Nature vs. Nurture in Complex (and Not-so-Complex) Systems

Daniel Stein

Departments of Physics and Mathematics

New York University

Brookhaven National Laboratory

April 24, 2018

Collaborators: Reza Gheissari (NYU), Jon Machta (UMass, Amherst),

Chuck Newman (NYU), Lily Wang (NYUSH,Cornell), Jing Ye (NYU, Princeton),

Partially supported by US National Science Foundation Grant DMR1207678

A.J. Bray, ``Theory of Phase-Ordering Kinetics'', *Adv. Phys*. **51**, 481-587 (2002).

``Part of the fascination of the field, and the reason why it remains a challenge more than three decades after the first theoretical papers appeared, is that, in the thermodynamic limit, final equilibrium is never achieved! This is because the longest relaxation time diverges with the system size in the ordered phase, reflecting the broken ergodicity. Instead, a network of domains of the equilibrium phases develops, and the typical length scale associated with these domains increases with time t.''

True, but a little misleading (and not so useful).

Introduce a stronger concept: *local* equilibration or nonequilibration.

C.M. Newman and DLS, *J. Stat. Phys*. **94**, 709 (1999); S. Nanda, C.M. Newman and DLS, *Amer. Math. Soc. Transl.* **2**, 198 (2000);C.M. Newman and DLS, *J. Phys.: Cond. Mat*. **20**, 244132 (2008).

Dynamical evolution following a deep quench introduces a rich array of physical and mathematical questions and problems.

To name just a (very) few:

Dynamics of domain growth: curvature, timescales, annihilation, merging, …

Aging and memory effects

Persistence

Damage spreading

We've introduced some new questions that we're particularly interested in:

Connection between nonequilibrium dynamics and pure state multiplicity

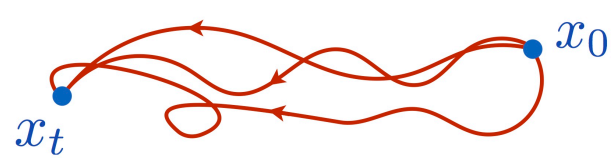
Does local equilibration occur for different systems?

Predictability: to what extent is the state at time t influenced by the initial conditions and to what extent by the system history? Nature vs. nurture.

Example of a Complex System

or

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



?

Dynamical Evolution of Ising Model Following a Deep Quench

Consider the stochastic process $\sigma^t = \sigma^t(\omega)$ with

$$\sigma^t \in \{-1,+1\}^{Z^d}$$

corresponding to the zero-temperature limit of Glauber dynamics for an Ising model with Hamiltonian

$$\mathcal{H} = -\sum_{\|x-y\|=1} J_{xy}\,\sigma_x \sigma_y$$

We are particularly interested in σ^0's chosen from the infinite-temperature Gibbs measure P_{σ^0}; i.e., each spin is ± 1 with equal probability, independently of the others.

Following the quench, the system evolves according to (continuous-time) zero-temperature dynamics. That is, the dynamics are described by independent, rate-1 Poisson processes at each *x* when a spin flip ($\sigma_x^{t+0} = -\sigma_x^{t-0}$) is *considered*. If the change in energy

$$\mathcal{H}_x(\sigma) = 2 \sum_{y:\|x-y\|=1} J_{xy}\sigma_x\sigma_y$$

is negative (or zero or positive) then the flip occurs with probability 1 (or ½ or 0). We denote by P_ω the probability distribution on the realizations ω of the dynamics and by $P_{\sigma^0,\omega} = P_{\sigma^0} \times P_\omega$ the joint distribution of the σ^0's and ω's.

The time evolution of such a model is known as *coarsening, phase separation,* or *spinodal decomposition.*

There are three sources of randomness in the problem:

- Realization of initial configuration: σ^0
- Realization of dynamics: ω

Two components to ω: order in which spins are selected, and (in case of ties), outcome of tie-breaking coin flips.

Two questions

1) For a.e. σ^0 and ω, does $\sigma^\infty(\sigma^0, \omega)$ exist? (Or equivalently, for every *x* does $\sigma^t_x(\sigma^0, \omega)$ flip only finitely many times?)

2) As *t* gets large, to what extent does $\sigma^t(\sigma^0, \omega)$ depend on σ^0 (``nature'') and to what extent on $\omega$ (``nurture'')?

Phrasing (2) more precisely depends on the answer to (1).

We will consider two kinds of models:

- the homogeneous ferromagnet where J_{xy}=+1 for all {x,y}.
- disordered models where a realization *J* of the J_{xy}'s is chosen from the independent product measure P_J of some probability measure on the real line.

Simplest case: $d = 1$

Theorem (Arratia `83, Cox-Griffeath `86): For the $d = 1$ homogeneous ferromagnet, $\sigma^{\infty}_{x}(\sigma^{0}, \omega)$ does not exist for a.e. σ_0 and ω and every x.

Sketch of Proof:

- Translation-invariance in problem, combined with spin-flip symmetry, rules out uniform final state.

- Therefore, if a final state exists, there must be an infinite number of domain walls (``kinks'') confined to fixed regions.

x ++++++++++++++++++ - - - - - - - - - - - - - - - - - x'

- But under the dynamical rules, the probability is zero that domain wall will be restricted to a finite region in an infinite time.

What about higher dimensions?

R. Arratia, *Ann. Prob.* **11**, 706-713 (1983); J.T. Cox and D. Griffeath, *Ann. Prob.* **14**, 347-370 (1986).

Theorem (NS99, NNS00): In the $d = 2$ homogeneous ferromagnet, for a.e. σ^0 and ω and for every x in $\mathbf{Z}^2$, $\sigma_x^t(\omega)$ flips infinitely often.

Higher dimensions: remains open. Older numerical work (Stauffer '94) suggests that every spin flips infinitely often for dimensions 3 and 4, but a positive fraction (possibly equal to 1) of spins flips only finitely often for $d \geq 5$.

We'll return to the question of predictability in homogeneous Ising ferromagnets, but first we'll look at the behavior of $\sigma^\infty(\sigma^0,\omega)$ for *disordered* Ising models.

C.M. Newman and DLS, *J. Stat. Phys*. **94**, 709 (1999); S. Nanda, C.M. Newman, and D.L. Stein, pp. 183—194, in *On Dobrushin's Way (from Probability Theory to Statistical Physics)*, eds. R. Minlos, S. Shlosman, and Y. Suhov, Amer. Math. Soc. Trans. (2) 198 (2000).

D. Stauffer, *J. Phys. A* **27**, 5029—5032 (1994).

In some respects, this case is *simpler* than the homogeneous one.

Recall that the J_{xy}'s are chosen from the independent product measure P_J of some probability measure on the real line. Let μ denote this measure.

Theorem (NNS '00): If μ has finite mean, then for a.e. J, σ^0 and ω, and for every x, there are only finitely many flips of σ_x^t that result in a *nonzero* energy change.

It follows that a spin lattice in any dimension with continuous coupling disorder having finite mean (e.g., Gaussian) has a limiting spin configuration at all sites.

But the result holds not only for systems with continuous coupling disorder. It holds also for discrete distributions and even homogeneous models where each site has an odd number of nearest neighbors (e.g., hexagonal lattice in 2D).

Will provide proof in one dimension.

Proof (*1D* only): Consider a chain of spins with couplings $J_{x,x+1}$ chosen from a continuous distribution (which in *1D* need not have finite mean). Consider the doubly infinite sequences x_n of sites where $|J_{x_n,x_n+1}|$ is a *strict* local maximum and y_n in the interval (x_n, x_{n+1}) where $|J_{y_n,y_n+1}|$ is a *strict* local minimum:

$$\left|J_{x_n,x_n+1}\right| > \left|J_{x_n-1,x_n}\right|, \left|J_{x_n+1,x_n+2}\right|$$

$$\left|J_{y_n,y_n+1}\right| < \left|J_{y_n-1,y_n}\right|, \left|J_{y_n+1,y_n+2}\right|$$

That is, the coupling magnitudes are strictly increasing from y_{n-1} to x_n and strictly decreasing from x_n to y_n.

Now notice that the coupling $|J_{x_n,x_n+1}|$ is a ``bully''; once it's satisfied (i.e., $J_{x_n,x_n+1}\sigma^0_{x_n}\sigma^0_{x_n+1} > 0$, the values of σ_{x_n} and σ_{x_n+1} can never change thereafter, regardless of what's happening next to them. For all other spins in $\{y_{n-1}+1, y_{n-1}+2, \ldots, y_n\}$, σ_y^∞ exists and its value is determined so that $J_{x,y}\sigma_x^\infty\sigma_y^\infty > 0$ for x and y=x+1 in that interval.

In other words, there is a *cascade of influence* to either side of $\{x_n, x_n+1\}$ until $y_{n-1}+1$ and y_n, respectively, are reached.

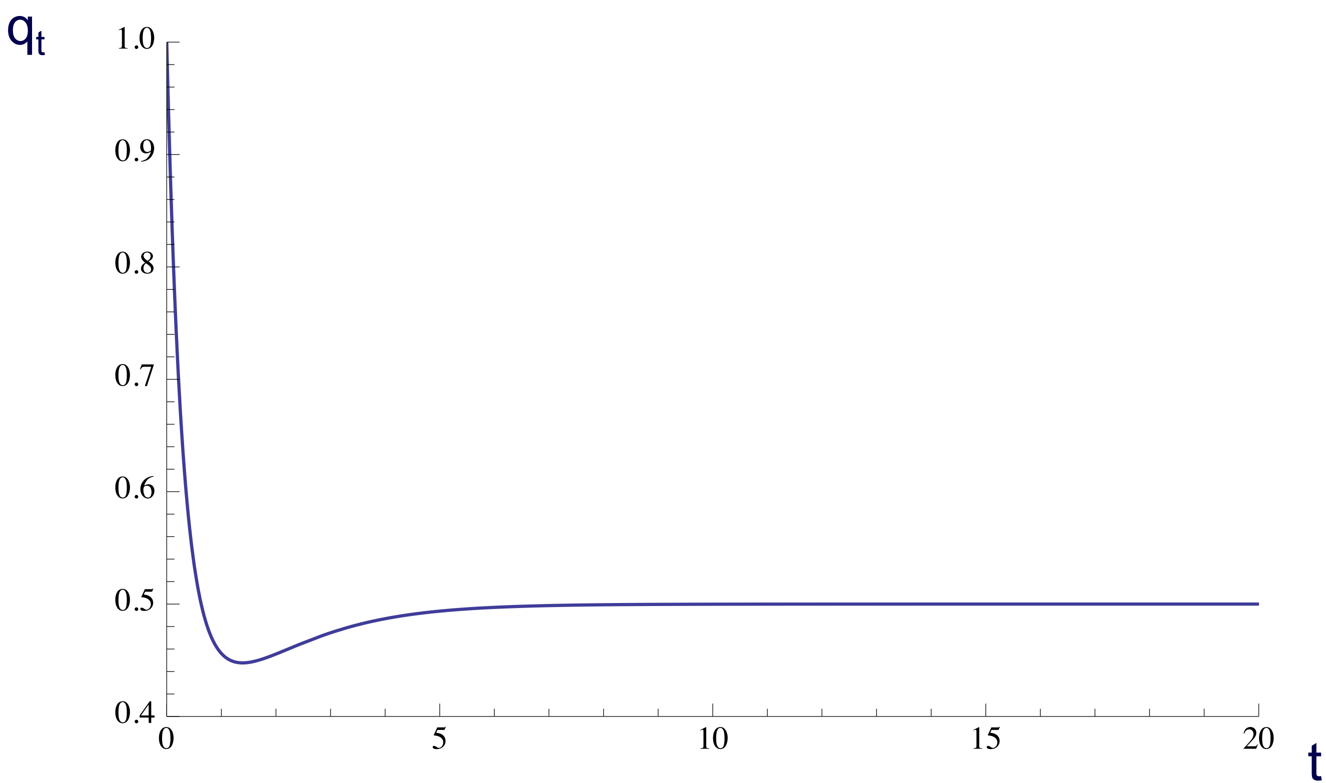
Question 2: Predictability

Define ``order parameter'' $q_D = \lim_{t\to\infty} q^t$, where

$$q^t = \lim_{L\to\infty}(2L+1)^{-d}\sum_{x\in\Lambda_L}(\langle\sigma_x\rangle_t)^2 = E_{J,\sigma^0}(\langle\sigma_x\rangle_t)^2$$

Theorem (NNS '00): For the one-dimensional spin chain with continuous coupling disorder, $q_D=½$.

Proof: Choose the origin as a typical point of **Z** and define X=X(*J*) to be the x_n such that that 0 lies in the interval $\{y_{n-1}+1, y_{n-1}+2, \ldots, y_n\}$. Then σ_0^∞ is completely determined by (*J* and) σ^0 if $J_{X,X+1}\sigma_X^0\sigma_{X+1}^0 > 0$ (so that $<\sigma_0>^\infty = +1$ or -1) and otherwise is completely determined by ω (so that $<\sigma_0>^\infty = 0$). Thus q_D is the probability that $\sigma_X^0\, \sigma_{X+1}^0 = \mathrm{sgn}(J_{X,X+1})$, which is ½.



q_t
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
t

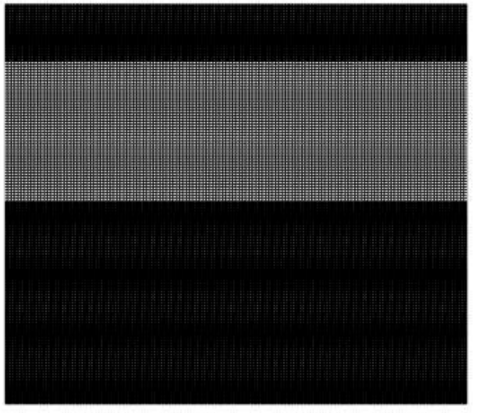
Numerical work

J. Ye, J. Machta, C.M. Newman and D.L. Stein, Phys. Rev. E **88**, 040101 (2013): simulations on L x L square lattice with

$$E = -\sum_{|x-y|=1} S_x S_y$$

Have to use finite-size scaling approach.

``Stripe states" occur roughly 1/3 of the time (V. Spirin, P.L. Krapivsky, and S. Redner, *Phys. Rev. E* **63**, 036118 (2000)).



Also P.M.C. de Oliveira, CMN, V. Sidoravicious, and DLS, *J. Phys. A* **39**, 6841-6849 (2006).

To distinguish the effects of nature vs. nurture, we simulated a *pair* of Ising lattices with identical initial conditions (i.e., ``twins'') (*cf.* damage spreading).

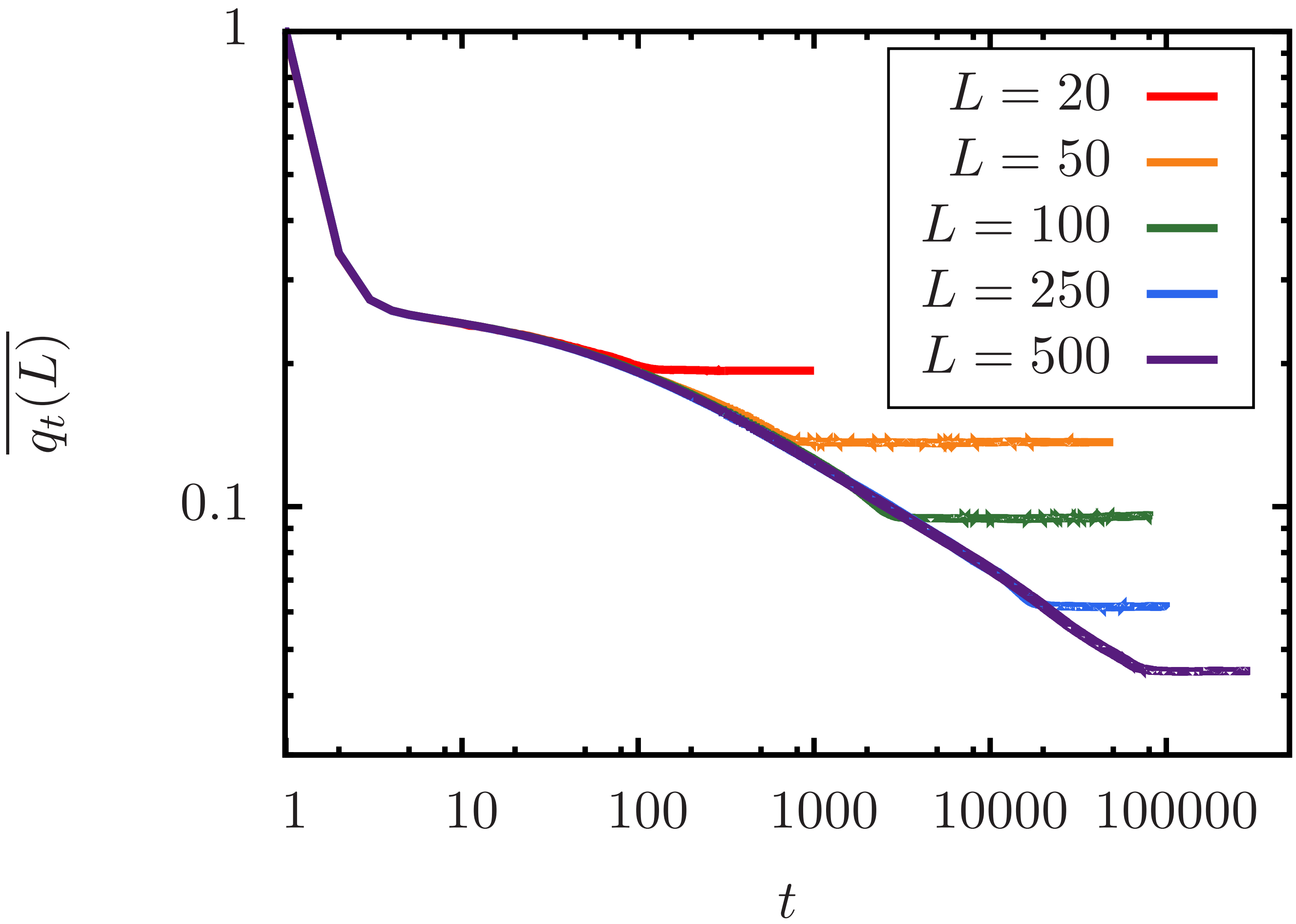
Examine the overlap q between a pair of twins at time t:

$$q_t(L) = \frac{1}{N}\sum_{i=1}^{L^2} S_i^1(t) S_i^2(t)$$

We are interested in the time evolution of the mean $\overline{q}_t$ and its final value $\overline{q}_\infty$ when the twins have reached absorbing states.

Looked at 21 lattice sizes from L = 10 to L = 500. For each size studied 30,000 independent twin pairs out to their absorbing states (or almost there).

Look at $\overline{q_t(L)}$ vs. *t* for several *L*.



The plateau value decreases from small to large *L*. A power law fit of the form $\overline{q_t} = dt^{-\theta_h}$ for the largest sizes gives a ``heritability exponent'' $\theta_h = 0.22 \pm 0.02$.

Finite size scaling ansatz

Use the fact that during coarsening, the typical domain size grows as $t^{1/z}$, with z = 2 for zero-temperature Glauber dynamics in the 2D ferromagnet.

Postulate the finite-size scaling form $\overline{q_t(L)} \approx t^{-\theta_h} f(\frac{t^{1/z}}{L})$, where the function f(x) is expected to behave as

$$f(x) \approx \begin{cases} 1 \quad for\ x << 1 \\ x^{z\theta_h} \quad for\ \ x >> 1 \end{cases}$$

So the t->∞ behavior is $\overline{q_\infty(L)} \approx L^{-z\theta_h}$, giving $b = z\theta_h = 2\theta_h$.

A.J. Bray, *Adv. Phys.* **43**, 357-459 (1994).

Summary so far

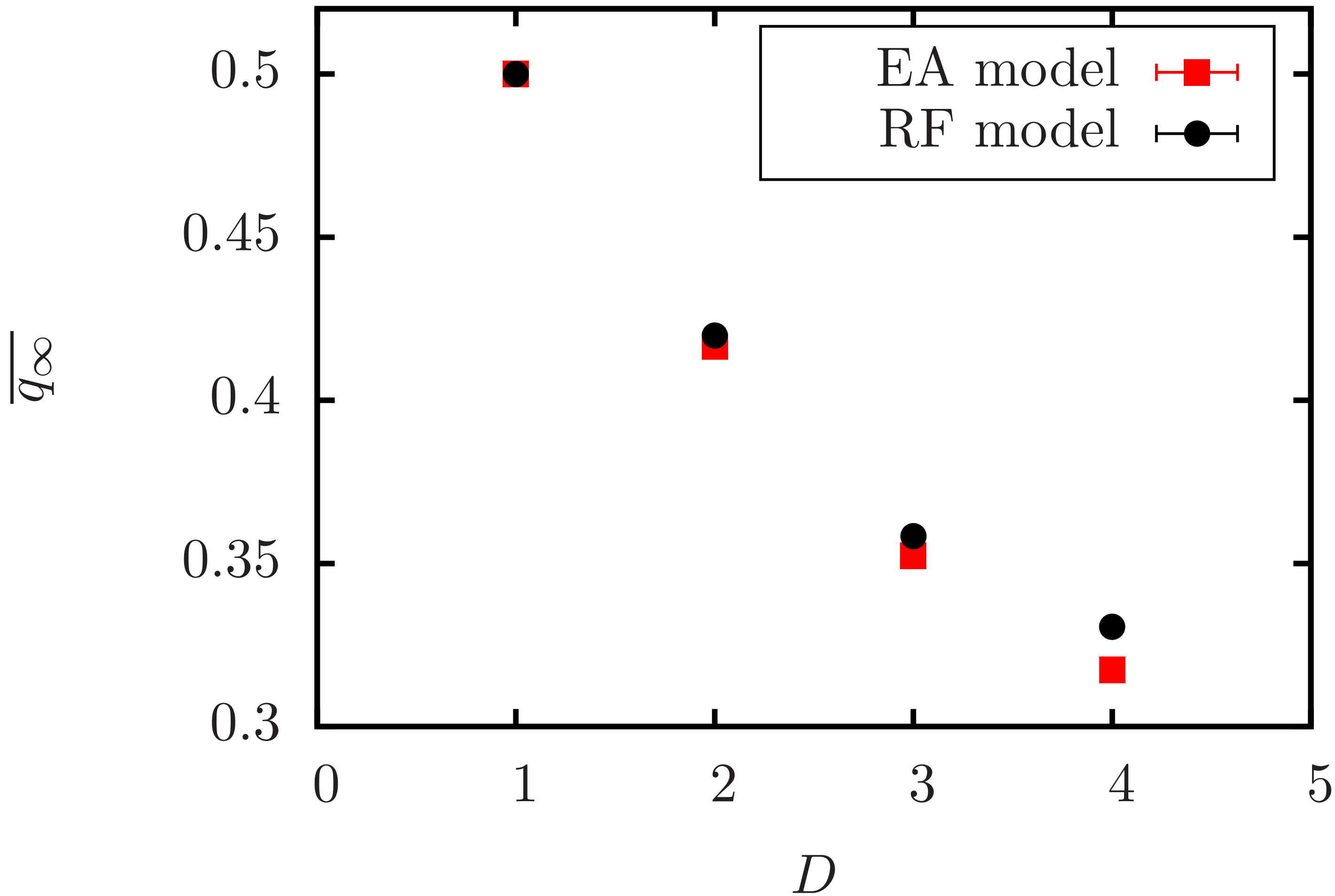
- For finite *L*, there are limiting absorbing states and overlaps $\overline{q_\infty(L)} \approx aL^{-b}$ with *b* = 0.46 ± 0.02.

- $\overline{q_t(L)}$ appears to approach $\overline{q_t} = dt^{-\theta_h}$ as $L \to \infty$ with $\theta_h = 0.22 \pm 0.02$.

- A finite size scaling analysis suggests that $b = 2\theta_h$, consistent with our numerical results.

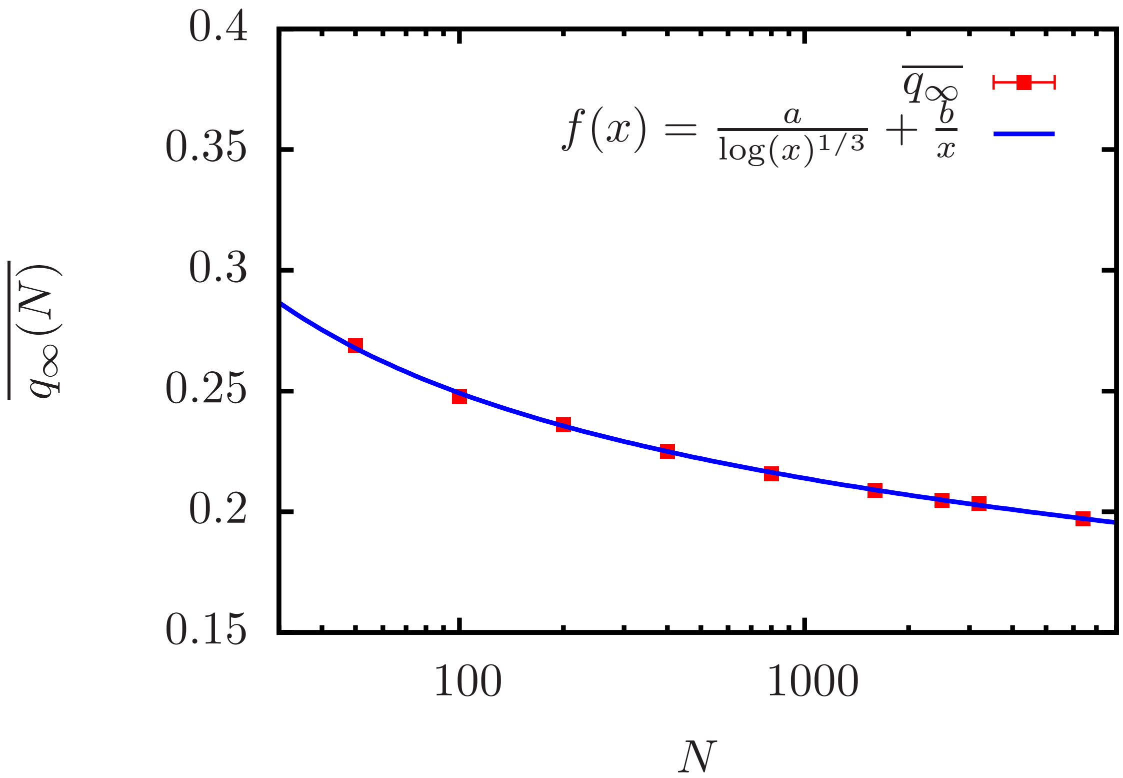
- Since $\theta_h > 0$, the *2D* Ising model displays weak LNE. But given the smallness of θ_h, information about the initial state decays slowly.

Disordered systems: random ferromagnets and spin glasses



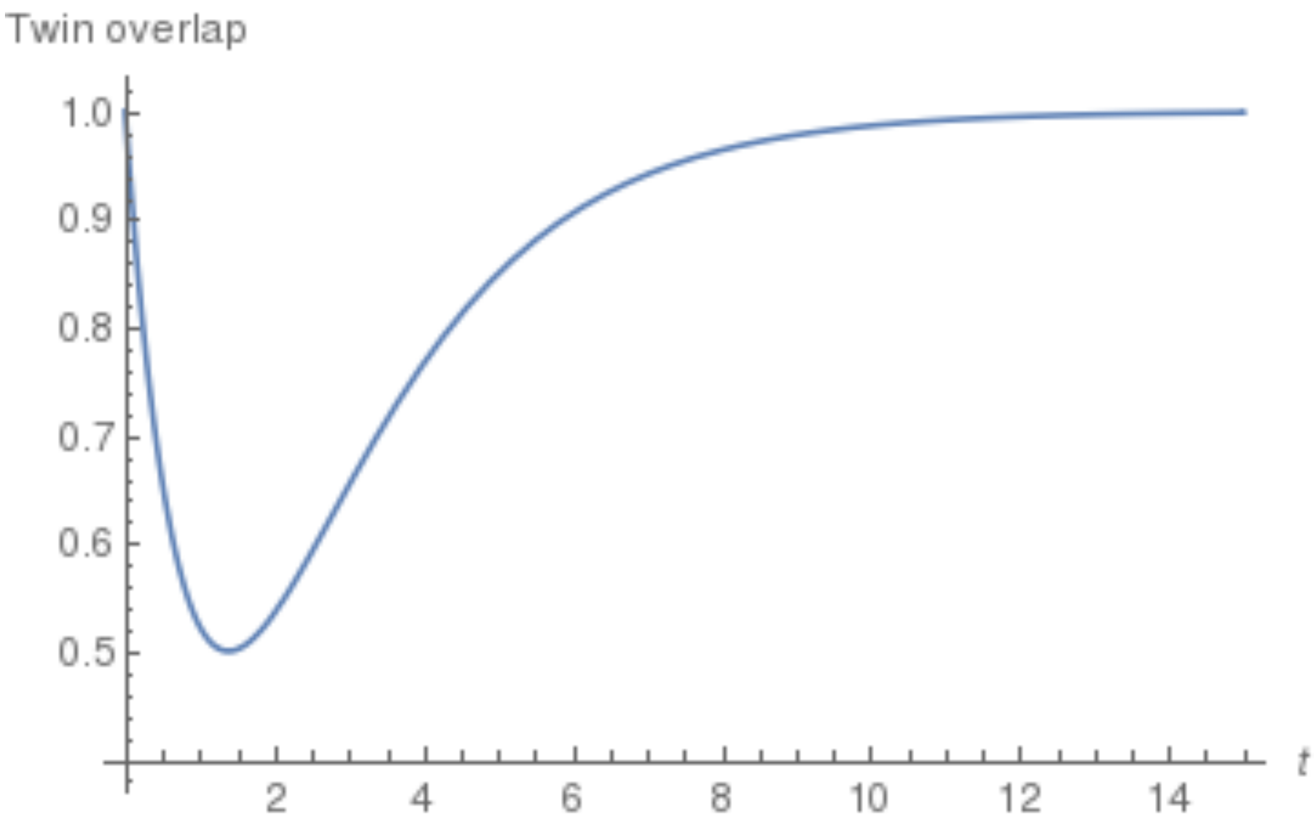
J. Ye, R. Gheissari, J. Machta, C.M. Newman, and DLS, *Phys. Rev. E* **95**, 042101 (2017).

But something funny is going on ...



... although not for spin glasses.

It's random ferromagnets that are behaving funny: q_D=1 for Curie-Weiss and randomly quenched Curie-Weiss.



R. Gheissari, C.M. Newman, and D.L. Stein, arXiv:1707.08875; L. Wang, RG, CMN, and DLS, in preparation.

Where do we go from here?

- Uniform ferromagnets in d≧3
- Random ferromagnets and spin glasses as d→∞
- Connection(s) to persistence, aging, damage spreading …