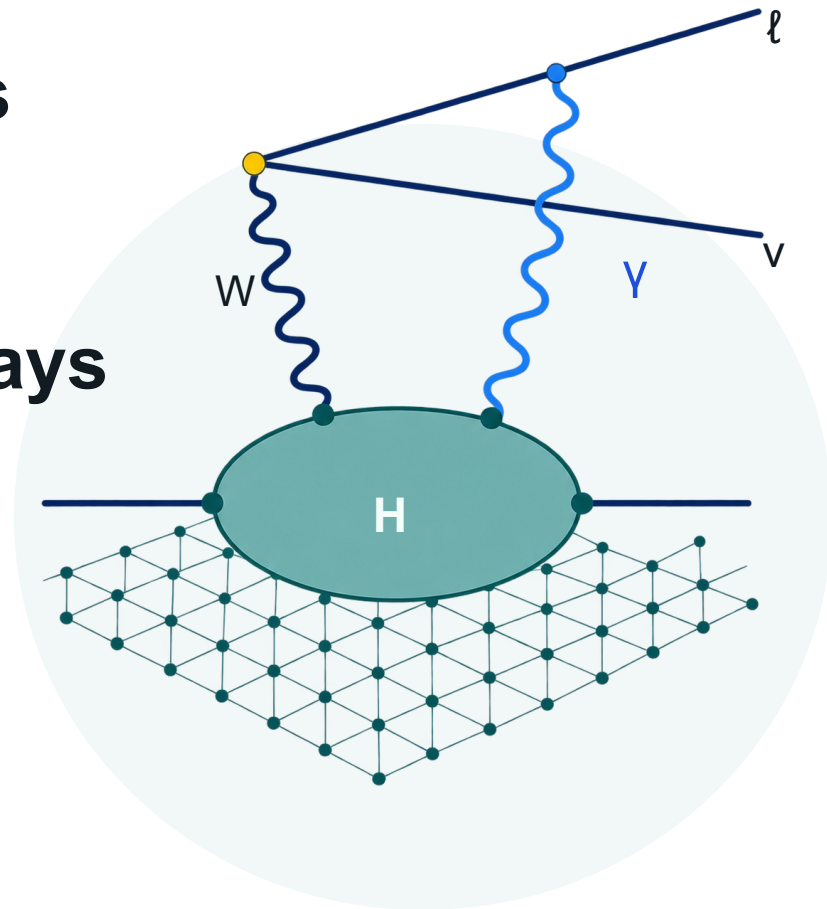
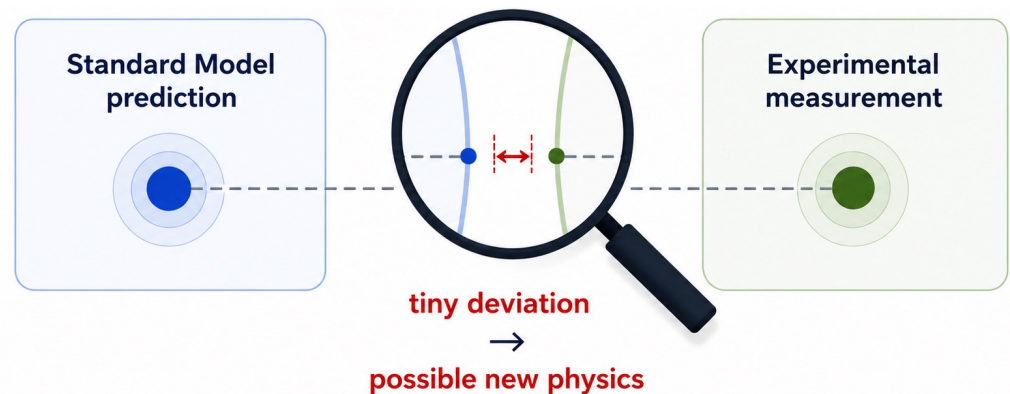

Toward Lattice Calculations of Radiative Corrections in Beta and Semileptonic Decays

Xin-Yu Tuo



Why radiative corrections matter

Precision frontier: indirect
probe of new physics



Key challenge: radiative corrections

small

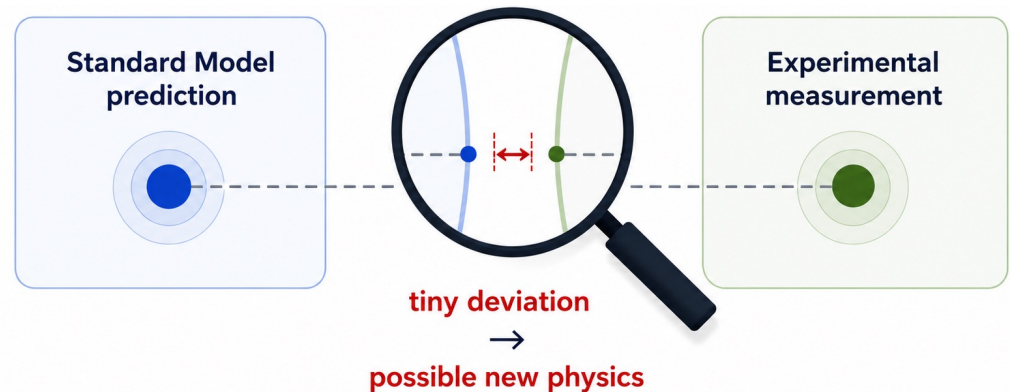
$$\alpha_{QED} \approx \frac{1}{137} \approx 0.7\%$$

but important

precision frontier enters
the sub-percent era

Why radiative corrections matter

Precision frontier: indirect probe of new physics



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small

$$\alpha_{QED} \approx \frac{1}{137} \approx 0.7\%$$

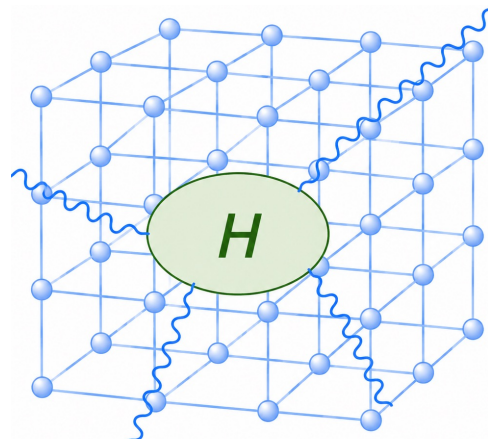
but important

precision frontier enters the sub-percent era

QED coupled to
nonperturbative QCD



major source of
theory uncertainty



**first-principles
lattice QCD+QED**

RCs: precision tests of the SM flavor structure

first row's CKM unitarity: **2~3 σ tension**

precision
 $\sim \mathbf{O(0.1\%)}$

RC size
 $\sim \mathbf{O(1\%)}$

$$\boxed{|V_{ud}|^2} + \boxed{|V_{us}|^2} + |V_{ub}|^2 = 0.9983(6)_{V_{ud}}(4)_{V_{us}}$$

RCs: precision tests of the SM flavor structure

first row's CKM unitarity: **$2\sim 3\sigma$ tension**

superallowed
 $0^+ \rightarrow 0^+$ nuclear β
decays

neutron β decay

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$\pi^+ \rightarrow \pi^0 e^+ \nu$


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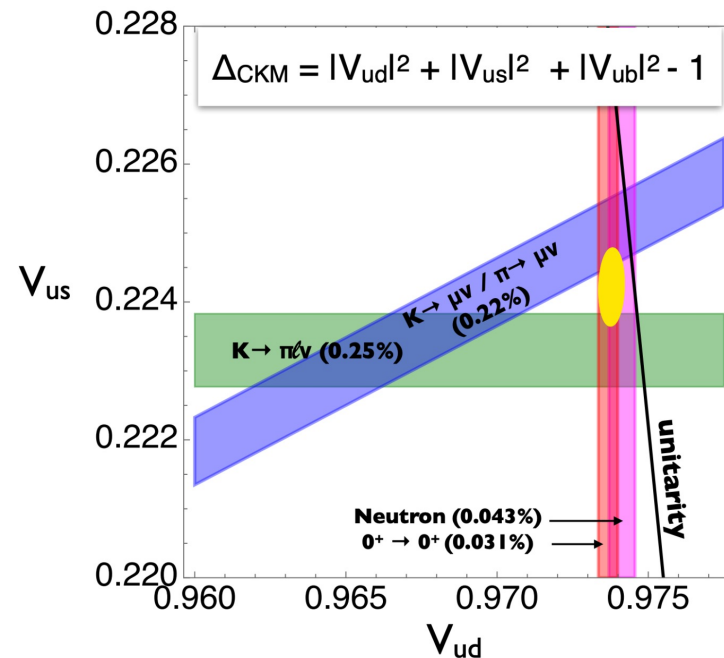
RC size
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$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9983(6)V_{ud}(4)V_{us}$$

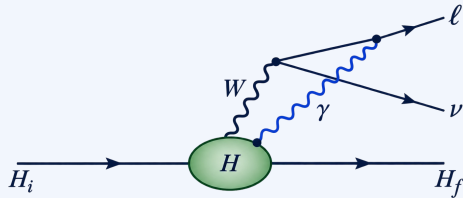
$\bigcirc K \rightarrow \mu \nu / \pi \rightarrow \mu \nu$

$\bigcirc K \rightarrow \pi \ell \nu (K_{\ell 3})$



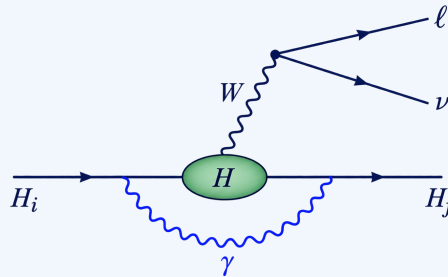
What is non-perturbative in a radiative correction?

γW box



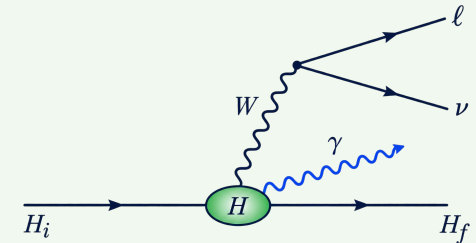
photon between
hadron and lepton

vertex correction



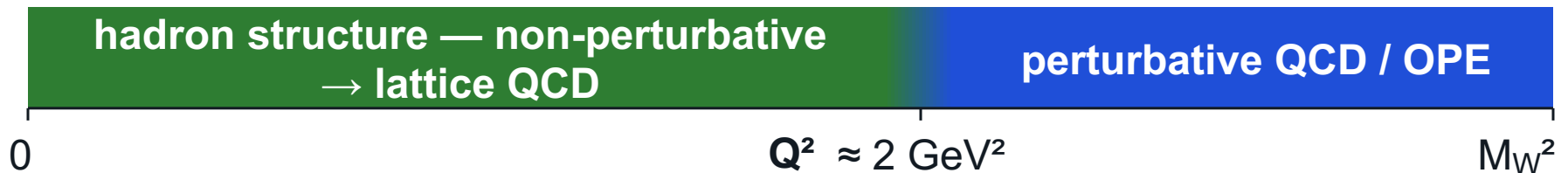
photon on both
hadron legs

real emission



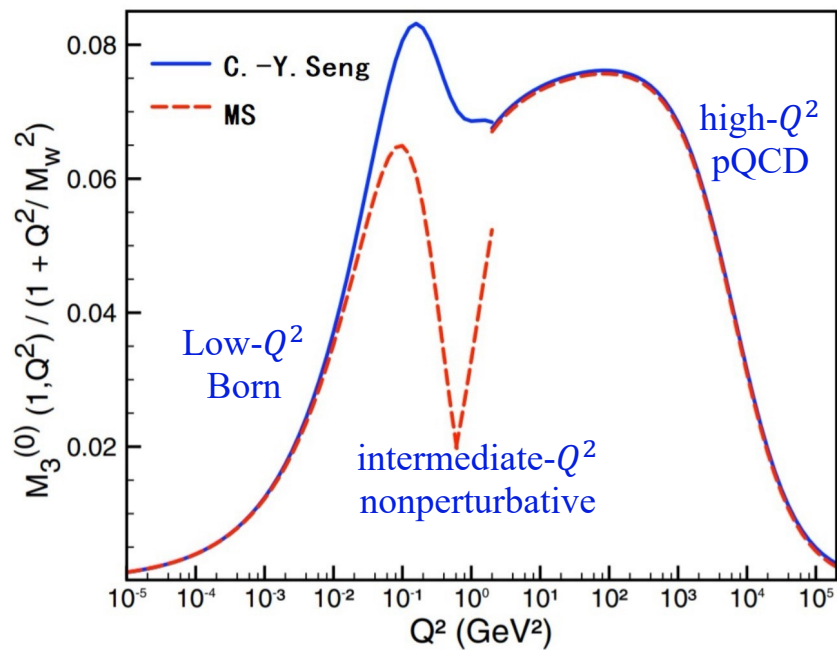
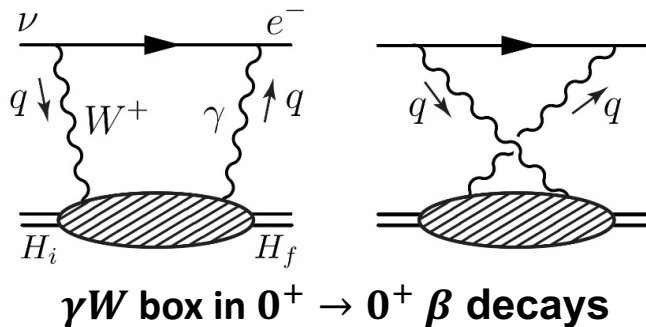
structure-dependent
emission

Energy scales:



Three kinds of RC — all need the hadronic function at low Q^2 .

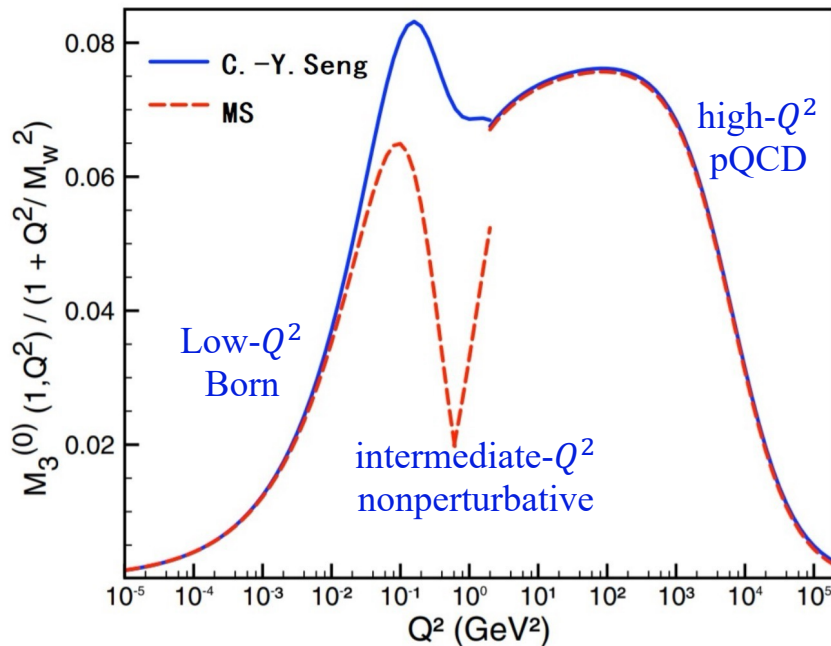
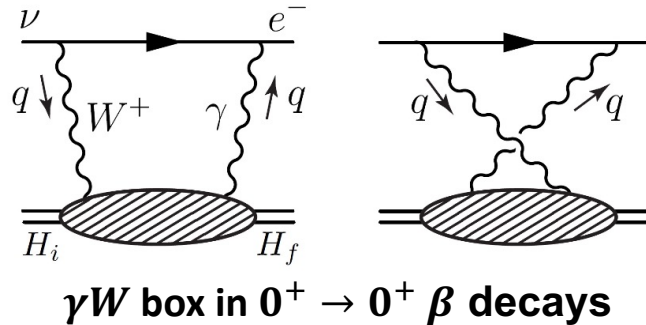
A lesson: one hadronic input moved V_{ud}



>

Non-perturbative inputs must be confirmed from first principles

A lesson: one hadronic input moved V_{ud}



2006: piecewise interpolation

Hadronic regions interpolated
High Q^2 : OPE + pQCD



2018: dispersive analysis

Hadronic input constrained by data
High Q^2 : pQCD

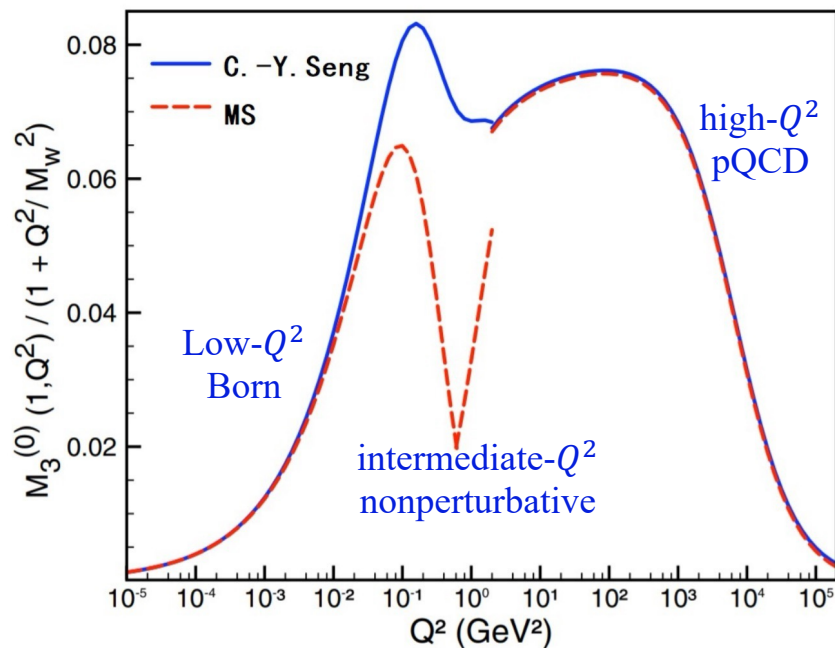
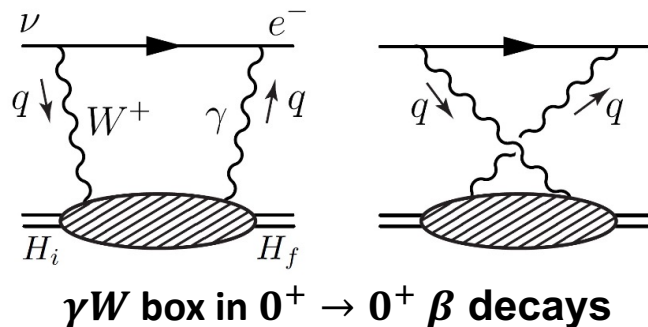
Marciano & Sirlin, PRL96, 032002 (2006)

Seng et.al. PRL 121, 241804 (2018)

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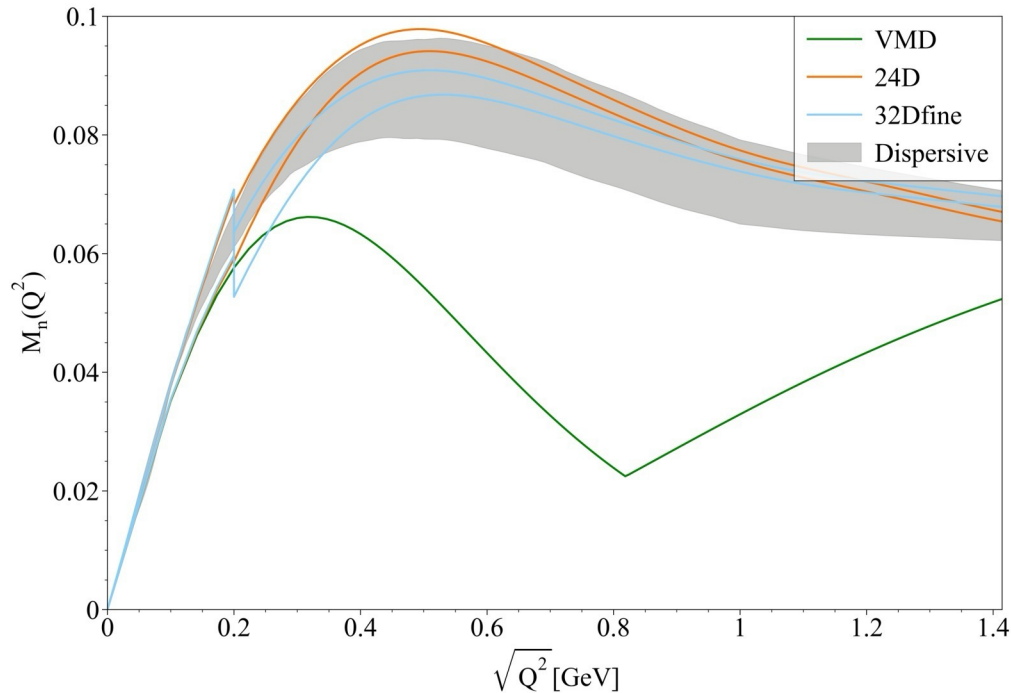
V_{ud}	PDG 2019 0.97420(21)	→	PDG 2020 0.97370(14)
Δ_{CKM}	PDG 2019 -0.00061(47)	→	PDG 2020 -0.00149(45)

Non-perturbative inputs must be confirmed from first principles

Lattice results: γW box in $0^+ \rightarrow 0^+ \beta$ decays

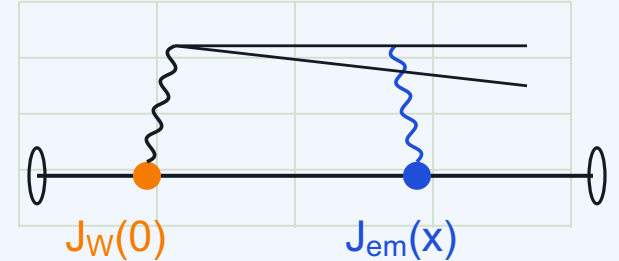
lattice QCD: $Q^2 \lesssim 2 \text{ GeV}^2$

pQCD: $Q^2 \gtrsim 2 \text{ GeV}^2$



P. Ma, et al. PRL 132 (2024) 191901

nucleon: $M_N(Q^2)$ from lattice QCD (24D, 32Dfine) vs dispersive analysis (band) and VMD



$$H_{\mu\nu}(x) = \langle H_f | T [J_\mu^{\text{em}}(x) J_\nu^W(0)] | H_i \rangle$$



$$\Box_{\gamma W} = \frac{3\alpha}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_H(Q^2)$$

$|V_{ud}|$

$0.97386(11)_{\text{exp.}} \text{ (9)}_{\text{RC}} \text{ (27)}_{\text{NS}}$

Δ_{CKM}

$2.1\sigma \rightarrow 1.8\sigma$

Non-perturbative inputs must be confirmed from first principles

Lattice γW boxes so far — all in the forward limit



all three: hadron at rest, equal masses

π · 2020



$$2.830(11)(26) \times 10^{-3}$$

K (SU(3) limit) · 2021



$$2.437(44) \times 10^{-3}$$

nucleon · 2024



$$3.65(8)(1) \times 10^{-3}$$

π and K confirmed independently (Yoo et al. 2023: 2.810(26), 2.389(17))

Feng et al. PRL 124 (2020); Ma et al. PRD 103 (2021); Ma et al. PRL 132 (2024)

Three cases are computed in the forward limit.

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Feng et al. PRL 124 (2020); Ma et al. PRD 103 (2021); Ma et al. PRL 132 (2024)

$K \rightarrow \pi \ell \nu$ breaks all three conditions

~~hadron at rest~~



π moves with momentum p

~~equal masses~~



$m_K \neq m_\pi$

~~no lepton recoil~~

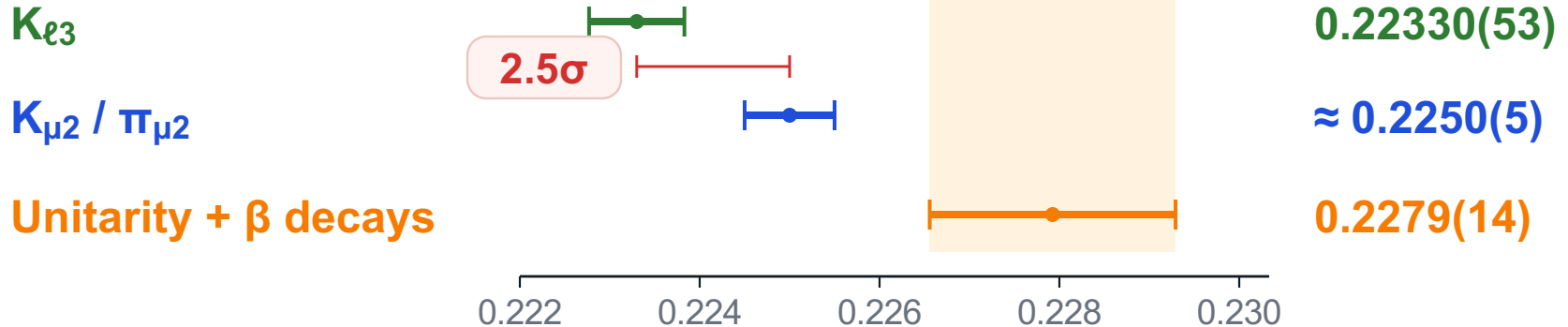


leptons recoil: integrate over phase space

Three cases are computed in the forward limit.

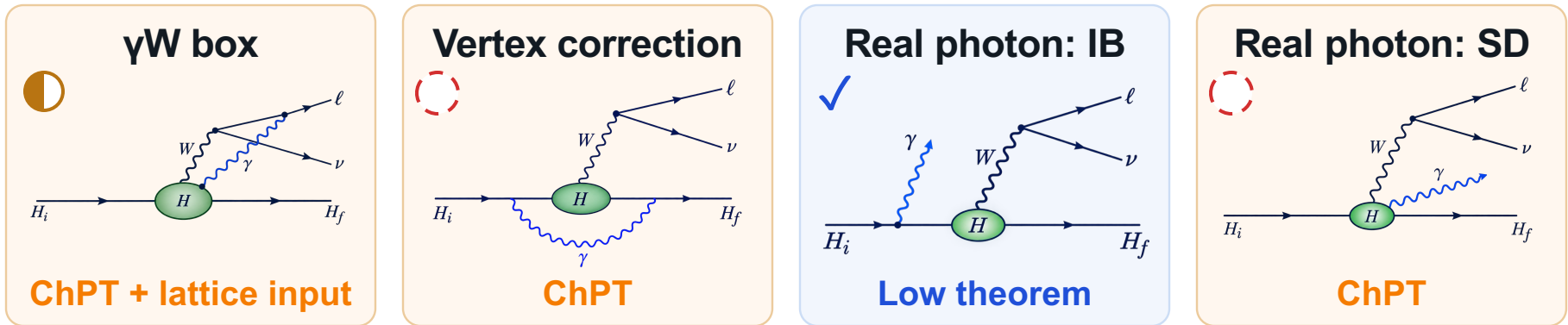
$K_{\ell 3}$: tensions with $K_{\ell 2}$ and unitarity

Three determinations of $|V_{us}|$



Unitarity band: $|V_{ud}| = 0.97367(32)$, from superallowed β decays

Hadronic radiative corrections in $K_{\ell 3}$



Box: lattice input in the $SU(3)$ limit; physical non-forward correction remains.

Both tensions motivate a complete first-principles RC calculation.

Neutron g_A vertex correction: exp. vs. lattice

Neutron g_A free from nuclear-structure uncertainties

Experiment

$$\lambda = g_A / g_V \\ = 1.2756(13)$$

PDG 2024 precision $\approx 0.1\%$



QED correction
to $g_A = ?$

Lattice QCD (pure QCD)

$$g_A^{\text{QCD}} = 1.263(10)$$

FLAG 2024 precision $\approx 1\%$

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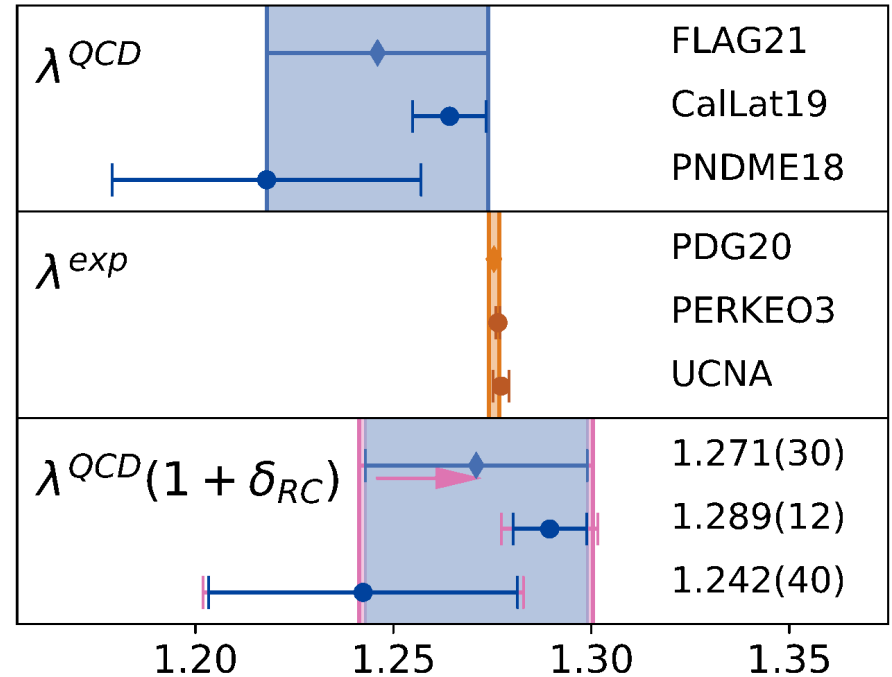


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2022 illustration: QED moves the lattice prediction
Cirigliano et al., PRL 129, 121801 (2022)

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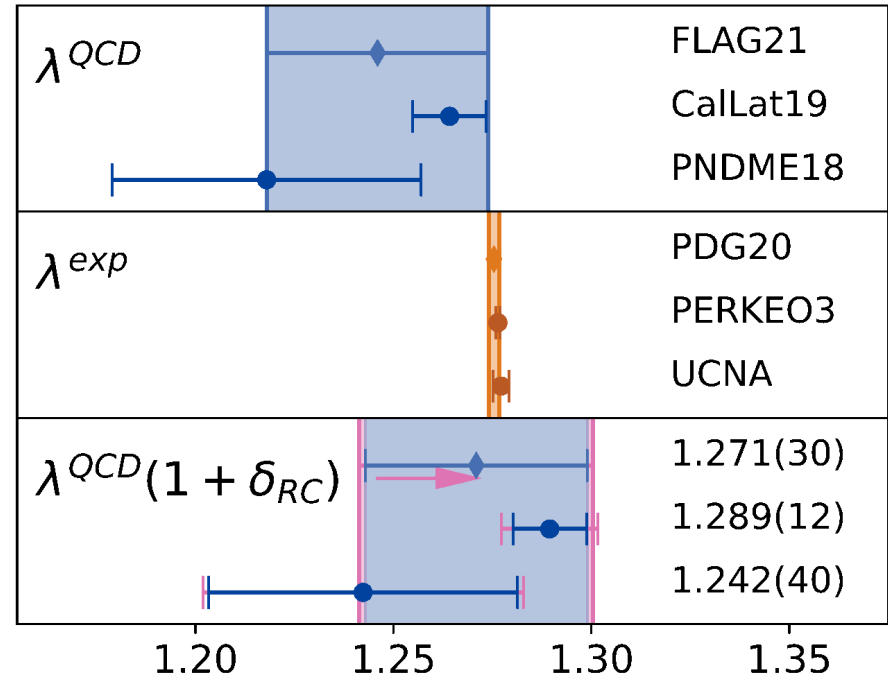


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FLAG 2024 precision $\approx 1\%$



2022 illustration: QED moves the lattice prediction
Cirigliano et al., PRL 129, 121801 (2022)

A missing Standard Model shift can masquerade as a right-handed current.

$$|\lambda_{\text{exp}}| = g_A^{\text{QCD}} [1 + \delta_{\text{RC}} - 2 \text{Re } \epsilon_R]$$

ϵ_R : right-handed weak
current

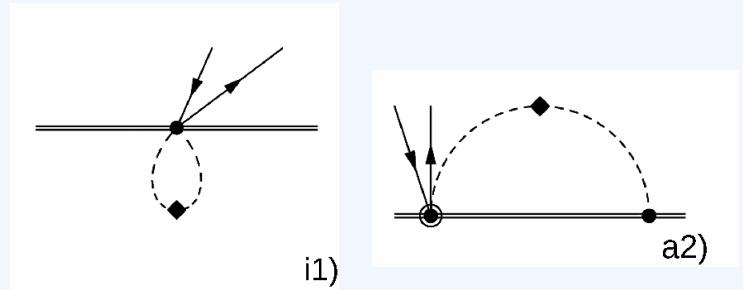
The pion cloud exposed a missing vertex correction

Earlier vertex correction: elastic states only

Nucleon states alone gave **zero vertex term** in the isospin limit.



2022: the missing pion cloud



Dashed line: pion ♦: EM pion-mass insertion
Cirigliano et al., PRL 129, 121801 (2022)

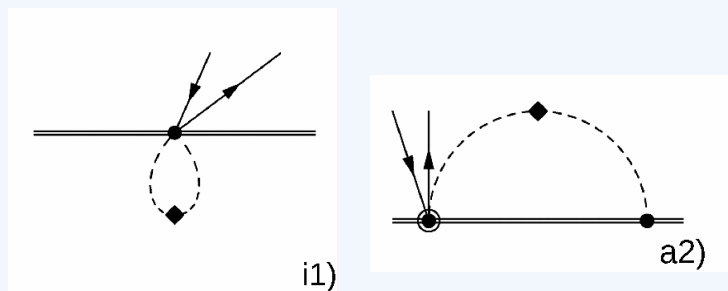
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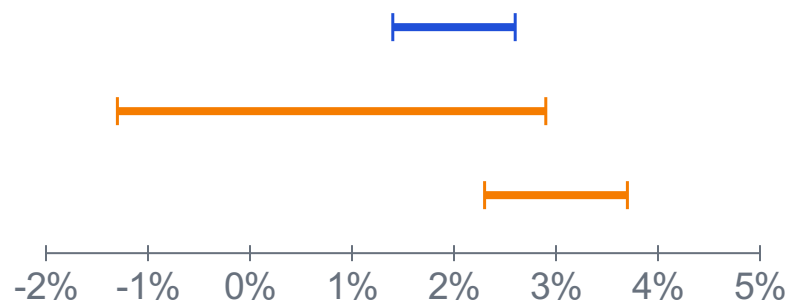
Previous estimations

Cirigliano et al. 2022: **1.4 – 2.6 %**

Tomalak & Yang 2026: lattice + πN input **0.8(2.1) %**

Tomalak & Yang 2026: πN data only **3.0(0.7) %**

Tomalak & Yang : Universe 12, 109 (2026), arXiv:2603.08596



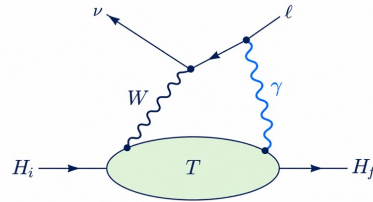
This correction must be confirmed from lattice QCD

$$\langle p | T \{ J_{\text{em}}^\mu(x) J_A^\rho(0) J_{\text{em}}^\nu(y) \} | n \rangle$$

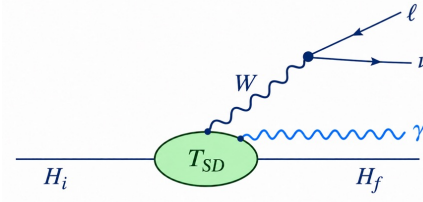
2 EM + 1 axial current
Five-point lattice correlator

The map: what is needed vs. what the lattice has done

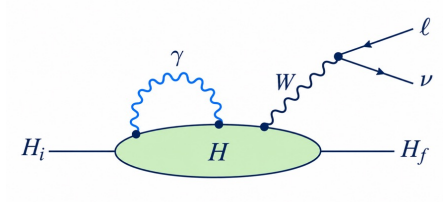
- ✓ Lattice result
- 🕒 Limited lattice input
- ⋯ No lattice result



γW box



Real photon (SD)



Vertex correction

$\pi^+ \rightarrow \pi^0 e^+ \nu$



2020, 2023



IB dominates¹



ChPT estimate

$0^+ \rightarrow 0^+ \beta$



Nucleon: 2024



IB dominates¹



Protected (CVC)

$K \rightarrow \pi \ell \nu$
($K_{\ell 3}$)



SU(3) only (2021)
Physical: missing



ChPT estimate



ChPT estimate

Neutron g_A



2024



IB dominates¹



ChPT estimate

¹ Real radiation exists in all β decays; this column tracks SD only.

Research sequence: 1 this talk; 2–5 outlook

Full physical corrections still need lattice input beyond the existing box results.

Real-photon decay: a first step to RC in $K_{\ell 3}$

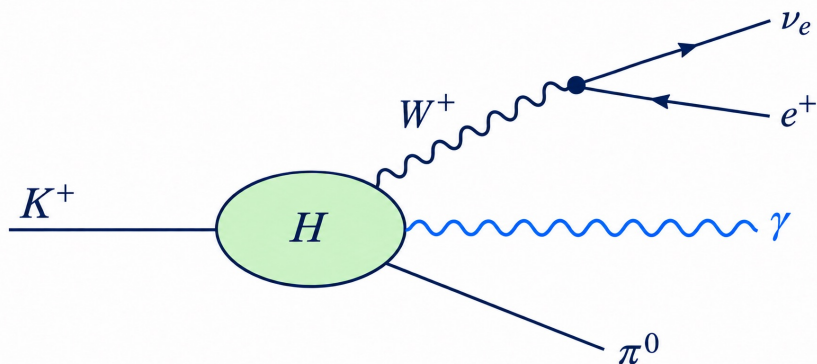
One hadronic input, two processes

$$H_{\mu\nu}(p, q) = \int d^4x e^{iq \cdot x} \langle \pi(p) | T [J_\mu^{\text{em}}(x) J_\nu^W(0)] | K(P) \rangle$$

Kaon at rest, pion with momentum p

Real photon

$q^2 = 0$

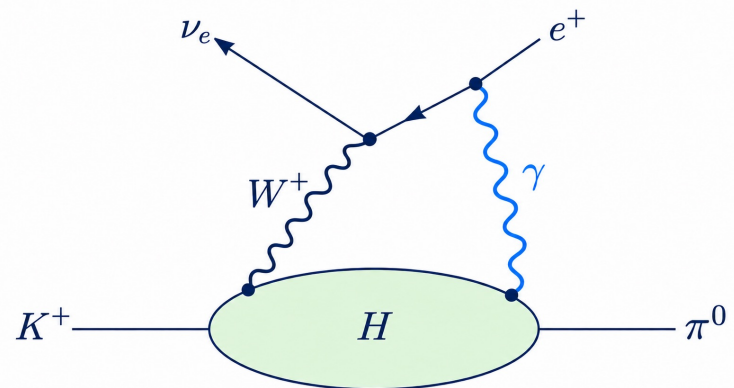


$K^+ \rightarrow \pi^0 e^+ \nu \gamma$

NA62: $\approx 1\%$ precision

Virtual photon

Integrate over q



γW box in $K_{\ell 3}$

Radiative correction to $|V_{us}|$

$K \rightarrow \pi e \nu \gamma$ and $K_{\ell 3} \gamma W$ box: same lattice data, same difficulty from moving π .

Tensions between NA62 and ChPT

$$R_j = \frac{\Gamma(K^+ \rightarrow \pi^0 e^+ \nu_e \gamma; \text{cuts}_j)}{\Gamma(K^+ \rightarrow \pi^0 e^+ \nu_e (\gamma))}$$

NA62 lies $\approx 5\%$ below
this ChPT prediction

• ChPT $\mathcal{O}(p^6)$

• NA62 (2023)

$10^2 R_j$

R₁ $E_\gamma > 10 \text{ MeV}$
 $\theta_{e\gamma} > 10^\circ$



R₂ $E_\gamma > 30 \text{ MeV}$
 $\theta_{e\gamma} > 20^\circ$



R₃ $E_\gamma > 10 \text{ MeV}$
 $0.6 < \cos \theta_{e\gamma} < 0.9$



Kaon rest frame cuts. NA62, JHEP 09 (2023) 040.

Tensions between NA62 and ChPT

$$R_j = \frac{\Gamma(K^+ \rightarrow \pi^0 e^+ \nu_e \gamma; \text{cuts}_j)}{\Gamma(K^+ \rightarrow \pi^0 e^+ \nu_e (\gamma))}$$

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• ChPT $O(p^6)$

• NA62 (2023)

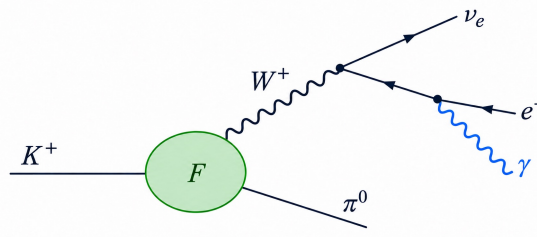
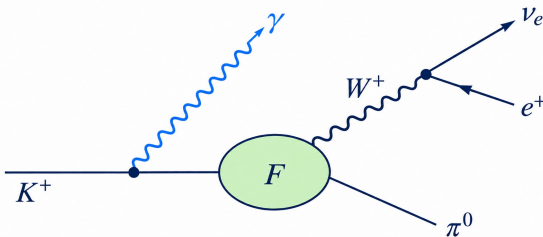
$10^2 R_j$



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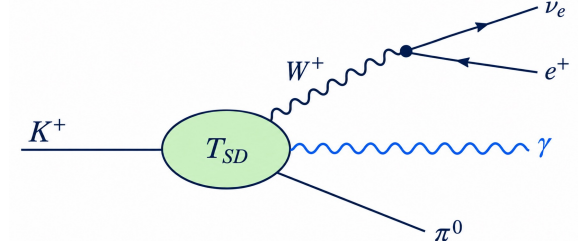
Inner bremsstrahlung

K^+ and e^+ emission; Low theorem fixes the full IB amplitude



Structure-dependent

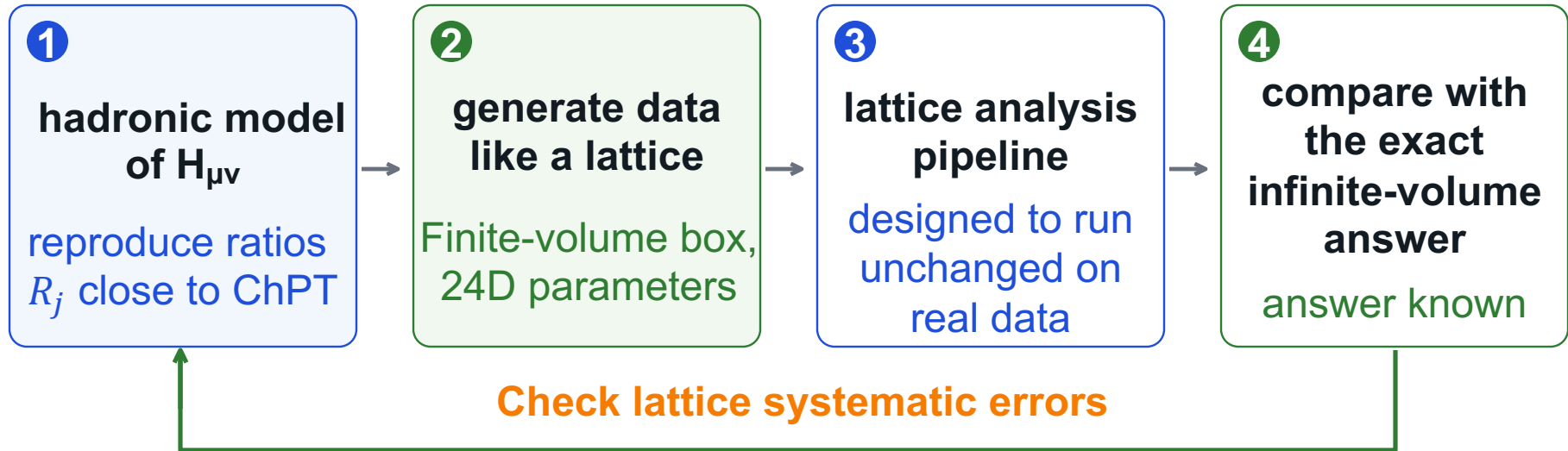
Hadronic structure enters here



🎯 Possible explanations: 1. SD part; 2. $O(\alpha^2)$ collinear radiation.

Strategy in methodology development

a “synthetic” lattice laboratory



Hadronic model content

inner bremsstrahlung (Low theorem)
resonances: ρ , a_1 , K^* , K_1
chiral anomaly (WZW)
pole-shaped $K_{\ell 3}$ form factors
reproduces ChPT $O(p^6)$ ratios R_j to 0.5 – 1 %

Lattice-like data

$m_\pi = 143$ MeV, $a^{-1} = 1.02$ GeV
(RBC/UKQCD 24D)
volumes $L/a = 16 \dots 256$

🌀 Every step is tested against a known answer — check the lattice pipeline.

The lattice analysis pipeline

$$C_{\mu\nu}(\mathbf{r}, T; \mathbf{x}, \tau) = \langle 0 | \phi_\pi(\mathbf{r}, T) T [J_\mu^{\text{em}}(\mathbf{x}, \tau) J_\nu^W(0)] | K \rangle$$
$$H_{\mu\nu}(p, q) = \underbrace{\frac{e^{E_p T}}{A} \int d^3 r e^{-i\mathbf{p}\cdot\mathbf{r}}}_{\text{pion}} \underbrace{\int d^3 x e^{-i\mathbf{q}\cdot\mathbf{x}}}_{\text{photon}} \underbrace{\int d\tau e^{\omega\tau}}_{\text{time}} C_{\mu\nu}(\mathbf{r}, T; \mathbf{x}, \tau)$$

1 Pion and photon integrals

difficulties:

- 1. moving pion, FV effects
- 2. SD signal is small

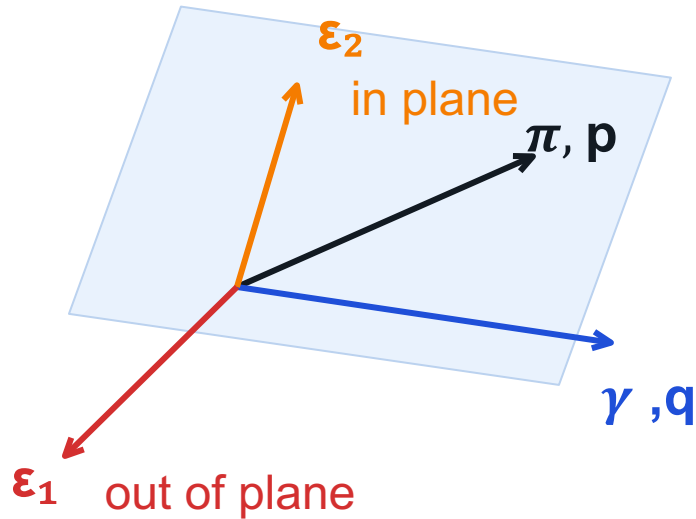
2 Time integral and soft limit

long tail for soft photons
→ infinite-volume reconstruction

3 Observables

difficulty:
3. complicated hadronic tensor structure

Difficulty 1: complicated tensor structures



Scalar projections:

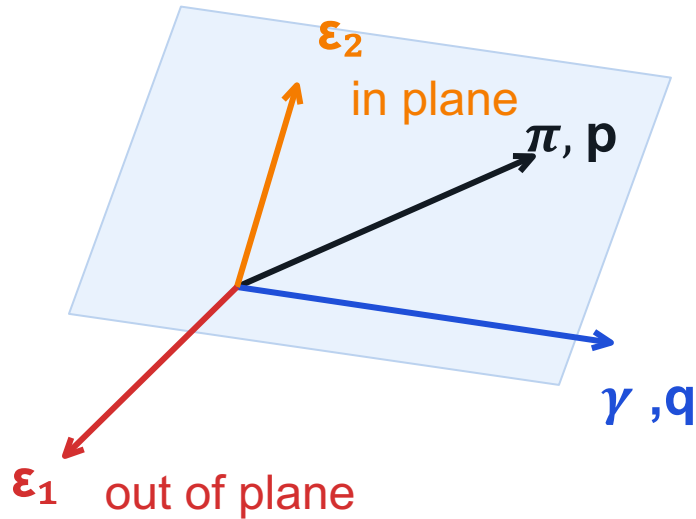
$$I_{\lambda Y} = \epsilon_{\lambda}^{\mu} H_{\mu\nu} Y^{\nu}, \quad Y \in \{\epsilon_1, \epsilon_2, q, P\}$$

	ϵ_1	ϵ_2	q	P
ϵ_1	V	A	A	A
ϵ_2	A	V	V	V

V / A: vector / axial; parity selects the nonzero sector



Difficulty 1: complicated tensor structures



K^+ channel
 $\epsilon \cdot P = 0 \rightarrow$ kaon pole removed

K^0 channel
 $\epsilon_1 \cdot p = \epsilon_1 \cdot q = 0 \rightarrow$ pion pole removed in the ϵ_1 row

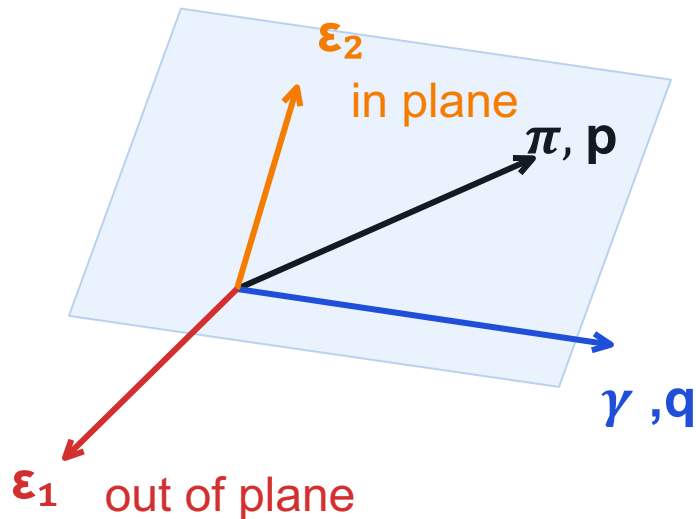
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	ϵ_1	ϵ_2	q	P
ϵ_1	V	A	A	A
ϵ_2	A	V	V	V

V / A: vector / axial; parity selects the nonzero sector

Construction of decay amplitude

$$\mathcal{M}_\lambda = \sum_Y I_{\lambda Y} a_Y + \epsilon_\lambda \cdot R$$

known coefficients electron radiation



The lattice delivers 8 numbers per (p, q) ; the amplitude is a linear combination of them.

Difficulty 2: A moving pion in a finite box

Example: tree-level $K \rightarrow l \nu$

infinite volume:	$h(\mathbf{p}) = \frac{e^{E_p T}}{A} \int d^3 r \, e^{-i\mathbf{p} \cdot \mathbf{r}} C(\mathbf{r}, T)$	exact in infinite volume
lattice:	$\hat{h}(\mathbf{p}) = \frac{e^{E_p T}}{A} \sum_{\mathbf{r} \in \text{box}} e^{-i\mathbf{p} \cdot \mathbf{r}} C_L(\mathbf{r}, T)$	the integral is cut at the box; periodicity folds the tail back

lattice data

$$C(\mathbf{r}, T) = \langle 0 | \phi_\pi(\mathbf{r}, T) J^W(0) | K \rangle$$

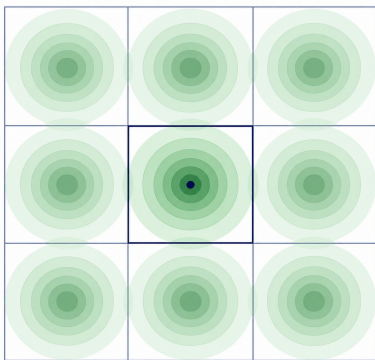
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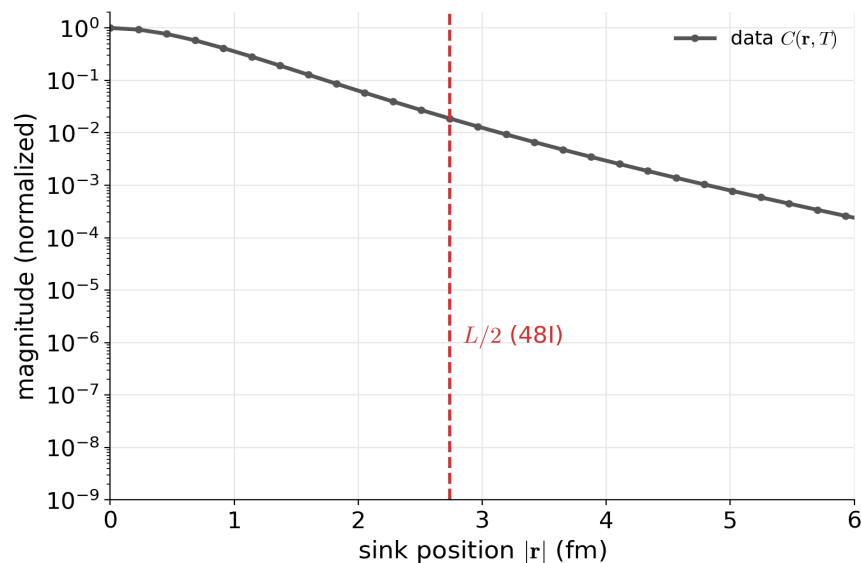
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lattice data

$$C(\mathbf{r}, T) = \langle 0 | \phi_\pi(\mathbf{r}, T) J^W(0) | K \rangle$$



wrap-around = finite-volume error



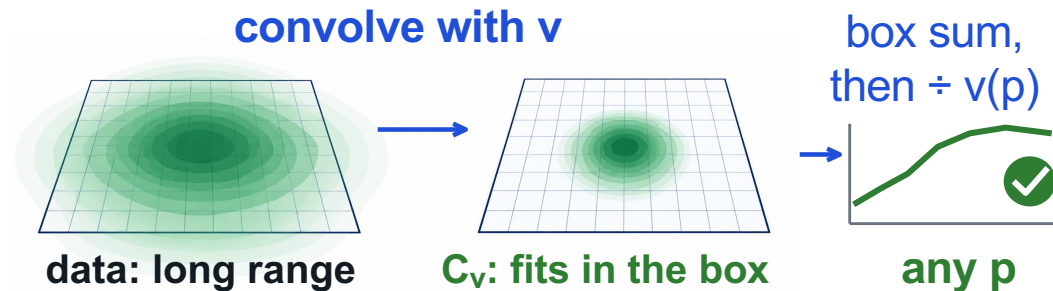
Finite-volume error = the part of the correlator that does not fit in the box.

Preconditioner method: make data fit inside the box

$$\hat{h}(\mathbf{p}) = \frac{e^{E_p T}}{A v(\mathbf{p})} \sum_{\mathbf{r} \in \text{box}} e^{-i\mathbf{p} \cdot \mathbf{r}} C_v(\mathbf{r}), \quad C_v = v * C$$

$$\tilde{C}_v(\mathbf{k}) = v(\mathbf{k}) \tilde{C}(\mathbf{k}) \quad \text{at the lattice momenta } \mathbf{k} = 2\pi\mathbf{n}/L \quad (\text{insert } 1 = v^{-1}v)$$

✓ $v(p)$ cancels exactly in infinite-volume limit, unbiased for any $v \neq 0$

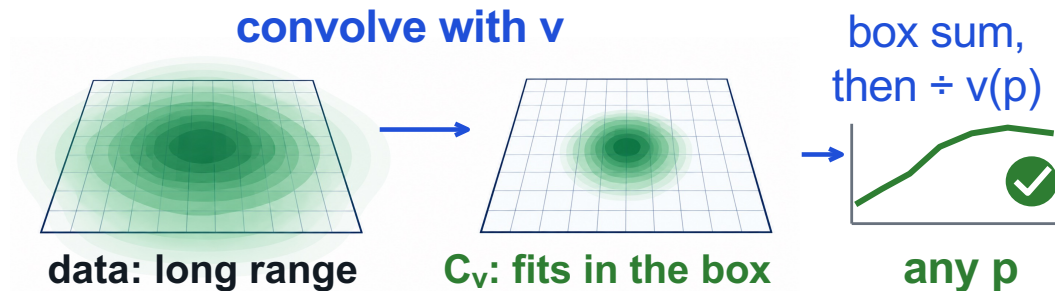


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Special choice: if $v(\mathbf{k}) \sim 1/\tilde{C}(\mathbf{k})$, then $C_v(\mathbf{r}) \sim \delta_{\mathbf{r}}$ (uncertainty principle)

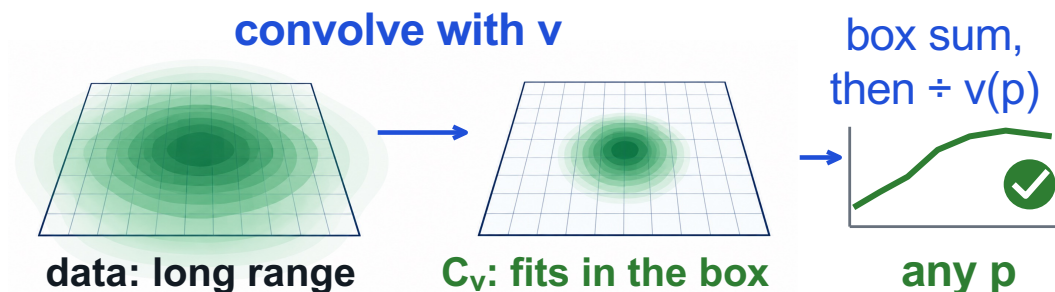
Practical calculation: we only need to do this on low modes

Preconditioner method: make data fit inside the box

$$\hat{h}(\mathbf{p}) = \frac{e^{E_p T}}{A v(\mathbf{p})} \sum_{\mathbf{r} \in \text{box}} e^{-i\mathbf{p} \cdot \mathbf{r}} C_v(\mathbf{r}), \quad C_v = v * C$$

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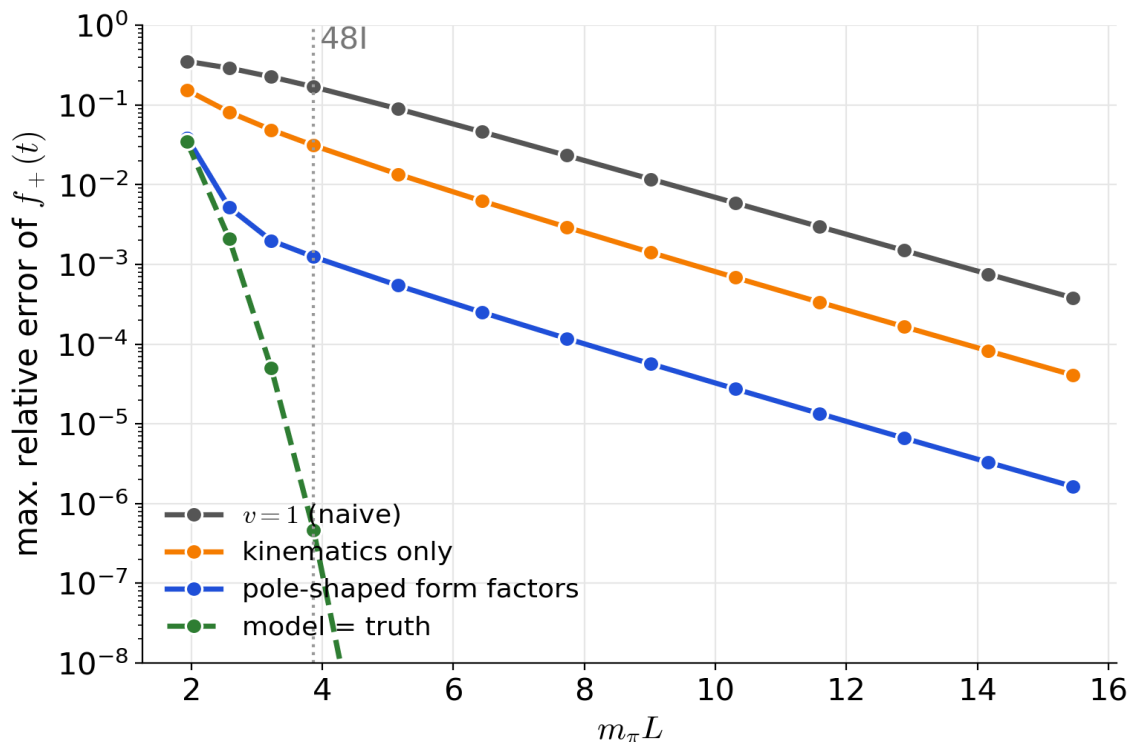
Special choice: if $v(\mathbf{k}) \sim 1/\tilde{C}(\mathbf{k})$, then $C_v(\mathbf{r}) \sim \delta_{\mathbf{r}}$ (uncertainty principle)

Practical calculation: we only need to do this on low modes

◎ The choice of low-mode model $v(k)$ only decides how small the finite-volume error is: better model, smaller FV effects

Tree-level $K_{\ell 3}$: test of the preconditioner

Hadronic model (“true” value): $K_{\ell 3}$ form factors with z-expansion



at the 48l volume
($m_\pi L \approx 3.9$)

17 %

no preconditioner

0.12 %

pole-shaped model

5×10^{-7}

model = truth
(closure test)

largest FV error on form factors in the whole phase space

error falls exponentially with the volume

A rough model already gains two orders of magnitude.

The preconditioner for $K_{l3\gamma}$: utilize the IB information

$$\hat{H}(p, q; \tau) = \underbrace{H_{\infty}^{\text{IB}}(p, q; \tau)}_{\text{add back, exact}}$$

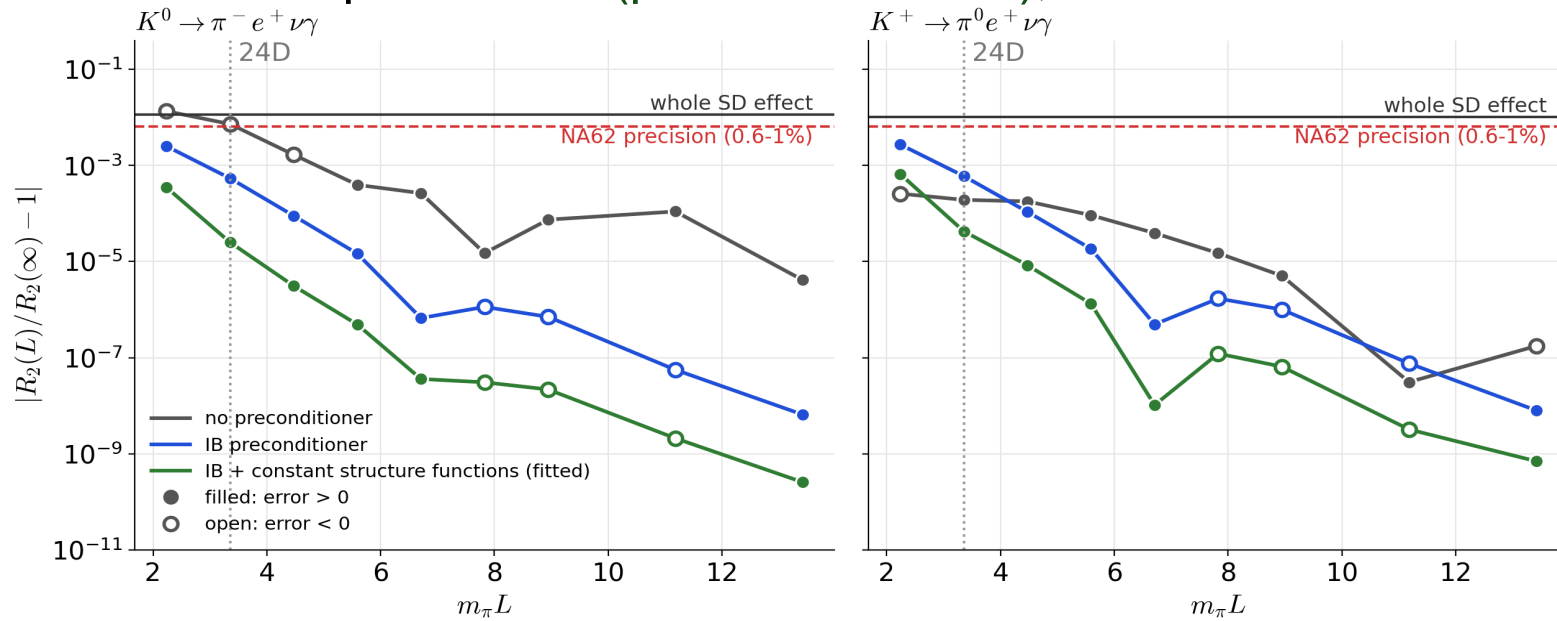
add back, exact

$$+ \frac{e^{E_p T}}{A v(\mathbf{p})} \sum_{\mathbf{r}, \mathbf{x} \in \text{box}} e^{-i\mathbf{p} \cdot \mathbf{r} - i\mathbf{q} \cdot \mathbf{x}} \underbrace{\left[v *_{\mathbf{r}} (C - C^{\text{IB}}) \right]}_{\text{data minus IB model, same box}}(\mathbf{r}; \mathbf{x}, \tau)$$

$$v(\mathbf{k}) = w(\mathbf{k}) \frac{e^{E_k T}}{A}$$

IB model:
subtracted,
not divided

Comparison: IB (point-like meson), IB+constant form factors



why not
divide?

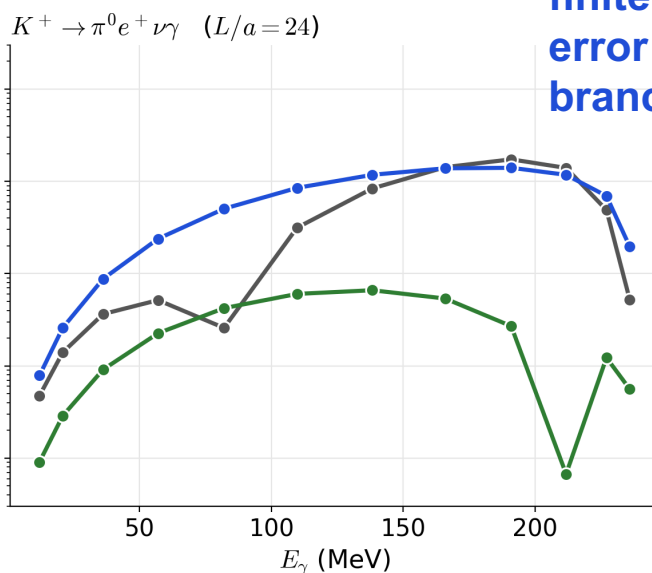
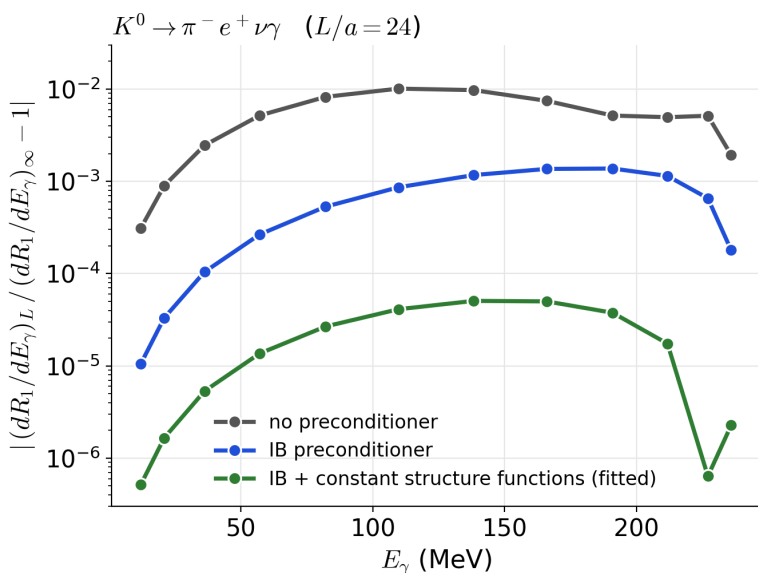
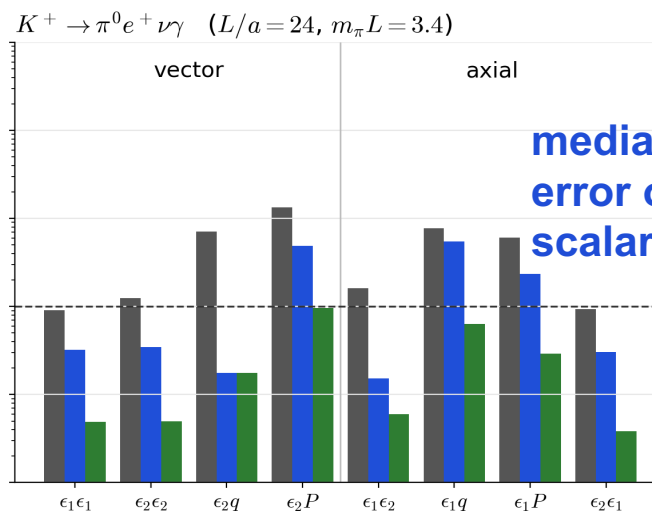
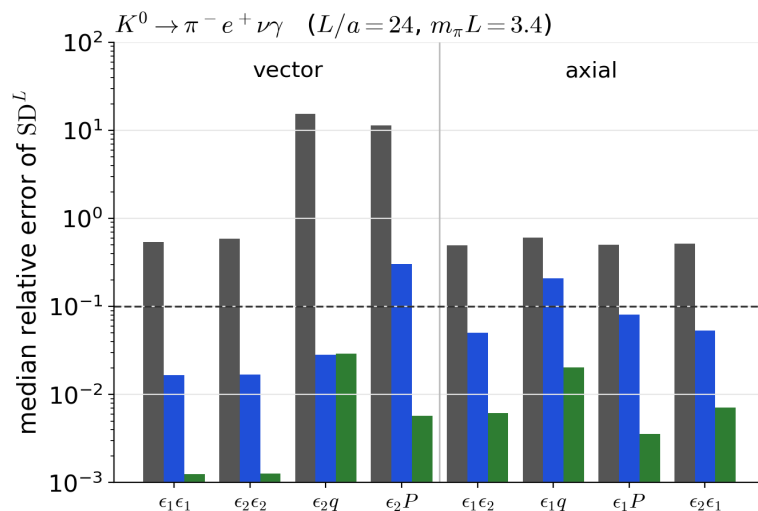
components
change sign
→ ratio
blows up



The choice of low-mode model only decides how small the finite-volume error is: better model, smaller FV effects

Individual SD functions and photon spectrum

no preconditioner IB preconditioner IB + constant structure functions (fitted)



Difficulty 3: the small structure-dependent part

Known from $K_{\ell 3}$: Low theorem

$1/q$

external charged-line pole

q^0

form factors $f_{\pm}(t)$ and slopes

$O(q)$: hadronic structure
→ calculate on the lattice

$$SD^L = \epsilon \cdot [H(q) - H(0)] \cdot Y$$

$H(q)$, $H(0)$: **the same lattice data** at photon momentum q and at $q = 0$,
cancellation of statistical and systematic errors.

$$\text{amplitude} = \underbrace{IB^L}_{\text{analytic, exact}} + \underbrace{SD^L}_{\text{lattice}}$$

Lattice errors



affect SD only

⊙ Analytical known IB contribution; compute the small SD remainder.

Outlook: the γW box with a moving pion

$$\delta\Gamma_{\gamma W} = 2 \operatorname{Re} \int d^4 X \int d^3 r \mathcal{W}^{\mu\nu}(\mathbf{r}, X) C_{\mu\nu}(\mathbf{r}, T; X)$$

known weight: QED loop $\times$ lepton phase
space $\times$ pion extraction

already in hand from $K \rightarrow \pi e \nu \gamma$ $\longrightarrow$ new for the γW box

■ moving pion from a finite box:
preconditioner

[] IB / SD split from the Low theorem

✓ synthetic-data laboratory

∫ Renormalization: lattice at long
distance, OPE at short distance

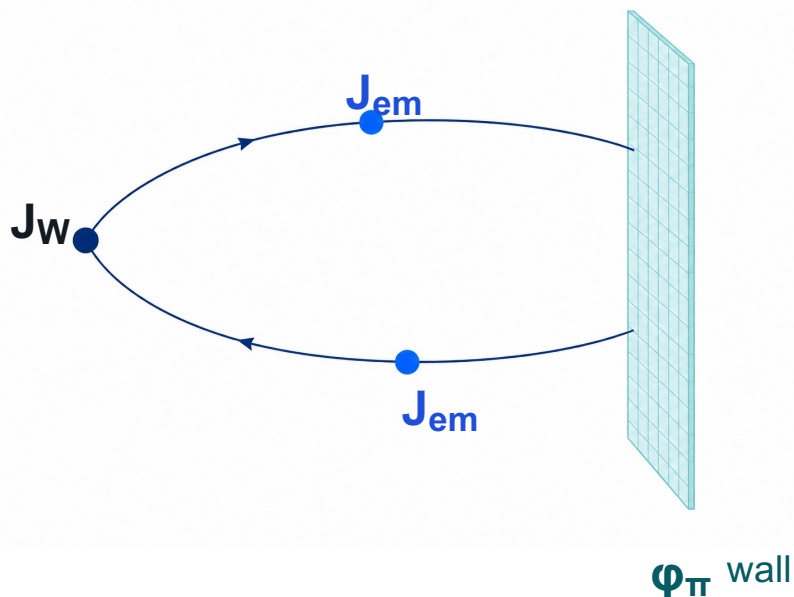
↔ infrared divergence cancels against
real emission

Many technical difficulties are already solved in $K \rightarrow \pi e \nu \gamma$.

Outlook: reuse the existing contraction method

K_{e2} project

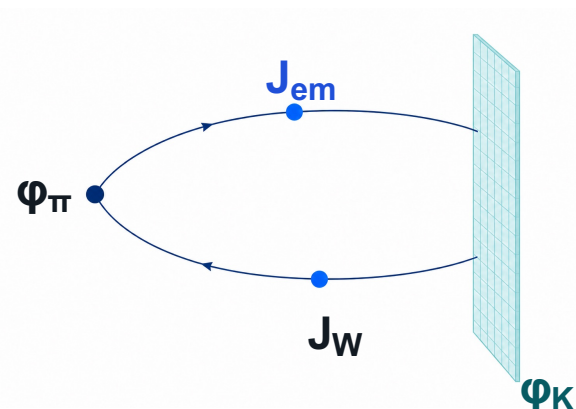
$$\langle J_{em} J_{em} J_W \phi_\pi \rangle$$



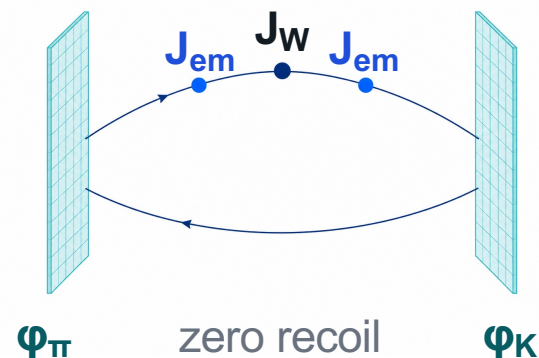
3 local points (2 free points) + wall

Same local contraction structure

γW box in K_{e3}



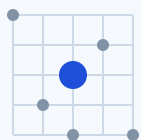
Vertex correction



Reuse the contraction method developed for the K_{e2} project.

Unbiased importance sampling of point pairs

1 Propagator storage



Sink points live on sparse sites

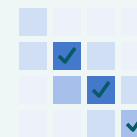
2 Score pairs cheaply

$$s(a, b) = \mathcal{K}(a - b) \mathcal{N}(a) \mathcal{N}'(b)$$



photon kernel $\times$
contraction norms

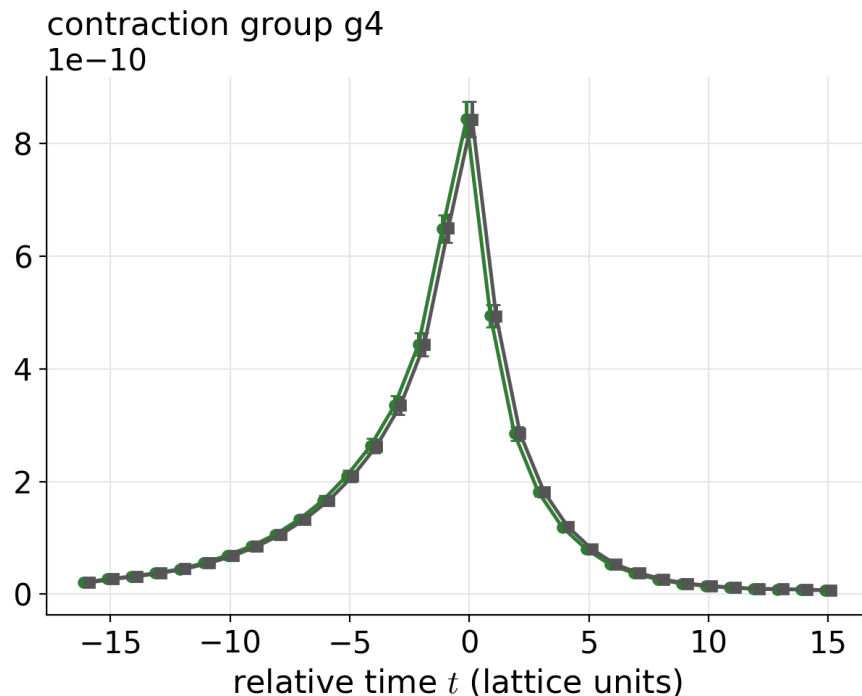
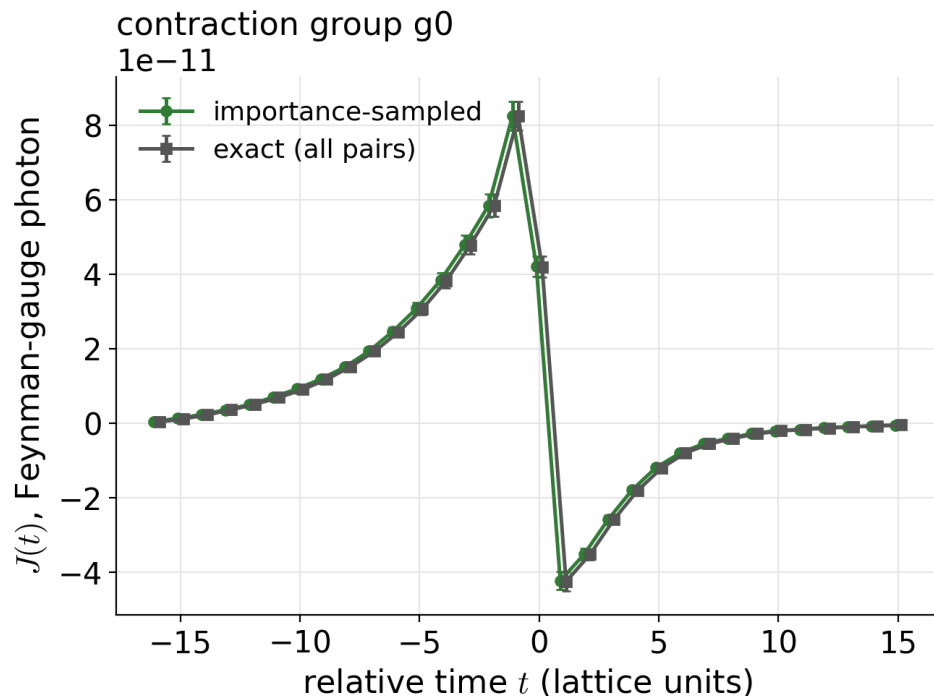
3 Sample & reweight



<0.1%

of pairs selected

Much faster contractions



A staircase to the nucleon

1 $K \rightarrow \pi e \nu \gamma$

moving pion, real photon
method closed on synthetic data

2 $K_{\ell 3}$ γW box, non-forward

photon loop, renormalization,
Infrared divergence

3 meson vertex correction

five-point function pair sampling

4 neutron g_A vertex correction

nucleon noise, $N\pi$ states

The lattice status map

	γW box	Real photon (SD)	Vertex correction
$\pi^+ \rightarrow \pi^0$	✓	—	⊘
$0^+ \rightarrow 0^+ \beta$	✓	—	—
$K_{\ell 3}$	◐	⊘	⊘
Neutron g_A	✓	—	⊘

✓ Lattice

◐ Limited

⊘ Missing

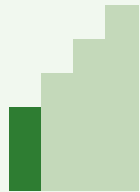
🎯 Reuse the tools step by step; the immediate test is real lattice data.

Summary



Precision for first-row CKM unitarity test is limited by the hadronic uncertainties of radiative corrections.

Most of them has no lattice input yet.



$K \rightarrow \pi e \nu \gamma$ is the first step, and has ChPT and NA62 tensions.

Preconditioner, scalar functions and the Low split are closed end-to-end on synthetic data.



The same tools lead to the $K_{\ell 3}$ box and to vertex corrections.

Efficient pair sampling makes the five-point function affordable.