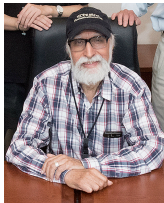


Strong CP Problem and Lattice nEDM

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HET lunch seminar



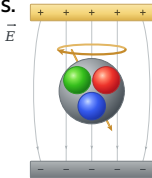
Why search for CP-odd interactions?

1.1-A Motivations for CP-odd interactions

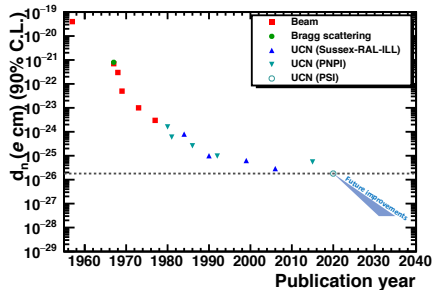
- **A baryon-asymmetric universe:**
matter–antimatter asymmetry motivates CP violation beyond CKM.
CP violation is one of Sakharov's three conditions.
- **EDM searches:** neutrons, atoms, molecules and charged particles are complementary probes.
- **Beyond the θ term:** quark EDMs/chromo-EDMs, the Weinberg operator and CP-odd four-fermion interactions.

$$H_{\text{EDM}} = -d_n \boldsymbol{\sigma} \cdot \mathbf{E}$$

P, T violation $\Rightarrow$ CP violation
assuming CPT.



Neutron EDM: historical limits



$$|d_n| < 1.8 \times 10^{-26} e \text{ cm}$$

Historical trend redrawn from Alarcon et al., Fig. 3.
Older points approximate; 2020 bound exact. No forecasts.

Alarcon et al., arXiv:2203.08103, Fig. 3; Abel et al., PRL 124 (2020) 081803; Engel et al., arXiv:1303.2371. [1:R1–R3]

Strong and weak CP violation

1.1-B A dimension-four CP-odd interaction

A local, gauge-invariant interaction allowed by QCD:

$$\mathcal{L}_\theta = \theta q_{\text{top}}(x), \quad q_{\text{top}}(x) = \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}, \quad \tilde{G}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^a.$$

The anomalous chiral transformation relates the gluonic angle to mass phases:

$$\bar{\theta} = \theta + \arg \det(M_u M_d).$$

- **Two independent CP phases:** the CKM phase and the strong phase $\bar{\theta}$.
- **A total derivative, but not automatically irrelevant:** invisible in ordinary perturbation theory around the trivial vacuum; effects are nonperturbative.
- **Lowest-dimension CP-odd term:** mixing with higher-dimensional CP-odd operators makes its response necessary for their analysis.

Ai, arXiv:2609.00620; Williams, arXiv:2601.07165v3. [1:R4,R10]

The conventional strong CP problem

1.2-A An unexplained small parameter

For small $\bar{\theta}$, the θ -induced neutron EDM is

$$d_n^{(\theta)} = c_n \bar{\theta} + O(\bar{\theta}^3), \quad c_n = \left. \frac{\partial d_n}{\partial \theta} \right|_0.$$

The conventional order-of-magnitude estimate gives

$$\underbrace{|c_n| \sim 10^{-16} \text{ e cm}}_{\text{QCD response}} \quad \underbrace{|d_n| \lesssim 10^{-26} \text{ e cm}}_{\text{experiment}} \quad \Rightarrow \quad \boxed{|\bar{\theta}| \lesssim 10^{-10}}.$$

Assuming the θ contribution is not hidden by cancellations with other CP-odd sources.

**Why is the strong CP phase so small ?
(weak interactions already violate CP)**

An unexplained small parameter—not an inconsistency of QCD.

Abel et al., arXiv:2001.11966; Engel et al., arXiv:1303.2371; Hook, arXiv:1812.02669. [1:R2,R3,R5]

Possible resolutions—or a different question?

1.2-B Mechanisms versus premises

- **Peccei–Quinn / QCD axion**

A spontaneously broken, QCD-anomalous $U(1)_{\text{PQ}}$ makes the strong phase dynamical.

- **Massless up quark**

A genuinely massless up quark permits phase removal; not supported by lattice phenomenology.

- **CP / parity in the UV**

Symmetry and controlled breaking suppress strong CP while allowing weak CP violation.

QCD axion

$$\theta_{\text{eff}}(x) = \bar{\theta} + \frac{a(x)}{f_a},$$

$$V(a) = E_{\text{QCD}} \left(\bar{\theta} + \frac{a}{f_a} \right)$$

$$\boxed{\langle \theta_{\text{eff}} \rangle = 0 \pmod{2\pi}}.$$

Neglecting other explicit PQ breaking and additional CP-odd sources.

Creutz's question: what is a nonperturbatively meaningful definition of $m_u = 0$?

Is an independent physical strong CP phase unavoidable?

Different proposals question different premises. The conventional θ -dependent theory is our baseline.

Hook; Grilli di Cortona et al.; Creutz / Srednicki; Alexandrou et al.; Williams; Benabou et al. [1:R5–R11]

Topological charge labels a history of CS number

2.1 A spatial configuration is not a spacetime history

$$N_{\text{CS}}[A(\tau)] = \int_{\Sigma_\tau} \omega_3(A), \quad q_E d^4x = d\omega_3,$$

$$q_E(x) = \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a.$$

No spatial boundary flux; consistent gauge patches:

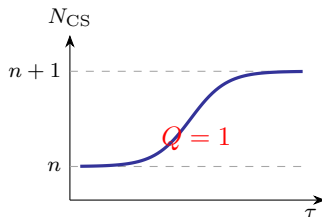
$$Q_{[\tau_i, \tau_f]} = N_{\text{CS}}(\tau_f) - N_{\text{CS}}(\tau_i).$$

N_{CS} is generally real; large gauge transformations give

$$N_{\text{CS}}[A^U] = N_{\text{CS}}[A] + \nu(U), \quad \nu(U) \in \mathbb{Z}.$$

**Euclidean instantons describe tunnelling semiclassically;
Euclidean time is not real-time evolution.**

Ai, arXiv:2609.00620, Secs. 3.2–3.5; Williams, arXiv:2601.07165v3, Sec. 6.6. [2:R1,R2]



Isolated-instanton schematic
(not a Monte Carlo time history)

$$n_i \xrightarrow{Q} n_f, \quad Q = n_f - n_i.$$

Topological sectors of Path integral config. space

2.2 Integer topology needs a specified global setting

1/2

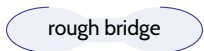
For smooth $SU(3)$ connections on a closed four-manifold,
or on $\mathbb{R}^4$ with suitable finite-action asymptotics,

$$Q = \frac{1}{8\pi^2} \int \text{tr}(\mathcal{F} \wedge \mathcal{F}) \in \mathbb{Z}.$$

This does not apply to an arbitrary open subvolume.

At finite lattice spacing, the unrestricted Wilson link space

$$\mathcal{U} = \prod_{\ell} SU(3) \quad \text{is connected.}$$



Unrestricted link space



Admissible subsets (schematic)

A continuous clover charge need not be integer; Q need not classify every component one-to-one.

Lüscher (1982); Phillips–Stone; Adams; Creutz; Hernández–Jansen–Lüscher; Cè et al.; Lüscher (2021). [2:R3–R9]

Topological sectors of Path integral config. space

2.2 Integer topology needs a specified global setting

2/2

Construction	What it provides	What must be controlled
Admissibility / Lüscher geometry	Integer Q_{geom} ; stable under allowed deformations	Plaquette bound and geometric construction
Overlap index	Integer $Q_{\text{ov}} = \text{index } D_{\text{ov}}$	Kernel choice, M_5 , zero crossings, locality
Gradient flow	Smooth positive-flow-time observables	Fixed physical flow time; continuum extrapolation

Creutz's question: how is topology assigned on rough fields, and is the continuum limit independent of the prescription?

Finite- a dependence on a definition is not, by itself, a surviving continuum ambiguity.

Classical-continuum index theorem: Adams. Quantum susceptibility: flow / density-chain agreement.

Lüscher (1982); Phillips–Stone; Adams; Creutz; Hernández–Jansen–Lüscher; Cè et al.; Lüscher (2021). [2:R3–R9]

Cluster decomposition and factorization: the θ vacuum

2.3 Separate factorization, unitarity and integer quantization

1/3

Widely separated local experiments should decorrelate.

$$\langle O_A(x) O_B(y) \rangle_c \longrightarrow 0 \quad (|\mathbf{x} - \mathbf{y}| \rightarrow \infty).$$



A property of the vacuum—not exact at finite distance, and not the same as microcausality.

Assume an additive history label and additional weights depending only on it:

$$Z = \sum_Q w(Q) Z_Q, \quad Q_{12} = Q_1 + Q_2.$$

Independent composition requires

$$w(Q_1 + Q_2) = w(Q_1) w(Q_2).$$

Williams, arXiv:2601.07165v3, Sec. 6.4, Eqs. (159)–(166), and Sec. 10.1; Jackiw–Rebbi (1976). [2:R2,R10]

Cluster decomposition and factorization: the θ vacuum

2.3 Separate factorization, unitarity and integer quantization

2/3

Input

Consequence

Factorization, $Q \in \mathbb{Z}$, $w(0) = 1$

$$w(Q) = z^Q, z \neq 0$$

Unitary topological character / Hermiticity

$$|z| = 1, z = e^{i\theta}; \text{real } \theta$$

Integer Q in the standard normalization

$$\theta \sim \theta + 2\pi$$

$$Z(\theta) = \sum_{Q \in \mathbb{Z}} e^{i\theta Q} Z_Q.$$

These conditions do not select $\theta = 0$.

Nor do they alone establish nontrivial θ dependence of observables.

Williams, arXiv:2601.07165v3, Sec. 6.4, Eqs. (159)–(166), and Sec. 10.1; Jackiw–Rebbi (1976). [2:R2,R10]

Cluster decomposition and factorization: the θ vacuum

2.3 Separate factorization, unitarity and integer quantization

3/3

Conventional canonical realization

A phase convention consistent with $Q = n_f - n_i$ and $e^{+i\theta Q}$:

$$|\theta\rangle \propto \sum_n e^{-in\theta} |n\rangle, \quad U_m |\theta\rangle = e^{im\theta} |\theta\rangle.$$

$|n\rangle$ denotes the conventional vacuum-type states,
not eigenstates of a conserved physical charge.

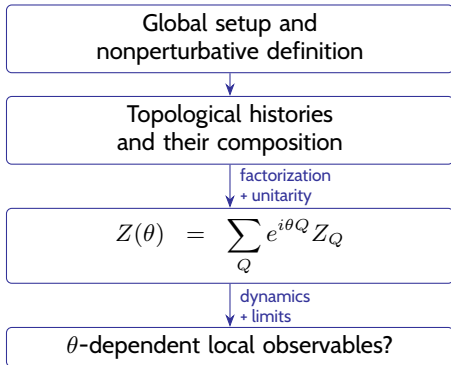
$$Z(\theta) = \sum_{Q \in \mathbb{Z}} e^{i\theta Q} Z_Q.$$

**The vacuum family, Nature's θ , and observable dependence
are separate questions.**

Williams, arXiv:2601.07165v3, Sec. 6.4, Eqs. (159)–(166), and Sec. 10.1; Jackiw–Rebbi (1976). [2:R2,R10]

Strong CP problem: which step is being questioned?

2.4 Separate the global definition, the allowed weights and the dynamics



Williams

Is the conventional global construction—and an independent physical θ —compulsory?

Ai–Cruz–Garbrecht–Tamarit

Which Hilbert space, and which ordering of the volume limit and sector sum?

Kaplan–Sen

Does a proposed 5D regulator of the chiral SM have the same global physics as standalone QCD?

Different claims, not one common mechanism. Counterarguments belong alongside them in Chapter 3.

What can a lattice determination of $d'_n(0)$ establish about this chain?

Williams; Ai–Garbrecht–Tamarit; Kaplan–Sen; Benabou et al. (counterargument). [2:R2,R12–R14]

Williams: Is a physical θ required, or merely allowed?

3.1 The status of the parameter

Local structure he accepts

- Topological density $q(x)$ and the axial anomaly.
- Local correlation functions and susceptibility.
- Clustering of widely separated local experiments.

What he questions

- Must the full quantum theory adopt the conventional global sector construction?
- Does a source response require an independent coupling chosen by Nature?

$$Z[J] = \int \mathcal{D}\Phi e^{-S_0 + i \int J(x) q(x) d^4x} \quad \not\Rightarrow_{\text{Williams}} \quad \text{mandatory physical } \theta.$$

Allowed does not mean required: this is the claim, not a proof that nonzero- θ QCD is inconsistent.

A. G. Williams, arXiv:2601.07165v3, Secs. 10–12; arXiv:2607.10272v2.

ACGT: Which order of limits defines the theory?

3.2 The order of the sector sum and the volume limit

Define a regulated, **normalized** expectation value:

$$\mathcal{R}_{V,K}(\theta) = \frac{\sum_{Q=-K}^K e^{i\theta Q} Z_Q(V) \langle O \rangle_{Q,V}}{\sum_{Q=-K}^K e^{i\theta Q} Z_Q(V)}.$$

Conventional prescription

$$\lim_{V \rightarrow \infty} \lim_{K \rightarrow \infty} \mathcal{R}_{V,K}(\theta)$$

Sum topological histories before taking the thermodynamic limit.

ACGT proposal

$$\lim_{K \rightarrow \infty} \lim_{V \rightarrow \infty} \mathcal{R}_{V,K}(\theta)$$

Fixed-sector infinite-volume limits first: the authors argue for phase alignment and no strong CP violation.

Schematic prescriptions. The canonical Hilbert-space argument is separate.

Claimed CP conservation does not remove local topological fluctuations.

Ai–Cruz–Garbrecht–Tamarit, arXiv:2001.07152; Ai–Garbrecht–Tamarit, arXiv:2404.16026v2, 2403.00747; reply arXiv:2511.04216v1.

Other mechanisms: decoupling or vacuum selection?

3.3 Two distinct proposals, not a common mechanism

Schierholz: decoupling

- Confinement and topological zero modes.
- Vanishing zero-mode density in the infinite-volume limit.
- Claimed decoupling from finite-size hadrons, removing their strong-CP response.

The last implication is the author's dynamical claim.

Strocchi: vacuum selection

- A central operator replaces an arbitrary c-number vacuum angle.
- Mass terms and the thermodynamic limit select a vacuum phase.
- Claimed alignment with the mass phase removes strong CP violation.

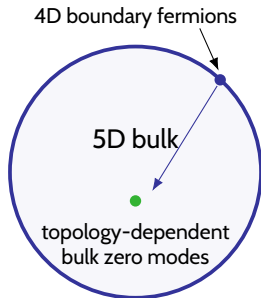
The QCD construction uses modified APS boundary conditions.

These are claims about dynamics or vacuum selection, not the necessity of a parameter.

G. Schierholz, arXiv:2403.13508v2, J. Phys. G 52 (2025) 04LT01; F. Strocchi, arXiv:2404.19400v3.

Kaplan–Sen: Can the regulator change the global theory?

3.4 A proposed chiral-gauge regulator, not ordinary vector-like DWF



Schematic of the proposed boundary/bulk construction.

- Heavy bulk fields need not decouple from **global topology**: additional zero modes appear in nontrivial backgrounds.
- Boundary observables can resemble a theory restricted to zero global topological charge.
- Their proposed mechanism removes the singlet mass phase while retaining local anomalous dynamics.

Conditional on this regulator construction. A real-time Minkowski formulation remains a task.

Not a claim that standalone 4D lattice QCD was computed incorrectly.

D. B. Kaplan and S. Sen, arXiv:2412.02024v4, Secs. V–VI. Diagram: schematic drawn for this talk from their construction.

Counterarguments: spectra, instantons, and chiral dynamics

3.5 A controlled example, then the QCD arguments

Quantum rotor: an exact benchmark

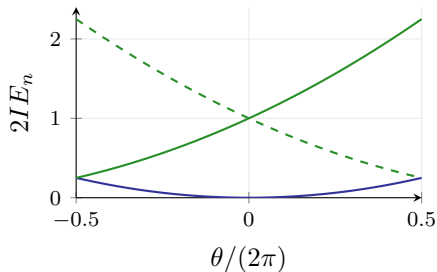
$$E_n(\theta) = \frac{1}{2I} \left(n + \frac{\theta}{2\pi} \right)^2.$$

$$Z(\theta, \beta) = \sum_{n \in \mathbb{Z}} e^{-\beta E_n(\theta)}.$$

The winding sum must reproduce the canonical spectrum in this model.

- Aghaie–Sato and Albanea et al.: changing the order loses this θ dependence.
- An exact check of the model, **not an automatic identification with QCD**.

Aghaie–Sato, arXiv:2601.18248v2, Sec. 2; Albanea–Catumba–Ramos, arXiv:2402.17518v2.



Analytic spectrum, redrawn from the formula;
 $n = 0, \pm 1$.

Counterarguments: spectra, instantons, and chiral dynamics

3.5 A controlled example, then the QCD arguments

	Standard-side objection	ACGT response / issue to separate
Spectra	The rotor must recover its known θ -dependent energies.	What does the proposed prescription retain as a local observable?
Instantons	Explicit semiclassical amplitudes retain topological phases.	The sector sum and infinite-volume definition are precisely disputed.
Chiral dynamics	Susceptibility, η' physics and matching constrain the theory.	Local susceptibility need not equal a fixed-global- Q variance; EFT matching is disputed.

Compare specified observables and limiting procedures, rather than the slogan “no strong CP problem.”

Benabou et al., arXiv:2510.18951v1; Khoze, arXiv:2512.06827v2; Bhattacharya, arXiv:2512.10127v1; Ai–Garbrecht–Tamarit reply, arXiv:2511.04216v1.

What can a lattice nEDM calculation actually test?

3.6 From the debate to a calculable response

Question	What must be established?
A physical parameter?	Which global theory and boundary/limiting prescription define QCD?
A hadronic response?	Within that theory, does a CP-odd perturbation induce a neutron EDM?
Why small in Nature?	How is the response related to experimental limits and other CP-odd sources?

$$c_n = \left. \frac{\partial d_n}{\partial \bar{\theta}} \right|_{\bar{\theta}=0}$$

Next: determine this coefficient nonperturbatively in the lattice theory.

Logical organization of this talk; not a test of every proposed formulation.

Lattice calculation: Blum et al., arXiv:2607.10606v1 and 2608.22587v1.

"The Neutron Electric Dipole Moment from Lattice QCD using a Background Electric Field"
Thomas Blum, Fangcheng He, Taku Izubuchi, Luchang Jin, Hiroshi Ohki, Sergey Syritsyn
arXiv:2607.10606

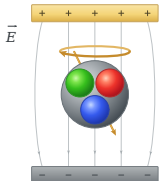
"Calculation of neutron electric dipole moment from Lattice QCD" Thomas Blum,
Fangcheng He, Taku Izubuchi, Luchang Jin, Hiroshi Ohki, Sergey Syritsyn
arXiv:2607.10606

The nEDM as a mixed response to electric field and θ

4.1 Nonperturbative QCD; perturbative external sources

For spin $\sigma = \pm 1$, the Euclidean spin-dependent energy shift is

$$\bar{E} \quad E_\sigma(\mathcal{E}_z, \bar{\theta}) = E_{\text{even}}(\mathcal{E}_z) + i\sigma \mathcal{E}_z c_n \bar{\theta} + \cdots, \quad d_n = c_n \bar{\theta} + O(\bar{\theta}^3).$$



$$c_n = \frac{1}{i\sigma} \left. \frac{\partial^2 E_\sigma}{\partial \bar{\theta} \partial \mathcal{E}_z} \right|_0$$

- **Interpretation:** first order in each of the two sources, $\bar{\theta}$, $\mathcal{E}$.
- **Implementation:** θ derivative at $\bar{\theta} = 0$; put a small, constant quantized U(1) electric field, in a unit of $\mathcal{E} = 3 \times 2\pi / (L_T L_Z)$, in the valence propagators.
- **Electro-quenched:** no background-field coupling in the sea determinant; test the field-strength dependence.

Convention bridge: this chapter uses the paper's $e^{-S_0 - i\bar{\theta}Q_P}$ and $q_P(t)$. With the Chapter 2 orientation, $Q_P = -Q_{Ch2}$. The electric field includes $|e|$: quoted EDMs restore the unit e .

A spectral representation in ordinary QCD

4.2 The states below are those of $\mathcal{E}_z = \bar{\theta} = 0$ QCD

The electric field mixes opposite-parity states through $\Delta H = i\mathcal{E}_z \mathcal{D}_z$:

$$|\mathcal{N}_0^R\rangle = |N_0\rangle + i\mathcal{E}_z \sum_i \frac{\langle N_i^* | \mathcal{D}_z | N_0 \rangle}{m_{N_0} - m_{N_i^*}} |N_i^*\rangle + O(\mathcal{E}_z^2).$$

The nEDM is a product of two transition matrix elements:

$$c_n = 2 \sum_i \frac{\text{Im}[\langle N_0 | q_P | N_i^* \rangle \langle N_i^* | \mathcal{D}_z | N_0 \rangle]}{m_{N_i^*} - m_{N_0}}$$

- $q_P(t) \sim \int F \tilde{F} d^3x$ is a **spatial integral on one time slice**, not the global $Q_P = \int q_P(t) dt$.
- Nonperturbative QCD matrix elements; perturbation theory only in the sources.
- We do **not** reconstruct this entire spectral sum numerically: Feynman–Hellmann and a GEVP extract the response.

The physical parity admixture belongs to the polarized ground state.

Non-Hermitian perturbation and PT symmetry

4.3 At $\bar{\theta} = 0$: left and right states need not be Hermitian conjugates

$$H = H_0 + i\mathcal{E}_z\mathcal{D}_z, \quad H|n_R\rangle = E_n|n_R\rangle, \quad \langle n_L|H = E_n\langle n_L|.$$

In the parity basis, the correlation matrix has the structure

$$C(t) = \begin{pmatrix} C_{++} & C_{+-} \\ -C_{+-}^* & C_{--} \end{pmatrix}, \quad \boxed{C^\dagger = SC S^{-1}}, \quad S = \text{diag}(1, -1).$$

- Pseudo-Hermiticity implies **real eigenvalues or complex-conjugate pairs**.
- For a real generalized eigenvalue, one can choose $v_L = S v_R$.
- Weak-field state extraction at $\bar{\theta} = 0$ is distinct from the **imaginary $\bar{\theta}\mathcal{E}_z$ EDM energy shift**.

The symmetry allows a real weak-field spectrum; it does not guarantee reality for arbitrary field strength.

Non-Hermitian perturbation and PT symmetry

4.3 At $\bar{\theta} = 0$: left and right states need not be Hermitian conjugates

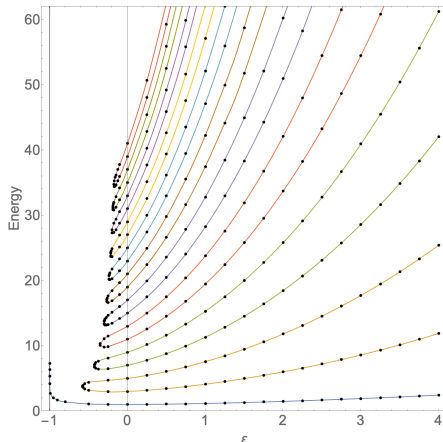
A familiar PT -symmetric example

$$H = p^2 + x^2(ix)^\varepsilon.$$

In Bender–Hook Fig. 4:

- Real levels for $\varepsilon \geq 0$.
- Pairwise merging and complex-conjugate pairs for $\varepsilon < 0$.
- An illustration of the spectral possibilities, not a model of the neutron.

The right-hand two-level illustration displays the real / complex-conjugate alternatives.



Bender–Hook Fig. 4.

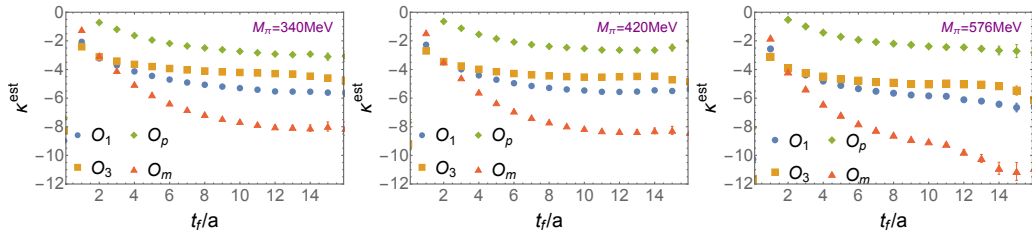
C. M. Bender and D. W. Hook, Rev. Mod. Phys. 96, 045002 (2024), arXiv:2312.17386, Fig. 4.

What the κ estimator teaches us

4.4 Anomalous magnetic moment of nucleon from wave function of nucleon with electric field background. κ^{est} is not generally the Pauli form factor $F_2(0)$

Detmold, Tiburzi, Walker-Loud arXiv:0904.1586, 1001.1131]

$$\kappa^{\text{est}}(t_f) = i \frac{2m_N^2}{\mathcal{E}_z} \frac{\text{Tr}[\gamma_3 \gamma_4 C_{2,\mathcal{E}}(t_f)]}{\text{Tr}[T + C_{2,\mathcal{E}}(t_f)]}.$$



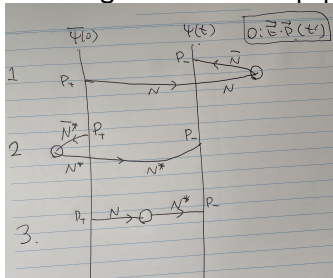
- The estimator depends on interpolator overlaps and on **both nucleon and vacuum polarization**.
- This is not merely a finite- t_f artifact, nor the old F_2/F_3 mixing problem.

Even a large-time limit must be interpreted in the field-deformed state.

What the κ estimator teaches us

4.4 Anomalous magnetic moment of nucleon from wave function of nucleon with electric field background. κ^{est} is not generally the Pauli form factor $F_2(0)$

In the large-time limit, the paper obtains



$$\kappa^{\text{est}} \longrightarrow 2m_N^2 \text{Im} \left[\underbrace{\sum_i \frac{Z_{N_i^*}}{Z_{N_0}} \frac{\langle N_i^* | \mathcal{D}_z | N_0 \rangle}{m_{N_0} - m_{N_i^*}}}_{\text{nucleon polarization (3)}} - \underbrace{\sum_n \frac{Z_{N_0}^{n-}}{Z_{N_0}} \frac{\langle \Omega | \mathcal{D}_z | n^- \rangle}{E_n}}_{\text{vacuum polarization (1,2)}} \right].$$

$Z_{N_i} = \langle \Omega | \psi_N | N_i \rangle$, $Z_{N_i^*} = \langle \Omega | \psi_{N^*} | N_i^* \rangle$, and $Z_{N_0}^{n-} = \langle n^- | \psi_{N^*} | N_0 \rangle$. Spin-up components are understood.

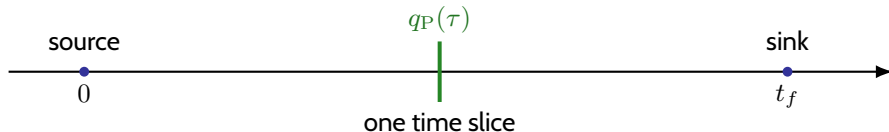
For the nEDM, extract the energy response of the actual polarized ground state instead.

Feynman–Hellmann: from global Q to local $q(t)$

4.5 A forward matrix element rather than an all-time insertion

$$\left. \frac{\partial E_{\uparrow}}{\partial \bar{\theta}} \right|_0 = i \langle \mathcal{N}_0^L | q_P | \mathcal{N}_0^R \rangle_{\mathcal{E}},$$

$$c_n = \lim_{\mathcal{E}_z \rightarrow 0} \frac{\langle \mathcal{N}_0^L | q_P | \mathcal{N}_0^R \rangle_{\mathcal{E}}}{\mathcal{E}_z}.$$



$$\frac{1}{\mathcal{E}_z} \frac{C_{3,\mathcal{E}}[q_P](t_f, \tau)}{C_{2,\mathcal{E}}(t_f)} \longrightarrow c_n + O(\mathcal{E}_z^2), \quad \tau, t_f - \tau \rightarrow \infty.$$

A local ground-state matrix element avoids the noisy all-time Q_P insertion.

The state projection and finite-field extrapolation are essential parts of this statement.

A baryon GEVP for the polarized neutron

4.6 Keep the physical parity admixture; suppress unwanted propagation

Use both parities for each interpolator:

$$\mathcal{B} = \{O_1^+, O_1^-, \dots, O_4^+, O_4^-\}, \quad C_{ij}(t) = \langle \mathcal{B}_i(t) \mathcal{B}_j^\dagger(0) \rangle_{\mathcal{E}}.$$

$$C(t)v_{R,n} = \lambda_n C(t_0)v_{R,n},$$

$$v_{L,n}^\dagger C(t) = \lambda_n v_{L,n}^\dagger C(t_0).$$

$$c_n^{\text{est}}(t_f, \tau) = \frac{1}{\mathcal{E}_z} \frac{v_{L,0}^\dagger C_3[q_P](t_f, \tau) v_{R,0}}{v_{L,0}^\dagger C_2(t_f) v_{R,0}}$$

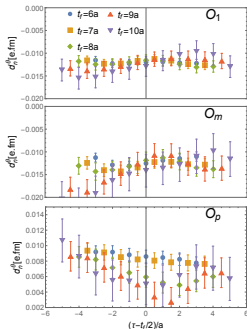
- Independent left/right projections for the non-Hermitian matrix.
- Vary the interpolator basis and the source–sink separation.

The target is not the zero-field positive-parity state.

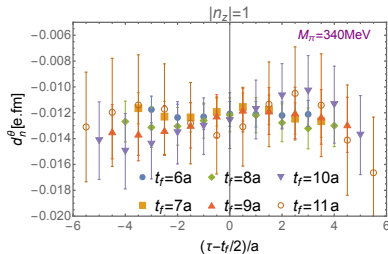
A baryon GEVP for the polarized neutron

4.6 Keep the physical parity admixture; suppress unwanted propagation

Single-operator choices



Full basis $\{O_1, O_2, O_3, O_4\}$



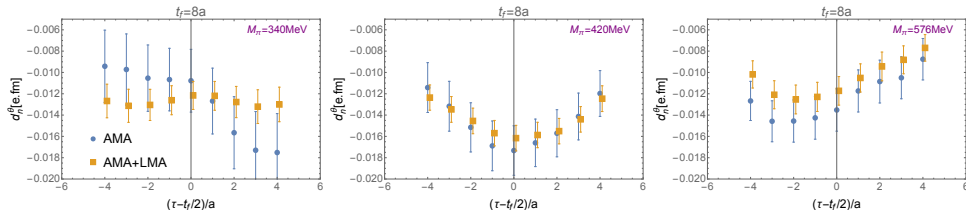
24I-005, $m_\pi = 340$ MeV, $|n_z| = 1$. The source-sink separations are varied.

Operator covariance alone does not guarantee a clean ground-state matrix element.

Low modes and topology: physics-guided variance reduction

4.7 AMA + full-volume LMA with 200 preconditioned low modes

$$\hat{C}_{\text{imp}} = \underbrace{\overline{C_{\text{low}}}^V}_{\text{full-volume LMA}} + \underbrace{(\overline{C_{\text{sl}} - C_{\text{low}}})^{N_{\text{sl}}}}_{\text{sampled remainder}} + \underbrace{(\overline{C_{\text{ex}} - C_{\text{sl}}})^{N_{\text{ex}}}}_{\text{AMA bias correction}}.$$



The corrected estimator—not the low-mode approximation—is the observable.

Topology through gluons and quarks

4.8 Global charge agreement is not equality of local time profiles

The axial Ward identity inside the nucleon correlator

$$C_3[q_P^G](t_f, \tau) = m C_3[P](t_f, \tau) - \frac{Z_A}{2} \partial_\tau C_3[A_4](t_f, \tau)$$

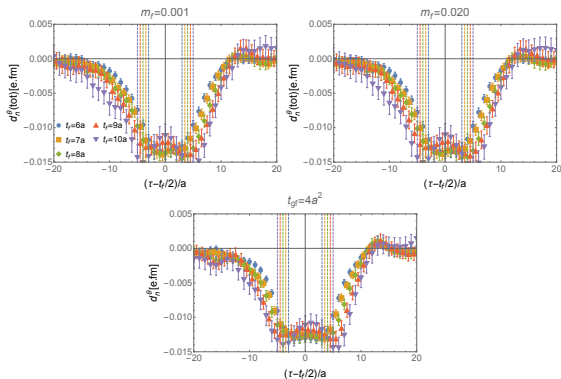
$$P(\tau) = \sum_{\mathbf{x}} J_5(\tau, \mathbf{x}), \quad A_4(\tau) = \sum_{\mathbf{x}} A_4(\tau, \mathbf{x}), \quad m = m_f + m_{\text{res}}.$$

- A time integral can remove a derivative term; a **time-local insertion cannot simply drop it**.
- The derivative vanishes in an exact forward ground-state matrix element, but not in finite-time excited-state terms.
- Compare flow times and loop masses; inspect pseudoscalar and axial terms separately and together.

The final nEDM values use the flowed gluonic definition.

Topology through gluons and quarks

4.8 Global charge agreement is not equality of local time profiles



Pseudoscalar + axial

Top: $m_f = 0.001$ and 0.020 .

Flowed gluonic density

Bottom: $t_{gf} = 4a^2$.

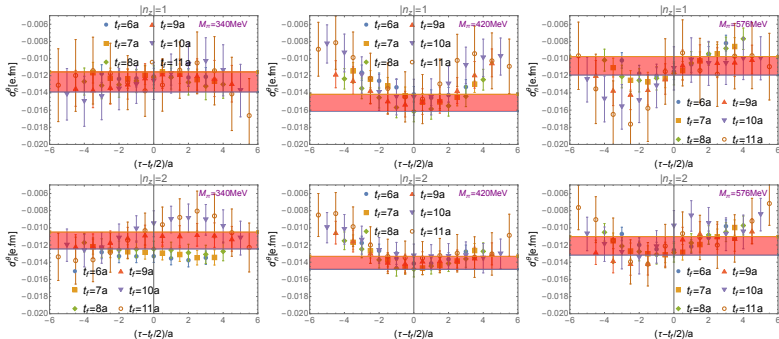
- Corrected profiles agree across loop masses.
- Strong cancellations outside the source–sink interval.

24I-005, O_1 , $|n_z| = 2$; several t_f values.

Does the signal survive changing the analysis?

4.9 Stability tests, not a substitute for every continuum systematic

Vary source–sink separation and electric field strength



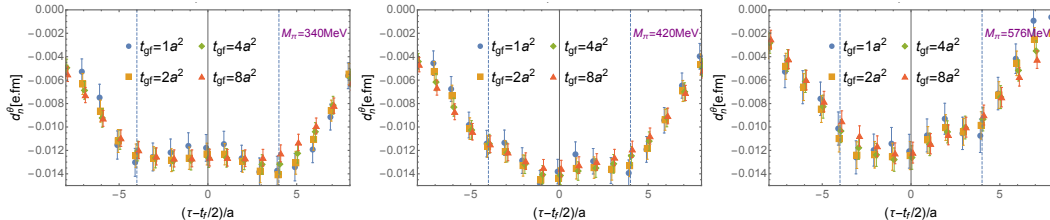
Columns: $m_\pi = 340, 420, 576$ MeV. Rows: $|n_z| = 1, 2$. Full four-interpolator GEVP. Shaded bands are the chosen ground-state fits.

Use variation with t_f and $\mathcal{E}_z$ to quantify the corresponding analysis uncertainty.

Does the signal survive changing the analysis?

4.9 Stability tests, not a substitute for every continuum systematic

Vary the smoothing scale of the gluonic density



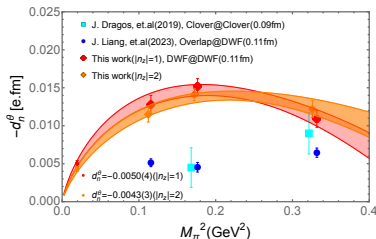
GEVP row of the flow-time comparison. Columns: $m_\pi = 340, 420, 576$ MeV. $O_1, |n_z| = 2, t_f = 8a$.

The GEVP result is compared across flow times on all three ensembles.

The result—and the remaining systematics

4.10 A nonzero response, with a precisely delimited error budget

$$c_n = \left. \frac{\partial d_n}{\partial \bar{\theta}} \right|_0 = -0.0050(4)_{\text{stat}}(8)_{\text{sys}} \text{ e fm}$$



Physical-point extrapolation; the two bands correspond to $|n_z| = 1, 2$.

A nonzero local response is resolved; the physical parameter is a separate question.

Calculation

2 + 1-flavor DWF; $a \simeq 0.11$ fm;
 $m_\pi = 340, 420, 576$ MeV; electro-quenched.

Quoted systematic error

Fit-range and field-strength dependence.

Still to be controlled

Continuum limit, finite volume, chiral extrapolation and sea-field coupling.

A nonzero nEDM from local topology

5.1 What the new method measures and resolves

A statistically resolved response to the QCD θ term

$$c_n \equiv \left. \frac{\partial d_n}{\partial \theta} \right|_0 = -0.0050(4)_{\text{stat}}(8)_{\text{sys}} \text{ e fm.}$$

Extrapolated to the physical pion mass from $m_\pi = 340, 420, 576 \text{ MeV}$.

One lattice spacing, $a \simeq 0.11 \text{ fm}$; 2 + 1-flavor DWF; electro-quenched background field.

Signal Local $q_{\text{top}}(t)$ through Feynman–Hellmann, with AMA / low-mode averaging.

State A non-Hermitian baryon GEVP resolves the electrically polarized, parity-mixed ground state.

Checks Interpolating-operator variation and gluonic / fermionic topology linked by the axial Ward identity.

Quoted systematic error: fit-range and electric-field-strength dependence—not a complete error budget.

Strong CP implications— and the next calculations

5.2 Conditional interpretation and the next calculations

Implication for strong CP

$$d_n^{(\theta)} = c_n \bar{\theta} + O(\bar{\theta}^3).$$

If $\bar{\theta}$ is an **independent physical parameter** and its contribution is **not concealed by cancellations**, EDM limits constrain it at the 10^{-11} level.

Next steps

- **Physical QCD:** lighter pion masses, finer lattice spacings, and larger volumes.
- **Beyond $\bar{\theta}$:** extend the method to the Weinberg three-gluon operator and CP-odd four-quark interactions.

Local CP-odd matrix elements provide a practical route from nonperturbative QCD to EDM phenomenology.

Backup

Derivations, conventions and references

Naturalness versus radiative stability

B1.1 Two different questions

- **QCD alone:** for real positive quark masses, CP is restored at $\bar{\theta} = 0$.
- **Full SM:** setting only $\bar{\theta} = 0$ does not restore CP in the presence of the CKM phase.
- **Radiative stability:** CKM-induced corrections in the minimal SM are extremely small.

Origin of a small value $\neq$ stability of a small value .

Avoid describing this as cancellation of large radiative corrections at every order.

“Generic $O(1)$ ” is a naturalness expectation, not a numerical SM prediction.

Hook, arXiv:1812.02669. [1:R5] Based on the naturalness note in the supplied Chapter 1 draft.

Massless up quark: what is the question?

B1.2 A mechanism, a definition issue and a lattice test

A genuinely massless up quark permits an anomalous field redefinition that removes the strong phase.

- **Creutz:** with other quarks massive, $m_u = 0$ need not be a physical singular point; nonperturbative additive ambiguities raise a definition question.
- **Srednicki:** an explicit counterargument—Creutz's conclusion is not presented here as a settled theorem.
- **Lattice test:** Alexandrou et al. constrain the topological mass contribution and do not support the massless-up scenario.

Separate the definition question from the phenomenological test.

Creutz, hep-ph/0312225; Srednicki, hep-ph/0503051; Alexandrou et al., arXiv:2002.07802. [1:R7–R9]

Chern–Simons normalization

B2.1-A Differential forms and component conventions

Hermitian generators: $\text{tr}(T^a T^b) = \delta^{ab}/2$, $\mathcal{A} = g_s A_\mu^a T^a dx^\mu$, $D = d - i\mathcal{A}$.

$$\mathcal{F} = d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}, \quad \omega_3 = \frac{1}{8\pi^2} \text{tr} \left(\mathcal{A} \wedge d\mathcal{A} - \frac{2i}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right).$$

$$d\omega_3 = \frac{1}{8\pi^2} \text{tr}(\mathcal{F} \wedge \mathcal{F}) = q_E d^4x.$$

Oriented slab: $d^4x = d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$, $\epsilon^{\tau 123} = +1$.

$$N_{\text{CS}} = \frac{g_s^2}{16\pi^2} \int_{\Sigma} d^3x \epsilon^{ijk} \left(A_i^a \partial_j A_k^a + \frac{g_s}{3} f^{abc} A_i^a A_j^b A_k^c \right).$$

Check the factor of two: an expression using $A_i F_{jk}$ carries a different prefactor from this $A \partial A$ form.

Throughout: $e^{-S_{E,0} + i\theta Q}$, so $S_E = S_{E,0} - i\theta Q$.

Conventions fixed in B2.1 of the supplied draft; Ai; Williams; Jackiw–Rebbi. [2:R1,R2,R10]

Boundary terms and periodic Euclidean time

B2.1-B A nonzero global charge needs consistent patching

- Stokes' theorem gives the slab charge as a CS difference **plus any spatial-boundary / patch contributions**.
- N_{CS} changes by integer shifts under allowed large gauge transformations; an open-subvolume charge need not be integer.
- A nonzero Chern number on a torus is compatible with periodic lattice links; it does not imply one globally smooth, strictly periodic continuum potential.
- In temporal gauge, the final connection may be glued to the initial one by a large transformation.

Do not infer $Q = 0$ by assuming a periodic CS branch while omitting gluing data.

A real Euclidean instanton is not generally a real classical Minkowski solution obtained by merely relabeling τ as t .

Ai; Williams; Jackiw-Rebbi. [2:R1,R2,R10] Source: Chapter 2 draft, B2.1.

Overlap index and the M_5 dependence

B2.2-A Integer-valued does not mean kernel-independent at finite a

For Wilson-kernel overlap ($r = 1$), use dimensionless operators:

$$H_W(M_5) = \gamma_5(aD_W - M_5), \quad \hat{D}_{\text{ov}} = 1 + \gamma_5 \text{sgn } H_W(M_5).$$

When H_W has no zero eigenvalues,

$$Q_{\text{ov}} = \text{Tr } \gamma_5 \left(1 - \frac{\hat{D}_{\text{ov}}}{2} \right) = -\frac{1}{2} \text{Tr } \text{sgn } H_W(M_5) \in \mathbb{Z}.$$

- Here index $D = n_+ - n_-$; match γ_5 and orientation to gluonic Q .
- Varying M_5 changes Q_{ov} only at a kernel zero crossing.
- Admissibility is **sufficient**, not necessary, for the gap/locality conditions.
- At an exact kernel zero the sign function is undefined; failure of a plaquette bound alone does not establish this.

M_5 is a regulator parameter, not the physical quark mass.

Adams, hep-lat/0001014; Creutz, hep-lat/0208026 and arXiv:1306.1245; Hernández et al., hep-lat/9808010. [2:R4–R7]

Creutz's question and continuum input

B2.2-B Separate classical configurations from quantum observables

Creutz's question

How is topology assigned to rough fields, and is the continuum limit independent of the prescription?

- **Classical-continuum statement:** fine discretizations of a fixed smooth connection reproduce its topological charge in the physical Wilson branch $0 < M_5 < 2$, away from endpoints.
- **Quantum-observable statement:** gradient-flow and density-chain definitions of QCD topological susceptibility agree in the continuum (in the asymptotically free flavour range).
- Neither statement asserts equality of all finite- a , per-configuration definitions.

Do not turn a finite- a prescription dependence into a proof of arbitrary continuum χ_t .

Creutz, arXiv:1306.1245, Sec. 5; Adams, hep-lat/0001014; Cè et al., arXiv:1506.06052; Lüscher, arXiv:2109.07965. [2:R4,R6,R8,R9]

If the additive label were real

B2.3-A Characters of $\mathbb{R}$ versus characters of $\mathbb{Z}$

Assume Q ranges over all of $\mathbb{R}$, w is continuous, $w(0) = 1$, and

$$w(Q_1 + Q_2) = w(Q_1)w(Q_2).$$

Then

$$w(Q) = e^{cQ}, \quad c \in \mathbb{C}, \quad \text{unitarity} \implies c = i\theta, \quad \theta \in \mathbb{R}.$$

Charge set	Unitary weight	Period in θ
$\mathbb{Z}$	$e^{i\theta Q}$	2π
$q_0\mathbb{Z}$	$e^{i\theta Q}$	$2\pi/q_0$
$\mathbb{R}$	$e^{i\theta Q}$	No forced nonzero period

A noninteger clover measurement is not a continuous homotopy label.

An allowed character need not label physically inequivalent theories; observable dependence is a further question.

Algebraic discussion in the supplied Chapter 2 draft, B2.3.

Can θ have an imaginary part?

B2.3-B Factorization alone is weaker than unitarity

With the fixed convention $e^{i\theta Q}$,

$$\theta = \theta_R + i\theta_I \quad \implies \quad w(Q) = e^{i\theta_R Q - \theta_I Q}.$$

- Factorization permits this extra real exponential.
- It is not a unitary topological character when $\theta_I \neq 0$.
- Imaginary- θ simulations and analytic continuation are useful deformations, not evidence for a complex physical strong phase in ordinary Hermitian Minkowski QCD.

$$\underbrace{w(Q)w(-Q) = 1}_{\text{composition}} \quad + \quad \underbrace{w(-Q) = w(Q)^*}_{\text{Hermitian evolution}} \quad \implies \quad |w(Q)| = 1.$$

This uses Hermiticity—not CP conservation.

Algebraic organization in the supplied Chapter 2 draft, Sec. 2.3 and B2.3; compare Williams, Sec. 6.4. [2:R2]

Fixed Q and the cluster limit

B2.4 Finite-volume constraints versus infinite-volume local physics

For large Euclidean four-volume V , fixed $Q = O(1)$, $\chi_t V \gg 1$, and the standard $\theta = 0$ expansion, the large-distance limit is

$$\langle q(x)q(0) \rangle_Q \longrightarrow \frac{1}{V} \left(\frac{Q^2}{V} - \chi_t - \frac{c_4}{2\chi_t V} \right) + O(V^{-3}) + \text{decaying terms}.$$

Since $\langle q \rangle_Q = Q/V$,

$$\boxed{\langle q(x)q(0) \rangle_{Q,c} \simeq -\frac{\chi_t}{V} + \dots}.$$

- The global constraint produces a finite-volume plateau.
- It can disappear if $V \rightarrow \infty$ is taken first for local observations.
- Spacelike commutativity forbids signalling; it does not erase all correlations in all states.

The Moon analogy does not settle the order-of-limits question.

Aoki–Fukaya–Hashimoto–Onogi, PRD 76 (2007) 054508, arXiv:0707.0396, Eq. (39). [2:R11]

Left and right perturbations: the same $+i$

Backup Spectral representation in the paper's convention

For $H = H_0 + i\mathcal{E}_z\mathcal{D}_z$, the first-order expressions are

$$|\mathcal{N}_0^R\rangle = |N_0\rangle + i\mathcal{E}_z \sum_i \frac{\langle N_i^*|\mathcal{D}_z|N_0\rangle}{m_0 - m_i^*} |N_i^*\rangle,$$

$$\langle\mathcal{N}_0^L| = \langle N_0| + i\mathcal{E}_z \sum_i \frac{\langle N_0|\mathcal{D}_z|N_i^*\rangle}{m_0 - m_i^*} \langle N_i^*|.$$

The left state is not obtained by Hermitian conjugating the first line.

$$c_n = \lim_{\mathcal{E}_z \rightarrow 0} \frac{\langle\mathcal{N}_0^L|q_P|\mathcal{N}_0^R\rangle}{\mathcal{E}_z} = 2 \sum_i \frac{\text{Im}[\langle N_0|q_P|N_i^*\rangle \langle N_i^*|\mathcal{D}_z|N_0\rangle]}{m_i^* - m_0}.$$

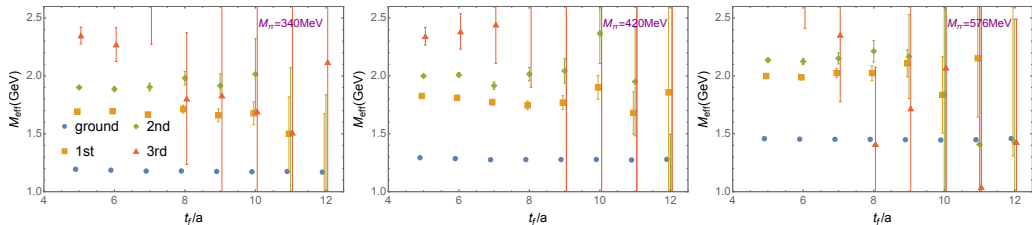
The anti-Hermitian perturbation and Euclidean operator conventions must be treated together.

The interpolator basis and its spectrum

Backup Four operators, each with two parity components

$$O_1 = \epsilon^{abc}(d_a^T C \gamma_5 u_b) d_c, \quad O_2 = -\epsilon^{abc}(d_a^T C u_b) \gamma_5 d_c,$$

$$O_3 = \epsilon^{abc}(d_a^T C \gamma_4 \gamma_5 u_b) d_c, \quad O_4 = \epsilon^{abc}(d_a^T C \gamma_4 u_b) \gamma_5 d_c.$$



First four effective masses in a two-interpolator $\{O_1, O_2\}$ GEVP. Columns: $m_\pi = 340, 420, 576$ MeV; $|n_z| = 2$.

Enlarge the basis while keeping both parities and the left/right structure.

The sign bridge between Chapters 2 and 4

Backup Keep the orientation and Wick-rotation conventions explicit

	Chapter 2	nEDM paper notation
Euclidean weight	$e^{-S_0 + i\bar{\theta} Q_{\text{Ch2}}}$	$e^{-S_0 - i\bar{\theta} Q_{\text{P}}}$
Charge orientation	Q_{Ch2}	$Q_{\text{P}} = -Q_{\text{Ch2}}$
Single time slice	$\int d^3x q_{\text{Ch2}}(x)$	$q_{\text{P}}(t)$ in the paper's convention

- A change of the Euclidean orientation is not a different physical theory.
- The paper's spectral formula and signed final result are displayed together in its convention.
- Do not change the sign in the path-integral weight without changing the operator/convention consistently.

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