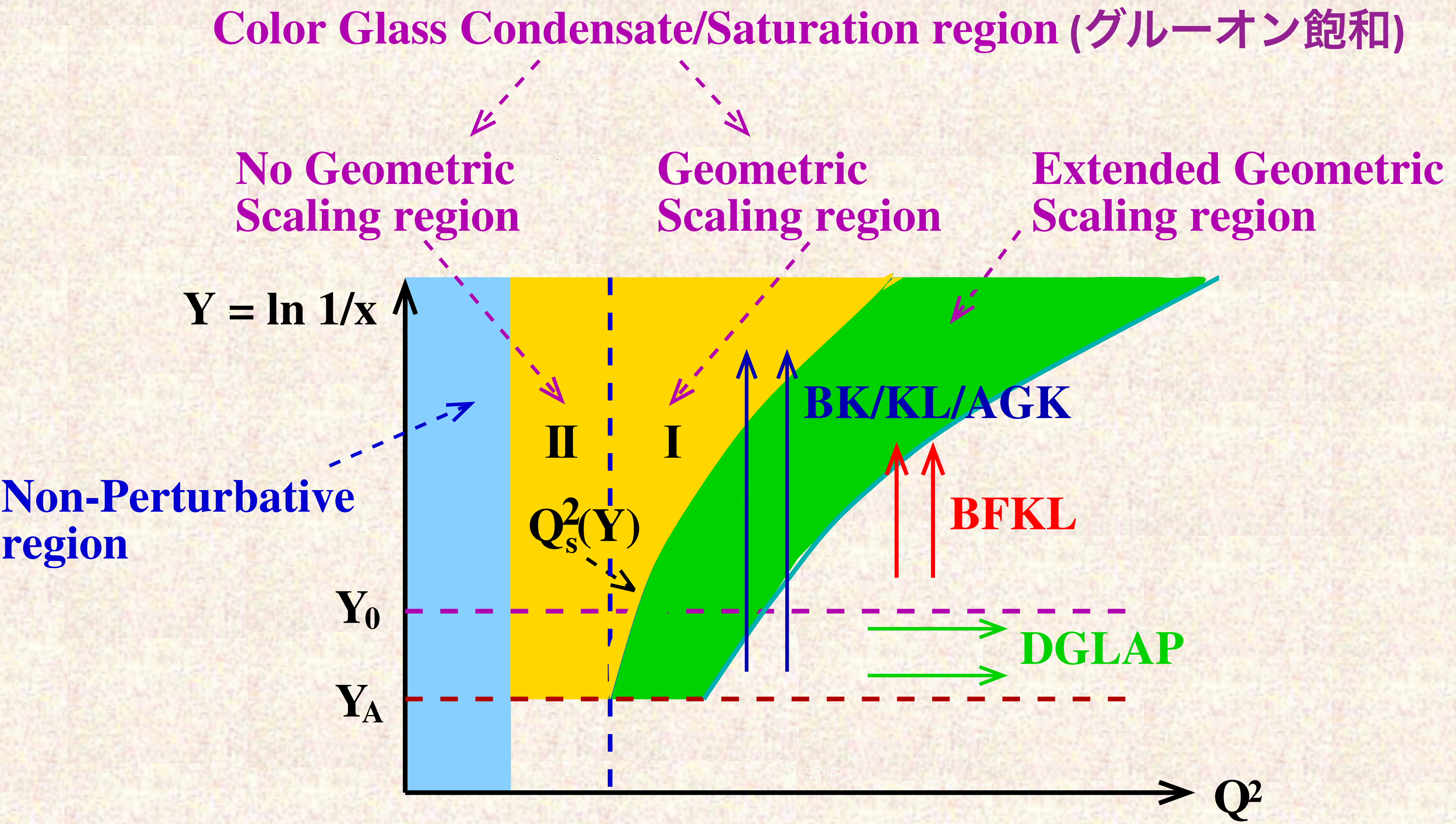
$$\left. - N\left(Y; \mathbf{x}_{01}, \mathbf{b}\right) - N\left(Y; \mathbf{x}_{12}, \mathbf{b}\right) N\left(Y; \mathbf{x}_{02}, \mathbf{b}\right)\right]$$



with initial condition $N\left(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}\right) = 1 - \exp\left(-\frac{1}{4}\, x_{01}^2 \, Q_s^2(Y_A, \mathbf{b})\right)$

- **the nonlinear BK evolution equation describe the transition into the saturation region, along with the physics inside that region.**
- **The solution of the BK equation give us the scattering amplitude. To date, there is no exact solution. But we know a number of approximate analytical solutions**

The scattering amplitude



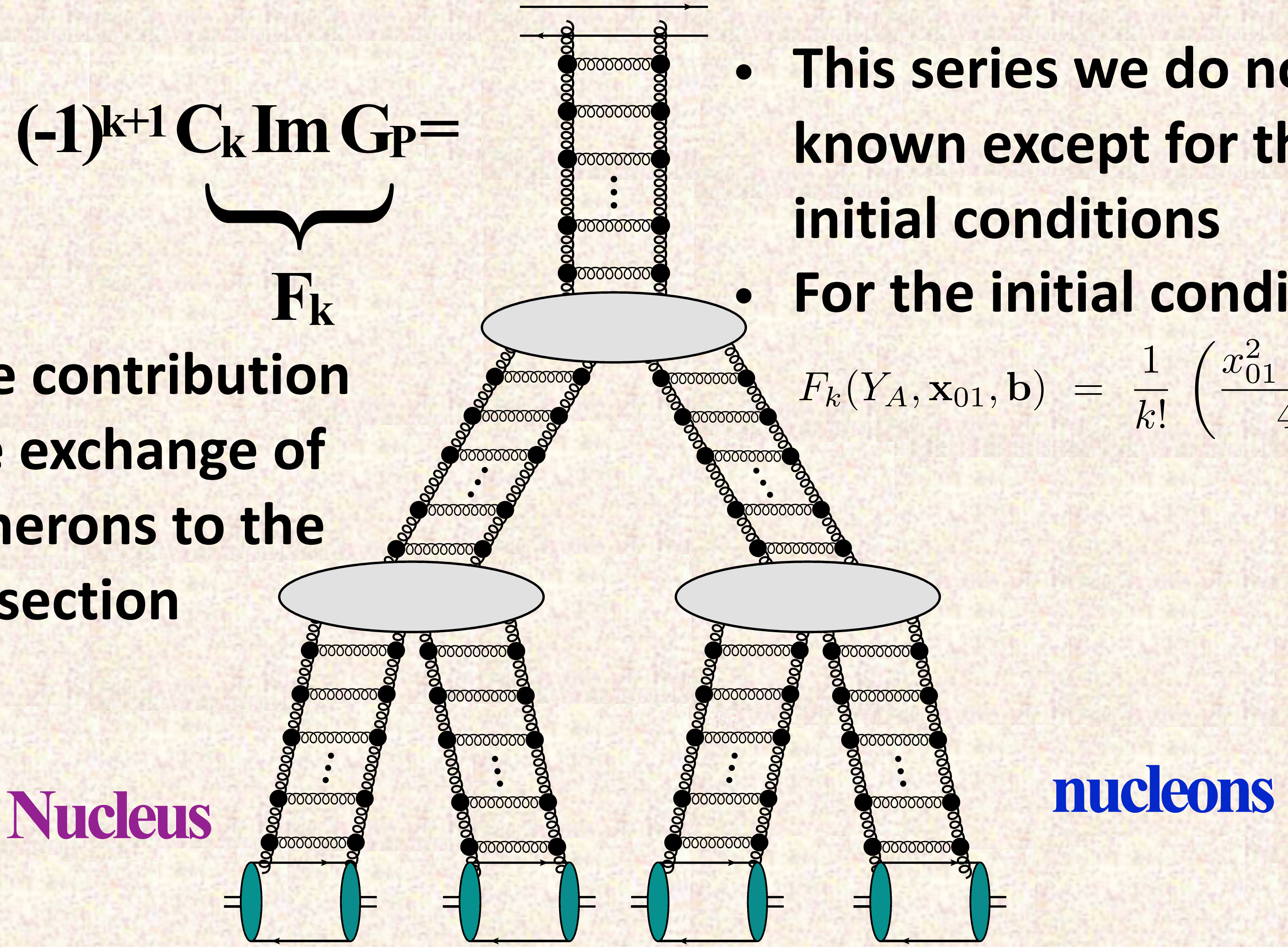
- **Three regions: pQCD ($x_{01}^2\, Q_s^2 < 1$), vicinity of saturation region ($x_{01}^2\, Q_s^2 \sim 1$) and the saturation region ($x_{01}^2\, Q_s^2 > 1$)**

The scattering amplitude

- The scattering amplitude, can also be calculated as the sum of the "fan" diagrams of the BFKL Pomeron calculus.

$$N = \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{C_k \, \mathrm{Im}\, G_P}_{F_k} =$$

F_k: the contribution of the exchange of k-Pomerons to the cross section



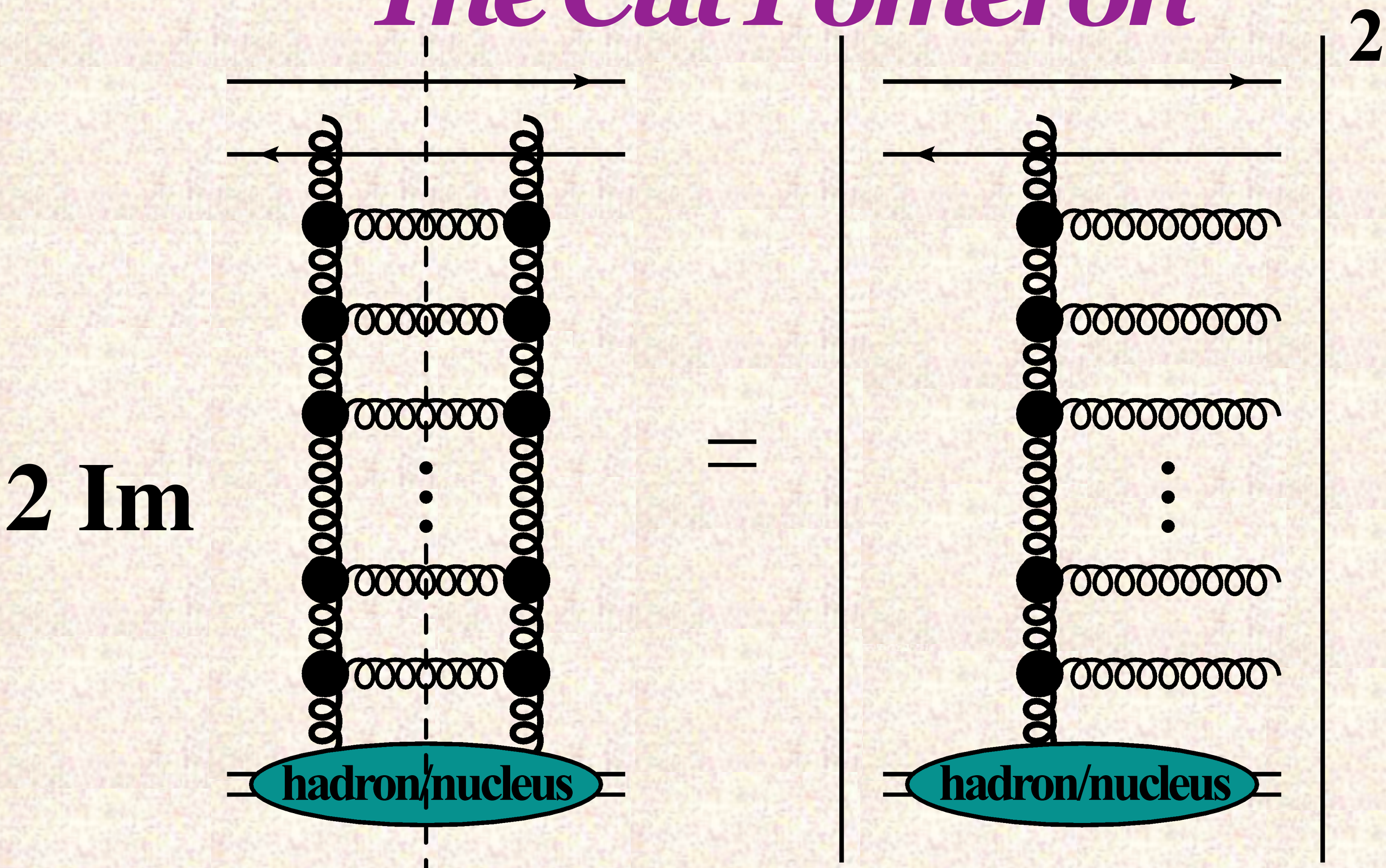
- This series we do not known except for the initial conditions
- For the initial condition

$$F_k(Y_A, \mathbf{x}_{01}, \mathbf{b}) \;=\; \frac{1}{k!}\left(\frac{x_{01}^2\, Q_{s0}^2}{4}\right)^k$$

The Cut Pomeron

- **We see the following equivalence: multiple Pomerons <-> non-linear evolution.**
- **In our treatment of the multiplicity distributions, we are going to explore the equivalence between the pomeron theory of high energy strong interactions and the dipole approach to QCD, which appear to (almost always) work in QCD in the LLA.**

The Cut Pomeron



- The optical theorem for the Pomeron: it gives the "cut pomeron", and, hence, particle production
- The cut Pomeron produces any number of gluons, though the average multiplicity is $\Delta_P Y$ due to Poission statistics.

The AGK cutting rules

- **The Abramovsky-Gribov-Kancheli (AGK) cutting rules allows us to calculate the contributions of n-cut Pomerons if we know Fk: the contribution of the exchange of k-Pomerons to the cross section**

$$\sigma_n^{\mathrm{AGK}}\left(Y;\mathbf{r},\mathbf{b}\right) \;=\; \sum_{k=n}^{\infty}\sigma_n^k\left(Y;\mathbf{r},\mathbf{b}\right)$$

where

$$\sigma_n^k\left(Y;\mathbf{r},\mathbf{b}\right) \;=\; \begin{cases} (-1)^k\left(2^k \;-\; 2\right)F_k(Y;\mathbf{r},\mathbf{b}) & \text{for }\; n=0 \\ (-1)^{k-n}\dfrac{k!}{(k-n)!\,n!}\,2^k\,F_k(Y;\mathbf{r},\mathbf{b}) & \text{for }\; n\geq 1 \end{cases}$$

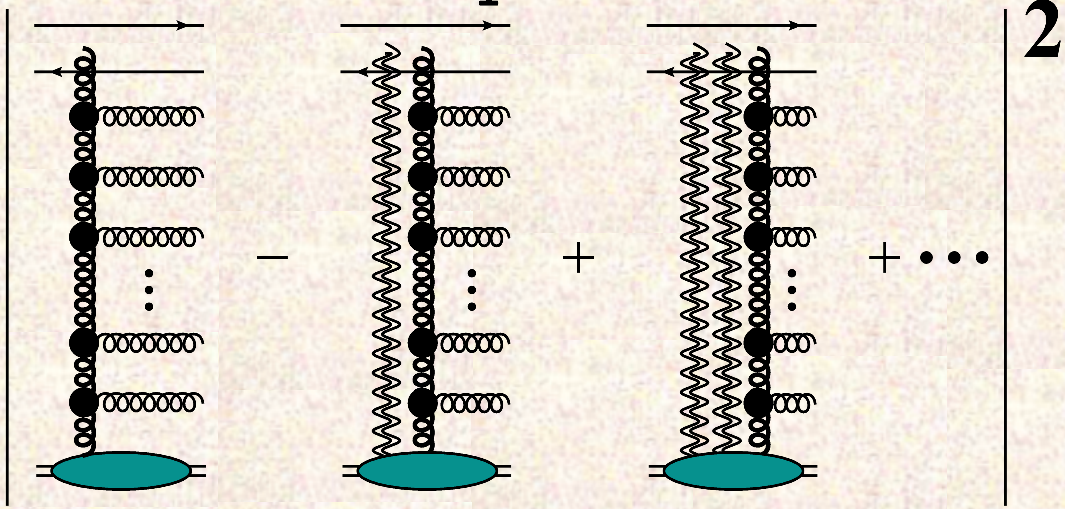
- **Using** $F_k(Y_A,\mathbf{x}_{01},\mathbf{b}) \;=\; \frac{1}{k!}\left(\frac{x_{01}^2\,Q_{s0}^2}{4}\right)^k$**, we can find the initial conditions for the cross section of n-cut BFKL Pomerons:**

$$\sigma_n^{\mathrm{AGK}}\left(Y=Y_A;\mathbf{x}_{01},\mathbf{b}\right) \;=\; \frac{\left(\frac{1}{2}\,x_{01}^2\,Q_{s0}^2\right)^n}{n!}\exp\left\{-\frac{1}{2}x_{01}^2\,Q_{s0}^2\right\}$$

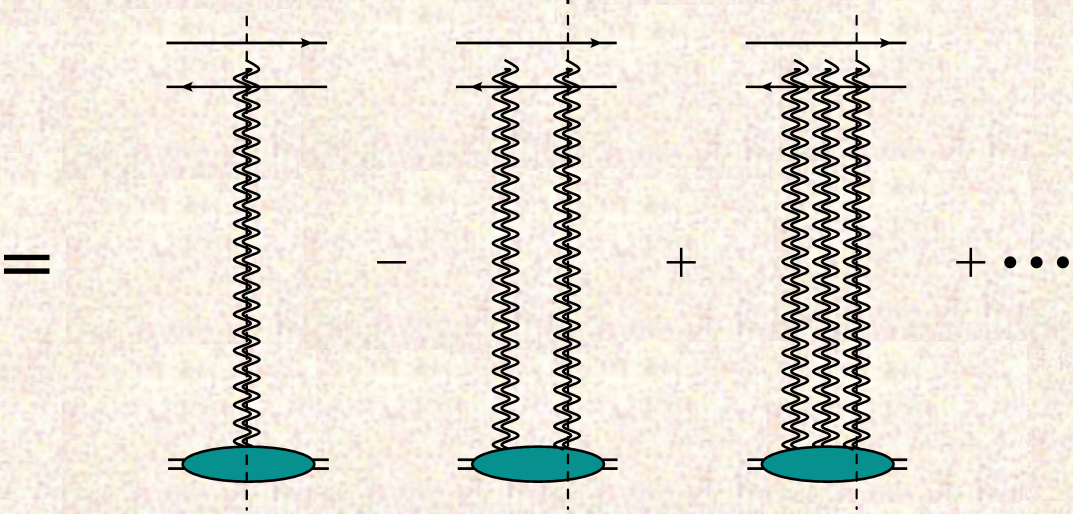
The AGK cutting rules

- **Again, we don't know how to sum this series except for the initial condition. For n=1 (σ_1), it takes the form:**

$$\sigma_1(Y;\mathbf{x}_{01},\mathbf{b}) \;=$$



$$=$$

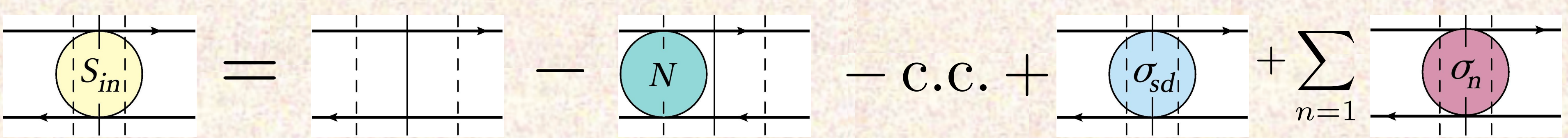


$$= C_1\,\sigma_{in}^{I\!P} \;-\; C_2\,2\,(2\,\mathrm{Im}\,G_{I\!P})\,\sigma_{in}^{I\!P} \;+\; C_3\,3\,(2\,\mathrm{Im}\,G_{I\!P})^2\,\sigma_{in}^{I\!P} \;+\;\cdots$$

The particle production S-matrix

- **We wish to find the equations of collisions for the n-cut Pomeron cross section. This has been done actually (Kormilitzin, Levin and Prygarin), but they employed a generalization of the generating functional Z to find them.**
- **We can give a more simple derivation of the equations of σ_n, by defining a new quantity – the particle production S-matrix "S_{in}": it includes σ_n along with the no-interaction term (=1) and all the interaction terms (N and σ_{sd}) on either side (or both sides) of the final state cut:**

$$S_{in} = 1 - 2N + \sigma_{SD} + \sum_{n=1} \sigma_n$$



The particle production S-matrix

- Replacing $S_{in}=1-2N+\sigma_{SD}+\Sigma_n\sigma_n$ in the equation for the S_{in}-matrix, we obtain the BK, the KL and the AGK/KLP evolution equations:

$$\frac{\partial}{\partial Y}\,\sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right) \;=\; \frac{\bar{\alpha}_S}{2\,\pi}\int d^2x_2\,\frac{x^2_{01}}{x^2_{02}\,x^2_{12}}$$

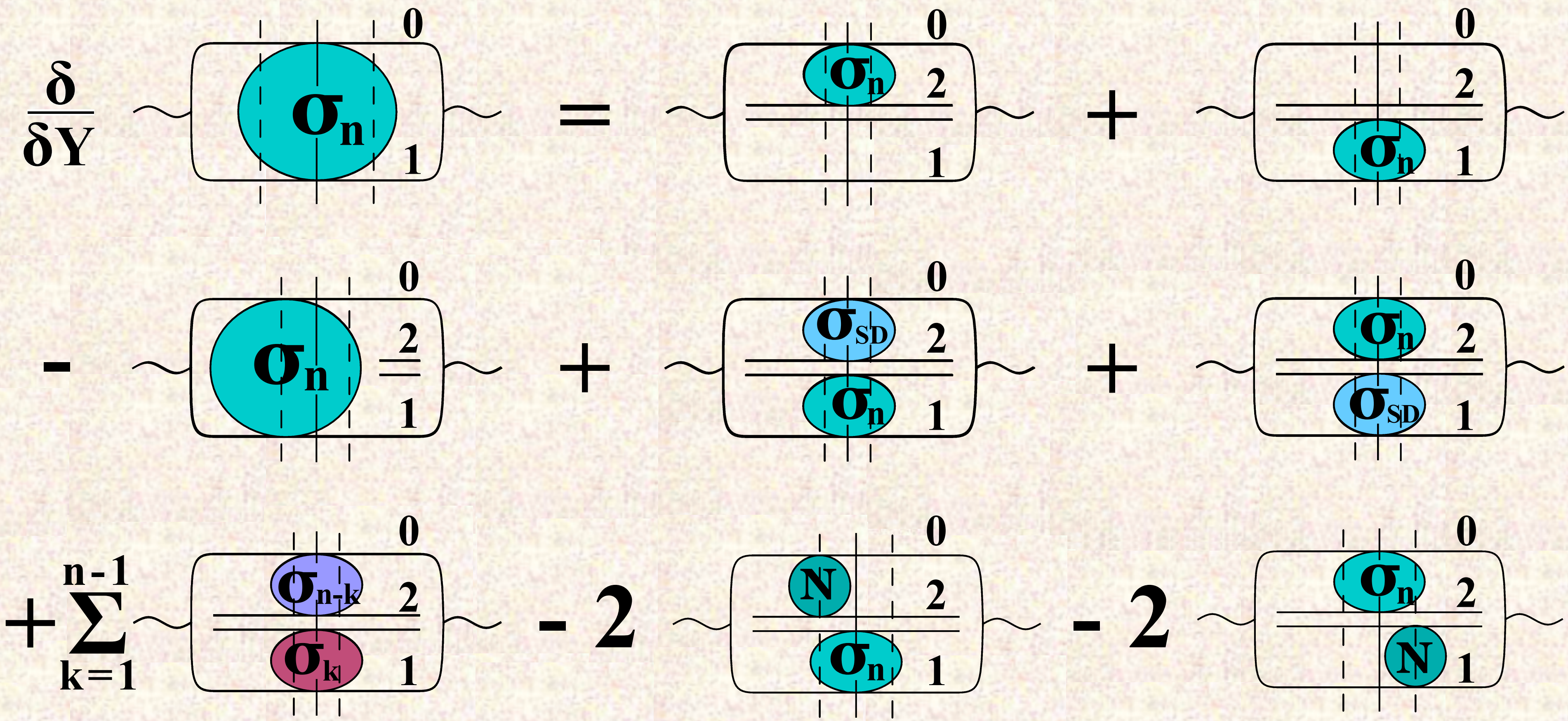
$$\times\left[\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\;+\;\sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\;-\;\sigma_n\left(Y;\mathbf{x}_{01},\mathbf{b}\right)\right.$$

$$+\,\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\;+\;\sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_{SD}\left(Y;\mathbf{x}_{12},\mathbf{b}\right)$$

$$+\;\sum_{k=1}^{n-1}\sigma_{n-k}\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\sigma_k\left(Y;\mathbf{x}_{12},\mathbf{b}\right)$$

$$\left.-\;2\,\sigma_n\left(Y;\mathbf{x}_{12},\mathbf{b}\right)N\left(Y;\mathbf{x}_{02},\mathbf{b}\right)\;-\;2\,\sigma_n\left(Y;\mathbf{x}_{02},\mathbf{b}\right)N\left(Y;\mathbf{x}_{12},\mathbf{b}\right)\right]$$

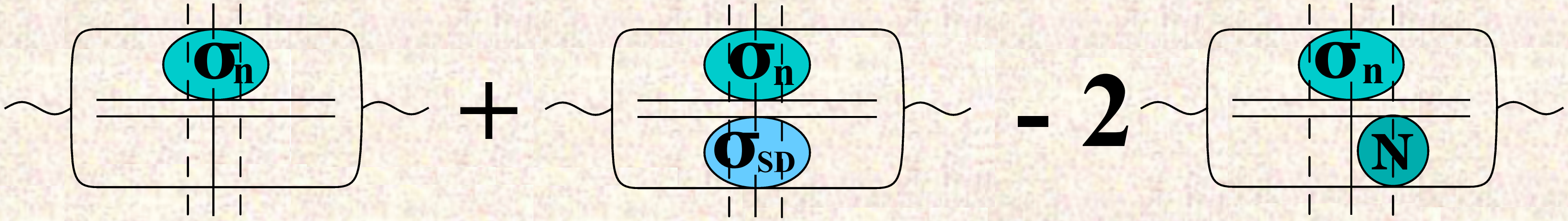
Evolution of σ_n at large N_c



- **We can see the equation is linear but with complicated kernel (K_{BFKL} + 2 - 4 N) and non-homogeneous**

Evolution of σ_n at large N_c

- **Lets note that (for dipoles 02 and 12):**



$$= \sigma_n \left(1 + \sigma_{sd} - 2\,N \right) = \sigma_n\,\Delta$$

$$\frac{\partial}{\partial Y}\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right) = \frac{\bar{\alpha}_S}{2\,\pi}\int d^2 x_2\,\frac{x^2_{01}}{x^2_{12}\,x^2_{02}}$$

$$\times\Big[-\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right) + \sigma_n\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\Delta\left(Y,\mathbf{x}_{02},\mathbf{b}\right)$$

$$+\,\Delta\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\sigma_n\left(Y,\mathbf{x}_{02},\mathbf{b}\right) + \sum_{k=1}^{n-1}\sigma_{n-k}\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\sigma_k\left(Y,\mathbf{x}_{12},\mathbf{b}\right)\Big]$$

Evolution of σ$_n$ at large N$_c$

- **Now we take the evolution kernel with leading logarithmic terms only (** $\begin{matrix} x_{01} \approx x_{02} \gg x_{12} \gg 1/Q_s \\ x_{01} \approx x_{12} \gg x_{02} \gg 1/Q_s \end{matrix}$ **)**

$$\frac{\bar{\alpha}_S}{2\,\pi}\int d^2 x_2\, \frac{x^2_{01}}{x^2_{12}\, x^2_{02}} \,\rightarrow\, \frac{\bar{\alpha}_S}{2}\int\limits_{Q_s^{-2}(Y,b)}^{x^2_{01}} \frac{d x^2_{02}}{x^2_{02}} \,+\, \frac{\bar{\alpha}_S}{2}\int\limits_{Q_s^{-2}(Y,b)}^{x^2_{01}} \frac{d x^2_{12}}{x^2_{12}} \,\equiv\, \int\limits_{-\xi_s}^{\xi} K_{LT}\left(\xi,\xi'\right) d\xi'$$

- **The equations for σ$_n$ for the leading twist BFKL kernel take the form:**

$$\frac{\partial}{\partial Y}\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right) \;=\; \bar{\alpha}_S\Bigg\{ -\,\ln\left(x^2_{01}\,Q^2_s(Y,\mathbf{b})\right)\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)$$

$$+\,\sigma_n\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)} d\ln\left(x^2_{02}\right)\,\Delta\left(Y,\mathbf{x}_{02},\mathbf{b}\right) \,+\, \Delta\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)} d\ln\left(x^2_{02}\right)\,\sigma_n\left(Y,\mathbf{x}_{02},\mathbf{b}\right)$$

$$+\,\sum_{k=1}^{n-1}\int\limits_{\ln\left(Q_s^{-2}(Y,\mathbf{b})\right)}^{\ln\left(x^2_{01}\right)} d\ln\left(x^2_{02}\right)\,\sigma_{n-k}\left(Y,\mathbf{x}_{02},\mathbf{b}\right)\,\sigma_k\left(Y,\mathbf{x}_{01},\mathbf{b}\right)\Bigg\}$$

Evolution of σ_n at large N_c

- **Defining the geometric scaling variables**

$$z \;=\; \ln\left(x^2_{01}\,Q^2_s\left(Y,\mathbf{b}\right)\right)\,;\;\; z' \;=\; \ln\left(x^2_{02}\,Q^2_s\left(Y,\mathbf{b}\right)\right)$$

and the rescaled variable $\delta\tilde{Y} \;=\; \bar{\alpha}_S\left(Y \;-\; Y_A\right)$**, we have**

$$\frac{\partial}{\partial\delta\tilde{Y}}\,\sigma_n\left(\delta\tilde{Y},z\right) \;+\; \kappa\,\frac{\partial}{\partial z}\,\sigma_n\left(\delta\tilde{Y},z\right) \;=\; -\,z\,\sigma_n\left(\delta\tilde{Y},z\right)$$

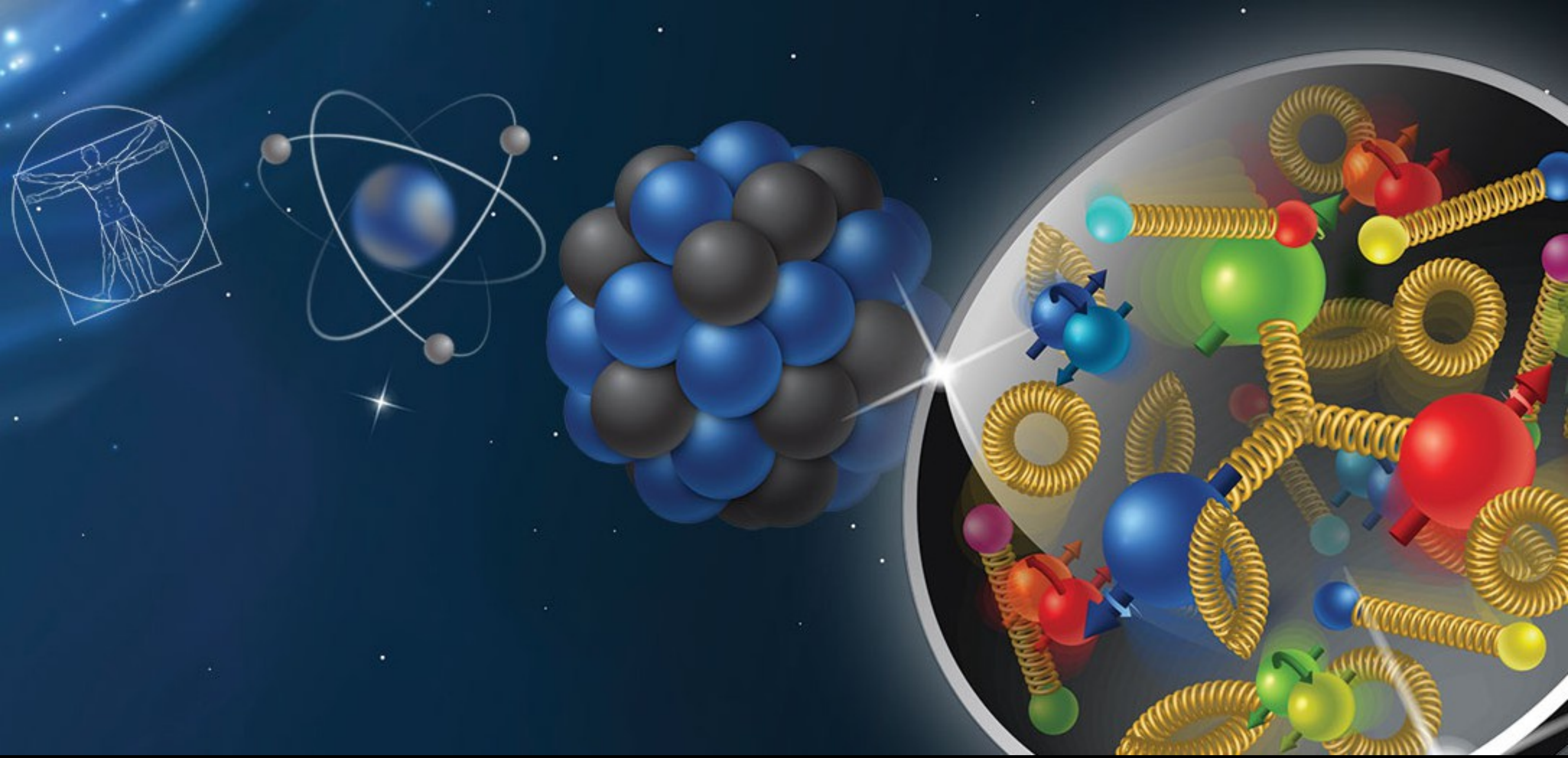
$$+\,\sigma_n\left(\delta\tilde{Y},z\right)\int\limits_{\xi^A_0}^{z} dz'\,\Delta\left(\delta\tilde{Y},z'\right) \;+\; \Delta\left(\delta\tilde{Y},z\right)\int\limits_{\xi^A_0}^{z} dz'\,\sigma_n\left(\delta\tilde{Y},z'\right)$$

$$+\,\sum_{k=1}^{n-1}\int\limits_{\xi^A_0}^{z} dz'\,\sigma_{n-k}\left(\delta\tilde{Y},z'\right)\,\sigma_k\left(\delta\tilde{Y},z\right)$$

where we used $Q^2_s(Y,\mathbf{b}) \;=\; Q_{s0}\,e^{\bar{\alpha}_S\,\kappa\,(Y\;-\;Y_A)}$ **with** $\kappa \;\equiv\; \dfrac{\chi\left(\gamma_{cr}\right)}{1-\gamma_{cr}}$

- **The minimum z is at Y=Y$_A$ (δỸ=0), i.e.**

$$z \;=\; \kappa\,\delta\tilde{Y} \;+\; \xi;\;\; z \overset{\delta\tilde{Y}\,\to\,0}{=} \xi$$ **or for GS case** $z = \xi^A_0$



$$\sigma_n(Y;\boldsymbol{x}_{01},\boldsymbol{b})$$

σ_1: region I (GS)

$$\kappa \frac{d\sigma_1(z)}{dz} = -z\,\sigma_1(z) + \sigma_1(z)\int_{\xi_0^A}^{z} dz'\,\Delta(z') + \Delta(z)\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')$$

$$= -\underbrace{\left(z - \int_{\xi_0^A}^{z} dz'\,\Delta(z')\right)}_{T(z)}\sigma_1(z) + \Delta(z)\underbrace{\int_{\xi_0^A}^{z} dz'\,\sigma_1(z')}$$

$$\int_{\xi_0^A}^{z} dz'\,\sigma_1(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz'\,\sigma_1(z')}_{=\sigma_{0,1}} - \underbrace{\int_{z}^{\infty} dz'\,\sigma_1(z')}_{=\tilde{\Sigma}_1(z)}$$

$$= -T(z)\,\sigma_1(z) + \sigma_{0,1}\,\Delta(z) - \Delta(z)\,\tilde{\Sigma}_1(z)$$

$$\Rightarrow \underbrace{\kappa\frac{d\sigma_1(z)}{dz} + T(z)\,\sigma_1(z) - \sigma_{0,1}\,\Delta(z)}_{\mathcal{L}[\sigma_1]} + \underbrace{\Delta(z)\,\tilde{\Sigma}_1(z)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

σ_1: region I (GS)

- **We have divided the equation in two parts**

$$\mathcal{L}[\sigma_1] \;=\; \kappa \frac{d\sigma_1(z)}{dz} \;+\; T(z)\,\sigma_1(z) \;-\; \sigma_{0,1}\,\Delta(z)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] \;=\; \Delta(z)\,\tilde{\Sigma}_1(z)$$

- **As a solution, we introduce**

$$\sigma_1 \;=\; \sigma_1^{(0)} \;+\; p\,\sigma_1^{(1)} \;+\; p^2\,\sigma_1^{(2)} \;+\; \cdots$$

such that $\mathcal{L}[\sigma_1^{(p)}] \;+\; p\,\mathcal{N}_{\mathcal{L}}[\sigma_1^{(p)}] \;=\; 0$

- **The 0th iteration is**

$$\mathcal{L}[\sigma_1^{(0)}];\;\; \kappa \frac{d\sigma_1^{(0)}(z)}{dz} \;=\; -T(z)\,\sigma_1^{(0)}(z) \;+\; \sigma_{0,1}\,\Delta(z)$$

σ_1: region I (GS)

- **The solution is a sum of the solution to the homogeneous part of the equation and the particular solution of the non-homogeneous one. We obtain**

$$\sigma_1^{(0,I)}(z) = \underbrace{\exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\, T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \underbrace{\left\{\frac{\sigma_{0,1}}{\kappa}\int_{\xi_0^A}^{z} dz'\, \Delta(z') \exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\, T(z'')\right) + C_\phi^{(1)}\right\}}_{\tilde{\sigma}_1^{(0,I)}}$$

where $C_\phi^{(1)} = \frac{1}{2} e^{\xi_0^A} \exp\left(-\frac{1}{2} e^{\xi_0^A}\right)$

- **For the next homotopy iteration we need to account for the linear terms in p:** $\mathcal{L}[\sigma_1^{(0)} + p\,\sigma_1^{(1)}] + p\mathcal{N}_{\mathcal{L}}[\sigma_1^{(0)}] = 0$
- **The equation takes the form**

$$\kappa \frac{d\sigma_1^{(1,I)}(z)}{dz} = -T(z)\,\sigma_1^{(1,I)}(z) - \Delta(z)\,\tilde{\Sigma}_1^{(0)}(z)$$

σ_1: region I (GS)

where $\tilde{\Sigma}_1^{(0)}(z) = \int\limits_z^{\infty} dz' \sigma_1^{(0)}(z')$. **The general solution is**

$$\sigma_1^{(1,I)}(z) = -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int\limits_{\xi_0^A}^{z} dz'\, T(z')\right)\int\limits_{\xi_0^A}^{z} dz' \Delta(z')\, \tilde{\Sigma}_1^{(0)}(z') \exp\left(\frac{1}{\kappa}\int\limits_{\xi_0^A}^{z'} dz''\, T(z'')\right)$$

- **In general, the equation for the p-iteration (p≥1) in region I takes the form**

$$\kappa \frac{d\sigma_1^{(p,I)}(z)}{dz} = -T(z)\,\sigma_1^{(p,I)}(z) - \Delta(z)\,\tilde{\Sigma}_1^{(p-1)}(z)$$

with solution

$$\sigma_1^{(p,I)}(z) = -\frac{1}{\kappa}\exp\left(-\frac{1}{\kappa}\int\limits_{\xi_0^A}^{z} dz'\, T(z')\right)\int\limits_{\xi_0^A}^{z} dz'\, \Delta(z')\, \tilde{\Sigma}_1^{(p-1)}(z') \exp\left(\frac{1}{\kappa}\int\limits_{\xi_0^A}^{z'} dz''\, T(z'')\right)$$

σ_1: region II (no GS)

$$\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial\delta\tilde{Y}} + \kappa\,\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial z} =$$

$$-\underbrace{\left(z - \int_{\xi_0^A}^{z} dz'\,\Delta\left(\delta\tilde{Y},z'\right)\right)}_{T(\delta\tilde{Y},z)}\,\sigma_1\left(\delta\tilde{Y},z\right) + \Delta\left(\delta\tilde{Y},z\right)\underbrace{\int_{\xi_0^A}^{z} dz'\,\sigma_1\left(\delta\tilde{Y},z'\right)}$$

$$\int_{\xi_0^A}^{z} dz'\,\sigma_1\left(\delta\tilde{Y},z'\right) = \underbrace{\int_{\xi_0}^{\infty} dz'\,\sigma_1\left(z'\right)}_{=\sigma_{0,1}} + \underbrace{\int_{\xi_0^A}^{\infty} dz'\left(\sigma_1\left(\delta\tilde{Y},z'\right) - \sigma_1\left(z'\right)\right)}_{=\delta\Sigma_1\left(\delta\tilde{Y}\right)} - \underbrace{\int_{z}^{\infty} dz'\sigma_1\left(\delta\tilde{Y},z'\right)}_{=\tilde{\Sigma}_1\left(\delta\tilde{Y},z\right)}$$

$$\underbrace{\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial\delta\tilde{Y}} + \kappa\,\frac{\partial\sigma_1\left(\delta\tilde{Y},z\right)}{\partial z} + T\left(\delta\tilde{Y},z\right)\sigma_1\left(\delta\tilde{Y},z\right)}_{\mathcal{L}[\sigma_1]} - \sigma_{0,1}\,\Delta\left(\delta\tilde{Y},z\right) + \Delta\left(\delta\tilde{Y},z\right)\delta\Sigma_1\left(\delta\tilde{Y}\right)$$

$$\underbrace{-\,\Delta\left(\delta\tilde{Y},z\right)\tilde{\Sigma}_1\left(\delta\tilde{Y},z\right)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

σ_1: region II (no GS)

- **In this case, the solution is more complex. After some algebra, the general solution takes the form**

$$\sigma_1^{(0,II)}\left(\delta\tilde{Y},z\right) = \Phi_1\left(-\kappa\,\delta\tilde{Y}+z\right)\sigma_1^{0'}\left(\delta\tilde{Y},z\right) + \sigma_1^{0'}\left(\delta\tilde{Y},z\right)\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right)$$

where

$$\sigma_1^{0'}\left(\delta\tilde{Y},z\right) = \exp\left(-\int_0^{\delta\tilde{Y}} d\delta\tilde{Y}'\,T\left(\delta\tilde{Y}',-\kappa\left(\delta\tilde{Y}-\delta\tilde{Y}'\right)+z\right)\right)$$

$$\tilde{\sigma}_1^{0'}\left(\delta\tilde{Y},z\right) = \frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,\frac{\sigma_{0,1}}{\sigma_1^{0'}\left(\delta\tilde{Y}-\frac{z-z'}{\kappa},z'\right)}\Delta\left(\delta\tilde{Y}-\frac{z-z'}{\kappa},z'\right) + C_{\sigma_1}$$

$$\Phi_1(\xi) = \frac{\frac{1}{2}e^{\xi}\exp\left(-\frac{1}{2}e^{\xi}\right)}{\sigma_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)} - \tilde{\sigma}_1^{0'}\left(\delta\tilde{Y}=0,z=\xi\right)$$

- **For other iterations, we just modify our definition of the non-homogeneous term by adding the contribution from N_L**

σ_n

- **Now we need to add to our definition of L[σ_1] the non-homogeneous term**

$$\mathcal{L}[\sigma_n] = \kappa \frac{d\sigma_n(z)}{dz} + T(z)\,\sigma_n(z) - \underbrace{\sigma_{0,n}\,\Delta(z) - \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)}_{-U_n(z)}$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_n] = \Delta(z)\,\tilde{\Sigma}_n(z)$$

- **We solve the equation in a similar way to σ_1, yielding**

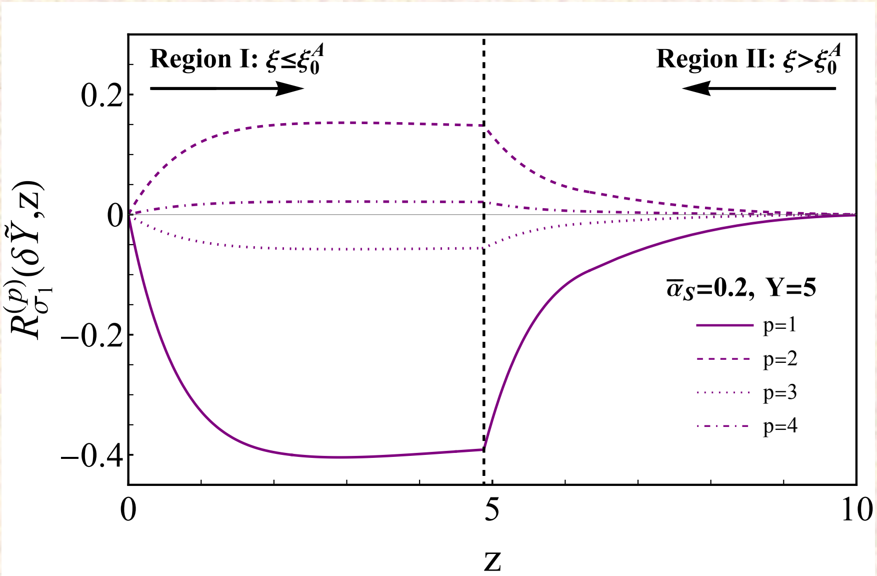
$$\sigma_n^{(0,I)}(z) = \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right) + C_\phi^{(n)}\right\}$$

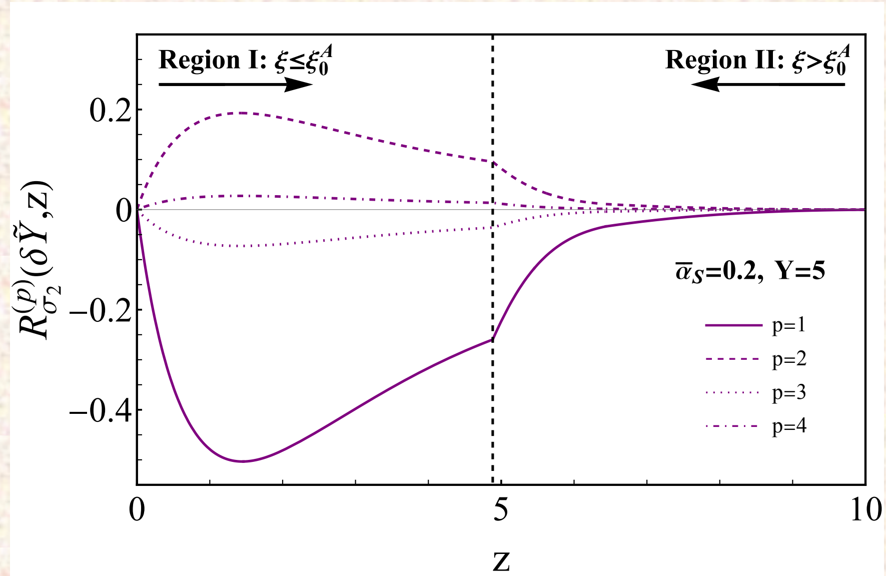
where $C_\phi^{(n)} = \frac{\left(\frac{1}{2}e^{\xi_0^A}\right)^n}{n!}e^{-\frac{1}{2}e^{\xi_0^A}}$**. The formula for the p-iteration is**

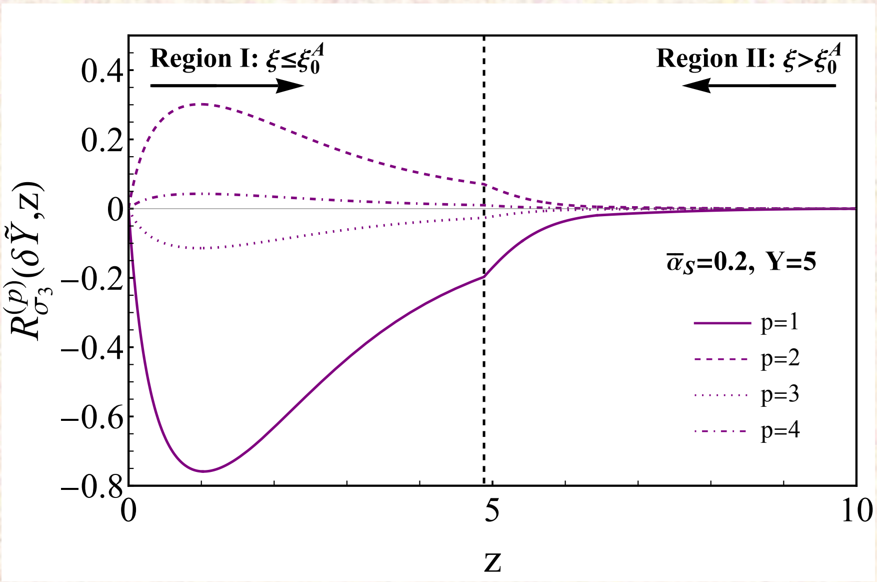
$$\sigma_n^{(p,I)}(z) = \exp\left(-\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,T(z')\right)\left\{\frac{1}{\kappa}\int_{\xi_0^A}^{z} dz'\,U_n(z')\,\tilde{\Sigma}_n^{(p-1)}(z')\,\exp\left(\frac{1}{\kappa}\int_{\xi_0^A}^{z'} dz''\,T(z'')\right) + C_\phi^{(n)}\right\}$$

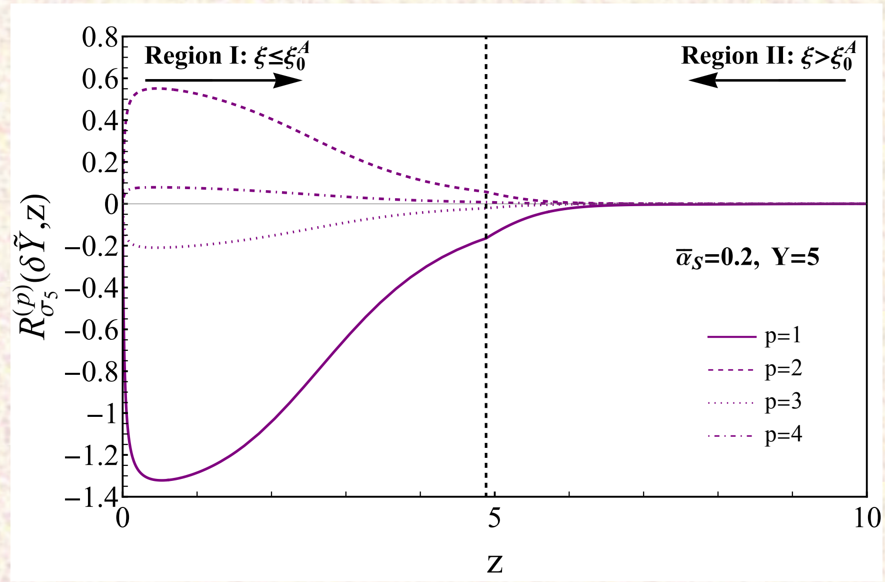
The ratios of the corrections

- **Here the plots for the ratios of σ_1, σ_2, σ_3 and σ_5**



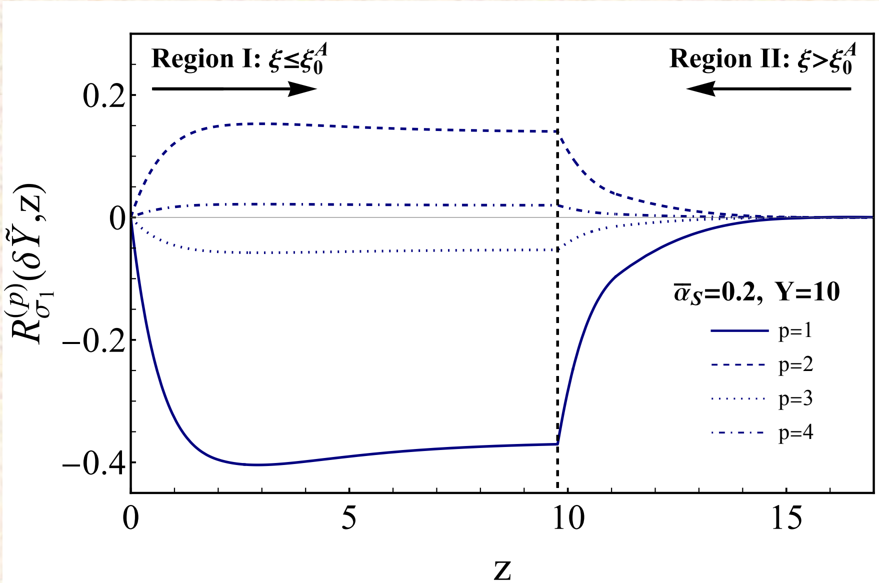


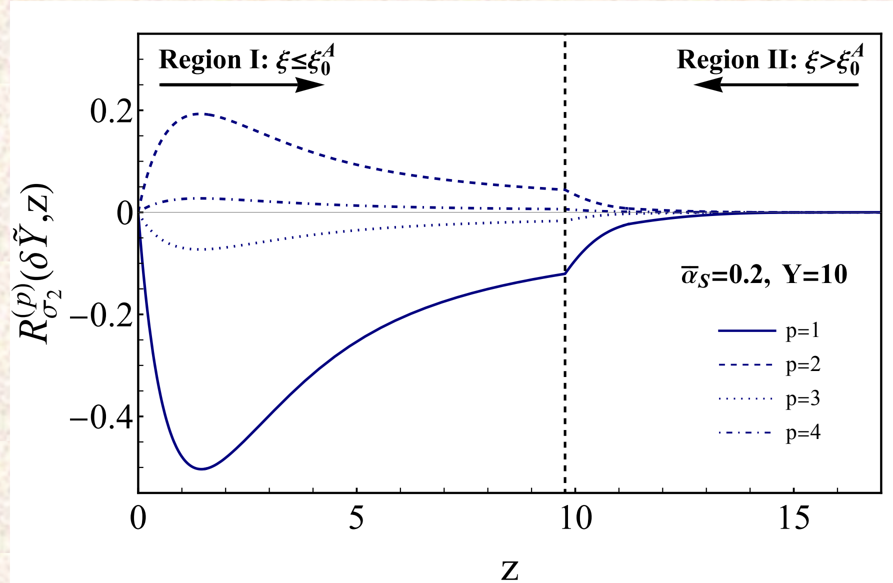


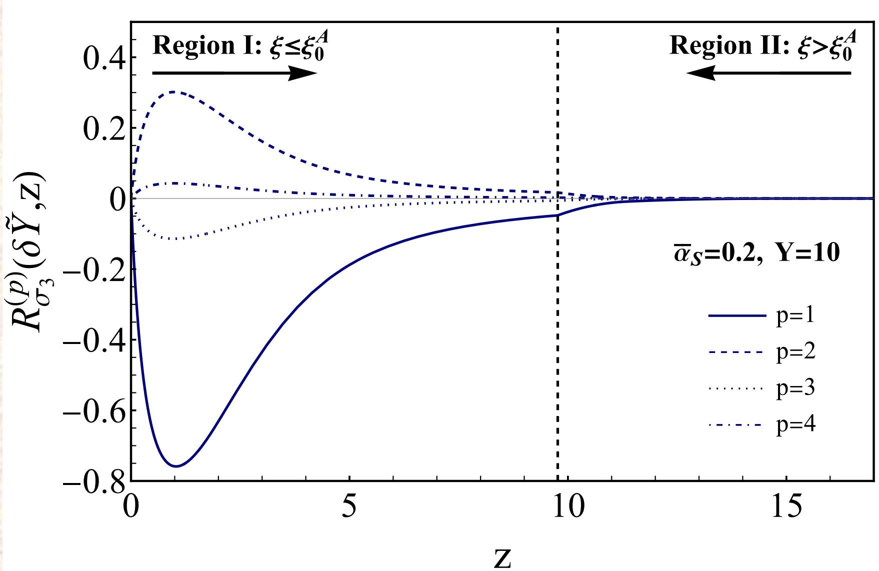


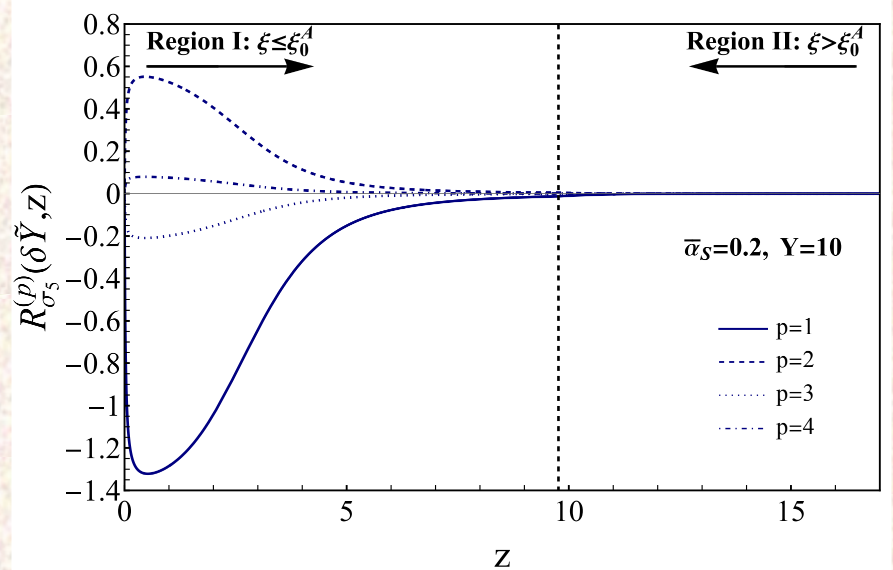
The ratios of the corrections

- **Here the plots for the ratios of $\sigma_1, \sigma_2, \sigma_3$**









The ratios of the corrections

- **One can see that the first iteration turns out to be rather large in the small-z region which corresponds to region I.**
- **Despite the fact that after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n. For this reason, we use this method only for small values of n.**
- **We solve the equation in the large-n limit, and study the possible matching with this solution.**

The large n≳N Solution

- **We start with our equation**

$$\kappa \frac{d\sigma_n(z)}{dz} = -\left(z - \Sigma_\Delta(z)\right)\sigma_n(z) + \Delta(z)\,\Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\sigma_k(z)$$

where $\Sigma_n(z) = \int^z dz' \sigma_n(z')$ **and** $\Sigma_\Delta(z) = \int^z dz' \Delta(z')$

- **We suggest that the solution has the following form**

$$\Sigma_n(z) = \phi(z)\,\exp\left(-n\,\Phi(z)\right)$$

- **Functions Φ (z) and φ (z) we will find from the equation. Noticing that**

$$\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\sigma_k(z) = \frac{1}{2}\frac{d}{dz}\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\,\Sigma_k(z) = \frac{n-1}{2}\frac{d}{dz}\phi^2(z)\exp\left(-n\Phi(z)\right)$$

- **The equation can be rewritten as**

$$\kappa \frac{d^2\Sigma_n}{dz^2} = -\frac{d}{dz}\left((z - \Sigma_\Delta(z))\Sigma_n(z)\right) + \Sigma_n(z) + \frac{n-1}{2}\frac{d}{dz}\left(\phi(z)\Sigma_n(z)\right)$$

The large n≳N Solution

- **Introducing**

$$\mathcal{S}_n(z) = \int^{z} dz'\,\Sigma_n(z')$$

and assuming that all functions decreases at large z, we have

$$\kappa\frac{d\Sigma_n}{dz} = -(z - \Sigma_\Delta(z))\Sigma_n(z) + \mathcal{S}_n(z) + \frac{n-1}{2}\phi(z)\,\Sigma_n(z)$$

- **Using** $\frac{d\Sigma_n}{dz} = (\phi'(z) - n\,\phi(z)\Phi'(z))\exp(-n\,\Phi(z))$

we can find now all unknown functions Φ, φ and S.

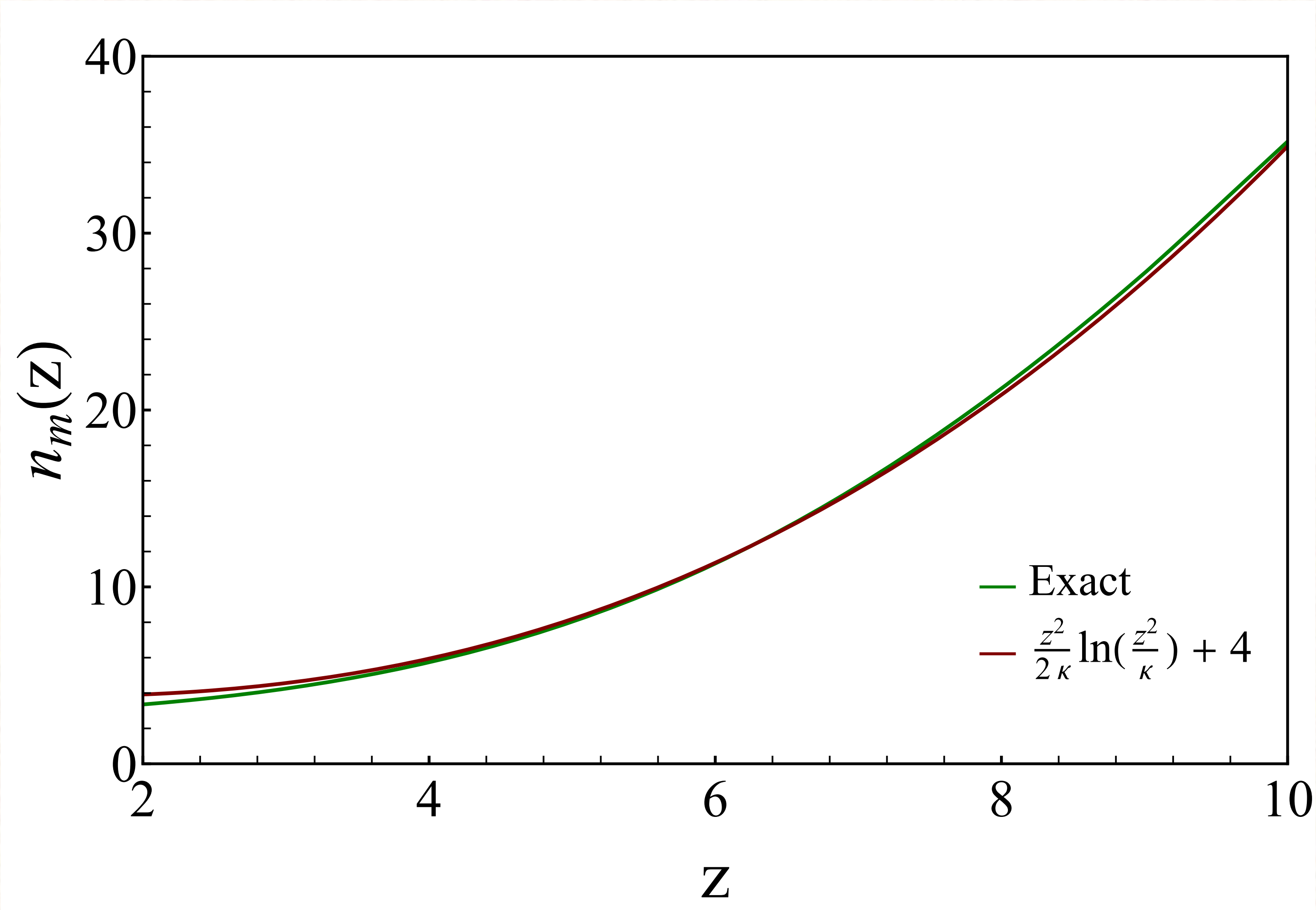
- **After some algebra and taking the large z and n limit, we find**

$$\sigma_n(z) = \left(\phi'(z) - n\phi(z)\,\Phi'(z)\right)\exp(-n\Phi(z)) \xrightarrow{n\geq N(z)} \frac{2}{\kappa}\frac{z^2}{N(z)}\Psi\left(\xi = \frac{n}{N(z)}\right)$$

where $\Psi(\xi) = \xi\,e^{-\xi}$ **and** $N(z) = \frac{2\,z}{C_\phi}\,e^{\frac{(z-z_\Delta)^2}{2\,\kappa}}$

Matching of the solutions

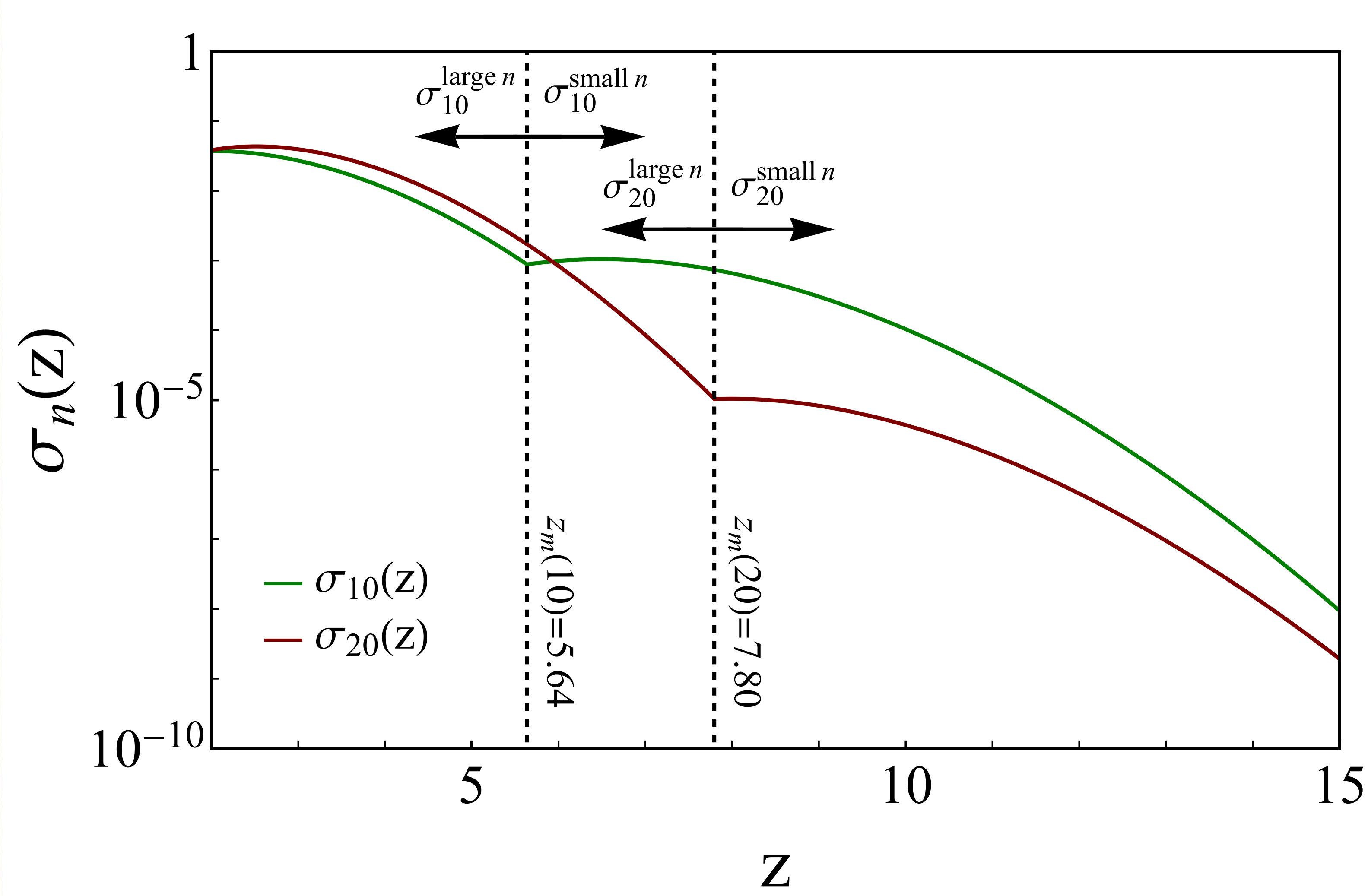
$$\sigma_{\text{small n}} = \sigma^{(0)}_{n_m}(z)\left(1 + \sum_{p=1}^{p_{\max}} \frac{\sigma^{(p)}_{n_m}(z)}{\sigma^{(0)}_{n_m}(z)}\right) = \sigma_{\text{large n}} = \frac{2}{\kappa}\frac{z^2}{N(z)}\Psi\left(\frac{n_m}{N(z)}\right)$$



Matching of the solutions

- **From the previous plot, we can invert n to obtain the matching z**

$$\sigma_n(z) = \sigma_{\text{large n}}(z)\,\Theta(z_m - z) + \sigma_{\text{small n}}(z)\,\Theta(z - z_m)$$




Quantum entanglement in
high-energy collisions

Entropy of produced gluons

- **The probability to have n-cut BFKL Pomerons in the final sate is equal to**

$$\mathcal{P}_n^{\mathrm{AGK}}(z) \;\equiv\; \frac{\sigma_n^{\mathrm{AGK}}(z)}{\sigma_{in}^{\mathrm{AGK}}(z)} \;=\; \frac{2}{\kappa}\,\frac{z^2}{N(z)}\,\Psi\left(\frac{n}{N(z)}\right)$$

- **The entropy content of multiplicity distributions is defined by**

$$S_E(z) \;=\; -\sum_n \mathcal{P}_n^{\mathrm{AGK}}(z)\,\ln\left(\mathcal{P}_n^{\mathrm{AGK}}(z)\right)$$

$$=\; \frac{(z - z_\Delta)^2}{2\,\kappa} \;-\; \ln z \;-\; \ln\left(\frac{C_\phi}{\kappa}\right) \;+\; \frac{2}{\kappa}\,\ln 2$$

$$\xrightarrow{z \gg 1}\; \frac{z^2}{2\,\kappa}$$

The conclusions of this talk

- We calculated the entropy and the multiplicity distributions directly from QCD evolution equations and the s-channel unitarity constraints. We obtained a much larger entropy that other QCD models ($\propto z^2$). We plan to clarify this discrepancy in our further publications.
- \sigma_1, \sigma_2 and \sigma_3 are cross sections for 1, 2 and 3 cut Pomeron production which are closely related to <n>, <n^2-n>, <n^3 - 3 n^2+3 n> moments of the multiplicity distributions for DIS experiment. So we need keep investigating how to connect with experiment.

Thanks/ありがとう!

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