Entanglement Entropy and Decoupling in the Universe

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Density matrix and Entropy

In quantum mechanics, a physical state is described by a vector which belongs to the Hilbert space of the total system.

However, in many situations, it is more convenient to consider quantum mechanics of a subsystem.

In a finite temperature system which contacts with heat bath,

Ignore the part of heat bath and consider a physical state given by a statistical average (Mixed state).

Density matrix

A physical observable $\langle O \rangle = \mathrm{Tr}_{\mathcal{H}_{\mathrm{tot}}}[O \cdot \rho_{\mathrm{tot}}]$

Density matrix and Entropy

Density matrix of a pure state $\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$ ➡ $\langle O\rangle = \langle\Psi|O|\Psi\rangle$

Density matrix of canonical ensemble

$$\rho_{\text{tot}} = \frac{e^{-\beta H}}{Z} \qquad Z = \text{Tr}_{\mathcal{H}_{\text{tot}}} e^{-\beta H} = e^{-\beta F} \qquad \beta = 1/T$$

Free energy

Thermodynamic entropy

$$S_{\text{tot}} = -\frac{\partial F}{\partial T} = -\text{Tr}_{\mathcal{H}_{\text{tot}}}[\rho_{\text{tot}} \log \rho_{\text{tot}}]$$

von Neumann entropy

Entanglement entropy

Decompose a quantum many-body system into its subsystems A and B.

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

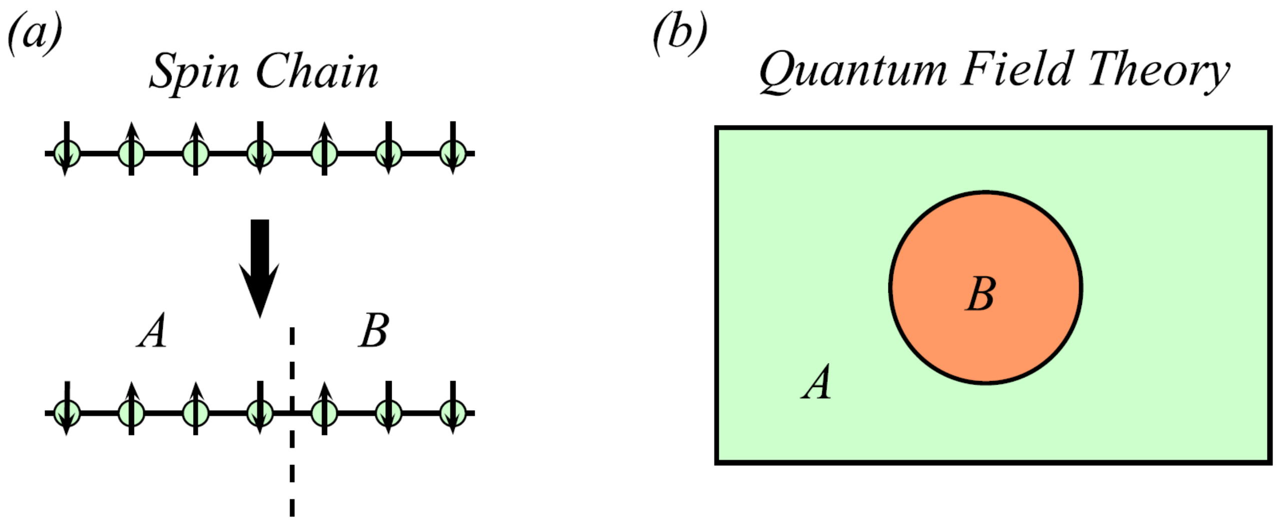
Trace out the density matrix over the Hilbert space of B.

The reduced density matrix

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}[\rho_{\mathrm{tot}}]$$

Entanglement entropy

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}[\rho_A \log \rho_A]$$



Nishioka, Ryu, Takayanagi (2009)

Entanglement entropy

Consider a system of two electron spins

$$|\Psi\rangle = \cos\theta|0\rangle_A|1\rangle_B + \sin\theta|1\rangle_A|0\rangle_B$$ ← Pure state

The reduced density matrix :

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}|\Psi\rangle\langle\Psi| = \cos^2\theta|0\rangle_A\langle 0|_A + \sin^2\theta|1\rangle_A\langle 1|_A$$ ← Mixed state

Entanglement entropy : $S_A = -\cos^2\theta \log\cos^2\theta - \sin^2\theta\log\sin^2\theta$

Maximum: $S_A = \log 2$

$$\cos^2\theta = \frac{1}{2}$$

EPR pair $|\Psi\rangle = \left(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B\right)/\sqrt{2}$

Quantum entanglement

Entanglement entropy

- ✓ **When the total system is a pure state,**

  The entanglement entropy represents the strength of quantum entanglement.

- ✓ **When the total system is a mixed state,**

  The entanglement entropy includes thermodynamic entropy.

Any application to particle phenomenology or cosmology ??

Decoupling in the Universe

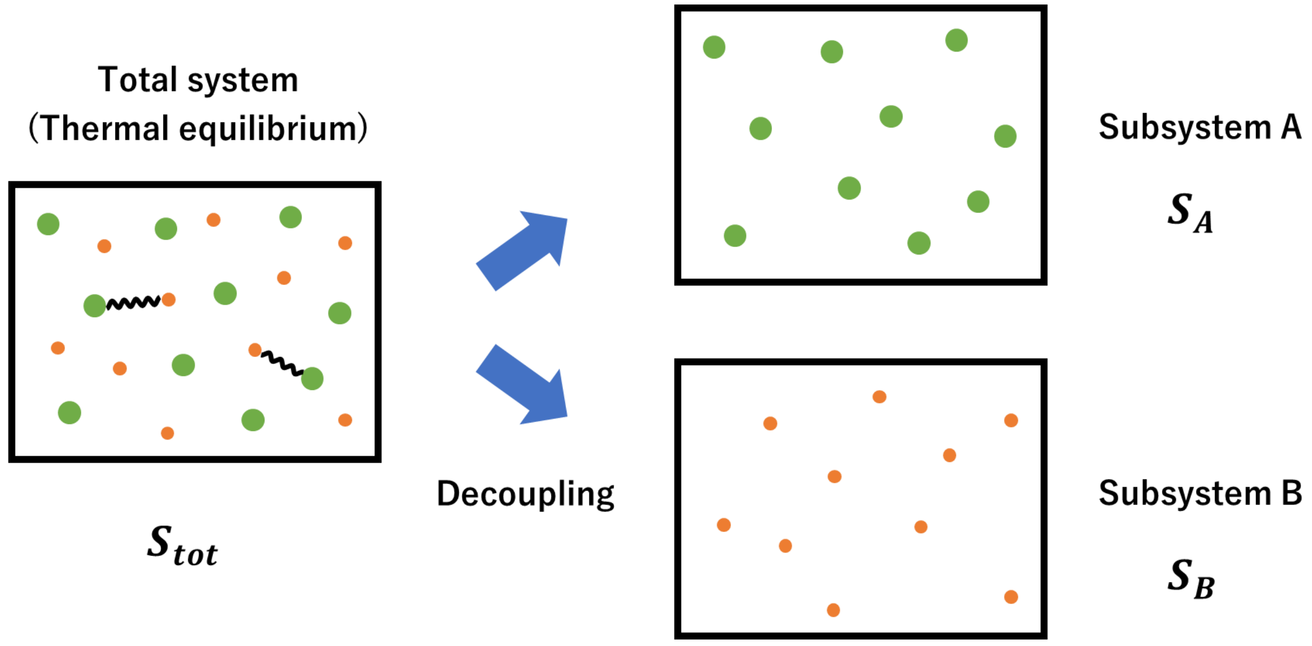
Particles A are no longer in thermal contact with particles B when the interaction rate is smaller than the Hubble expansion rate.

$$\Gamma_{AB} = n_B \langle \sigma_{AB} |v| \rangle < H \sim T^2 / M_{\rm Pl}$$

e.g. neutrino decoupling

Decoupled subsystems are treated separately.

Neutrino temperature drops independently from photon temperature.

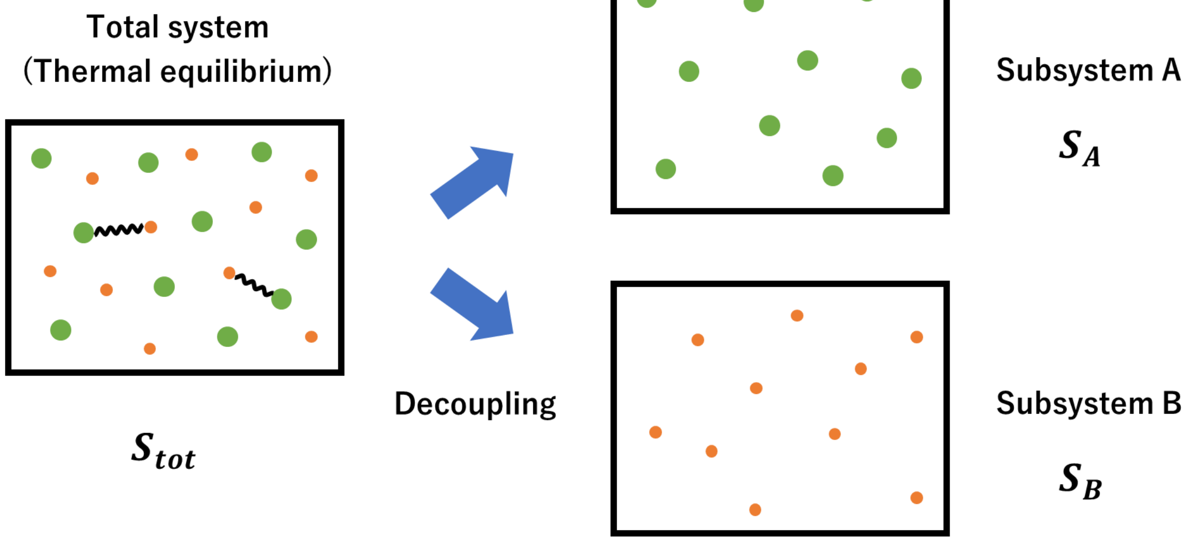


Thermodynamic entropy is expected to be conserved in each subsystem.

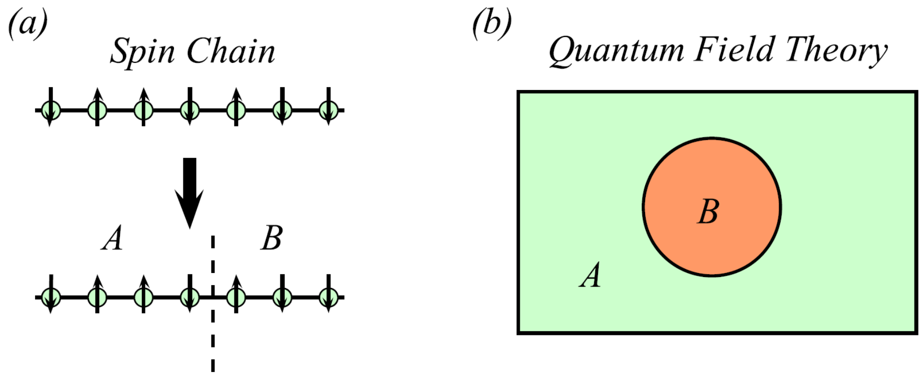
Decoupling in the Universe

However, this paradigm assumes that thermal equilibrium is maintained in the subsystems during and after the decoupling.

Can we refine the argument of decoupling?



Trace out the information of the other subsystem.



Thermodynamic entropy is generalized to entanglement entropy!

Entanglement entropy

Before decoupling, the total system is in thermal equilibrium and the density matrix is given by a grand canonical ensemble.

$$\rho_{\rm tot} = \frac{e^{-\beta(H_{\rm tot} - \mu_A \hat{N}_A - \mu_B \hat{N}_B)}}{Z_{\rm tot}}, \qquad Z_{\rm tot} = {\rm Tr}_{\mathcal{H}_{\rm tot}} e^{-\beta(H_{\rm tot} - \mu_A \hat{N}_A - \mu_B \hat{N}_B)}$$

After decoupling, in terms of the subsystem A, the system B can be seen as an environment and be traced out.

Entanglement entropy $S_A = -{\rm Tr}_{\mathcal{H}_A} [\rho_A \log \rho_A]$

A unitary evolution : $\rho_A(t_0) \to \rho_A(t) = e^{-iH_A(t-t_0)} \rho_A(t_0)\, e^{iH_A(t-t_0)}$

The entanglement entropy is conserved due to the nature of the trace!

Research background

- ✓ **Entanglement entropy has been extensively discussed for the case of a spatial submanifold.**
- ✓ **The case of field trace out has been paid attention to only in the context of condensed matter physics or CFT.**
- ✓ **A general formulation of perturbation theory is still lacking.**

Formulate perturbation theory and present Feynman rules!

Discuss its applications to cosmology!

Path integral formulation

Consider two interacting fields : $\phi_A(t,x)$ $\phi_B(t,x)$

The total system is in thermal equilibrium.

The usual finite-temperature field theory $\tau = it$

The partition function of the total system : $\beta = 1/T$

$$Z_{\mathrm{tot}}^{(\beta)} = \int \mathcal{D}\phi_A \mathcal{D}\phi_B \exp\left(-\int_0^\beta d\tau \int d^d x\, \mathcal{L}(\phi_A, \phi_B)\right)$$

The fields are (anti-)periodic in imaginary time $\tau \in (0,\beta)$

Path integral formulation

Trace out the density matrix over the Hilbert space of B.

The reduced density matrix :

$$\langle\phi_A|\,\rho_A\,|\phi'_A\rangle$$
$$= \frac{1}{Z_{\rm tot}^{(\beta)}}\int \mathcal{D}\phi_A\mathcal{D}\phi_B|_{\phi_A(0)=\phi'_A,\phi_A(\beta)=\phi_A}\exp\left(-\int_0^\beta d\tau\int d^dx\,\mathcal{L}(\phi_A,\phi_B)\right)$$

NOT the form of a grand canonical ensemble in general.

The entanglement entropy is defined as the von Neumann entropy.

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}\left[\rho_A\log\rho_A\right]$$

Path integral formulation

It is not easy to evaluate the trace of $\rho_A \log \rho_A$

We first calculate the Renyi entropy and take the limit $n \to 1$

$$S_A^{(n)} = \frac{1}{1-n} \log \left(\mathrm{Tr} \rho_A^n \right) \qquad \lim_{n \to 1} S_A^{(n)} = S_A$$

$$\mathrm{Tr} \rho_A^n = \frac{\widetilde{Z}_{\mathrm{tot}}^{(n\beta)}}{(Z_{\mathrm{tot}}^{(\beta)})^n} \equiv \frac{1}{(Z_{\mathrm{tot}}^{(\beta)})^n} \int \mathcal{D}\phi_A \mathcal{D}\phi_B \, \exp \left(- \sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta - \epsilon} d\tau \int d^d x \, \mathcal{L}^{(j)}(\phi_A, \phi_B^{(j)}) \right)$$

$\phi_A \qquad \tau \in (0, n\beta)$

$\mathcal{L}^{(j)}(\phi_A, \phi_B^{(j)}) = \mathcal{L}(\phi_A, \phi_B^{(j)})$

$\phi_B^{(j)} \; (j = 1, \cdots, n) \qquad \tau \in ((j-1)\beta, \, j\beta - \epsilon)$

Path integral formulation

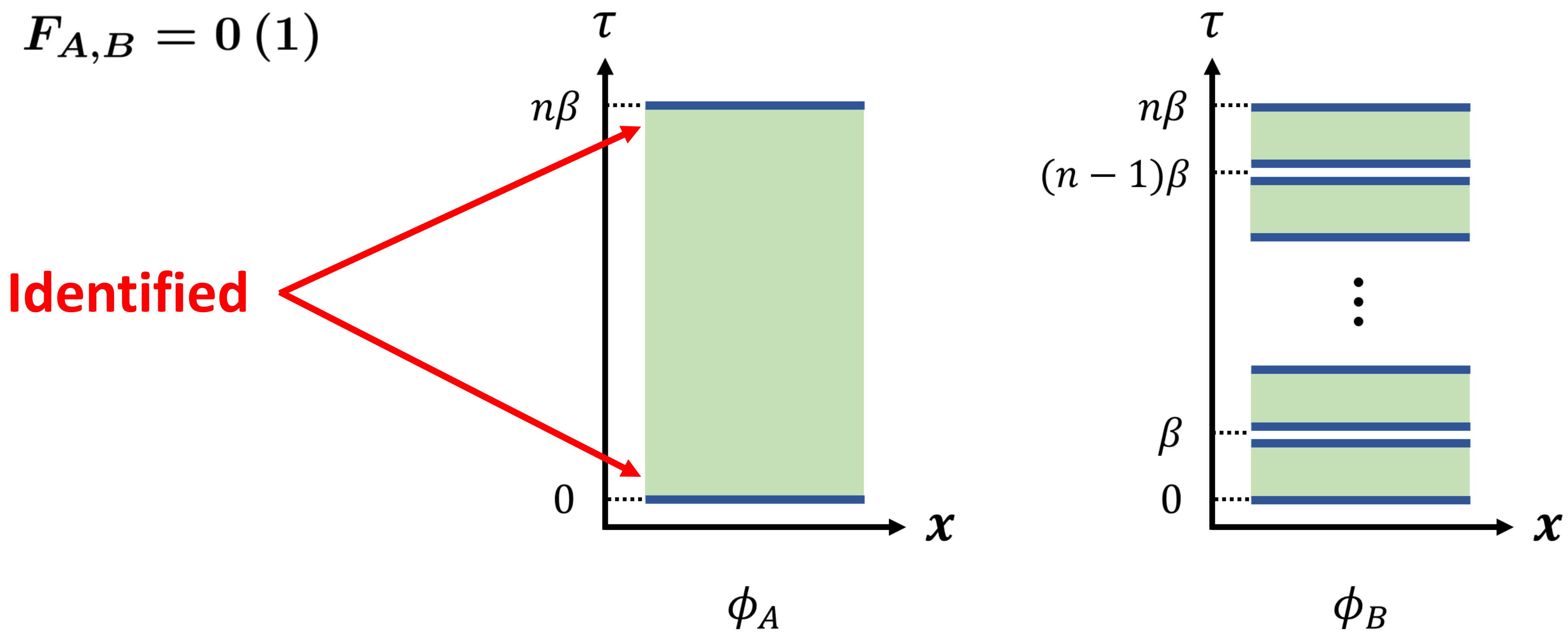
$$\phi_A(0) = (-1)^{F_A}\phi_A(n\beta),$$

ϕ_A has periodicity of $n\beta$.

$$\phi_B^{(j)}((j-1)\beta) = (-1)^{F_B}\phi_B^{(j)}(j\beta - \epsilon)$$

$\phi_B^{(j)}$ has periodicity of β.

$$F_{A,B} = 0\,(1)$$



Path integral formulation

Non-interacting part **Quantum correction**

$$S_A^{(n)} = S_{A,0}^{(n)} + \frac{1}{1-n}\left(\log \frac{\widetilde{Z}_{\text{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)}\,(Z_{B,0}^{(\beta)})^n} - n\log \frac{Z_{\text{tot}}^{(\beta)}}{Z_{A,0}^{(\beta)}\,Z_{B,0}^{(\beta)}}\right)$$

$$Z_{\alpha,0}^{(\beta)} = \int \mathcal{D}\phi_\alpha \,\exp\left(-\int_0^\beta d\tau \int d^dx\, \mathcal{L}_{\alpha,0}(\phi_\alpha)\right)$$

$(\alpha = A, B)$

The usual perturbation theory

$$S_{A,0}^{(n)} = \frac{1}{1-n}\log \frac{Z_{A,0}^{(n\beta)}}{(Z_{A,0}^{(\beta)})^n}$$

$n \to 1$

The thermodynamic entropy of a free field

Perturbative expansion

$$\widetilde{\mathcal{S}}^{(n\beta)} = \widetilde{\mathcal{S}}_0^{(n\beta)} + \widetilde{\mathcal{S}}_I^{(n\beta)}$$

$$\log \frac{\widetilde{Z}_{\text{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)} (Z_{B,0}^{(\beta)})^n}$$

$$= \int_0^{n\beta} d\tau \int d^d x \, \mathcal{L}_0(\phi_A) + \sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau \int d^d x \, \mathcal{L}_0^{(j)}(\phi_B^{(j)})$$

$$+ \sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau \int d^d x \, \mathcal{L}_I^{(j)}(\phi_A, \phi_B^{(j)})$$

We can consider this action as the theory of (n+1) scalar fields.

Propagator :

$$\widetilde{\omega}_m = 2\pi m T/n \; (m \in \mathrm{Z})$$

$$D_A^{(n\beta)}(\tau, x) = \frac{1}{n\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i(\widetilde{\omega}_m \tau + p \cdot x)}}{\widetilde{\omega}_m^2 + p^2 + M_A^2} \qquad (0 \leq \tau < n\beta),$$

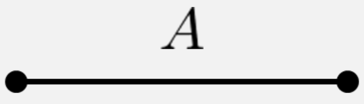
$$D_{B,j}^{(\beta)}(\tau, x) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i(\omega_m \tau + p \cdot x)}}{\omega_m^2 + p^2 + M_B^2} \qquad ((j-1)\beta \leq \tau < j\beta),$$

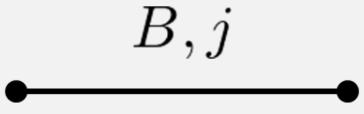
$$\omega_m = 2\pi m T$$

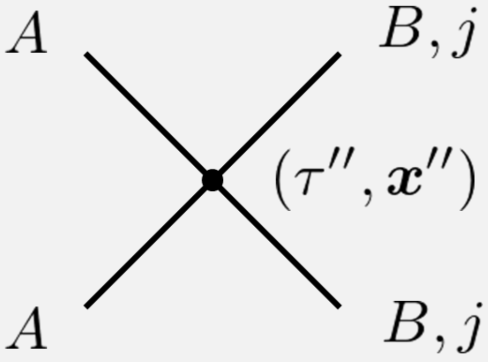
Feynman rules (Position space)

$$\log \frac{\widetilde{Z}^{(n\beta)}_{\mathrm{tot}}}{Z^{(n\beta)}_{A,0}\,(Z^{(\beta)}_{B,0})^n}$$

$$\mathcal{L}_I = \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda}{4}\phi_A^2\phi_B^2$$

1. $(\tau, \boldsymbol{x})$  $(\tau', \boldsymbol{x}')$ $= D_A^{(n\beta)}(\tau-\tau', \boldsymbol{x}-\boldsymbol{x}')$

2. $(\tau, \boldsymbol{x})$  $(\tau', \boldsymbol{x}')$ $= D_{B,j}^{(\beta)}(\tau-\tau', \boldsymbol{x}-\boldsymbol{x}')$

3.  $= -\lambda \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau'' \int d^d x''$

4. Divide by the symmetry factor.

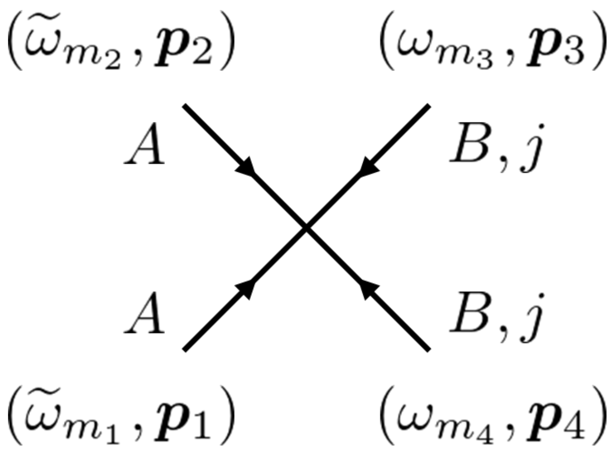
A scalar-scalar system in d+1 dimensions.

There exists a line for each $\phi_B^{(j)}$.

Lines with different *j*s do not directly connect with each other.

Only connected diagrams contribute.

Feynman rules (Momentum space)



$$\longleftrightarrow \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau'' \int d^d x''\, e^{i(\widetilde{\omega}_{m_1}\tau''+\boldsymbol{p}_1\cdot\boldsymbol{x}'')} e^{i(\widetilde{\omega}_{m_2}\tau''+\boldsymbol{p}_2\cdot\boldsymbol{x}'')} \times e^{i(\omega_{m_3}\tau''+\boldsymbol{p}_3\cdot\boldsymbol{x}'')} e^{i(\omega_{m_4}\tau''+\boldsymbol{p}_4\cdot\boldsymbol{x}'')}$$

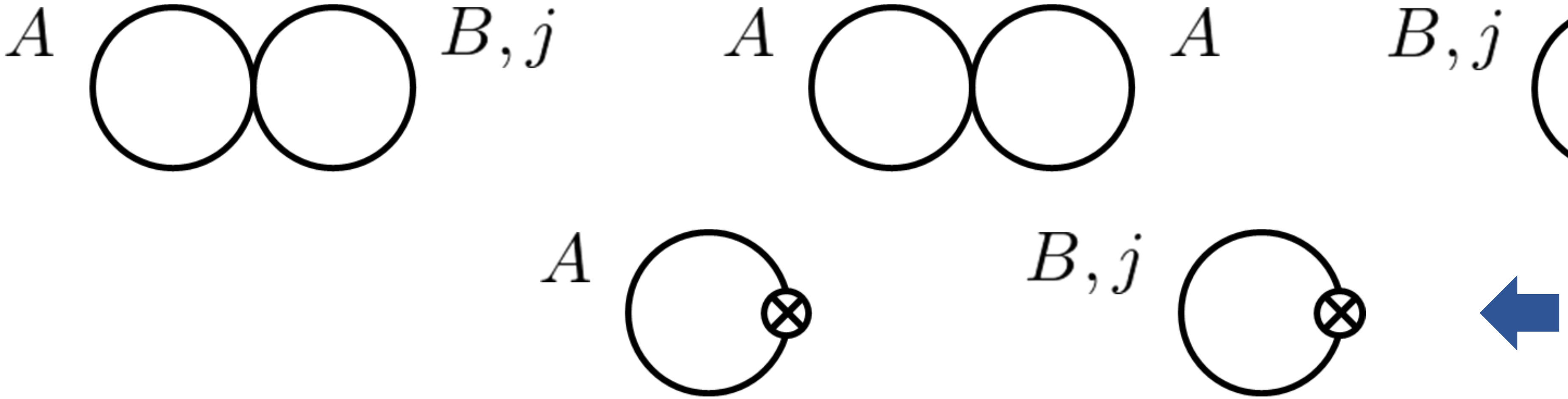
$$= (2\pi)^d \delta^{(d)}(\boldsymbol{p}_{\text{in}} - \boldsymbol{p}_{\text{out}})\, \beta\, e^{i(\omega_{\text{in}}-\omega_{\text{out}})(j-1)\beta} f_{\omega_{\text{in}},\omega_{\text{out}}}\,.$$

$$f_{\omega_{\text{in}},\omega_{\text{out}}} \equiv \frac{1}{\beta}\int_0^\beta d\tau''\, e^{i(\omega_{\text{in}}-\omega_{\text{out}})\tau''} = \begin{cases} 1 & \text{for } \omega_{\text{in}} = \omega_{\text{out}} \\ \dfrac{e^{i(\omega_{\text{in}}-\omega_{\text{out}})\beta}-1}{i(\omega_{\text{in}}-\omega_{\text{out}})\beta} & \text{for } \omega_{\text{in}} \neq \omega_{\text{out}} \end{cases}$$

$$\widetilde{\omega}_m = 2\pi m T/n \;\; (m \in \mathbf{Z})$$

➡ **Energy is not necessarily conserved at a vertex.**

The leading order diagrams



Counter terms

High temperature limit

$$S_A = VT^3 \left[\frac{2\pi^2}{45} - \frac{1}{12}\left(\frac{\lambda_A}{4!}\right) - \frac{1}{12}\left(\frac{\lambda}{4!}\right) + \cdots \right]$$

The correction terms are important when the couplings are sufficiently strong.

Dark radiation

Dark radiation is now explored by precise measurements.

Planck : $N_{\rm eff} = 3.15 \pm 0.23$ **CMB-S4 :** $\Delta N_{\rm eff} = 0.0156$

If the subsystem A enters thermal equilibrium again, …

The temperature of the subsystem A : $T_A = \left(\frac{45}{2\pi^2 g_A}\frac{S_A}{V}\right)^{1/3} \neq T_{A,0}$

A correction to the effective neutrino number

$$\Delta N_{\rm eff} \equiv \rho_A \left(2 \cdot \frac{7}{8} \cdot \frac{\pi^2}{30} \cdot T_\nu^4\right)^{-1} = \frac{8}{7}\frac{g_A}{2}\left(\frac{g_*(T_D)}{43/4}\right)^{-4/3}\left(\frac{T_A}{T_{A,0}}\right)^4$$

Summary

- ✓ **Thermodynamic entropy is generalized to entanglement entropy!**
- ✓ **Formulation of perturbation theory to derive the entanglement entropy of coupled quantum fields and Feynman rules.**
- ✓ **A possible effect on dark radiation.**
  (Its measurement is now becoming more and more precise!)
- ✓ **The correction may be important in circumstances of instantaneous decoupling.**

Thank you.

Extra slides

Perturbative expansion

Consider a scalar-scalar system in d+1 dimensions.

$$\mathcal{L}_I = \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda}{4}\phi_A^2\phi_B^2$$

Preserve two independent parities : $\phi_A \to -\phi_A \qquad \phi_B \to -\phi_B$

Formulate perturbation theory and present Feynman rules !

$$S_A^{(n)} = S_{A,0}^{(n)} + \frac{1}{1-n}\left(\underbrace{\log \frac{\widetilde{Z}_{\text{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)}\,(Z_{B,0}^{(\beta)})^n}}_{\text{(ii)}} - n\,\underbrace{\log \frac{Z_{\text{tot}}^{(\beta)}}{Z_{A,0}^{(\beta)}\,Z_{B,0}^{(\beta)}}}_{\text{(i)}}\right)$$

Zeroth-order contributions

Neutral scalar field

$$\mathcal{L}_0(\phi_i) = \frac{1}{2}\left[\left(\frac{\partial \phi_i}{\partial \tau}\right)^2 + (\nabla\phi_i)^2 + M_i^2\phi_i^2\right]$$

The result of the functional integration : See e.g. Kapusta, Gale (2011).

$$\log Z^{(\beta)}_{\phi_i,0} = V\int \frac{d^3p}{(2\pi)^3}\left\{-\frac{1}{2}\beta\omega - \log\left(1-e^{-\beta\omega}\right)\right\} \qquad \omega = \sqrt{p^2+M_i^2}$$

$$\Rightarrow \quad S_{A,0} = V\int \frac{d^3p}{(2\pi)^3}\left\{\frac{\beta\omega}{e^{\beta\omega}-1} - \log\left(1-e^{-\beta\omega}\right)\right\}$$

Perturbative expansion

The term (i) $\mathcal{S}^{(\beta)} = \mathcal{S}_0^{(\beta)} + \mathcal{S}_I^{(\beta)}$

Averaged over the unperturbed ensemble.

$$\log \frac{Z_{\rm tot}^{(\beta)}}{Z_{A,0}^{(\beta)}\, Z_{B,0}^{(\beta)}} = \log\left(1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int \mathcal{D}\phi_A \mathcal{D}\phi_B\, e^{-\mathcal{S}_0^{(\beta)}} (\mathcal{S}_I^{(\beta)})^l}{\int \mathcal{D}\phi_A \mathcal{D}\phi_B\, e^{-\mathcal{S}_0^{(\beta)}}}\right)$$

The usual finite temperature perturbation theory.

Propagator : $D_\alpha^{(\beta)}(\tau, x) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d}\, e^{i(\omega_m \tau + p\cdot x)}\, \widetilde{D}_\alpha^{(\beta)}(\omega_m, p)$

$(\alpha = A, B)$

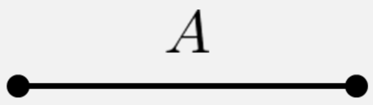
$$= \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i(\omega_m \tau + p\cdot x)}}{\omega_m^2 + p^2 + M_\alpha^2}$$

$\omega_m = 2\pi m T \quad (m \in \mathrm{Z})$

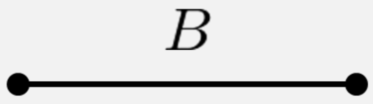
Feynman rules (Position space)

The term (i)

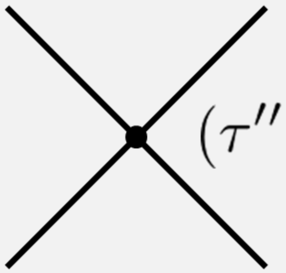
Kapusta, Gale (2011).

1. $(\tau, \boldsymbol{x})$ 

$(\tau', \boldsymbol{x}')$ $= D_A^{(\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

2. $(\tau, \boldsymbol{x})$ 

$(\tau', \boldsymbol{x}')$ $= D_B^{(\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

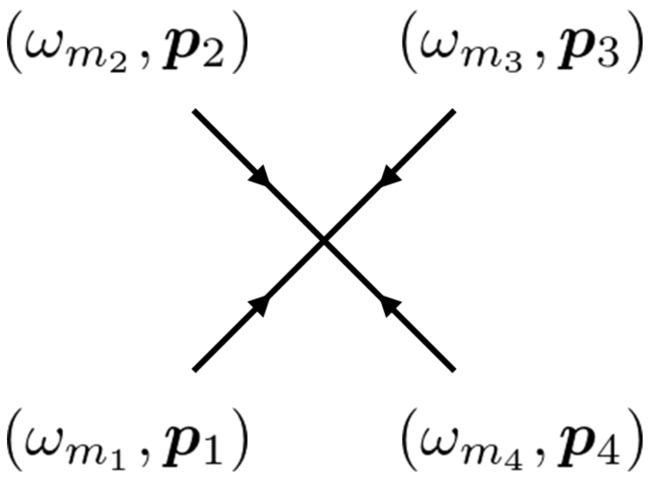
3. 
$(\tau'', \boldsymbol{x}'')$ $= -\lambda \int_0^\beta d\tau'' \int d^d x''$

4. Divide by the symmetry factor.

Draw all topologically inequivalent diagrams at a given order.

Only connected diagrams contribute.

Feynman rules (Momentum space)



$$\longleftrightarrow \quad \int_0^\beta d\tau'' \int d^d x''\, e^{i(\omega_{m_1}\tau'' + \boldsymbol{p}_1\cdot\boldsymbol{x}'')} e^{i(\omega_{m_2}\tau'' + \boldsymbol{p}_2\cdot\boldsymbol{x}'')} \times e^{i(\omega_{m_3}\tau'' + \boldsymbol{p}_3\cdot\boldsymbol{x}'')} e^{i(\omega_{m_4}\tau'' + \boldsymbol{p}_4\cdot\boldsymbol{x}'')} = (2\pi)^d \delta^{(d)}(\boldsymbol{p}_{\mathrm{in}} - \boldsymbol{p}_{\mathrm{out}})\, \beta \delta_{\omega_{\mathrm{in}},\, \omega_{\mathrm{out}}}\,.$$

Energy and momentum are conserved at each vertex.

1. For each propagator of $\phi_{A,B}$, assign a factor $\frac{1}{\beta}\sum_m \int \frac{d^d p}{(2\pi)^d}\, \widetilde{D}^{(\beta)}_{A,B}(\omega_m, p)$.
2. Include a factor $-\lambda(2\pi)^d \delta^{(d)}(p_{\mathrm{in}} - p_{\mathrm{out}})\, \beta\delta_{\omega_{\mathrm{in}},\, \omega_{\mathrm{out}}}$ for each vertex.
3. Divide by the symmetry factor.

Feynman rules (Momentum space)

The term (ii)

1. For each propagator of ϕ_A, assign a factor $\frac{1}{n\beta}\sum_m \int \frac{d^dp}{(2\pi)^d}\, \widetilde{D}_A^{(n\beta)}(\widetilde{\omega}_m, p)$.

2. For each propagator of $\phi_B^{(j)}$, assign a factor $\frac{1}{\beta}\sum_m \int \frac{d^dp}{(2\pi)^d}\, \widetilde{D}_{B,j}^{(\beta)}(\omega_m, p)$.

3. Include a factor $-\lambda(2\pi)^d\delta^{(d)}(p_{\rm in}-p_{\rm out})\,\beta\, e^{i(\omega_{\rm in}-\omega_{\rm out})(j-1)\beta} f_{\omega_{\rm in},\,\omega_{\rm out}}$ for each vertex of $\frac{\lambda}{4}(\phi_A\phi_B^{(j)})^2$.

4. Include a factor $-\lambda_A(2\pi)^d\delta^{(d)}(p_{\rm in}-p_{\rm out})\, n\beta\delta_{\omega_{\rm in},\,\omega_{\rm out}}$ for each vertex of $\frac{\lambda_A}{4!}\phi_A^4$.

5. Include a factor $-\lambda_B(2\pi)^d\delta^{(d)}(p_{\rm in}-p_{\rm out})\,\beta\delta_{\omega_{\rm in},\,\omega_{\rm out}}$ for each vertex of $\frac{\lambda_B}{4!}\left(\phi_B^{(j)}\right)^4$.

6. Divide by the symmetry factor.

The coupled ϕ^4 theory

Calculate the leading order correction to the thermodynamic entropy.

$$\mathcal{L}(\phi_A, \phi_B) = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{\text{counter}} \,,$$

$$\mathcal{L}_0(\phi_A, \phi_B) = \frac{1}{2}\left[(\partial_\mu \phi_A)^2 + M_A^2 \phi_A^2\right] + \frac{1}{2}\left[(\partial_\mu \phi_B)^2 + M_B^2 \phi_B^2\right],$$

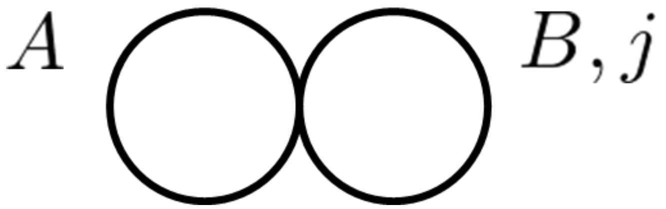
$$\mathcal{L}_I(\phi_A, \phi_B) = \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda}{4}\phi_A^2 \phi_B^2 \,, \qquad (\partial_\mu \phi)^2 \equiv (\partial_\tau \phi)^2 + (\nabla \phi)^2$$

$$\mathcal{L}_{\text{counter}}(\phi_A, \phi_B) = \frac{1}{2}\left[\delta_{Z_A}(\partial_\mu \phi_A)^2 + \delta_{M_A}\phi_A^2\right] + \frac{1}{2}\left[\delta_{Z_B}(\partial_\mu \phi_B)^2 + \delta_{M_B}\phi_B^2\right]$$

$$+ \frac{\delta_{\lambda_A}}{4!}\phi_A^4 + \frac{\delta_{\lambda_B}}{4!}\phi_B^4 + \frac{\delta_\lambda}{4}\phi_A^2 \phi_B^2$$

Divergence is renormalized by counter terms of the usual zero-temperature field theory.

The leading order contributions



The symmetry factor is F = 4.

$$\frac{1}{4}\sum_{j=1}^{n}\left(\frac{1}{n\beta}\sum_{m_A}\int\frac{d^3p_A}{(2\pi)^3}\,\widetilde{D}_A^{(n\beta)}(\widetilde{\omega}_{m_A},p_A)\right)$$
$$\times\left(\frac{1}{\beta}\sum_{m_B}\int\frac{d^3p_B}{(2\pi)^3}\,\widetilde{D}_{B,j}^{(\beta)}(\omega_{m_B},p_B)\right)\times(-\lambda)(2\pi)^3\delta^3(0)\,\beta$$

Accidentally, energy is conserved.

$$=-\frac{\lambda}{4}n\beta V\,D_A^{(n\beta)}(0,0)\,D_B^{(\beta)}(0,0)$$

⬅ Divergent

$$\widetilde{\omega}_{m_A}=2\pi m_A T/n\,,\quad \omega_{m_B}=2\pi m_B T$$

Renormalization conditions

Decompose the propagators into the $T=0$ part and $T\neq 0$ part.

$$\frac{1}{\beta}\sum_{m=-\infty}^{\infty}\mathcal{F}(p_0=i\omega_m=2\pi mTi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp_0\,\frac{1}{2}\left[\mathcal{F}(p_0)+\mathcal{F}(-p_0)\right]$$

$$+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}dp_0\,\left[\mathcal{F}(p_0)+\mathcal{F}(-p_0)\right]\frac{1}{e^{\beta p_0}-1}$$

$$\mathcal{F}(p_0)=\int\frac{d^3p}{(2\pi)^3}(-p_0^2+p^2+M_B^2)^{-1}$$

Perform a contour integral with a residue at $p_0=\omega\equiv\sqrt{p^2+M_B^2}$

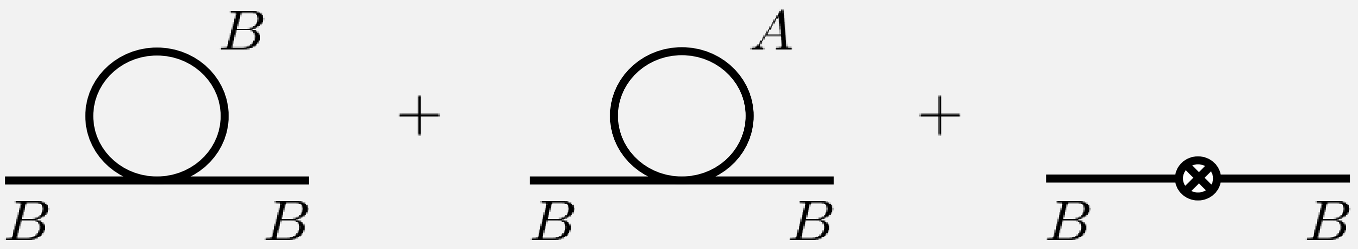
$$p_0\to -ip_4 \qquad D_B^{(\beta)}(0,0)=D_B^{\mathrm{vac}}+D_B^{\mathrm{mat}}(\beta)$$

$$\equiv\int\frac{d^4p}{(2\pi)^4}\frac{1}{p_4^2+p^2+M_B^2}+\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}$$

Renormalization conditions

Compute the sum of all 1PI insertions into the propagators.

$$-\Pi_B(p_E^2) =$$



$$= -\frac{\lambda_B}{2} D_B^{\rm vac} - \frac{\lambda}{2} D_A^{\rm vac} - \left(\delta_{Z_B} p_E^2 + \delta_{M_B}\right) .$$

Renormalization conditions

$$\Pi_B(p_E^2 = -M_B^2) = 0 \, , \qquad \frac{d}{dp_E^2} \Pi_B \Big|_{p_E^2 = -M_B^2} = 0$$

$$\Rightarrow \quad \delta_{Z_B} = 0 \, , \qquad \delta_{M_B} = -\frac{\lambda_B}{2} D_B^{\rm vac} - \frac{\lambda}{2} D_A^{\rm vac}$$

The leading order diagrams

Sum up the contributions from all the leading order diagrams.

$$\log \frac{\tilde{Z}_{\text{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)}(Z_{B,0}^{(\beta)})^n} = n\beta V \Big[-\frac{\lambda}{4}\left(D_A^{\text{vac}} + D_A^{\text{mat}}(n\beta)\right)\left(D_B^{\text{vac}} + D_B^{\text{mat}}(\beta)\right)$$
$$-\frac{\lambda_A}{8}\left(D_A^{\text{vac}} + D_A^{\text{mat}}(n\beta)\right)^2 - \frac{\lambda_B}{8}\left(D_B^{\text{vac}} + D_B^{\text{mat}}(\beta)\right)^2$$
$$+\frac{1}{4}\left(\lambda_A D_A^{\text{vac}} + \lambda D_B^{\text{vac}}\right)\left(D_A^{\text{vac}} + D_A^{\text{mat}}(n\beta)\right)$$
$$+\frac{1}{4}\left(\lambda_B D_B^{\text{vac}} + \lambda D_A^{\text{vac}}\right)\left(D_B^{\text{vac}} + D_B^{\text{mat}}(\beta)\right)\Big]$$

Renyi entropy :

$$S_A^{(n)} - S_{A,0}^{(n)} = \frac{n\beta V}{n-1}\Big[\frac{\lambda}{4}\left(D_A^{\text{mat}}(n\beta) - D_A^{\text{mat}}(\beta)\right) D_B^{\text{mat}}(\beta)$$
$$+\frac{\lambda_A}{8}\left(D_A^{\text{mat}}(n\beta)^2 - D_A^{\text{mat}}(\beta)^2\right)\Big]$$

The leading order diagrams

Take the limit $n \to 1$

Entanglement entropy :

$$S_A - S_{A,0} = -\beta^2 V \left[\frac{\lambda}{4} D_B^{\mathrm{mat}}(\beta) + \frac{\lambda_A}{4} D_A^{\mathrm{mat}}(\beta)\right]$$

$$\times \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{e^{\beta\tilde{\omega}} - 1} + \left(\frac{1}{e^{\beta\tilde{\omega}} - 1}\right)^2\right]$$

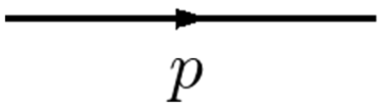
High temperature limit

$$\tilde{\omega} \equiv \sqrt{p^2 + M_A^2}$$

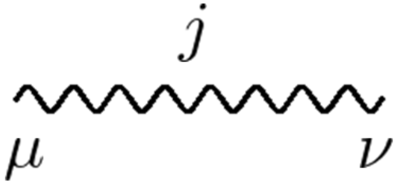
$$S_A = VT^3 \left[\frac{2\pi^2}{45} - \frac{1}{12}\left(\frac{\lambda_A}{4!}\right) - \frac{1}{12}\left(\frac{\lambda}{4!}\right) + \cdots\right]$$

The correction terms are important when the couplings are sufficiently strong.

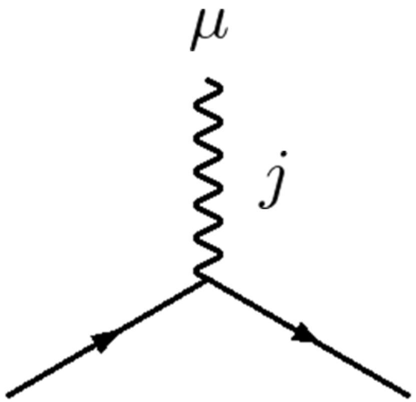
Quantum electrodynamics

1. 

$$= \frac{1}{n\beta}\sum_m \int \frac{d^4p}{(2\pi)^4}\, \widetilde{D}_\psi^{(n\beta)}(p)\,, \qquad \widetilde{D}_\psi^{(n\beta)}(p) = \frac{1}{\not{p} - M}$$

2. 

$$= \frac{1}{\beta}\sum_m \int \frac{d^3p}{(2\pi)^3}\, \widetilde{D}_{\gamma,\,j}^{(\beta)\mu\nu}(p)\,, \qquad \widetilde{D}_{\gamma,\,j}^{(\beta)\mu\nu}(p) = \frac{g^{\mu\nu}}{p^2}$$

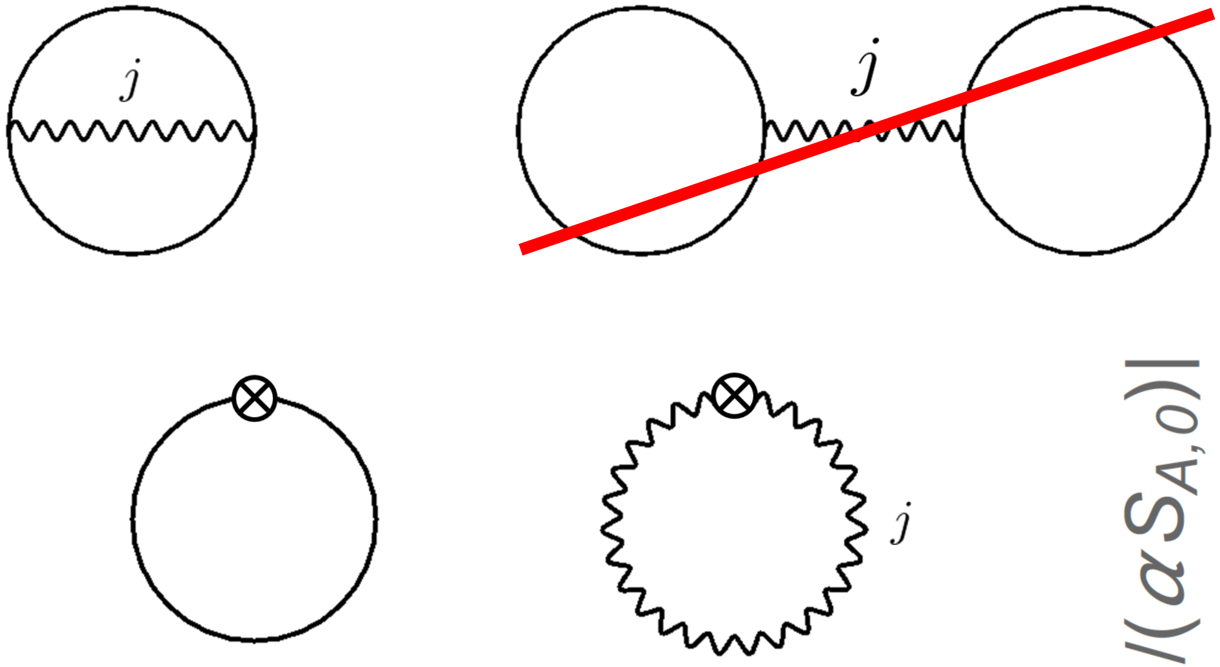
3. 

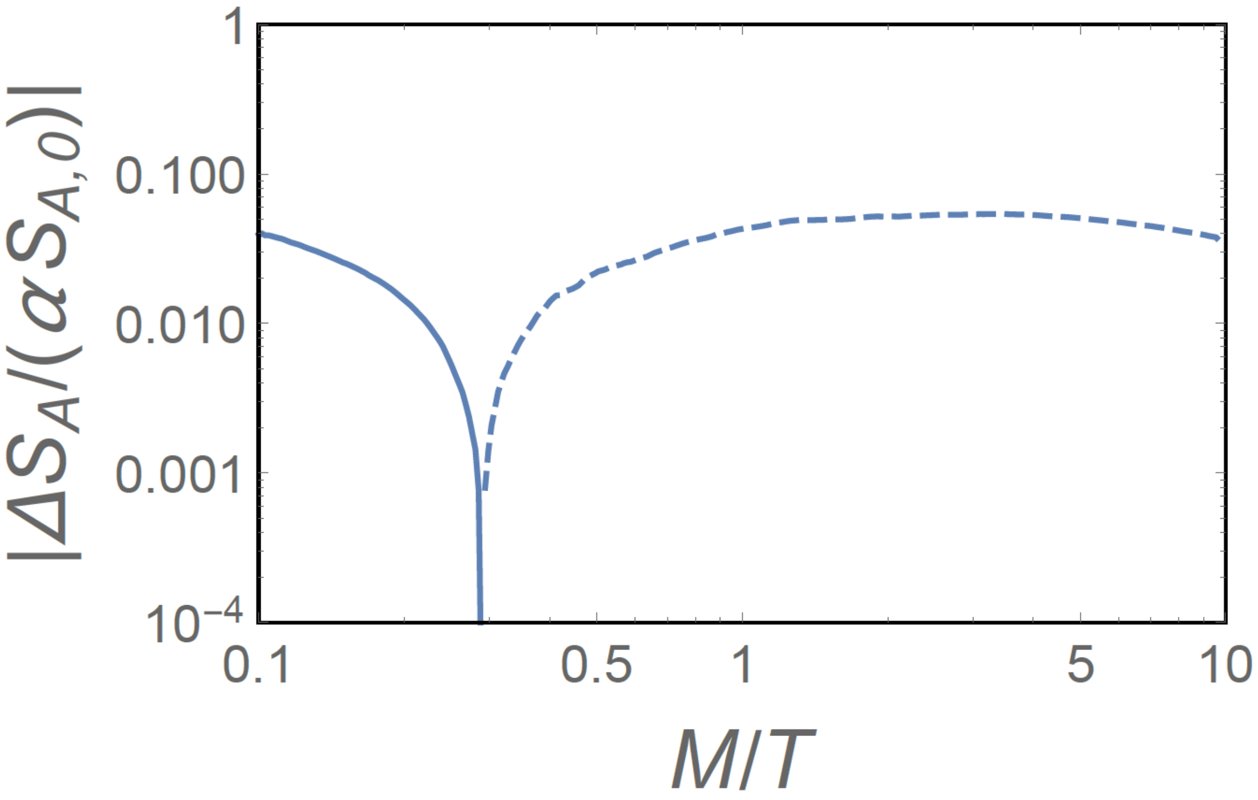
$$= -e\gamma^\mu (2\pi)^3 \delta^{(3)}(\boldsymbol{p}_{\text{in}} - \boldsymbol{p}_{\text{out}})\,\beta$$
$$\times\, e^{i(\omega_{\text{in}} - \omega_{\text{out}})(j-1)\beta} f_{\omega_{\text{in}},\omega_{\text{out}}}$$

4. Divide by the symmetry factor.

The photon field is traced out.

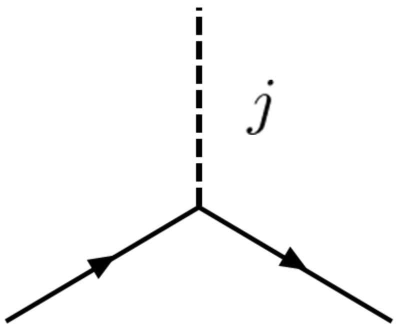
The leading order diagrams in QED





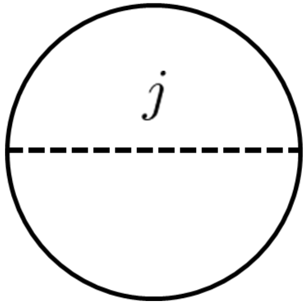
Energy is not conserved at a vertex.

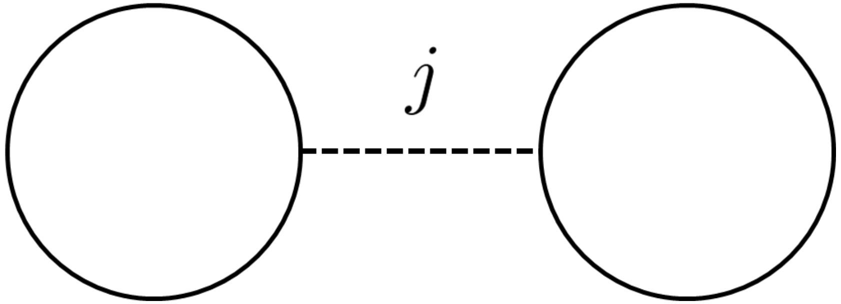
The Yukawa theory



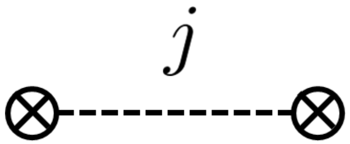
$$= -g(2\pi)^3\delta^{(3)}(\boldsymbol{p}_{\rm in} - \boldsymbol{p}_{\rm out})\,\beta$$
$$\times\, e^{i(\omega_{\rm in}-\omega_{\rm out})(j-1)\beta} f_{\omega_{\rm in},\omega_{\rm out}}$$

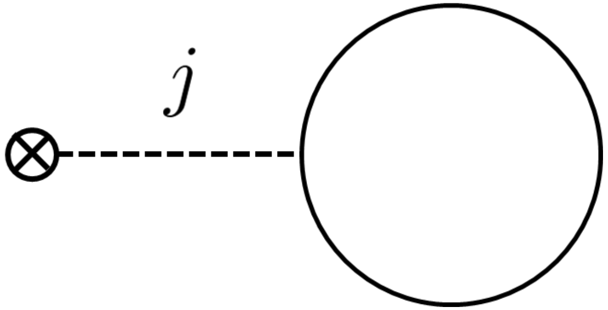
Nonzero





$$\boldsymbol{\mathcal{L}}_{\textbf{counter}}(\boldsymbol{\psi}, \boldsymbol{\phi}) \supset \boldsymbol{\delta_\phi \phi}$$





Instantaneous decoupling

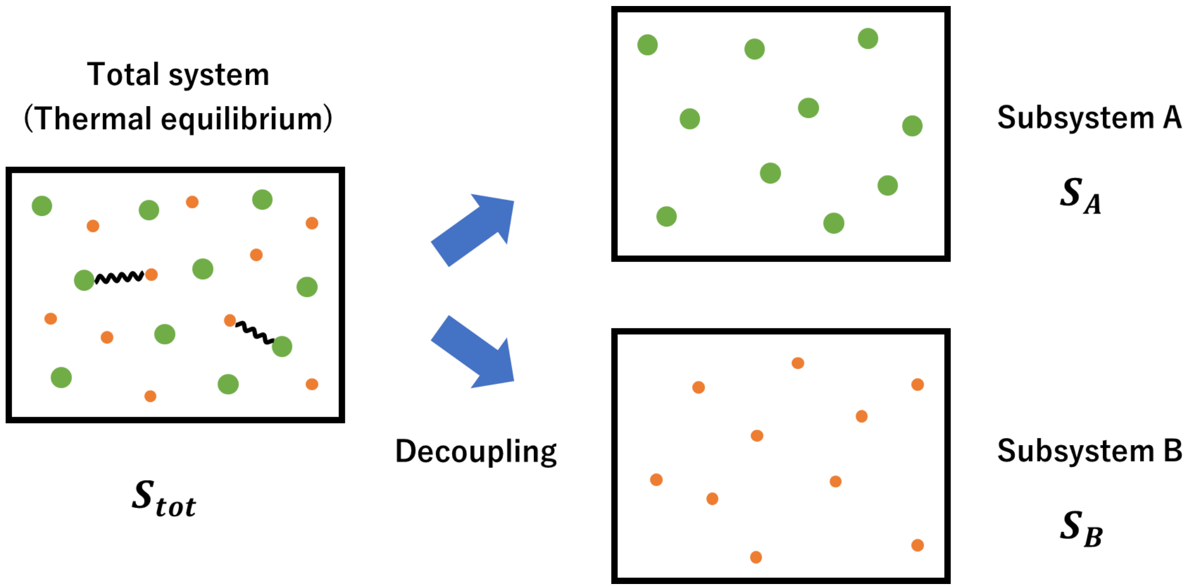
Decoupling may proceed much faster than the time scale of cosmic expansion and the time scale of interactions in a subsystem.

➡ **Thermal equilibrium cannot be maintained during the decoupling.**

Thermodynamic entropy is no longer a good fiducial quantity.

The form of the reduced density matrix is preserved before and after the decoupling.

Entanglement entropy is evaluated at the time just before the decoupling.



Instantaneous decoupling

The coupled ϕ^4 theory with $\lambda_A = \lambda_B = 0$

$$\phi_A \phi_B \leftrightarrow \phi_A \phi_B \qquad \Gamma_A = n_B \langle \sigma_{\phi_A \phi_B \to \phi_A \phi_B} v \rangle$$

Make ϕ_B massive dynamically and decay into some particles ψ_C.

$$\mathcal{L} = \frac{\lambda}{4}\phi_A^2 \phi_B^2 + \frac{\kappa}{2} X^2 \phi_B^2 + y \phi_B \bar{\psi}_C \psi_C + M \bar{\psi}_C \psi_C$$

$\langle X \rangle = v_X$ **at some temperature**

$\Gamma_{\phi_B \to \psi_C \bar{\psi}_C} \gg H$ ➡ **Instantaneous decoupling**