g-2 on the lattice

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October 12, 2017 – Brookhaven Forum

Collaborators in the RBC/UKQCD $g-2$ effort

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James Harrison (Southampton)
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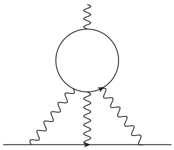
Luchang Jin (Connecticut)
Antonin Portelli (Edinburgh)

Theory status for a_μ – summary

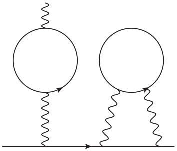
Contribution	Value $\times 10^{10}$	Uncertainty $\times 10^{10}$
QED (5 loops)	11 658 471.895	0.008
EW	15.4	0.1
HVP LO	692.3	**4.2**
HVP NLO	-9.84	0.06
HVP NNLO	1.24	0.01
Hadronic light-by-light	10.5	**2.6**
Total SM prediction	11 659 181.5	4.9
BNL E821 result	11 659 209.1	6.3
FNAL E989/J-PARC E34 goal		$\approx$ **1.6**

Reduce uncertainties on hadronic contributions!

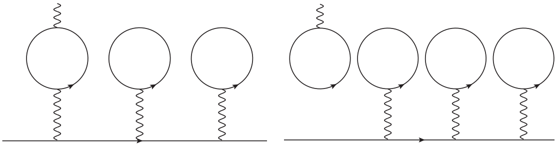
The Hadronic Light-by-Light contribution



Quark-connected piece (charge factor of up/down quark contribution: $\frac{17}{81}$)



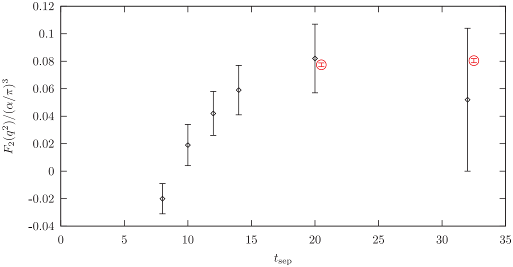
Dominant quark-disconnected piece (charge factor of up/down quark contribution: $\frac{25}{81}$)



Sub-dominant quark-disconnected pieces (charge factors of up/down quark contribution: $\frac{5}{81}$ and $\frac{1}{81}$)

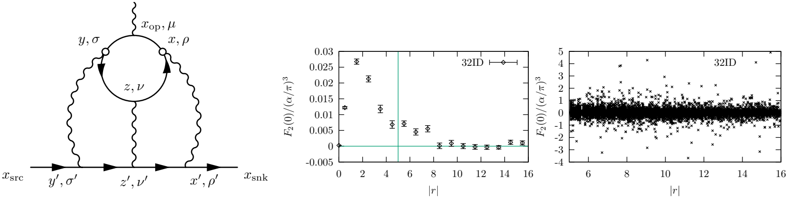
T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, and C.L., Phys. Rev. D 93, 014503 (2016)

Compute quark-connected contribution with new computational strategy



yields more than an order-of-magnitude improvement (red symbols) over previous method (black symbols) for a factor of ≈ 4 smaller cost.

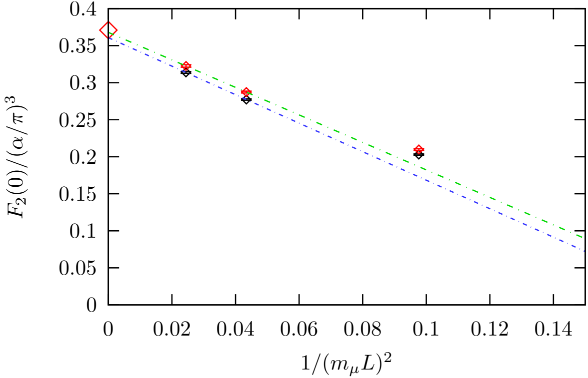
New stochastic sampling method



Stochastically evaluate the sum over vertices x and y:

- Pick random point x on lattice
- Sample all points y up to a specific distance $r = |x - y|$, see vertical red line
- Pick y following a distribution $P(|x - y|)$ that is peaked at short distances

Cross-check against analytic result where quark loop is replaced by muon loop



In Figure: $m_{\text{loop}} = m_{\text{line}}$

T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, and C.L., PRL118(2017)022005

$$a_\mu^{\mathrm{cHLbL}} = \left.\frac{g_\mu - 2}{2}\right|_{\mathrm{cHLbL}} = (0.0926 \pm 0.0077)\left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10} \quad (11)$$

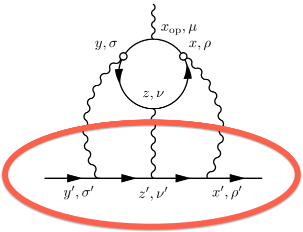
$$a_\mu^{\mathrm{dHLbL}} = \left.\frac{g_\mu - 2}{2}\right|_{\mathrm{dHLbL}} = (-0.0498 \pm 0.0064)\left(\frac{\alpha}{\pi}\right)^3 = (-6.25 \pm 0.80) \times 10^{-10} \quad (12)$$

$$a_\mu^{\mathrm{HLbL}} = \left.\frac{g_\mu - 2}{2}\right|_{\mathrm{HLbL}} = (0.0427 \pm 0.0108)\left(\frac{\alpha}{\pi}\right)^3 = (5.35 \pm 1.35) \times 10^{-10} \quad (13)$$

Makes HLbL an unlikely candidate to explain the discrepancy!

Next need to address finite-volume and lattice-spacing systematics and sub-leading diagrams

Finite-volume errors of the HLbL



Remove power-law like finite-volume errors by computing the muon-photon part of the diagram in infinite volume (C.L. talk at lattice 2015 and Green et al. 2015, PRL115(2015)222003; Asmussen et al. 2016, PoS,LATTICE2016 164)

Now completed PRD96(2017)034515 with improved weighting function.

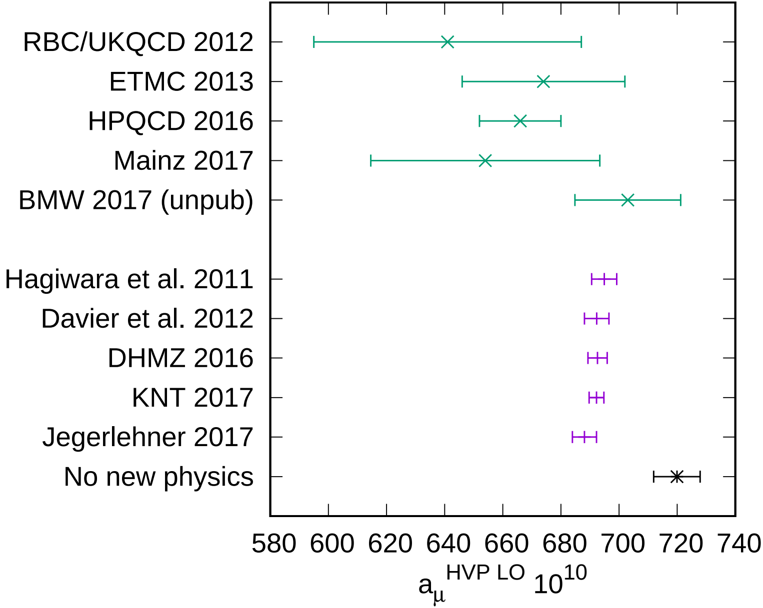
Hadronic light-by-light contribution – Summary and outlook

New methods allow for a complete first-principles calculation of the hadronic light-by-light contribution to the $(g-2)_\mu$.

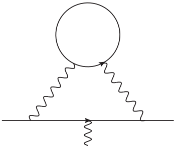
Control of all uncertainties on the 10% level over the next 1-2 years seems possible.

Even with current large systematic uncertainty, no new physics scenario (HLbL) unlikely!

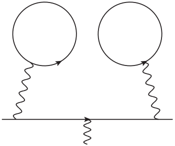
Theory status for a_μ – HVP LO



First-principles approach to HVP LO

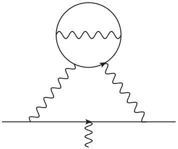


Quark-connected piece with by far dominant part from up and down quark loops, $\mathcal{O}(700 \times 10^{-10})$



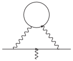
Quark-disconnected piece, $-9.6(4.0) \times 10^{-10}$

Phys.Rev.Lett. 116 (2016) 232002



QED corrections, $\mathcal{O}(10 \times 10^{-10})$

All results below are obtained using domain-wall fermions at physical pion mass with lattice cutoffs $a^{-1} = 1.73$ GeV and $a^{-1} = 2.36$ GeV.

 HVP quark-connected contribution

Starting from the vector current

$$J_\mu(x) = i \sum_f Q_f \overline{\Psi}_f(x) \gamma_\mu \Psi_f(x)$$

we may write

$$a_\mu^{\mathrm{HVP}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle$$

and w_t capturing the photon and muon part of the diagram (Bernecker-Meyer 2011).

A connection to the R-ratio data

We now have all ingredients to compare to the R-ratio data

We can connect $C(t)$ to the R-ratio data (Bernecker, Meyer 2011) as

$$\Pi(-Q^2) = \frac{1}{\pi}\int_0^\infty ds\,\frac{s}{s+Q^2}\sigma(s, e^+e^- \to \mathrm{had})$$

with

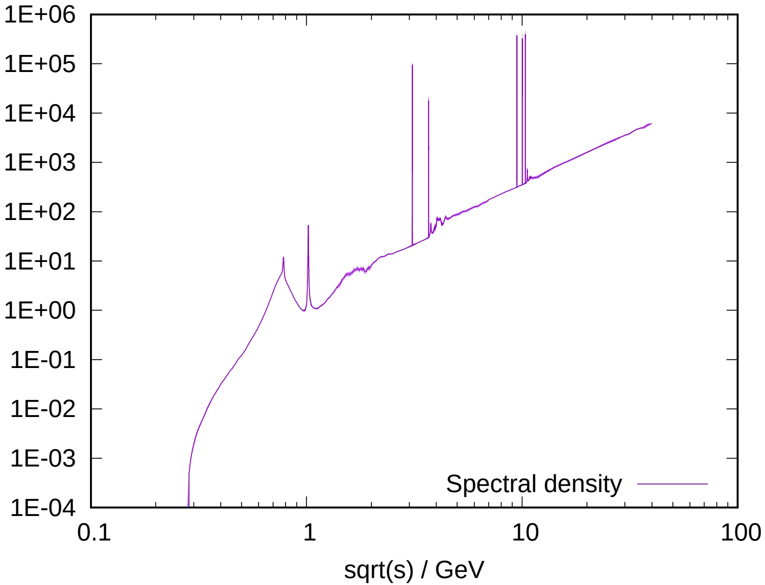
$$R(s) = \frac{\sigma(s, e^+e^- \to \mathrm{had})}{\sigma(s, e^+e^- \to \mu^+\mu^-, \mathrm{tree})} = \frac{3s}{4\pi\alpha^2}\sigma(s, e^+e^- \to \mathrm{had})\,.$$

A Fourier transform then gives

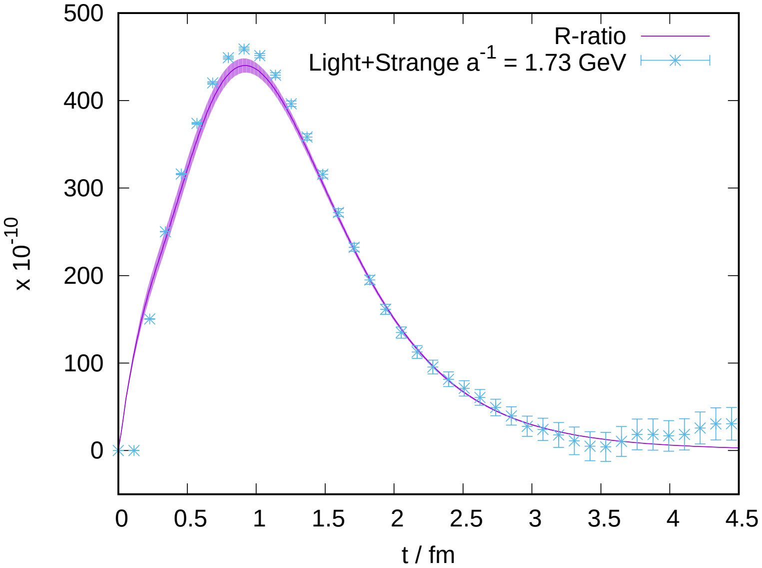
$$C(t) \propto \int_0^\infty d(\sqrt{s})R(s)se^{-\sqrt{s}t} \equiv \int_0^\infty d(\sqrt{s})\rho(\sqrt{s})e^{-\sqrt{s}t}$$

with spectral density $\rho(\sqrt{s})$.

Below the $R(s)$ is taken from Jegerlehner 2016:

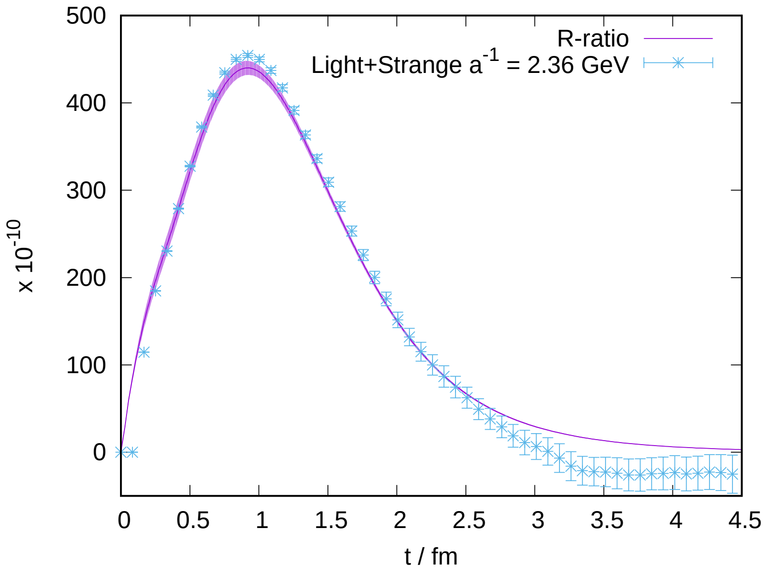


Lattice data agrees quite well with the R-ratio data



Lattice data is precise at shorter distances, R-ratio data is precise at longer distances

Lattice data agrees quite well with the R-ratio data



Lattice data is precise at shorter distances, R-ratio data is precise at longer distances

This allows us to devise a "Window method":

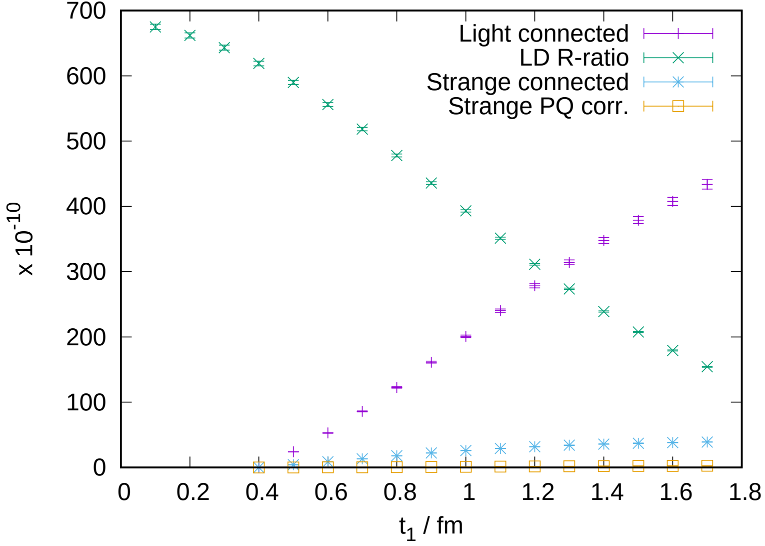
$$a_\mu = \sum_t w_t C(t) \equiv a_\mu^{\rm SD} + a_\mu^{\rm W} + a_\mu^{\rm LD}$$

with

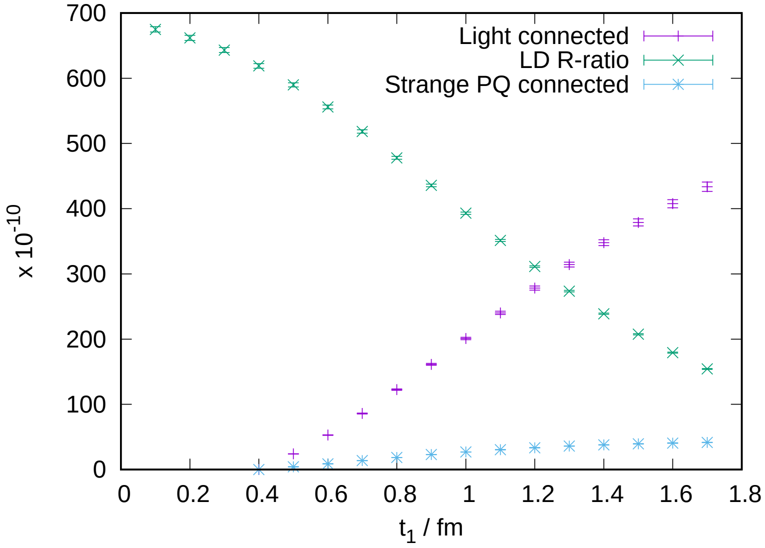
$$\begin{aligned} a_\mu^{\rm SD} &= \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)]\,, \\ a_\mu^{\rm W} &= \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)]\,, \\ a_\mu^{\rm LD} &= \sum_t C(t) w_t \Theta(t, t_1, \Delta) \end{aligned}$$

and each contribution accessible from both lattice and R-ratio data.

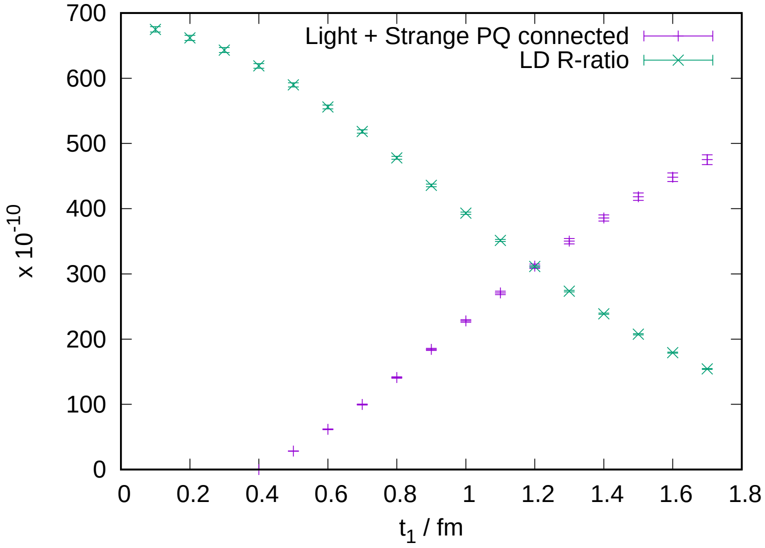
Re-combine a_μ^{W} from lattice with a_μ^{LD} from R-ratio:



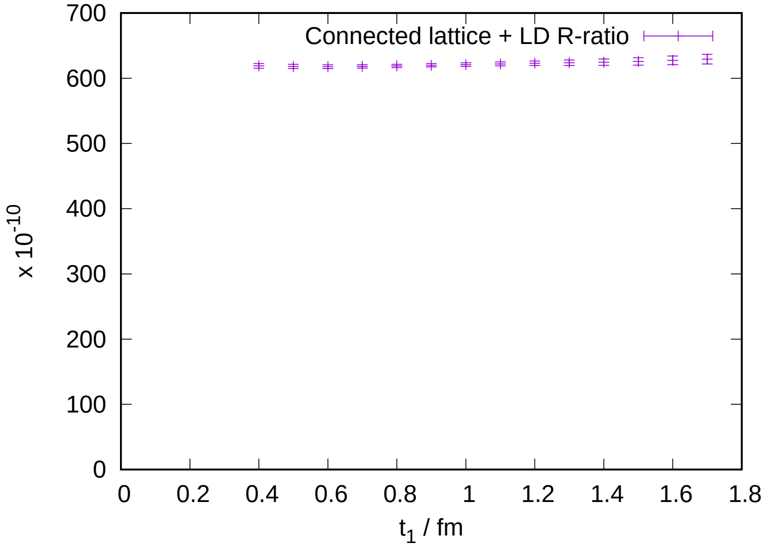
Re-combine a_μ^{W} from lattice with a_μ^{LD} from R-ratio:

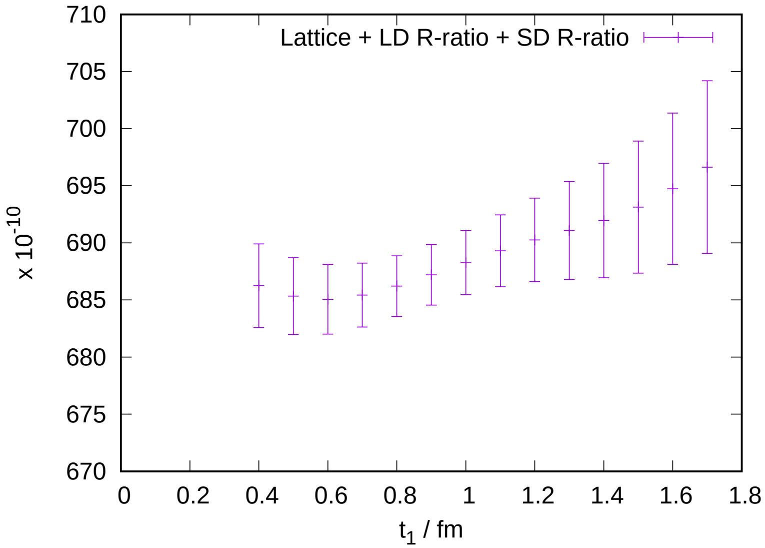


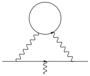
Re-combine a_μ^{W} from lattice with a_μ^{LD} from R-ratio:

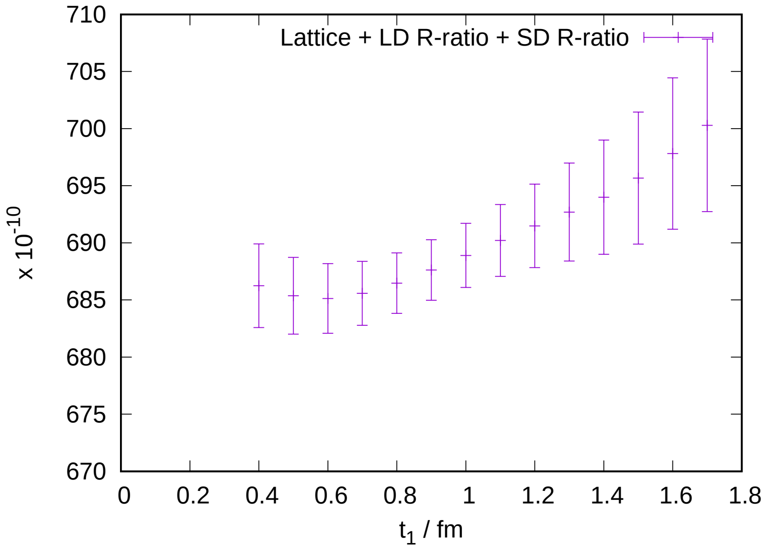


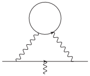
Re-combine a_μ^{W} from lattice with a_μ^{LD} from R-ratio:

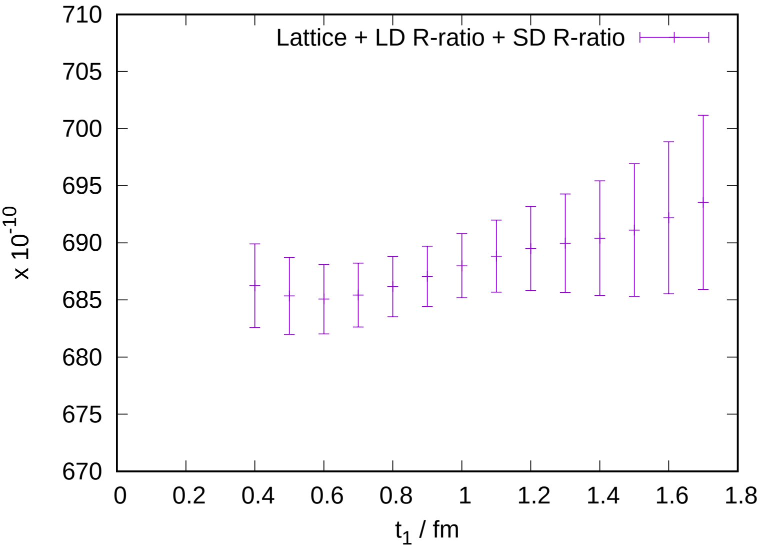


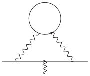
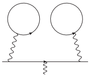

Lattice + LD R-ratio + SD R-ratio
710
705
700
695
690
685
680
675
670
x 10-10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
t1 / fm

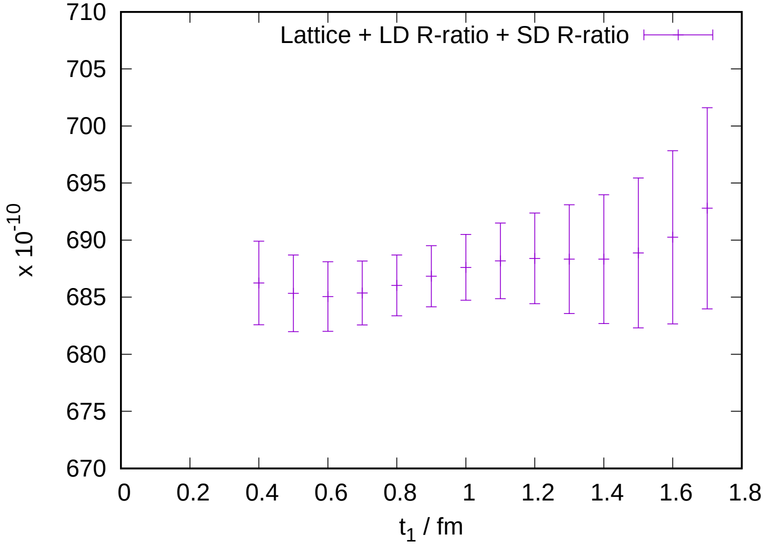
Lattice: 

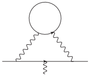
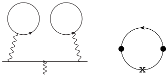


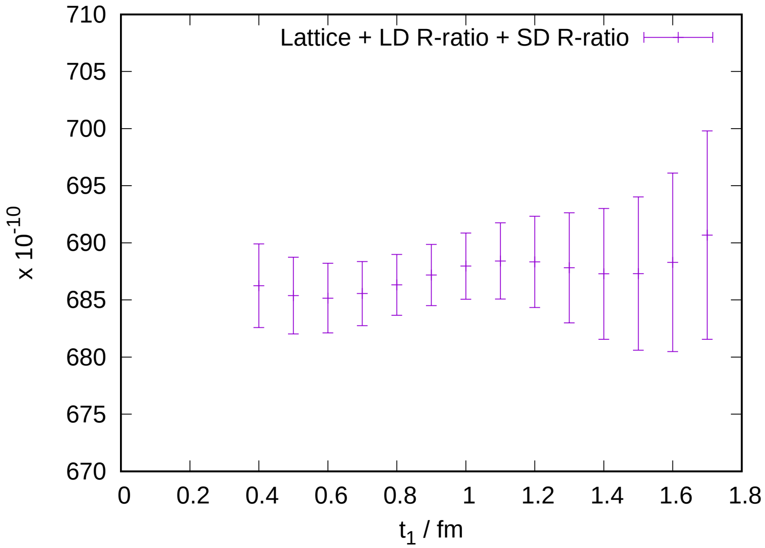
Lattice:  sQED FV correction

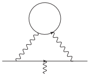
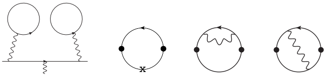


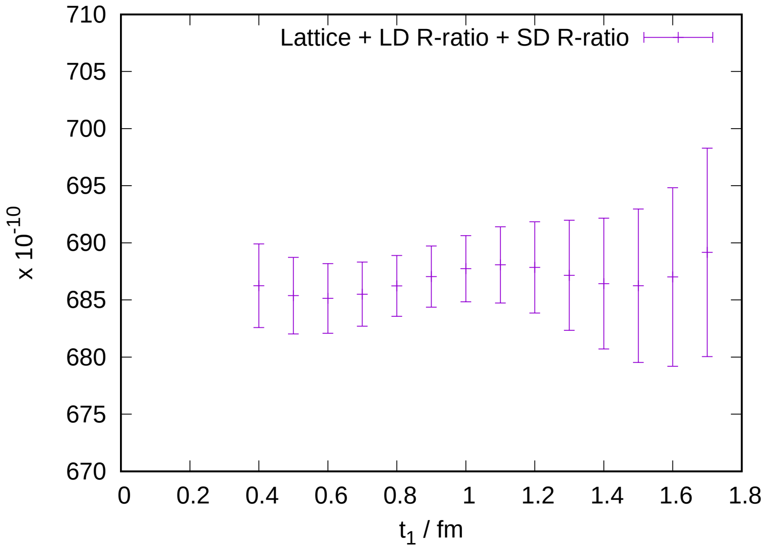
Lattice:  sQED FV correction 

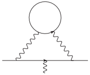
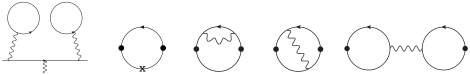


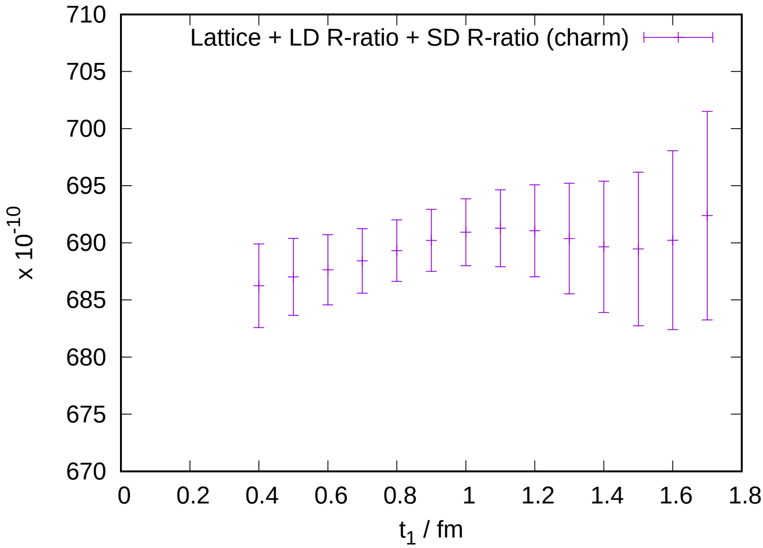
Lattice:  sQED FV correction 

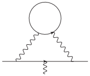
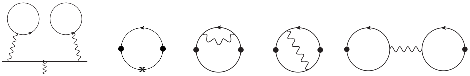


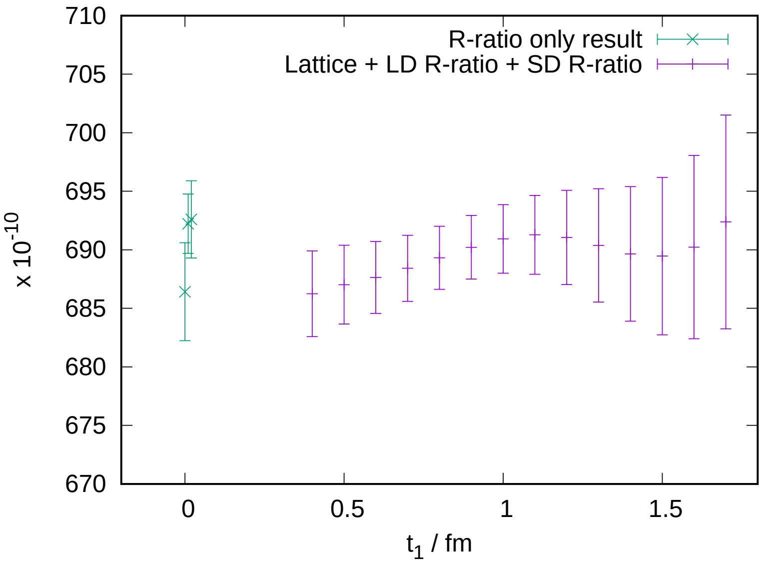
Lattice:  sQED FV correction 



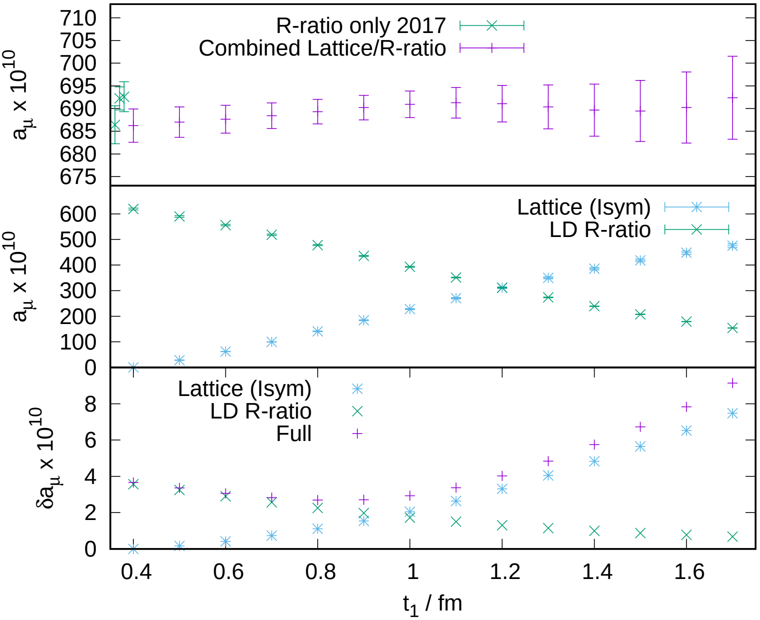
Lattice:  sQED FV correction 



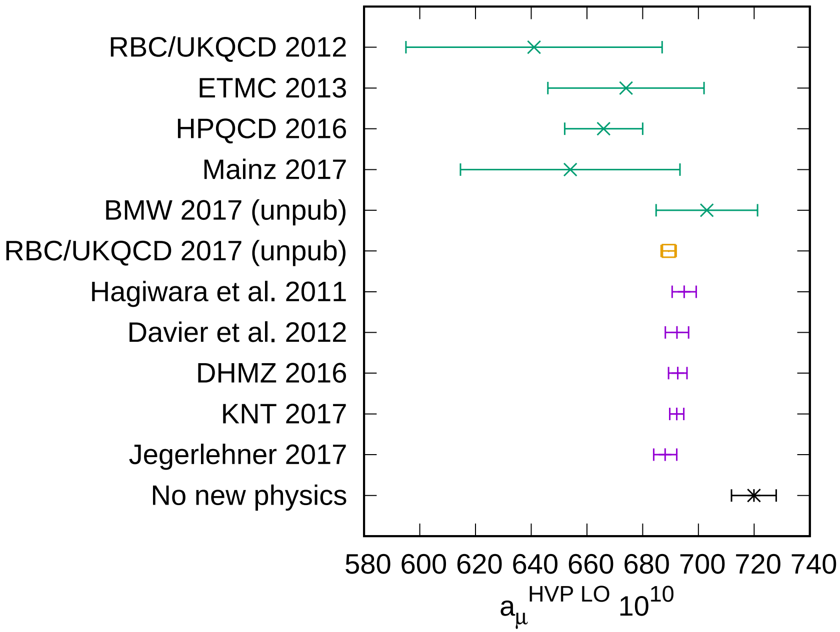
Lattice:  sQED FV correction 

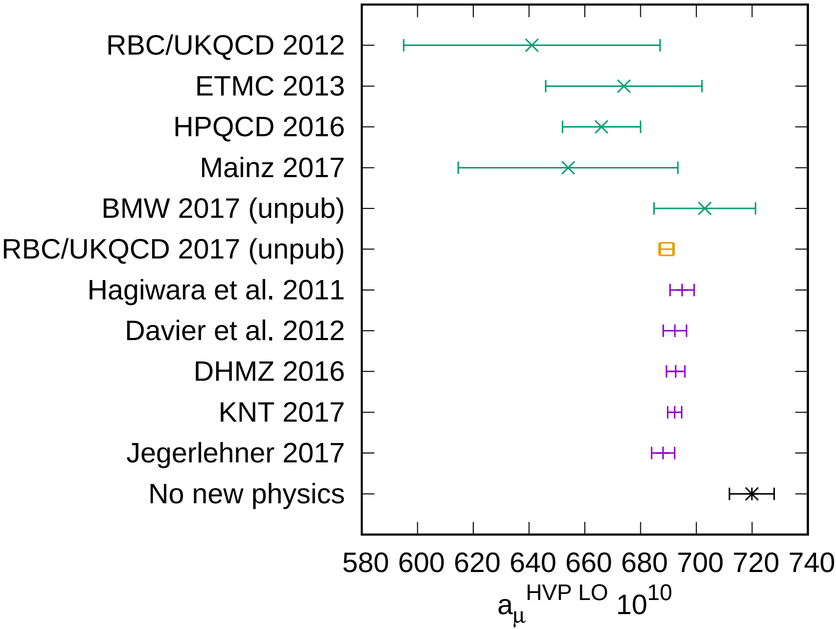


Note: combined lattice and R-ratio is more precise than R-ratio alone!
Error minimal for $t_1 = 1.2$ fm.



R-ratio only 2017
Combined Lattice/R-ratio
Lattice (Isym)
LD R-ratio
Lattice (Isym)
LD R-ratio
Full
$a_\mu \times 10^{10}$
$a_\mu \times 10^{10}$
$\delta a_\mu \times 10^{10}$
t_1 / fm
710
705
700
695
690
685
680
675
600
500
400
300
200
100
0
8
6
4
2
0
0.4
0.6
0.8
1
1.2
1.4
1.6


RBC/UKQCD 2012
ETMC 2013
HPQCD 2016
Mainz 2017
BMW 2017 (unpub)
RBC/UKQCD 2017 (unpub)
Hagiwara et al. 2011
Davier et al. 2012
DHMZ 2016
KNT 2017
Jegerlehner 2017
No new physics
580 600 620 640 660 680 700 720 740
$a_\mu^{\mathrm{HVP\ LO}}\ 10^{10}$



No new physics scenario (HVP) unlikely

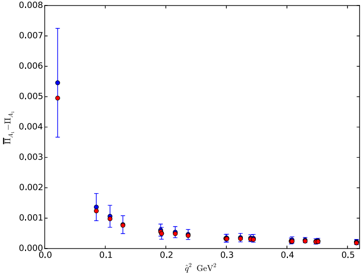
HVP – Summary and outlook

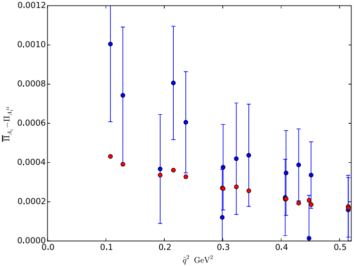
- QED corrections included
- Combine and cross-check lattice and R-ratio data: This method allows for further reduction in uncertainty over the already very precise R-ratio results.
- Here we used the results of Jegerlehner 2016 for a combined analysis and obtained a result with $\delta a_\mu^{\mathrm{HVP\ LO}} = 2.7 \times 10^{-10}$ (most precise HVP determination currently available!)
- No new physics scenario (HVP) unlikely, however, low-energy $\pi\pi$ states and $\pi\gamma$ states in R-ratio effectively not yet checked $\Rightarrow$ future work

Thank you

Addressing the finite-volume problem

From Aubin et al. 2015 (arXiv:1512.07555v2)



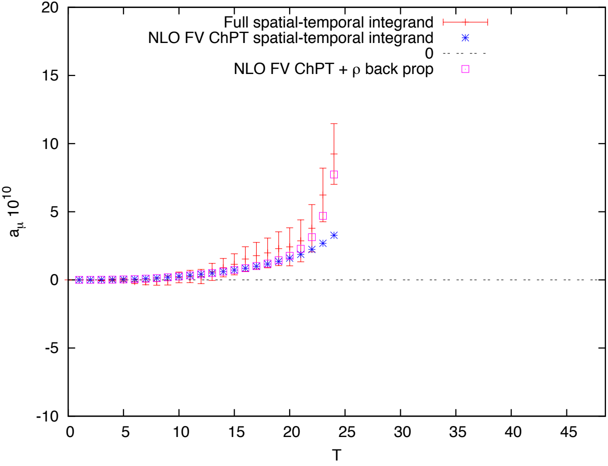


MILC lattice data with $m_\pi L = 4.2$, $m_\pi \approx 220$ MeV; Plot difference of $\Pi(q^2)$ from different irreps of 90-degree rotation symmetry of spatial components versus NLO FV ChPT prediction (red dots)

While the absolute value of a_μ is poorly described by the two-pion contribution, the volume dependence may be described sufficiently well to use ChPT to control FV errors at the 1% level; this needs further scrutiny

Aubin et al. find an $O(10\%)$ finite-volume error for $m_\pi L = 4.2$ based on the $A_1 - A_1^{44}$ difference (right-hand plot)

Compare difference of integrand of $48 \times 48 \times 96 \times 48$ (spatial) and $48 \times 48 \times 48 \times 96$ (temporal) geometries with NLO FV ChPT $(A_1 - A_1^{44})$:



$m_\pi = 140$ MeV, $p^2 = m_\pi^2/(4\pi f_\pi)^2 \approx 0.7\%$

Our efforts to control the finite-volume error:

- We have generated three additional lattices with physical pion mass and $L = 4.8$fm, 6.4fm, and 9.6fm; we have started first measurements on these lattices.

- We are currently tuning our new Multi-Grid Lanczos method on the largest volumes to continue to use our noise-reduction techniques for these studies. For these ensembles the improved Multi-Grid Lanczos is critical.

Addressing the long-distance noise problem

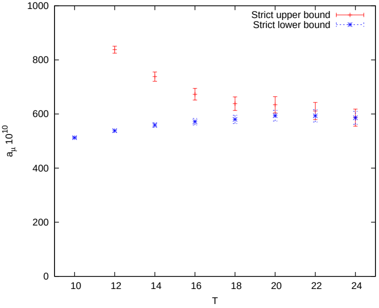
There are two general classes of solutions to the long-distance noise problem

- Statistics $\rightarrow$ Systematics: One can reduce statistical uncertainty at the cost of introducing an additional systematic uncertainty that then needs to be controlled; This requires additional care in estimating a potential systematic bias but may be overall beneficial.
- Statistics $\uparrow$: One can devise improved statistical estimators without additional systematic uncertainties

Concrete recent proposals:

- Replace $C(t)$ for large t with model, say multi-exponentials for $t \geq t^*$ HPQCD arXiv:1601.03071 (Statistics $\rightarrow$ Systematics)
- Define stochastic estimator for strict upper and lower bounds of a_μ which have reduced statistical fluctuations RBC/UKQCD 2015, BMWc arXiv:1612.02364 (Statistics $\uparrow$)

More details, e.g., talk C.L. at Rutgers 2015



Bound $C_l(t) \leq C(t) \leq C_u(t)$ with

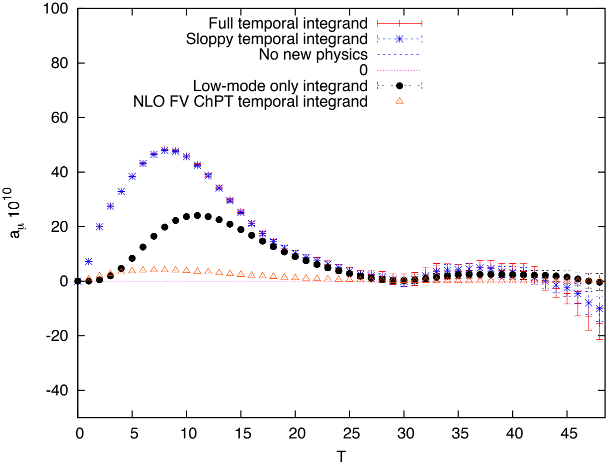
$$C_{l/u}(t) = \begin{cases} C(t) & t < T\,, \\ C(T)e^{-(t-T)\bar{E}_{l/u}} & t \geq T \end{cases}$$

with $\bar{E}_u$ being the ground state of the VV correlator and

$$\bar{E}_l = \log(C(T)/C(T+1))\,.$$

Concrete recent proposals (continued):

- RBC/UKQCD 2015 Improved stochastic estimator; hierarchical approximations including exact treatment of low-mode space DeGrand & Schäfer 2004: (Statistics ↑):



Concrete recent proposals (continued):

- Phase reweighting (Savage et al.) (Statistics $\rightarrow$ Systematics)

$$C(t) \rightarrow C(t)\,\mathrm{Sign}[C(t-\Delta)]$$

  extrapolate to $\Delta \rightarrow \infty$

- Multi-level gauge field generation (Ce/Giusti/Schafer) (Statistics $\uparrow$)
  - Action is local $\Rightarrow$ independent evolution of gauge fields in sub-domains possible
  - Recombination of independent samples over all subdomains may lead to exponential reduction of noise
  - We are currently investigating this method for the HVP (M. Bruno for RBC/UKQCD)

The setup:

$$C(t) = \frac{1}{3V} \sum_{j=0,1,2} \sum_{t'} \langle \mathcal{V}_j(t+t') \mathcal{V}_j(t') \rangle_{\mathrm{SU(3)}} \tag{1}$$

where V stands for the four-dimensional lattice volume,
$\mathcal{V}_\mu = (1/3)(\mathcal{V}_\mu^{u/d} - \mathcal{V}_\mu^s)$, and

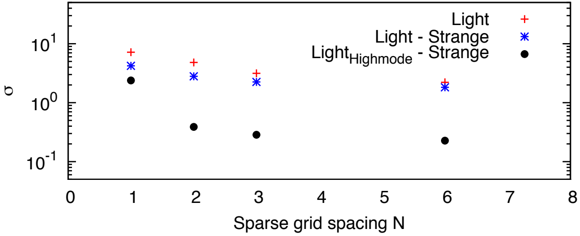
$$\mathcal{V}_\mu^f(t) = \sum_{\vec{x}} \mathrm{Im}\, \mathrm{Tr}[D^{-1}_{\vec{x},t;\vec{x},t}(m_f)\gamma_\mu]\,. \tag{2}$$

We separate 2000 low modes (up to around m_s) from light quark propagator as $D^{-1} = \sum_n v^n (w^n)^\dagger + D^{-1}_{\mathrm{high}}$ and estimate the high mode stochastically and the low modes as a full volume average Foley 2005.

We use a sparse grid for the high modes similar to Li 2010 which has support only for points x_μ with $(x_\mu - x_\mu^{(0)}) \bmod N = 0$; here we additionally use a random grid offset $x_\mu^{(0)}$ per sample allowing us to stochastically project to momenta.

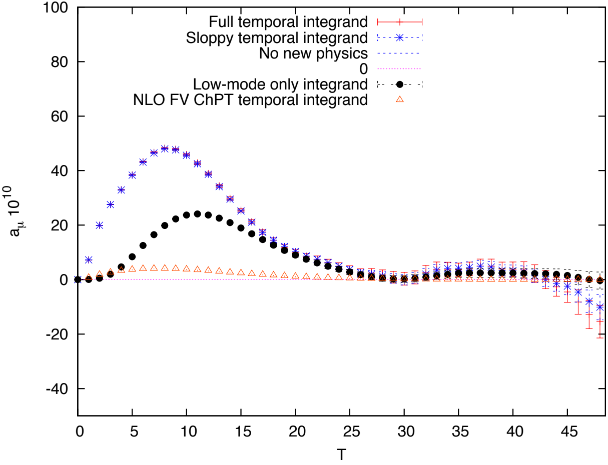
Combination of both ideas is crucial for noise reduction at physical pion mass!

Fluctuation of $\mathcal{V}_\mu$ (σ):

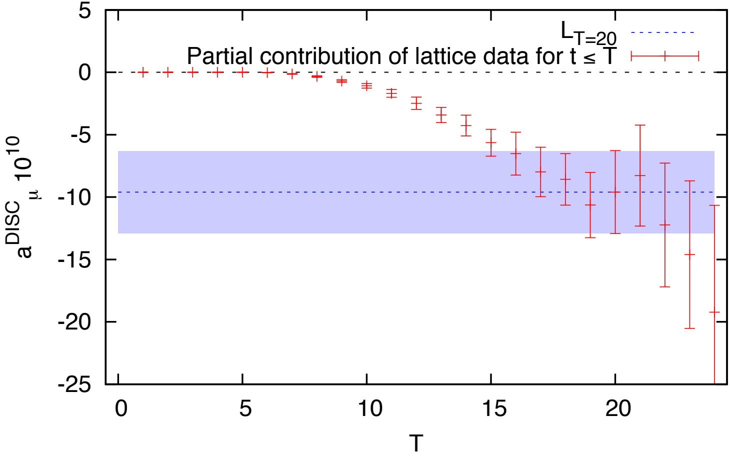


Since $C(t)$ is the autocorrelator of $\mathcal{V}_\mu$, we can create a stochastic estimator whose noise is potentially reduced linearly in the number of random samples, hence the normalization in the lower panel

Low-mode saturation for physical pion mass (here 2000 modes):

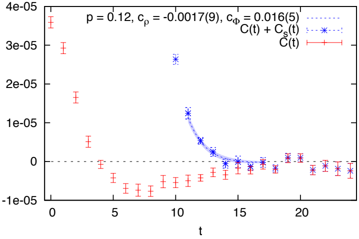


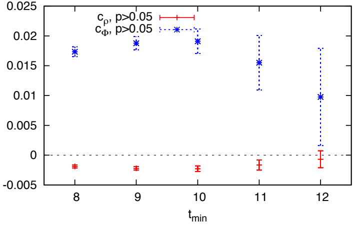
Result for partial sum $L_T = \sum_{t=0}^{T} w_t C(t)$:



For $t \geq 15$ $C(t)$ is consistent with zero but the stochastic noise is t-independent and $w_t \propto t^4$ such that it is difficult to identify a plateau region based only on this plot

Resulting correlators and fit of $C(t) + C_s(t)$ to $c_\rho e^{-E_\rho t} + c_\phi e^{-E_\phi t}$ in the region $t \in [t_{\min}, \ldots, 17]$ with fixed energies $E_\rho = 770$ MeV and $E_\phi = 1020$. $C_s(t)$ is the strange connected correlator.

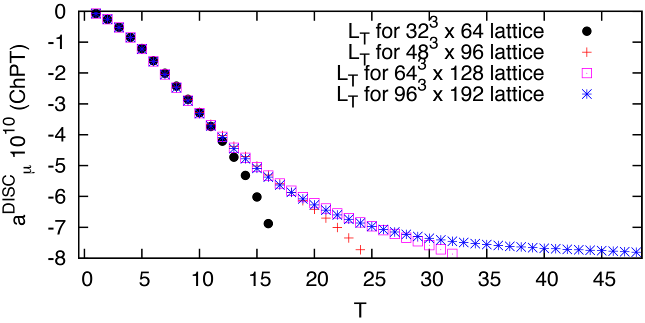




We fit to $C(t) + C_s(t)$ instead of $C(t)$ since the former has a spectral representation.

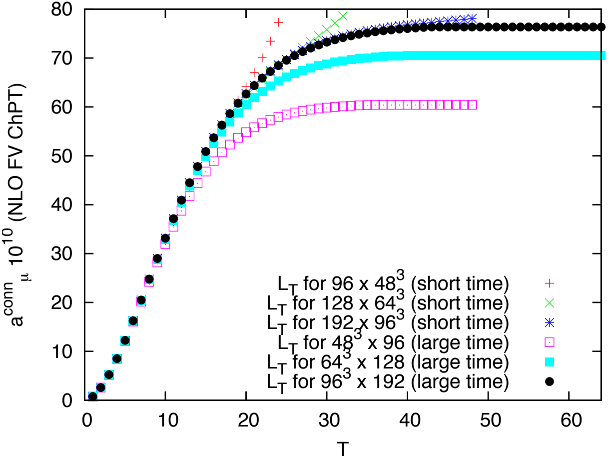
We could use this model alone for the long-distance tail to help identify a plateau but it would miss the two-pion tail

We therefore additionally calculate the two-pion tail for the disconnected diagram in ChPT:



A closer look at the NLO FV ChPT prediction (1-loop sQED):

We show the partial sum $\sum_{t=0}^{T} w_t C(t)$ for different geometries and volumes:



The dispersive approach to HVP LO

The dispersion relation

$$\begin{aligned}\Pi_{\mu\nu}(q) &= i\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)\\ \Pi(q^2) &= -\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{q^2-s}\,.\end{aligned}$$

allows for the determination of $a_\mu^{\rm HVP}$ from experimental data via

$$a_\mu^{\rm HVP\ LO} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2\left[\int_{4m_\pi^2}^{E_0^2} ds\,\frac{R_\gamma^{\rm exp}(s)\hat{K}(s)}{s^2} + \int_{E_0^2}^{\infty} ds\,\frac{R_\gamma^{\rm pQCD}(s)\hat{K}(s)}{s^2}\right],$$

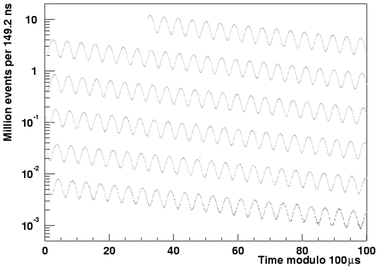
$$R_\gamma(s) = \sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})/\frac{4\pi\alpha^2}{3s}$$

Experimentally with or without additional hard photon (ISR: $e^+e^- \to \gamma^*(\to \text{hadrons})\gamma$)

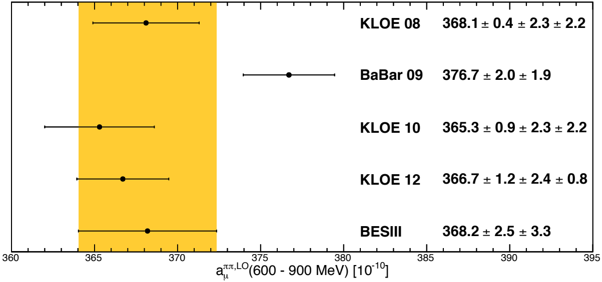
Experimental setup: muon storage ring with tuned momentum of muons to cancel leading coupling to electric field

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta} \times \vec{E}}{c}\right]$$

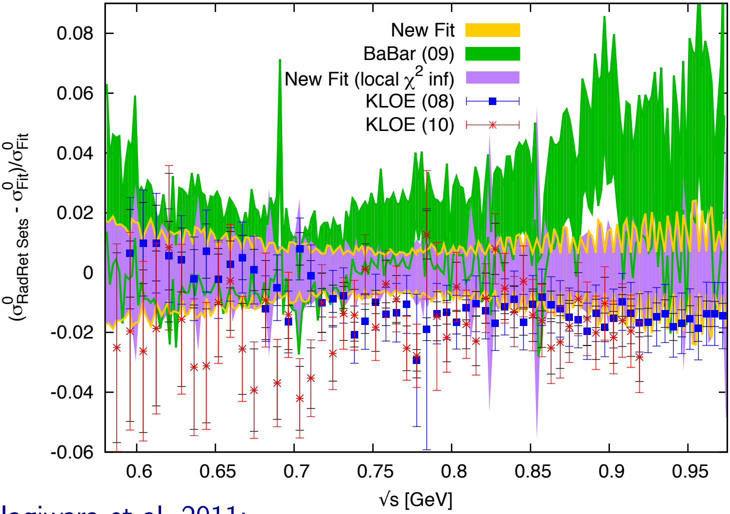
Because of parity violation in weak decay of muon, a correlation between muon spin and decay electron direction exists, which can be used to measure the anomalous precession frequency ω_a:



BESIII 2015 update:



BESIII 2015 update:



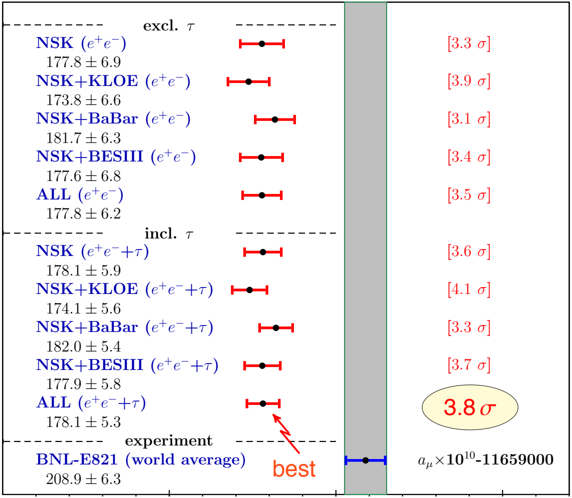
Hagiwara et al. 2011:

Jegerlehner FCCP2015 summary:

final state	range (GeV)	$a_\mu^{\mathrm{had}(1)} \times 10^{10}$ (stat) (syst) [tot]	rel	abs
ρ	(0.28, 1.05)	507.55 (0.39) (2.68)[2.71]	0.5%	39.9%
ω	(0.42, 0.81)	35.23 (0.42) (0.95)[1.04]	3.0%	5.9%
ϕ	(1.00, 1.04)	34.31 (0.48) (0.79)[0.92]	2.7%	4.7%
J/ψ		8.94 (0.42) (0.41)[0.59]	6.6%	1.9%
Υ		0.11 (0.00) (0.01)[0.01]	6.8%	0.0%
had	(1.05, 2.00)	60.45 (0.21) (2.80)[2.80]	4.6%	42.9%
had	(2.00, 3.10)	21.63 (0.12) (0.92)[0.93]	4.3%	4.7%
had	(3.10, 3.60)	3.77 (0.03) (0.10)[0.10]	2.8%	0.1%
had	(3.60, 9.46)	13.77 (0.04) (0.01)[0.04]	0.3%	0.0%
had	(9.46,13.00)	1.28 (0.01) (0.07)[0.07]	5.4%	0.0%
pQCD	(13.0,∞)	1.53 (0.00) (0.00)[0.00]	0.0%	0.0%
data	(0.28,13.00)	687.06 (0.89) (4.19)[4.28]	0.6%	0.0%
total		688.59 (0.89) (4.19)[4.28]	0.6%	100.0%

Results for $a_\mu^{\mathrm{had}(1)} \times 10^{10}$. Update August 2015, incl SCAN[NSK]+ISR[KLOE10,KLOE12,BaBar,BESIII]

Jegerlehner FCCP2015 summary ($\tau \leftrightarrow e^+e^-$):



Our setup:

$$C(t) = \frac{1}{3V} \sum_{j=0,1,2} \sum_{t'} \langle \mathcal{V}_j(t+t') \mathcal{V}_j(t') \rangle_{\mathrm{SU(3)}} \tag{3}$$

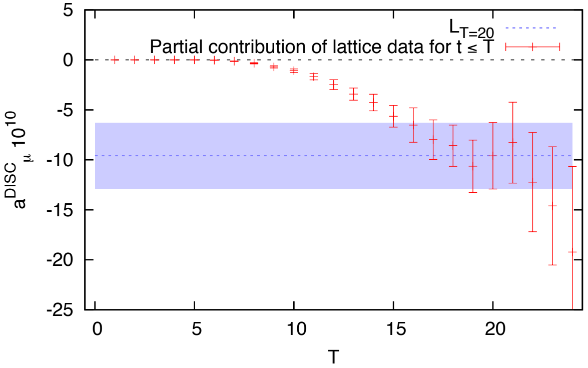
where V stands for the four-dimensional lattice volume, $\mathcal{V}_\mu = (1/3)(\mathcal{V}_\mu^{u/d} - \mathcal{V}_\mu^s)$, and

$$\mathcal{V}_\mu^f(t) = \sum_{\vec{x}} \mathrm{Im}\,\mathrm{Tr}[D^{-1}_{\vec{x},t;\vec{x},t}(m_f)\gamma_\mu]\,. \tag{4}$$

We separate 2000 low modes (up to around m_s) from light quark propagator as $D^{-1} = \sum_n v^n (w^n)^\dagger + D^{-1}_{\mathrm{high}}$ and estimate the high mode stochastically and the low modes as a full volume average Foley 2005.

We use a sparse grid for the high modes similar to Li 2010 which has support only for points x_μ with $(x_\mu - x_\mu^{(0)})$ mod $N = 0$; here we additionally use a random grid offset $x_\mu^{(0)}$ per sample allowing us to stochastically project to momenta.

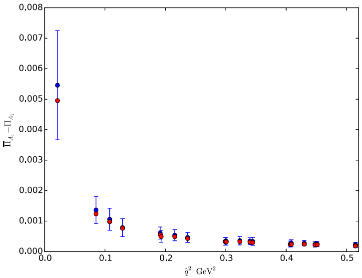
Study $L_T = \sum_{t=T+1}^{\infty} w_t C(t)$ and use value of T in plateau region (here $T = 20$) as central value. Use a combined estimate of a resonance model and the two-pion tail to estimate systematic uncertainty.

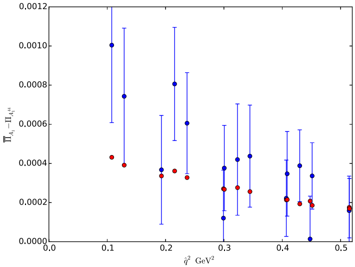


Combined with an estimate of discretization errors, we find

$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10}\,. \qquad (5)$$

From Aubin et al. 2015 (arXiv:1512.07555v2)



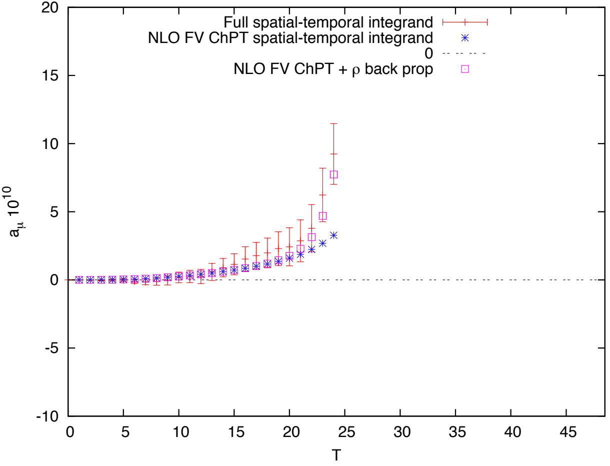


MILC lattice data with $m_\pi L = 4.2$, $m_\pi \approx 220$ MeV; Plot difference of $\Pi(q^2)$ from different irreps of 90-degree rotation symmetry of spatial components versus NLO FV ChPT prediction (red dots)

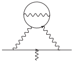
While the absolute value of a_μ is poorly described by the two-pion contribution, the volume dependence may be described sufficiently well to use ChPT to control FV errors at the 1% level; this needs further scrutiny

Aubin et al. find an $O(10\%)$ finite-volume error for $m_\pi L = 4.2$ based on the $A_1 - A_1^{44}$ difference (right-hand plot)

Compare difference of integrand of $48 \times 48 \times 96 \times 48$ (spatial) and $48 \times 48 \times 48 \times 96$ (temporal) geometries with NLO FV ChPT $(A_1 - A_1^{44})$:

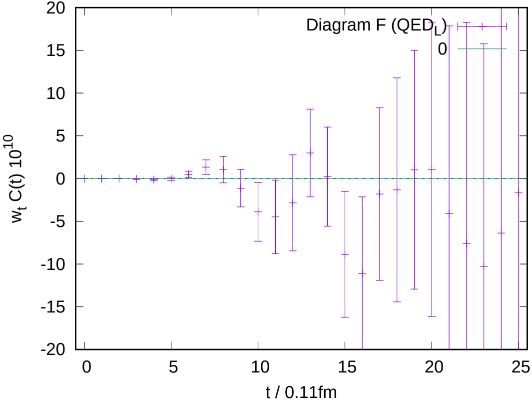


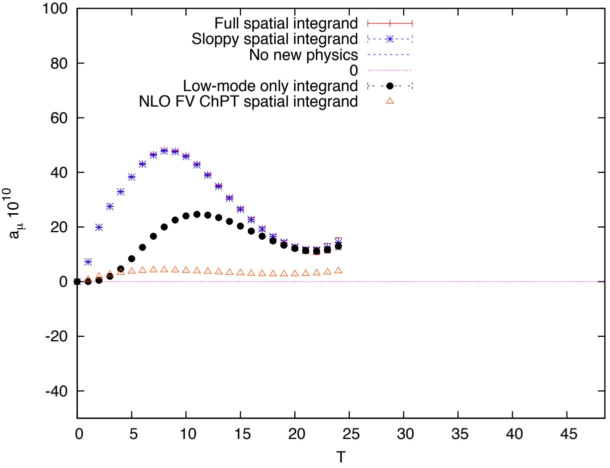
$m_\pi = 140$ MeV, $p^2 = m_\pi^2/(4\pi f_\pi)^2 \approx 0.7\%$



HVP QED+strong IB contributions

HVP QED diagram F





Full spatial integrand
Sloppy spatial integrand
No new physics
0
Low-mode only integrand
NLO FV ChPT spatial integrand
$a_\mu\ 10^{10}$
T
100
80
60
40
20
0
-20
-40
0
5
10
15
20
25
30
35
40
45