Computer Algebra Algorithms for Harmonic Sums and Polylogarithms and their Generalizations

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joint work with
J. Blümlein (DESY) and C. Schneider (RISC)

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"Background"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.

- These are related to the QCD anomalous dimensions and massive operator matrix elements.

- At 2-loop order all respective calculations are finished:

  M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;
  I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;
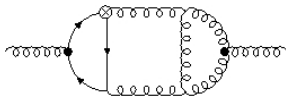 I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008) 1;

  and lead to representations in terms of harmonic sums.

Joint work with J. Blümlein (DESY), C. Schneider (RISC)
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram

(containing three massive fermion propagators)

⇓

Around 1000 sums have to be calculated

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

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$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\dfrac{\displaystyle\sum_{k=1}^{j}\dfrac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98;Blümlein/Kurth 98

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A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{(2N^2+6N+5)\,S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\cdots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$

$$+\ldots$$

where, e.g.,

145 S–sums occur

$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\displaystyle\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\displaystyle\sum_{k=1}^{j}\frac{\displaystyle\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

Generalized Harmonic Sums

For $a_i \in \mathbb{N}$ and $x_i \in \mathbb{R} \setminus \{0\}$ we define

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{a_1}} \cdots \frac{x_k^{i_k}}{i_k^{a_k}}.$$

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Example

$$S_{2,3,1}\left(2,\frac{1}{2},-1;n\right) = \sum_{k=1}^{n} \frac{2^k \sum_{j+1}^{k} \frac{\left(\frac{1}{2}\right)^j \sum_{i=1}^{j} \frac{(-1)^i}{i}}{j^3}}{k^2}$$

$$S_{2,3,1}(-1,1,-1;n) = \sum_{k=1}^{n} \frac{(-1)^k \sum_{j+1}^{k} \frac{\sum_{i=1}^{j} \frac{(-1)^i}{i}}{j^3}}{k^2} = S_{-2,3,-1}(n)$$

Algebraic Relations

$$S_{a_1,a_1,a_2}(x_1, x_2, x_3; n) =$$
$$S_{a_2}(x_3; n) S_{a_1,a_1}(x_1, x_2; n) + S_{a_1,a_1+a_2}(x_1, x_2x_3; n) - S_{a_1}(x_1; n)$$
$$S_{a_2,a_1}(x_3, x_2; n) - S_{a_2,2a_1}(x_3, x_1x_2; n) + S_{a_2,a_1,a_1}(x_3, x_2, x_1; n),$$

$$S_{a_1,a_1,a_2}(x_1, x_3, x_2; n) =$$
$$S_{a_2}(x_2; n) S_{a_1,a_1}(x_1, x_3; n) + S_{a_1,a_1+a_2}(x_1, x_2x_3; n) - S_{a_1}(x_1; n) S_{a_2,a_1}(x_2, x_3; n)$$
$$-S_{a_2,2a_1}(x_2, x_1x_3; n) + S_{a_2,a_1,a_1}(x_2, x_3, x_1; n),$$

$$S_{a_1,a_1,a_2}(x_2, x_1, x_3; n) =$$
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$$-S_{a_1+a_2,a_1}(x_1x_3, x_2; n) + S_{a_2,a_1,a_1}(x_3, x_1, x_2; n)$$

Differential Relations

- integral representation

$$S_{1,2,1}\left(2,\frac{1}{2},1;n\right) = \int_0^1 \frac{1}{x_1-1} \int_1^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-2} \int_2^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

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- differentiation

$$\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n\log(x_4)}{x_4-1}dx_4 dx_3 dx_2 dx_1$$

Differential Relations

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$$S_{1,2,1}\left(2,\frac{1}{2},1;n\right) = \int_0^1 \frac{1}{x_1-1} \int_1^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-2} \int_2^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

- differentiation

$$\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} = \int_0^1 \frac{1}{x_1-1} \int_1^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-2} \int_2^{x_3} \frac{x_4^n \log(x_4)}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

- integration of right hand side

$$\begin{aligned}
\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} = \ & -2H_{0,0,1,2,1}(1) - 4H_{0,0,2,1,1}(1) - H_{0,1,0,2,1}(1) - H_{0,1,2,0,1}(1) - 2H_{0,2,0,1,1}(1) \\
& -H_{0,2,1,0,1}(1) + (H_2(1)(H_{0,1,2}(1) + H_{0,2,1}(1)) + 2H_{0,0,1,2}(1) + 2H_{0,0,2,1}(1) \\
& +H_{0,1,0,2}(1))S_1(2;n) + S_2(\infty)\left(\frac{3(H_{0,1,2}(1)+H_{0,2,1}(1))}{2} + S_{1,2}\left(2,\frac{1}{2};n\right)\right) \\
& -H_2(1)S_{1,2,1}\left(2,\frac{1}{2},1;n\right) - S_{-1}(\infty)S_{1,2,1}\left(2,\frac{1}{2},1;n\right) - S_{1,2,2}\left(2,\frac{1}{2},1;n\right) \\
& -2S_{1,3,1}\left(2,\frac{1}{2},1;n\right) - S_{2,2,1}\left(2,\frac{1}{2},1;n\right)
\end{aligned}$$

Number of Basic S-Sums

We consider

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

with $x_i \in \{1, -1, 1/2, -1/2, 2, -2\}$.

Number of Basic S-Sums

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$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

with $x_i \in \{1,-1,1/2,-1/2,2,-2\}$.

Example (Harmonic Sums ($x_i \in \{1,-1\}$))

w	Number of							
	All	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Number of Basic Harmonic Sums

For $w \geq 2$ we have

$$
\begin{aligned}
All(w) &= 2 \cdot 3^{w-1} \\
N_A(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 3^d \\
N_D(w) &= 4 \cdot 3^{w-2} \\
N_H(w) &= 2 \cdot 3^{w-1} - 2^{w-1} \\
N_{AD}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 3^d - \frac{1}{w-1} \sum_{d|w-1} \mu\left(\frac{w-1}{d}\right) 3^d \\
N_{AH}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) \left[3^d - 2^d\right] \\
N_{DH}(w) &= 4 \cdot 3^{w-2} - 2^{w-2} \\
N_{ADH}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) \left[3^d - 2^d\right] - \frac{1}{w-1} \sum_{d|w-1} \mu\left(\frac{w-1}{d}\right) \left[3^d - 2^d\right]
\end{aligned}
$$

Number of Basic S-Sums

We consider

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

with $x_i \in \{1,-1,1/2,-1/2,2,-2\}$.

Number of Basic S-Sums

We consider

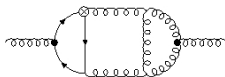$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

with $x_i \in \{1, -1, 1/2, -1/2, 2, -2\}$.

Example

Each of the indices $\{1/2, -1/2, 2, -2\}$ is allowed to appear just once in each sum.

Weight	Number of			
	All	N_A	N_D	N_{AD}
1	6	6	6	6
2	38	23	32	17
3	222	120	184	97
4	1206	654	984	543
5	6150	3536	4944	2882
6	29718	18280	23568	14744

⇓   C. Schneider's Sigma.m

Around 1000 sums are calculated containing in total 533 *S*–sums
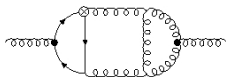
⇓ C. Schneider's Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

⇓ HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

Asymptotic Expansion of Harmonic Sums

We say that the function $f : \mathbb{R} \to \mathbb{R}$ is expanded in an asymptotic series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k}, \ x \to \infty,$$

where a_k are constants, if for all $K \geq 0$

$$R_K(x) = f(x) - \sum_{k=0}^{K} \frac{a_k}{x^k} = o\left(\frac{1}{x^K}\right), \ x \to \infty.$$

Why do we need these expansions of harmonic sums?

E.g.,

- for limits of the form

$$\lim_{n \to \infty} n \left(S_2(n) - \zeta_2 - S_{2,2}(n) + \frac{7 \zeta_2^2}{10} \right)$$

- for the approximation of the values of analytic continued harmonic sums at the complex plane

$$S_{2,-3}(-20 + 10i)$$

Basic Idea

$$S_{-1,3}(n) = (-1)^n \int_0^1 x^n \frac{H_{1,0,0}(x)}{1+x} dx + const$$

Basic Idea

$$S_{-1,3}(n) = (-1)^n \int_0^1 x^n \frac{\overbrace{H_{1,0,0}(x)}^{\varphi(x)}}{1 + x} \, dx + const$$

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

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$$S_{-1,3}(n) = (-1)^n \underbrace{\int_0^1 x^n \overbrace{\frac{H_{1,0,0}(x)}{1+x}}\, dx}_{\sum_{k=0}^{\infty} \frac{a_{k+1}k!}{n(n+1)\dots(n+k)}} + const$$

Basic Idea

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$$S_{-1,3}(n) = (-1)^n \underbrace{\int_0^1 x^n \overbrace{\frac{H_{1,0,0}(x)}{1+x}}\, dx}_{} + const$$

$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\ldots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

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$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\ldots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

$$b_1 = a_1$$

$$b_k = \sum_{l=0}^{k-2} (-1)^l S_{k-l}^l a_{k-l} (k-l)!$$

Basic Idea

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$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\dots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

$$b_1 = a_1$$

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$$S_{-1,3}(n) \sim (-1)^n \left(-\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6} \right) + \frac{3\ln 2\, z3}{4}$$

$$+ (-1)^n \left(\frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6} \right) z3 - \frac{19\, z2^2}{40}$$

Example:

In[1]:= **SExpansion[S[-1, 3, n], n, 10]**

Out[1]=
$$(-1)^n \left(-\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6} + \frac{31}{24n^7} + \frac{133}{96n^8} - \frac{169}{24n^9} - \frac{163}{16n^{10}} \right) + \frac{3 \ln2 \, z3}{4} + (-1)^n \left(\frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6} + \frac{17}{16n^8} - \frac{31}{4n^{10}} \right) z3 - \frac{19 \, z2^2}{40}$$
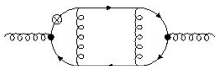
In[2]:= **GetApproximation[S[-1,3,n], {-2.5, 2}]**

Out[2]= $-0.795096 - 0.105476 \, i$

In[3]:= **HLimit[n*(S[2, n] - z2 - S[2, 2, n] + 7*z2^2/10), n]**

Out[3]= $-1 + z2$

We consider another diagram:

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Sigma ...

$$I_3 = \frac{C_3}{(N+1)(N+2)(N+3)} \left\{ \frac{1}{6} S_1^3 + \frac{N^2 + 12N + 16}{2(N+1)(N+2)} S_1^2 + \frac{4(2N+3)}{(N+1)^2(N+2)} S_1 \right.$$

$$+ \frac{8(2N+3)}{(N+1)^3(N+2)} + 2 \left[-2^{N+3} + 3 - (-1)^N \right] \zeta_3 - (-1)^N S_{-3} + \left[\frac{3N^2 + 40N + 56}{2(N+1)(N+2)} - \frac{1}{2} S_1 \right] S_2$$

$$- \frac{3N+17}{3} S_3 - 2(-1)^N S_{-2,1} - (N+3) S_{2,1} + 2^{N+4} S_{1,2} \left(\frac{1}{2}, 1; N \right)$$

$$\left. + 2^{N+3} S_{1,1,1} \left(\frac{1}{2}, 1, 1; N \right) \right\} + O(\epsilon)$$

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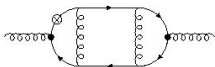
Sigma ...

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$$\left. + 2^{N+3} S_{1,1,1} \left(\frac{1}{2}, 1, 1; N \right) \right\} + O(\epsilon)$$

remaining sums:

$$S_1(n), S_2(n), S_3(n), S_{-3}(n), S_{2,1}(n), S_{-2,1}(n), S_{1,2}\left(\frac{1}{2}, 1; n\right), S_{1,1,1}\left(\frac{1}{2}, 1, 1; n\right)$$

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Asymptotic Expansion: Basic Idea

- again we start from the integral representation

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = \int_0^{\frac{1}{6}} \frac{1}{x_1-\frac{1}{6}} \int_{\frac{1}{6}}^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-\frac{1}{3}} \int_{\frac{1}{3}}^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

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- integration by parts leads to integrals of depth one:

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = -\left(H_{\frac{1}{3},0,\frac{1}{6}}(\frac{1}{6}) - H_{\frac{1}{3}}(\frac{1}{6})H_{0,\frac{1}{6}}(\frac{1}{6})\right)\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n - 1}{x - 1} dx + H_{0,\frac{1}{6}}(\frac{1}{6})\int_0^{\frac{1}{6}} \frac{x^n - 1}{x - 1} H_{\frac{1}{3}}(x) dx$$
$$- \int_0^{\frac{1}{6}} \frac{x^n - 1}{x - 1} H_{\frac{1}{3},0,\frac{1}{6}}(x) dx$$

Asymptotic Expansion: Basic Idea

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$$S_{1,2,1}\left(\frac{1}{3}, \frac{1}{2}, 1; n\right) = \int_0^{\frac{1}{6}} \frac{1}{x_1 - \frac{1}{6}} \int_{\frac{1}{6}}^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3 - \frac{1}{3}} \int_{\frac{1}{3}}^{x_3} \frac{x_4^n - 1}{x_4 - 1} dx_4 dx_3 dx_2 dx_1$$

- integration by parts leads to integrals of depth one:

$$S_{1,2,1}\left(\frac{1}{3}, \frac{1}{2}, 1; n\right) = -\left(H_{\frac{1}{3},0,\frac{1}{6}}(\frac{1}{6}) - H_{\frac{1}{3}}(\frac{1}{6})H_{0,\frac{1}{6}}(\frac{1}{6})\right) \int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n - 1}{x - 1} dx + H_{0,\frac{1}{6}}(\frac{1}{6}) \int_0^{\frac{1}{6}} \frac{x^n - 1}{x - 1} H_{\frac{1}{3}}(x) dx$$

$$- \int_0^{\frac{1}{6}} \frac{x^n - 1}{x - 1} H_{\frac{1}{3},0,\frac{1}{6}}(x) dx$$

- as before for harmonic sums we can expand these integrals and get

$$\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n - 1}{x - 1} dx \sim -H_1(\frac{1}{6}) + H_1(\frac{1}{3}) + 3^{-n}\left(-\frac{15}{n^5} + \frac{33}{8n^4} - \frac{3}{2n^3} + \frac{3}{4n^2} - 2^{-n}\left(-\frac{4074}{3125n^5} + \frac{366}{625n^4}\right.\right.$$

$$\left.\left. -\frac{42}{125n^3} + \frac{6}{25n^2} - \frac{1}{5n}\right) - \frac{1}{2n}\right)$$

$$\int_0^{\frac{1}{6}} \frac{x^n - 1}{x - 1} H_{\frac{1}{3}}(x) dx \sim H_{1,\frac{1}{3}}(\frac{1}{6}) + 6^{-n}\left(\log(2)\left(-\frac{4074}{3125n^5} + \frac{366}{625n^4} - \frac{42}{125n^3} + \frac{6}{25n^2} - \frac{1}{5n}\right) - \frac{10834}{625n^5}\right.$$

$$\left. +\frac{456}{125n^4} - \frac{22}{25n^3} + \frac{1}{5n^2}\right)$$

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) \sim$$

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;\infty\right) + 3^{-n}\left(2^{-n}\left(L(n)\left(\frac{42H_{0,2}(1)}{125n^3} - \frac{6H_{0,2}(1)}{25n^2} + \frac{H_{0,2}(1)}{5n} + \log(2)^2\left(\frac{21}{125n^3} - \frac{3}{25n^2} + \frac{1}{10n}\right)\right.\right.\right.$$

$$-\frac{21\zeta_2}{125n^3} + \frac{1}{5n^3} + \frac{3\zeta_2}{25n^2} - \frac{\zeta_2}{10n}\right) + \log(2)\left(-\frac{42H_{0,2}(1)}{125n^3} + \frac{6H_{0,2}(1)}{25n^2} - \frac{H_{0,2}(1)}{5n}\right) + \frac{42H_2(1)H_{0,2}(1)}{125n^3}$$

$$-\frac{2461H_{0,2}(1)}{1500n^3} + \frac{126H_{0,0,2}(1)}{125n^3} - \frac{42(H_{0,0,2}(1) + H_{2,2,2}(1))}{125n^3} - \frac{42(H_{0,0,2}(1) + H_{0,6,2}(1))}{125n^3} - \frac{42(H_{0,0,2}(1) + H_{2,0,2}(1))}{125n^3}$$

$$-\frac{42(H_{0,0,2}(1) + H_{6,0,2}(1))}{125n^3} - \frac{42H_{6,2,0}(1)}{125n^3} - \frac{6H_2(1)H_{0,2}(1)}{25n^2} + \frac{27H_{0,2}(1)}{50n^2} - \frac{18H_{0,0,2}(1)}{25n^2} + \frac{6(H_{0,0,2}(1) + H_{0,2,2}(1))}{25n^2}$$

$$+\frac{6(H_{0,0,2}(1) + H_{0,6,2}(1))}{25n^2} + \frac{6(H_{0,0,2}(1) + H_{2,0,2}(1))}{25n^2} + \frac{6(H_{0,0,2}(1) + H_{6,0,2}(1))}{25n^2} + \frac{6H_{6,2,0}(1)}{25n^2}$$

$$+\frac{H_2(1)H_{0,2}(1)}{5n} + \frac{3H_{0,0,2}(1)}{5n} - \frac{H_{0,0,2}(1) + H_{0,2,2}(1)}{5n} - \frac{H_{0,0,2}(1) + H_{0,6,2}(1)}{5n} - \frac{H_{0,0,2}(1) + H_{2,0,2}(1)}{5n}$$

$$-\frac{H_{0,0,2}(1) + H_{6,0,2}(1)}{5n} - \frac{H_{6,2,0}(1)}{5n} + \log(2)^3\left(-\frac{7}{125n^3} + \frac{1}{25n^2} - \frac{1}{30n}\right) + \log(2)^2\left(\frac{27}{100n^2} - \frac{2461}{3000n^3}\right)$$

$$+\frac{2461\zeta_2}{3000n^3} + \frac{63\zeta_3}{250n^3} - \frac{27\zeta_2}{100n^2} - \frac{9\zeta_3}{50n^2} + \frac{3\zeta_3}{20n}\right) - \frac{3H_2(1)H_{0,2}(1)}{2n^3} - \frac{9H_{0,0,2}(1)}{2n^3} + \frac{3(H_{0,0,2}(1) + H_{0,2,2}(1))}{2n^3}$$

$$+\frac{3(H_{0,0,2}(1) + H_{2,0,2}(1))}{2n^3} + \frac{3H_2(1)H_{0,2}(1)}{4n^2} + \frac{9H_{0,0,2}(1)}{4n^2} - \frac{3(H_{0,0,2}(1) + H_{0,2,2}(1))}{4n^2} - \frac{3(H_{0,0,2}(1) + H_{2,0,2}(1))}{4n^2}$$

$$-\frac{H_2(1)H_{0,2}(1)}{2n} - \frac{3H_{0,0,2}(1)}{2n} + \frac{H_{0,0,2}(1) + H_{0,2,2}(1)}{2n} + \frac{H_{0,0,2}(1) + H_{2,0,2}(1)}{2n}\right)$$

$$S_{\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}}(s_1,\ldots,s_l;n) =$$
$$\sum_{k_1=1}^{n} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\};\ldots;\{a_l,b_l,c_l\}}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}_+, b_i \in \mathbb{N}, \quad s_i = \pm 1, a_i > b_i$.

Cyclotomic Harmonic Sums

$$S_{\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}}(s_1,\ldots,s_l;n) =$$
$$\sum_{k_1=1}^{n} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\};\ldots;\{a_l,b_l,c_l\}}(s_2,\ldots,s_l;k_1),$$

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- algebraic relations

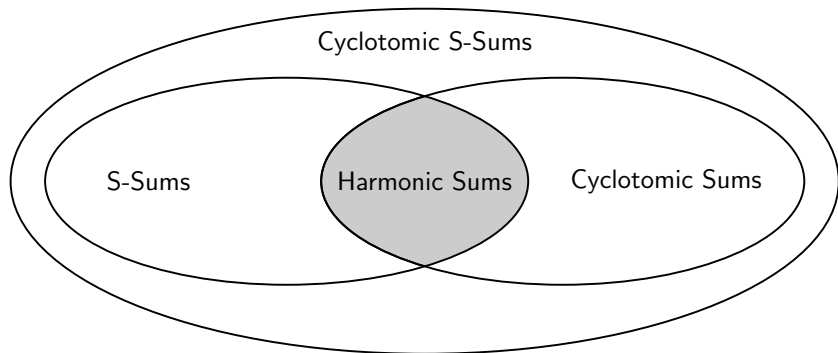# Cyclotomic Harmonic Sums

$$S_{\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}}(s_1,\ldots,s_l;n) =$$
$$\sum_{k_1=1}^{n} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\};\ldots;\{a_l,b_l,c_l\}}(s_2,\ldots,s_l;k_1),$$

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- algebraic relations
- differential relations

Cyclotomic Harmonic Sums

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- algebraic relations
- differential relations
- multiple argument relations

Cyclotomic Harmonic Sums

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- algebraic relations
- differential relations
- multiple argument relations
- asymptotic expansion

Cyclotomic Harmonic Sums

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with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}_+, b_i \in \mathbb{N}, \ \ s_i = \pm 1, a_i > b_i$.

- algebraic relations
- differential relations
- multiple argument relations
- asymptotic expansion
- allowing $s_i \in \mathbb{R}_+$ leads to Cyclotomic S-Sums.

The Package HarmonicSums

The package HarmonicSums offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations

The Package `HarmonicSums`

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- find algebraic and structural relations of harmonic sums and their generalizations
- compute the Mellin transform of harmonic sums and their generalizations, this leads to harmonic ploylogarithms and their generalizations

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- find algebraic and structural relations of harmonic polylogarithms and their generalizations
- apply algebraic and structural relations to harmonic sums
- calculate the asymptotic expansion of harmonic sums and their generalizations
- perform several other tasks not mentioned in this talk

Related work:

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