Recent Progress in Multi-Loop Calculations

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SFB TR9

LoopFest X

Outline

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1. Introduction
2. Heavy-Quark Correlators
3. Higgs Decays

Introduction

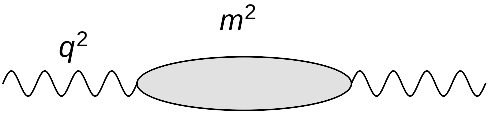
- much progress in multi-loop calculation
- but: still difficult to calculate processes with more than one mass-scale
  $\hookrightarrow$ only calculations with exceptional kinematics or expansion in small rations of parameters possible
- there are many single-scale problems where needed master integrals are not known

General procedure of multi-loop calculations

- reduce problem to scalar Feynman integrals, e.g. by use of suitable projectors
- expand in small ratios to reduce complexity
- reduce the problem to the calculation of a few basis integrals using integration-by-parts identities
- calculate the basis integrals $\hookrightarrow$ currently probably the biggest problem in multi-loop calculations

Important classes of integrals

There are two classes of integrals that are frequently encountered



- massive vacuum diagrams
  $\hookrightarrow$ result from expansion in q^2/m^2
- massless propagators
  $\hookrightarrow$ result from expansion in m^2/q^2
- everything that reduces to these classes of integrals is in principle doable at 4-loops.

Ways to expand

expansion by either

direct expansion on the level of diagrams

Ways to expand

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direct expansion on the level of diagrams

or

doing the calculation as far as possible keeping full kinematics

$\hookrightarrow$ calculate only expansion of master integrals

$\hookrightarrow$ more complicated but deeper expansion possible

Calculation of master integrals

Generate system of differential equations, e.g. by use of a master equation

$$\left(q^2\frac{\partial}{\partial q^2}+m^2\frac{\partial}{\partial m^2}-\frac{1}{2}D_m\right)M_i(q^2,m^2)=0$$

solve system of differential equations

- analytically
- numerically
- power series $\hookrightarrow$ leads to expansion corresponding to ansatz for power series

Expansion of master integrals

- start with system of differential equations obtained from master equation

$$\frac{\partial M_i}{\partial z} = \sum_j C_{ij} M_j$$

- insert ansatz for the integrals in form of a power series

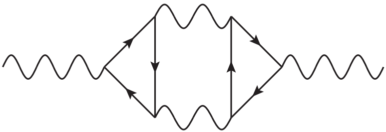
$$M_i = \sum_j D_{ij} z^j$$

- solve resulting system of linear equation for the D_{ij}

$\hookrightarrow$ very efficient way to obtain many terms in the expansion

Extension of the method

but, extension of method needed, consider e.g.



$\hookrightarrow$ not expandable in any limit in a simple power series
$\hookrightarrow$ extend ansatz to include non-integer powers

$$M_i = \sum_j D_{ij} z^j + D_{ij}^{(\epsilon)} z^{j-\epsilon} + \cdots$$

$\hookrightarrow$ terms with non-integer powers will generate the expected logarithms

Outline

Heavy-Quark Correlators: Motivation

- heavy-quark correlators related to $R(s)$ via optical theorem
- high-energy expansion $\hookrightarrow$ determination of α_s from measurement of $R(s)$ off threshold.
- low-energy expansion $\hookrightarrow$ determination of heavy quark masses from experimental data @ threshold using diagonal vector correlator.
- low-energy expansion $\hookrightarrow$ determination of heavy quark masses from lattice simulations $\hookrightarrow$ also non-diagnal and non-vector correlators interesting.

Heavy-Quark Correlators: Definition

diagonal vector correlator

$$\Pi_{\mu\nu}(q^2) = \int dx\, e^{iqx} \langle 0|T j_\mu(x) j_\nu(0)|0\rangle$$

non-diagonal vector correlator

$$\Pi_{\mu\nu}(q^2) = \int dx\, e^{iqx} \langle 0|T j_\mu(x) \tilde{j}_\nu(0)|0\rangle$$

$$j_\mu = \bar{\psi}\gamma_\mu\psi \qquad \tilde{j}_\mu = \bar{\psi}\gamma_\mu\chi$$

ψ: massive quark, χ: massless quark

$\hookrightarrow$ axial, scalar and pseudo-scalar correlators similarly defined

Heavy-Quark Correlators: Status

- at one and two loops known analytically [Källén, Sabry '55]
- low-energy expansion
  - diagonal correlators @ three loops : 30 terms are known [Chetyrkin et al; Boughezal et al; Maier et al]
  - diagonal correlators @ four loops : first three terms are known [Chetyrkin et al; Boughezal et al; Maier et al]
  - non-diagonal correlators @ three loops including light mass effects: 4 terms [Chetyrkin et al; Hoff et al]
- high-energy expansion
  - diagonal correlators @ three loops: leading terms [Chetyrkin et al; Harlander et al]
  - diagonal correlators @ four loops: leading terms [Baikov et al]
- vacuum polarization Π reconstructed at three and four loops using e.g. Padé approximations

[Chetyrkin et al; Hoang et al; Kiyo et al; Greynat et al]

Heavy-Quark Correlators: Calculation

Task: Calculate low- and high-energy expansion of all diagonal and non-diagonal correlators @ three loops

Heavy-Quark Correlators: Calculation

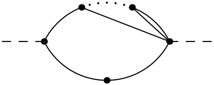
Task: Calculate low- and high-energy expansion of all diagonal and non-diagonal correlators @ three loops

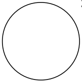
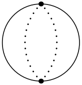
Calculation follows standard path:

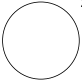
- Generate diagrams using `qgraf` [Nogueira]
- Calculation performed using `FORM` [Vermaseren]
- Reduce integrals to master integrals using integration-by-parts identities and the Laporta algorithm
- Expand master integrals in region of interest using Mathematica program

Checks: all known results reproduced

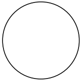
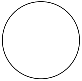
Master integrals: Example high-energy expansion

$M_{61}(q^2, m^2) =$  $= \sum C_k z^{-k} + C_k^{(\epsilon)} z^{-k-\epsilon} + C_k^{(2\epsilon)} z^{-k-2\epsilon} + C_k^{(3\epsilon)} z^{-k-3\epsilon}$,

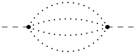
$\sum C_k z^{-k} = z^{-3}$  ^{3} $+ (3z^{-3} + z^{-2})$  $+ \ldots$,

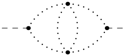
$\sum C_k^{(\epsilon)} z^{-k-\epsilon} = -6z^{-3}$   ^{2} $+ \ldots$,

$\sum C_k^{(2\epsilon)} z^{-k-2\epsilon} = (-8\epsilon^{-1} z^{-3} + 12\epsilon^{-1} z^{-2} - 2\epsilon^{-1} z^{-1} + 99z^{-3}$

$-48z^{-2} + 5z^{-1})$   $+ (4z^{-3} + 2z^{-2})$  ^{2}  $+ \ldots$,

$\sum C_k^{(3\epsilon)} z^{-k-3\epsilon} = (-36\epsilon^{-2} z^{-3} + 36\epsilon^{-2} z^{-2} - 6\epsilon^{-2} z^{-1} + 564\epsilon^{-1} z^{-3} - 294\epsilon^{-1} z^{-2}$

$+ 29\epsilon^{-1} z^{-1} - 2665z^{-3} + 508z^{-2} - 46z^{-1})$ 

$+ (-3z^{-3} - z^{-2} - z^{-1})$  $+ \ldots$.

Heavy-Quark Correlators: Results I

non-diagonal vector correlator, $\overline{\mathrm{MS}}$ scheme, low-energy

n	$\bar{C}_n^{(2),v}[1]$	$\bar{C}_n^{(2),v}[n_h]$	$\bar{C}_n^{(2),v}[n_l]$
1	−1.51195	0.0166564	0.337837
2	−0.840974	−0.0369934	0.187255
3	−0.665554	−0.0289884	0.135615
4	−0.475376	−0.0213752	0.104474
5	−0.309702	−0.0161172	0.0832408
6	−0.177592	−0.0125094	0.067963
7	−0.0753998	−0.00996452	0.0565728
8	0.00291844	−0.00811379	0.0478459
9	0.0628596	−0.00672995	0.0410083
10	0.108797	−0.00566985	0.0355496
11	0.144058	−0.00484063	0.031121
12	0.171144	−0.00418016	0.0274778
13	0.191926	−0.00364579	0.024444

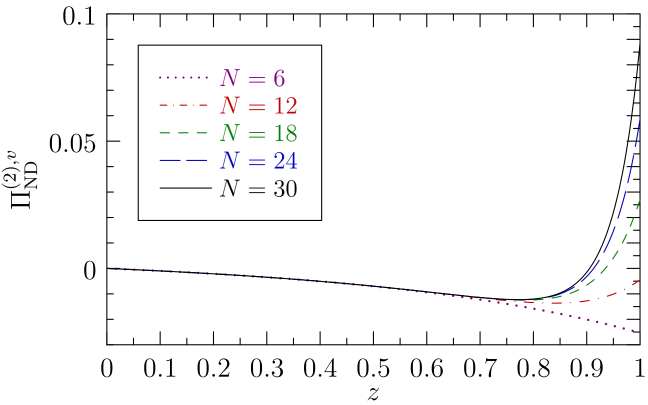
Heavy-Quark Correlators: Results II

non-diagonal vector correlator, $\overline{\mathrm{MS}}$ scheme, high-energy

n	$\bar{D}^{(2),v}_{n,0}$ [1]	$\bar{D}^{(2),v}_{n,1}$ [1]	$\bar{D}^{(2),v}_{n,2}$ [1]	$\bar{D}^{(2),v}_{n,3}$ [1]
1	−17.7072	25.1337	−16.3333	3.16667
2	−60.1359	47.0000	−8.00000	0
3	−41.3176	−15.4431	16.0772	−3.32442
4	4.27239	−20.8764	5.72599	0.659122
5	7.18683	−11.8251	0.822323	−0.129739
6	0.493132	−5.22086	2.92024	−0.566886
7	−16.8712	−10.5240	9.71602	−0.991008
8	−56.8660	−43.6415	29.4833	−1.70280
9	−164.387	−190.388	94.9042	−2.87364
10	−485.549	−831.419	339.294	−4.95108
11	−1537.17	−3780.76	1357.43	−8.63056
12	−5240.01	−18141.8	6001.46	−15.2756
13	−18953.8	−91894.0	28790.4	−27.3450

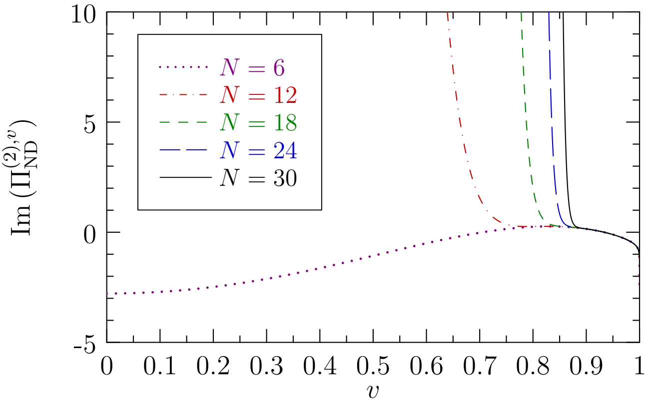
Heavy-Quark Correlators: Convergence I

non-diagonal vector correlator @ three loops
low-energy region: $z = q^2/m^2$



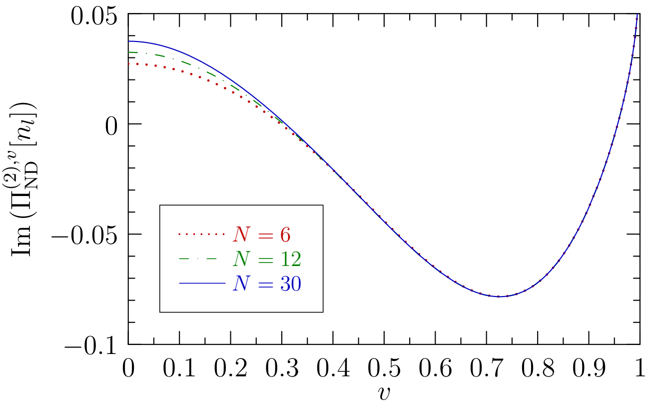
Heavy-Quark Correlators: Convergence II

non-diagonal vector correlator @ three loops, imaginary part
high-energy region: $v = \sqrt{1 - m^2/q^2}$



Heavy-Quark Correlators: Convergence III

non-diagonal vector correlator @ three loops,
imaginary part, n_l contribution
high-energy region: $v = \sqrt{1 - m^2/q^2}$



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3 Higgs Decays

Higgs decays

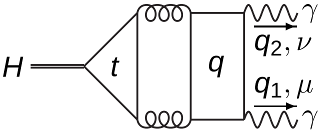
- $H \to \gamma\gamma$: very clean decay channel, possible detection channel for light Higgs @ LHC
- QCD corrections known analytically @ $\mathcal{O}(\alpha_s)$

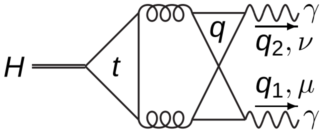
  [Fleischer et al '04, Harlander et al '05, Aglietti et al '06]

- three terms in the expansion in $m_H^2/(4m_t^2)$ known @ $\mathcal{O}(\alpha_s^2)$

  [Steinhauser '96]

  but: one class of contributions missing $\hookrightarrow$ singlet diagrams



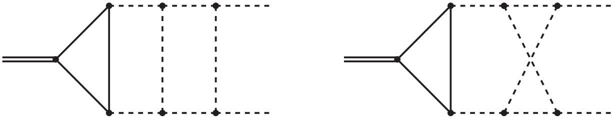


- possible large corrections due to large logarithms in light quark mass

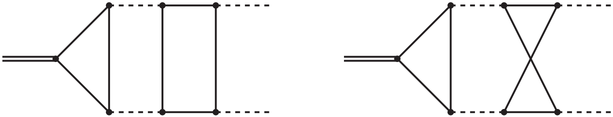
Higgs decays: Diagrams

Contributions from diagrams with

- massless quarks



- massive quarks



- fully analytic calculation not possible
  $\hookrightarrow$ calculate expansion in m_H/m_t

Higgs decays: Lorentz structure

- Structure of the amplitude given by

$$\mathcal{A}^{\mu\nu} = q_1^\nu q_2^\mu A + g^{\mu\nu} B + q_1^\mu q_2^\nu C + q_1^\mu q_1^\nu D + q_2^\mu q_2^\nu E$$

- Gauge invariance implies:
  - $B = -q_1 q_2 A$
  - $D = E = 0$

  $\hookrightarrow$ useful checks
- Contribution proportional to $q_1^\mu q_2^\nu$ vanishes after contraction with polarization vectors of the photon.

Higgs decays: Expansions

Calculation possible by either

Direct asymptotic expansion
$\hookrightarrow$ expansion depth limited

Higgs decays: Expansions

Calculation possible by either

Direct asymptotic expansion
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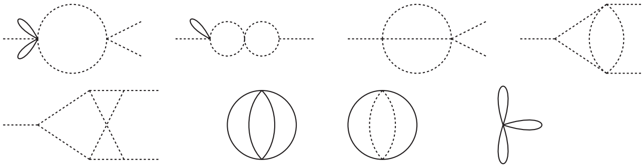
or

Calculation of expansion of master integrals
$\hookrightarrow$ more complicated but deeper expansion possible

Higgs decays: Technicalities

- Reduction using integration-by-parts identities leads to 3-loop master integrals of vertex type
  $\hookrightarrow$ (46|50|110|142) in the 4 topologies
- Master integrals have to be expanded in $m_h^2/(4m_t^2)$
  $\hookrightarrow$ boundary condition



- Extend calculation to non-singlet contribution as check and to improve known result.

Higgs decays: Results

- Are there any effects from light quark loops?

Higgs decays: Results

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$$
\begin{aligned}
A(H \to \gamma\gamma)|_{\text{light}-\text{singlet}} \propto & + \frac{L}{6} + \frac{2}{3}\left(\zeta(3) - \frac{13}{8}\right) \\
& + t\left(\frac{19\,L}{1620} + \frac{2}{3}\left(\frac{7\zeta\,(3)}{30} - \frac{3493}{32400}\right)\right) \\
& + t^2\left(\frac{2}{3}\left(\frac{2\zeta\,(3)}{21} - \frac{3953}{264600}\right) - \frac{L}{3780}\right) \\
& + t^3\left(\frac{2}{3}\left(\frac{26\zeta\,(3)}{525} - \frac{3668899}{1786050000}\right) - \frac{1696\,L}{1063125}\right) \\
& + t^4\left(\frac{2}{3}\left(\frac{512\zeta\,(3)}{17325} + \frac{207481}{736745625}\right) - \frac{136L}{91125}\right) \\
& + t^5\left(\frac{2}{3}\left(\frac{1216\zeta\,(3)}{63063} + \frac{611578464557}{939552540176250}\right) - \frac{7571576\,L}{6257426175}\right) + \cdots
\end{aligned}
$$

Higgs decays: Results

- Are there any effects from light quark loops? No!

$$
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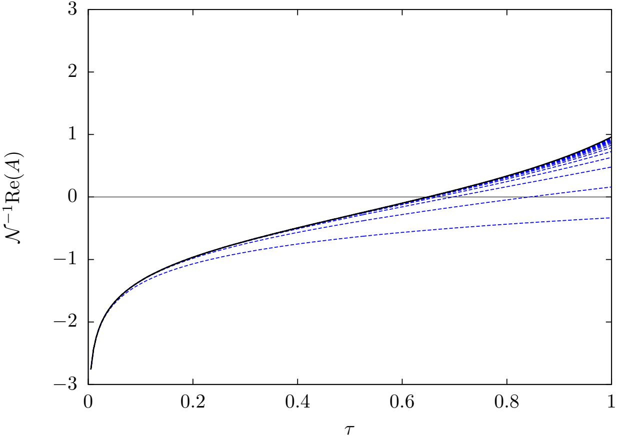
\end{aligned}
$$

- But: non-physical part C (vanishes after contraction with polarization vectors) is infrared divergent!
- Calculation reproduces known results for non-singlet contribution

Higgs decays: numerical results

three-loop singlet contribution

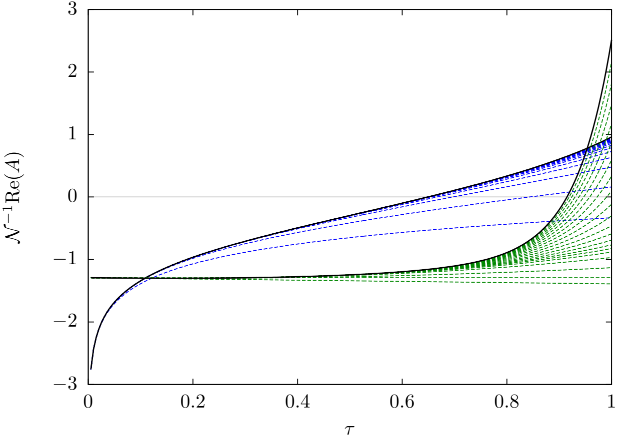
normalized to LO $\times \alpha_s^2$: $\tau = \frac{m_H^2}{4m_t^2}$



Higgs decays: numerical results

three-loop singlet + non-singlet contribution

normalized to LO $\times \alpha_s^2$: $\tau = \frac{m_H^2}{4m_t^2}$



Conclusions

- Heavy-quark correlators
  - Calculated 30 terms in the low- and high-energy expansion of all diagonal and non-diagonal correlators
  - Results needed for the determination of quark masses from lattice calculations
  - Possible improvement of the reconstruction of the full q^2 behaviour of the correlators at NNLO

Conclusions

- Heavy-quark correlators
  - Calculated 30 terms in the low- and high-energy expansion of all diagonal and non-diagonal correlators
  - Results needed for the determination of quark masses from lattice calculations
  - Possible improvement of the reconstruction of the full q^2 behaviour of the correlators at NNLO
- Higgs decays
  - Calculated 20 terms of the expansion in m_H^2/m_t^2 for the singlet contribution @ NNLO
  - Reproduced and improved the calculation of the non-singlet contribution @ NNLO
  - No logarithmic enhancement in singlet contribution
  - Singlet contribution of the same size as the non-singlet one