Generalized Parton Distributions and Angular Momentum

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June 4, 2018

People Involved

- Simonetta Liuti, University of Virginia
- Michael Engelhardt, New Mexico State University
- Aurore Courtoy, UNAM Mexico

AR, Engelhardt and Liuti arxiv:1709.05770

AR, Courtoy, Engelhardt and Liuti PRD 94 (2016)

Outline

- Spin Crisis !
- Orbital Angular Momentum
  - GTMD definition
  - GPD definition Ji
- What's the connection? Lorentz Invariance Relations
- Equation of Motion
- Quark Gluon Structure of Twist Three GPDs
- Conclusions

Proton Spin Crisis

$= g_1^P(x)$ Quark Spin Contribution

$$\frac{1}{2M}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda x}\left\langle P,S\left|\bar{\psi}\left(\frac{\lambda n}{2}\right)\gamma_\mu\gamma_5\psi\left(-\frac{\lambda n}{2}\right)\right|P,S\right\rangle = \Lambda g_1(x)p_\mu + g_T(x)S_{\perp_\mu}$$

Measured by EMC experiment in 1980s to be only 33% of total !!

Proton Spin Crisis

$$= \quad g_1^P(x)$$

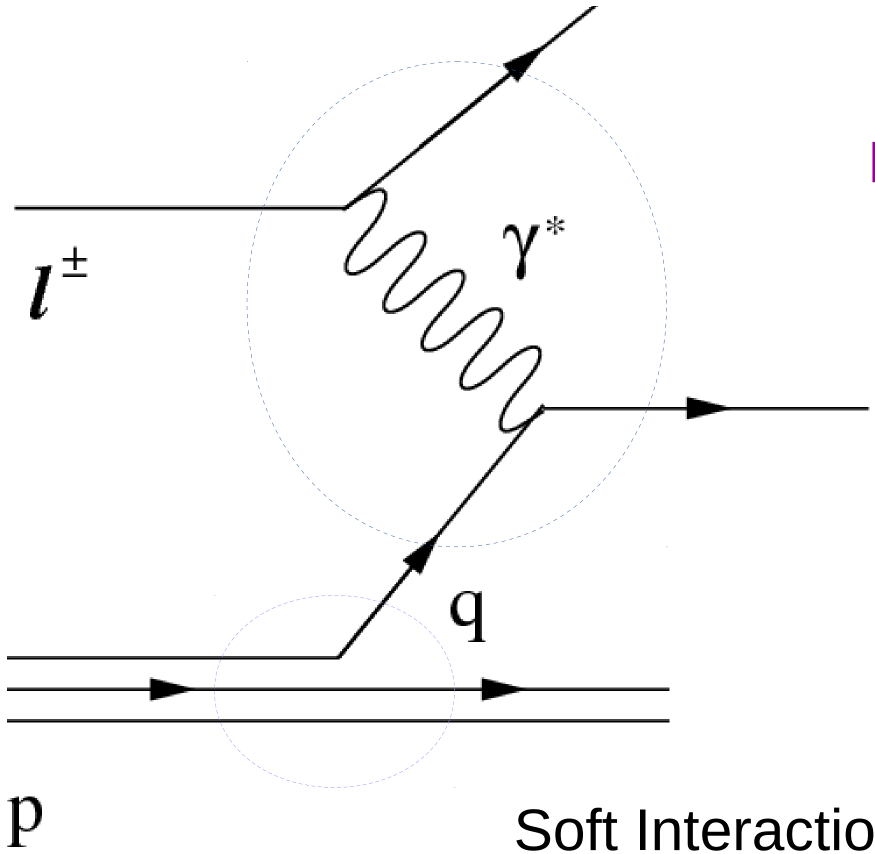
Quark Spin Contribution

$$\frac{1}{2M}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda x}\left\langle P,S\left|\bar{\psi}\left(\frac{\lambda n}{2}\right)\gamma_\mu\gamma_5\psi\left(-\frac{\lambda n}{2}\right)\right|P,S\right\rangle = \Lambda g_1(x)p_\mu + g_T(x)S_{\perp_\mu}$$

Measured by EMC experiment in
1980s to be only 33% of total !!

What are other sources ?
Partonic Orbital Angular Momentum

Hard and Soft Parts



Hard Interaction

Deep Inelastic Scattering

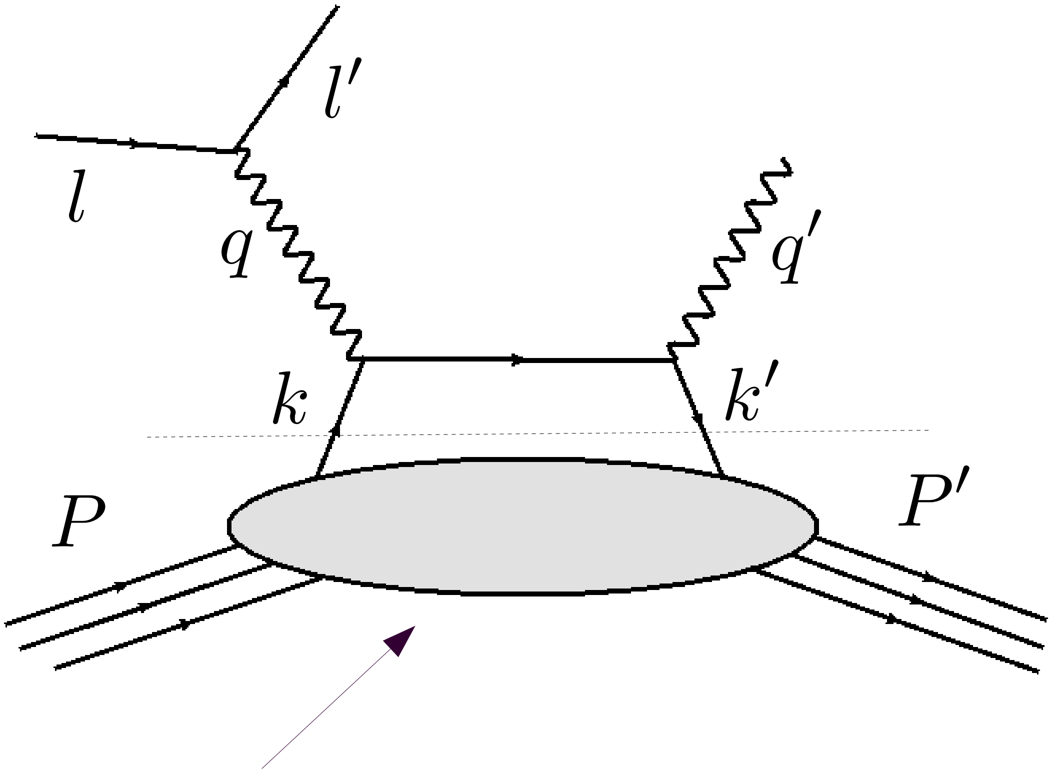
Soft Interaction

$$\int \frac{dz^-}{2\pi} e^{ik^+ z^-} \langle p, S \mid \bar{\psi}(-z/2)\gamma^+ \psi(z/2) \mid p, S\rangle_{z^+=z_T=0} = f_1(x)$$

$$a^{\pm} = \frac{a^0 \pm a^3}{\sqrt{2}}$$

Where are the partons located?

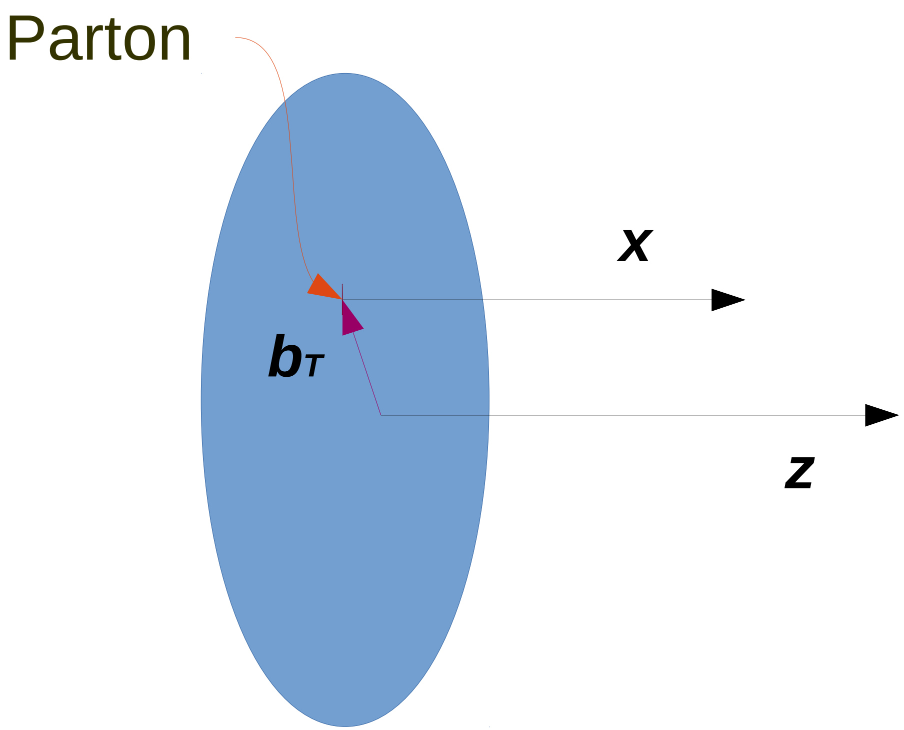
- Need a high energy photon to probe the partons
- The proton needs to remain intact to access spatial distribution
- <u>**Deeply Virtual Compton Scattering**</u>



Generalized Parton Distributions

Generalized Parton Distributions

- GPDs are the Fourier transform of the spatial distribution of partons in protons and neutrons.



GPD based definition of Angular Momentum

$$J_q = \frac{1}{2}\int_{-1}^{1} dx x(H_q(x,0,0) + E_q(x,0,0))$$

Xiangdong Ji, PRL 78.610,1997

To access OAM, we take the difference between total angular momentum and spin

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

Direct description of OAM

$$\int dx x G_2 = \int dx x(H + E) - \int dx \tilde{H}$$

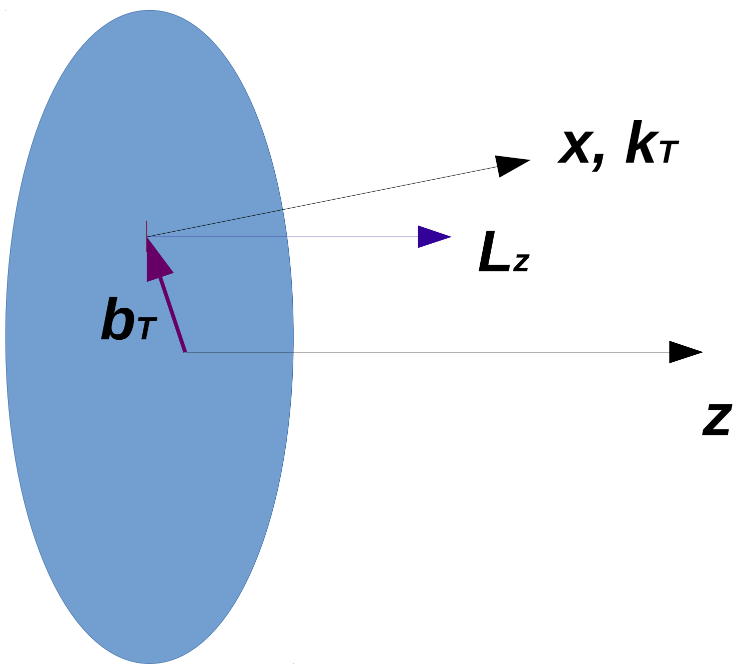
$$G_2 \equiv \tilde{E}_{2T} + H + E$$

Kiptily and Polyakov, Eur Phys J C 37 (2004)
Hatta and Yoshida, JHEP (1210), 2012

- The moment in x of the GPD G_2 shown to be OAM

Partonic Orbital Angular Momentum II

- Consider measuring both the intrinsic transverse momentum and the spatial distribution of partons
- $L_{q,z} = \mathbf{b_T} \times \mathbf{k_T}$



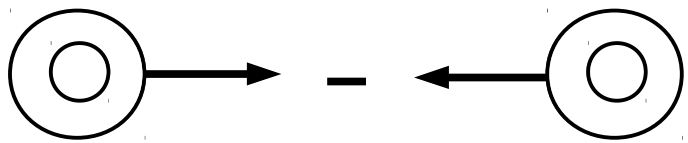
$$W_{\Lambda,\Lambda'}^{[\gamma^+]} = \frac{1}{2M}\bar{U}(p',\Lambda')[F_{11} + \frac{i\sigma^{i+}k_T^i}{\bar{p}_+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{\bar{p}_+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{M^2}F_{14}]U(p,\Lambda)$$

Generalized Transverse Momentum Distributions (related by Fourier transform to Wigner Distributions)

Meissner Metz and Schlegel,
JHEP 0908 (2009)

GTMDs that describe OAM

- How does F_{14} connect to OAM ?



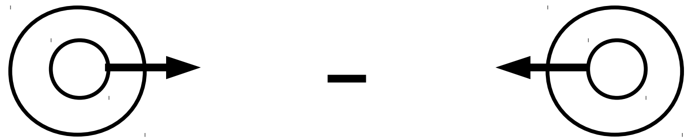
Unpolarized quark in a longitudinally polarized proton

$$\mathcal{W}(x, \mathbf{k}_T, \mathbf{b}) = \int \frac{d^2\Delta_T}{(2\pi)^2} e^{ib.\Delta_T} \left[W^{\gamma^+}_{++} - W^{\gamma^+}_{--} \right]$$

$$L = \int dx \int d^2k_T \int d^2\mathbf{b} (\mathbf{b} \times \mathbf{k}_T) \mathcal{W}(x, \mathbf{k}_T, \mathbf{b}) = -\int dx \int d^2k_T \frac{k_T^2}{M^2} F_{14}$$

Lorce et al PRD84, (2011)

- Another GTMD relevant to OAM



G_{11} describes a longitudinally polarized quark in an unpolarized proton. Measures spin orbit correlation.

The Two Definitions

- Weighted average of b_T X k_T

$$L_z = -\int dx \int d^2 k_T \frac{k_T^2}{M^2} F_{14}$$

- Difference of total angular momentum and spin

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

$$\frac{1}{2}\int_{-1}^{1} dx x (H_q + E_q)$$

$$\frac{1}{2}\int_{-1}^{1} dx \tilde{H}_q$$

The Two Definitions

- Weighted average of b_T X k_T

GTMD

$$L_z = -\int dx \int d^2 k_T \frac{k_T^2}{M^2} F_{14}$$

$$F_{14}^{(1)}$$

- Difference of total angular momentum and spin

GPD

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

$$\frac{1}{2}\int_{-1}^{1} dx x (H_q + E_q)$$

$$\frac{1}{2}\int_{-1}^{1} dx \tilde{H}_q$$

Is there a connection ?

- We find that

$$F_{14}^{(1)}(x) = \int_x^1 dy \left(\tilde{E}_{2T}(y) + H(y) + E(y)\right)$$

- This is a form of Lorentz Invariant Relation (LIR)
- This is a distribution of OAM in x
- Derived for a straight gauge link

Derivation of Generalized LIRs

To derive these we look at the parameterization of the quark quark correlator function at different levels

$$\int \frac{d^4z}{2\pi} e^{ik.z} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle$$

Generalized Parton Correlation Functions (GPCFS)

Integrate over k^-

Meissner Metz and Schlegel, JHEP 0908 (2009)

$$\int \frac{dz_- d^2 z_T}{2\pi} e^{ixP^+z^- - k_T.z_T} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=0}$$

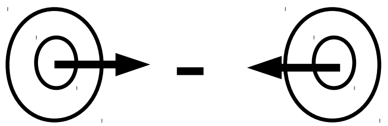
GTMDs

Integrate over k_T

$$\int \frac{dz_-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=z_T=0}$$

GPDs

Generalized Lorentz Invariance Relations



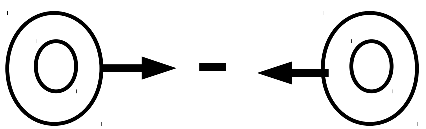
Axial Vector

$$\frac{dG_{11}^{e(1)}}{dx} = -\left(2\tilde{H}'_{2T} + E'_{2T}\right) - \tilde{H}$$

$$\frac{dG_{12}^{e(1)}}{dx} = H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} - \left(1 + \frac{\Delta_T^2}{2M^2}\right)\tilde{H}$$

Vector

$$\frac{dF_{14}^{e(1)}}{dx} = \tilde{E}_{2T} + H + E$$

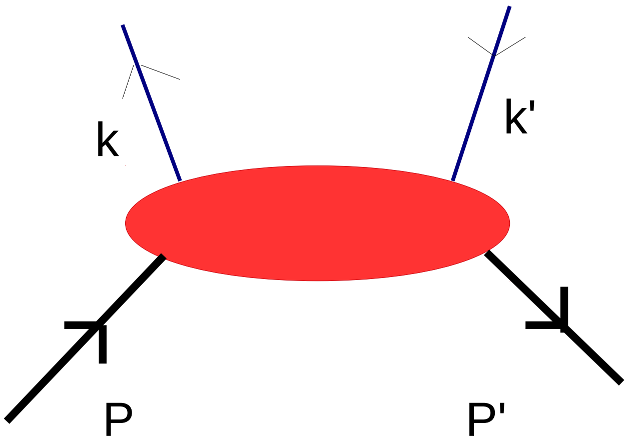


The GTMDs are complex in general.

$$X = X^e + iX^o$$

The imaginary part integrates to zero, on integration over k_T.

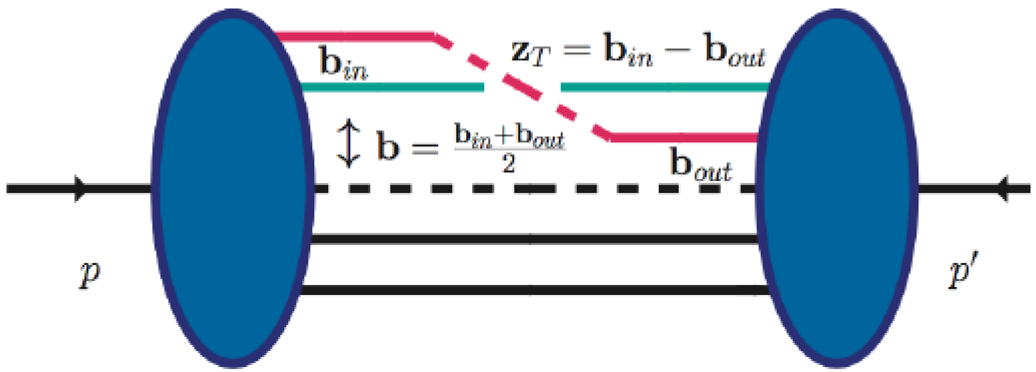
Intrinsic Momentum vs Momentum Transfer Δ

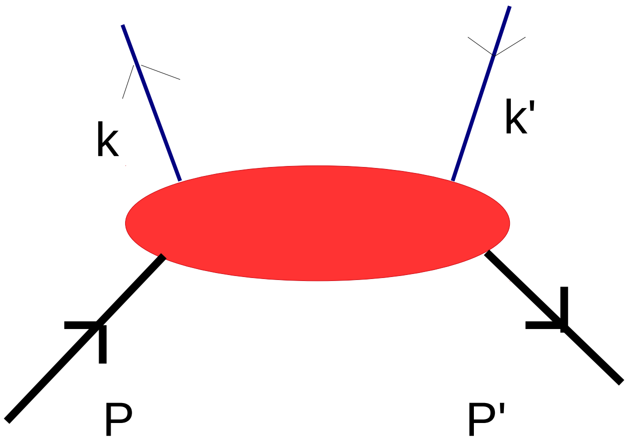


Courtoy et al PhysLett B731, 2013

Burkardt,Phys Rev D62, 2000

Intrinsic Momentum vs Momentum Transfer Δ





$$k \longleftrightarrow z$$

$$\Delta \longleftrightarrow b$$

Courtoy et al PhysLett B731, 2013

Burkardt,Phys Rev D62, 2000

Equations of Motion Relations

How do we obtain these ?

$$
\begin{aligned}
(i\not{D} - m)\psi(z_{out}) &= (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0, \\
\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) &= \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) = 0
\end{aligned}
$$

Equations of Motion Relations

How do we obtain these ?

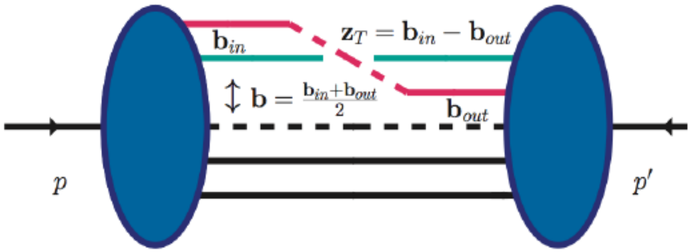
$$
\begin{aligned}
\mathcal{U} i\sigma^{i+}\gamma_5 (i\not{D} - m)\psi(z_{out}) &= \mathcal{U} i\sigma^{i+}\gamma_5 (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0, \\
\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma_5 \mathcal{U} &= \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = 0
\end{aligned}
$$

Equations of Motion Relations

How do we obtain these ?

$$\mathcal{U} i\sigma^{i+}\gamma_5 (i\not{D} - m)\psi(z_{out}) = \mathcal{U} i\sigma^{i+}\gamma_5 (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0,$$
$$\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = 0$$

$$b = \frac{z_{in} + z_{out}}{2}, \qquad z = z_{in} - z_{out}$$



$$\int db^- d^2 b_T e^{-ib\cdot\Delta} \int dz^- d^2 z_T e^{-ik\cdot z} \langle p', \Lambda' | \bar{\psi} \left[(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma^5 \pm i\sigma^{i+}\gamma^5 (i\overrightarrow{\not{D}} - m) \right] \psi | p, \Lambda \rangle = 0$$

Equations of Motion R

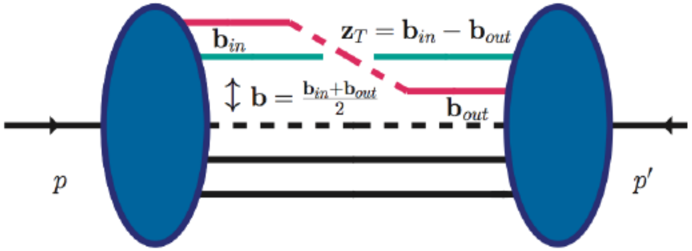
Crucial for understanding qgq contribution to GPDs!!

How do we obtain these ?

$$\mathcal{U} i\sigma^{i+}\gamma_5 (i\not{D} - m)\psi(z_{out}) = \mathcal{U} i\sigma^{i+}\gamma_5 (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0,$$

$$\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = 0$$

$$b = \frac{z_{in} + z_{out}}{2}, \qquad z = z_{in} - z_{out}$$



$$\int db^- d^2 b_T e^{-ib\cdot\Delta} \int dz^- d^2 z_T e^{-ik\cdot z} \langle p', \Lambda' | \bar{\psi} \left[(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma^5 \pm i\sigma^{i+}\gamma^5 (i\overrightarrow{\not{D}} - m) \right] \psi | p, \Lambda \rangle = 0$$

Use LIRs and Equation of Motion Relations to derive Wandzura Wilczek Relations

- The equations of motion connect k_T dependent quantities with collinear objects.
- These k_T dependent quantities are also connected to collinear objects by LIRs. This is independent of equation of motion relations.
- Use the LIR to eliminate the k_T dependent quantities in equation of motion relations. This results in the Wandzura Wilczek relations for twist 3 GPDs.

Wandzura Wilczek Relations

$$\tilde{E}_{2T} = -\int_x^1 \frac{dy}{y}(H + E) + \left[\frac{\tilde{H}}{x} - \int_x^1 \frac{dy}{y^2}\tilde{H}\right] + \left[\frac{1}{x}\mathcal{M}_{F_{14}} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{F_{14}}\right]$$

Twist three vector GPD

Twist two

Axial vector GPD contributes to a vector GPD

Genuine Tw 3

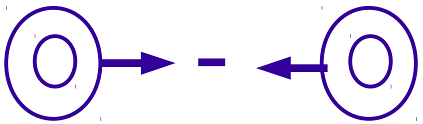
$$g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(x) + \bar{g}_2(x)$$

Twist three PDF

Twist two

Genuine Tw 3

Moments of twist three GPDs -Quark gluon structure



$$\int dx \widetilde{E}_{2T} = -\int dx (H+E) \qquad \Rightarrow \int dx \left(\widetilde{E}_{2T} + H + E \right) = 0$$

$$\int dx \underline{x} \widetilde{E}_{2T} = -\frac{1}{2} \int dx x (H+E) - \frac{1}{2} \int dx \widetilde{H}$$

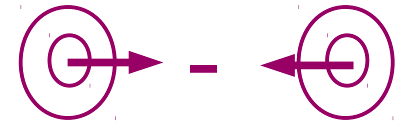
OAM

$$\int dx \underline{x}^2 \widetilde{E}_{2T} = -\frac{1}{3} \int dx x^2 (H+E) - \frac{2}{3} \int dx x \widetilde{H} - \boxed{\frac{2}{3} \int dx \, \mathcal{M}_{F_{14}} \Big|_{v=0}}$$

Genuine Twist Three

$$\int dx \, x \int d^2 k_T \, \mathcal{M}^{i,S}_{\Lambda' \Lambda} = \frac{ig}{4(P^+)^2} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ \gamma^5 F^{+i}(0) \psi(0) | p, \Lambda \rangle$$

Moments of twist three GPDs -Quark gluon structure



$$\int dx \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\int dx \widetilde{H} \qquad \Rightarrow \int dx \left(E'_{2T} + 2\widetilde{H}'_{2T} + \widetilde{H} \right) = 0$$

$$\int dx \underline{x} \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\frac{1}{2} \int dx x \widetilde{H} - \frac{1}{2} \int dx H + \boxed{\frac{m}{2M} \int dx (E_T + 2\widetilde{H}_T)}$$

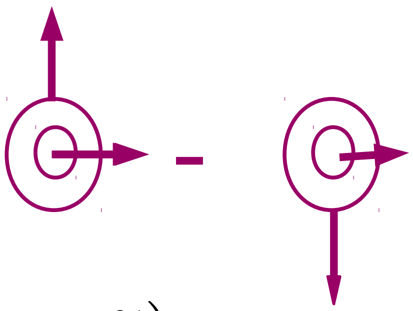
mass term

$$\int dx \underline{x}^2 \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\frac{1}{3} \int dx x^2 \widetilde{H} - \frac{2}{3} \int dx x H + \boxed{\frac{2m}{3M} \int dx x (E_T + 2\widetilde{H}_T)} - \boxed{\frac{2}{3} \int dx x \mathcal{M}_{G_{11}} \Big|_{v=0}}$$

Genuine Twist Three d_2

$$\int dx \, x \int d^2 k_T \, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \frac{g}{4(P^+)^2} \epsilon^{ij} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ F^{+j}(0) \psi(0) | p, \Lambda \rangle$$

Moments of twist three GPDs -Quark gluon structure

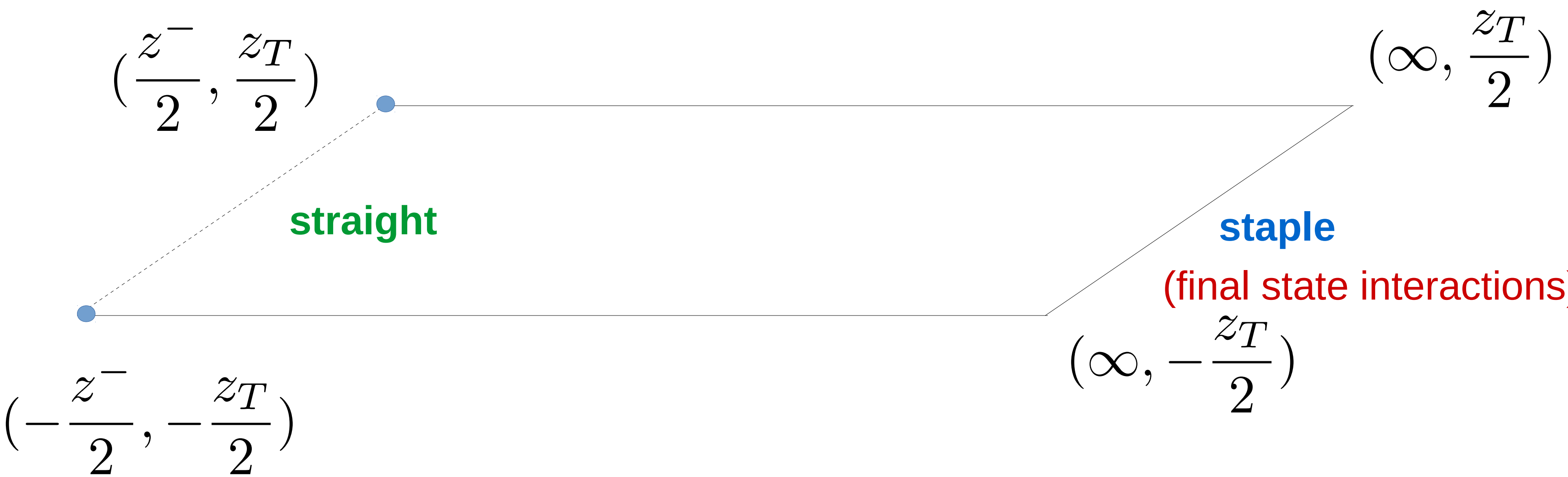


$$\int dx \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) = \left(1 + \frac{\Delta_T^2}{4M^2} \right) \int dx \widetilde{H} \quad \stackrel{\Delta_T \to 0}{\Longrightarrow} \int dx \left(H'_{2T} - \widetilde{H} \right)$$

$$\equiv \int dx \, g_2 = 0$$

Off forward extension of Burkhardt Cottingham

Quark gluon quark contributions



$$\int dx \int d^2k_T \, \mathcal{M}^{i,S}_{\Lambda'\Lambda} \quad = $$

$$i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0)\psi(0) | p, \Lambda\rangle$$

$$\int dx \int d^2k_T \, \mathcal{M}^{i,A}_{\Lambda'\Lambda} \quad = $$

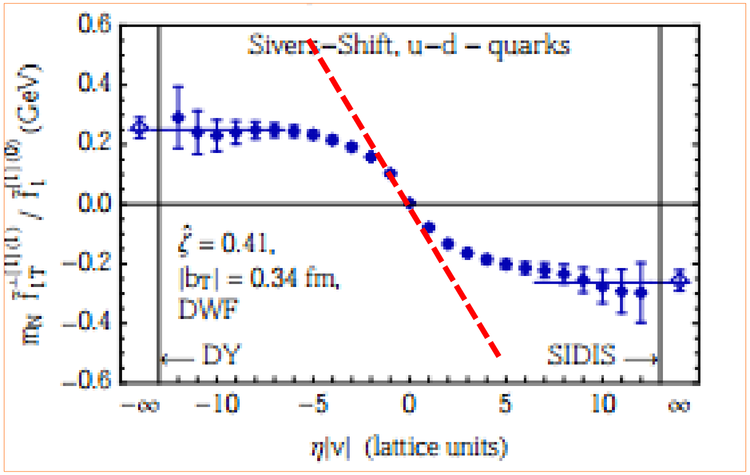
$$-g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 U(0, sv) F^{+i}(sv) U(sv, 0)\psi(0) | p, \Lambda\rangle$$

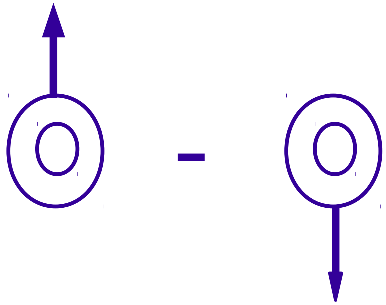
Calculating the force from Lattice data – Sivers function

$$\frac{d}{dv^-}\int dx F_{12}^{(1)}\Big|_{v^-=0} = \frac{d}{dv^-}\int dx \mathcal{M}_{F_{12}}\Big|_{v^-=0} = i(2P^+)\int dx\, x\frac{\Delta_i}{\Delta_T^2}\left(\mathcal{M}_{++}^{i,A} - \mathcal{M}_{--}^{i,A}\right) = \mathcal{M}_{G_{12}}^{n=3}$$

$$\frac{d}{dv^-}\int dx G_{12}^{(1)}\Big|_{v^-=0} = \frac{d}{dv^-}\int dx \mathcal{M}_{G_{12}}\Big|_{v^-=0} = i(2P^+)\int dx\, x\frac{\Delta_i}{\Delta_T^2}\left(\mathcal{M}_{++}^{i,S} + \mathcal{M}_{--}^{i,S}\right) = \mathcal{M}_{F_{12}}^{n=3}$$

The derivative with respect to the gauge link direction gives the force!



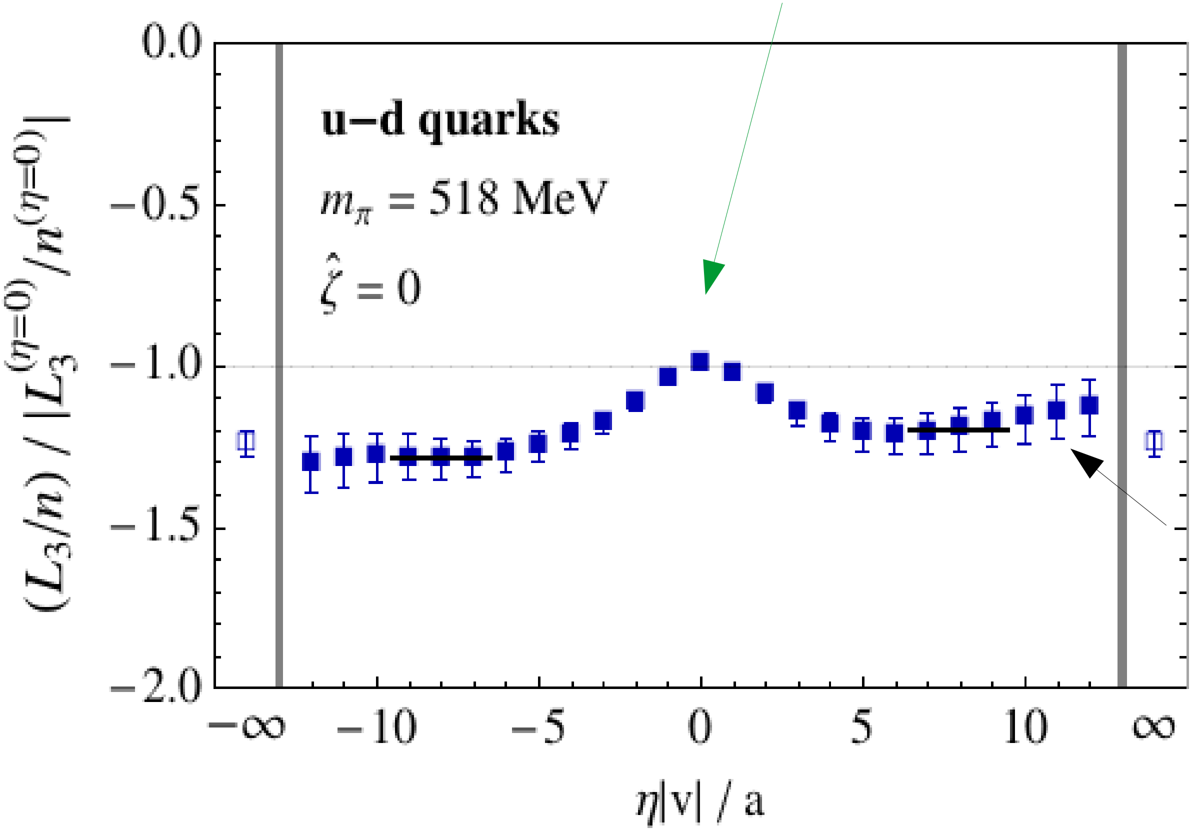


Transversely polarized proton

Calculating the torque from Lattice

Straight link

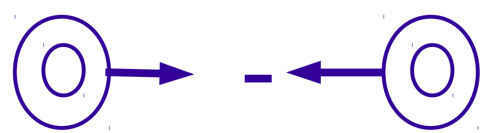
F_{14}



Staple

Michael Engelhardt

Phys. Rev. D95 (2017)



Longitudinally polarized proton

$$\mathcal{L}_{JM} - \mathcal{L}_{Ji} = \mathcal{T}$$

Torque!

Conclusions

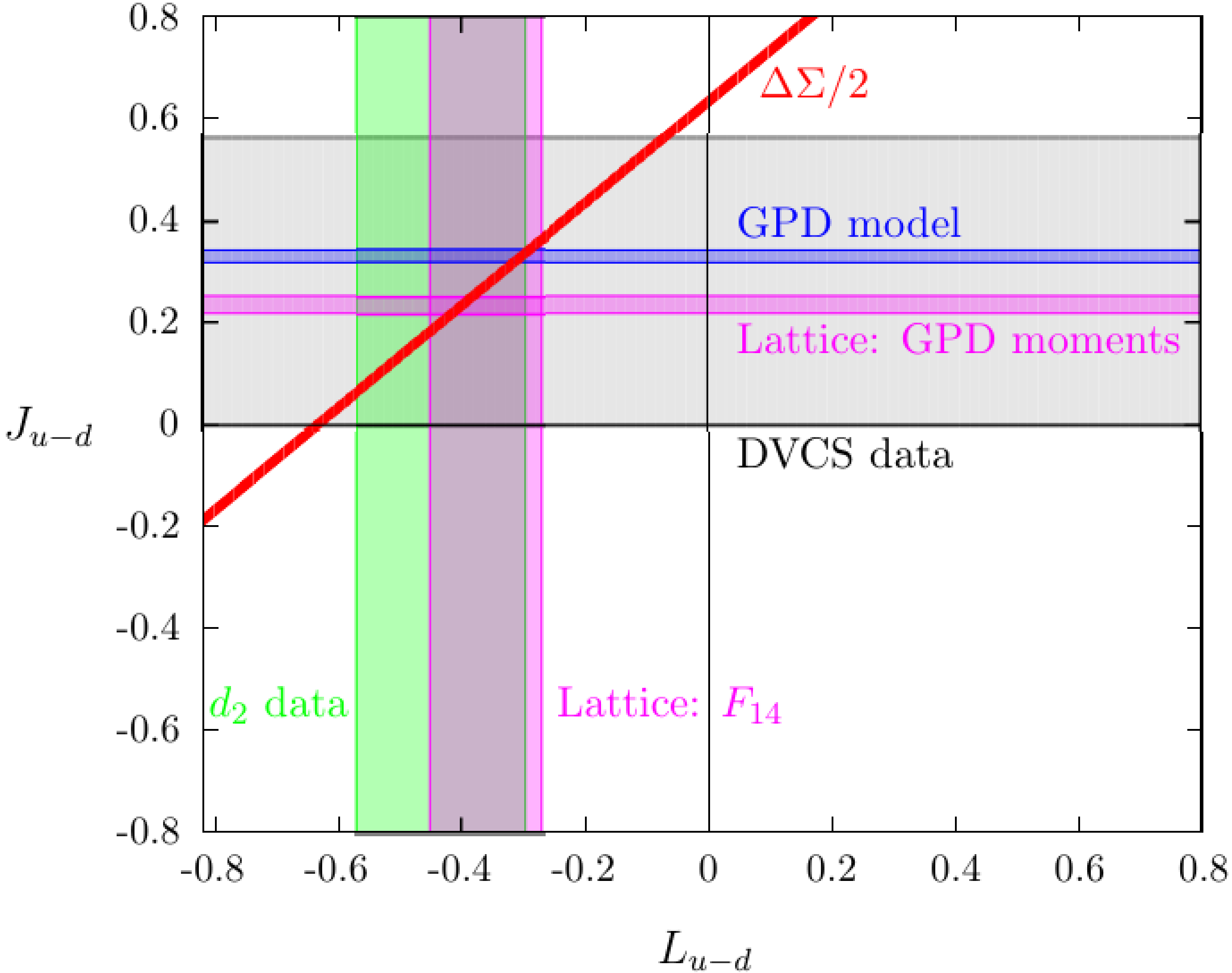
- Way of deriving the Wandzura Wilczek relations. Allows us to **write out precisely quark gluon contribution to twist 3. Study x dependence.**

- This also provides a way to measure effects that were solely associated with GTMDs by measuring the associated GPD.

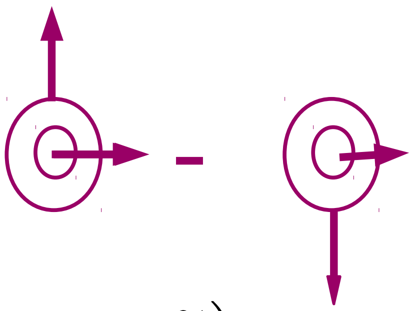
- Quark gluon quark interactions are at the heart of twist three effects.

Thank you!

Where do we stand experimentally?

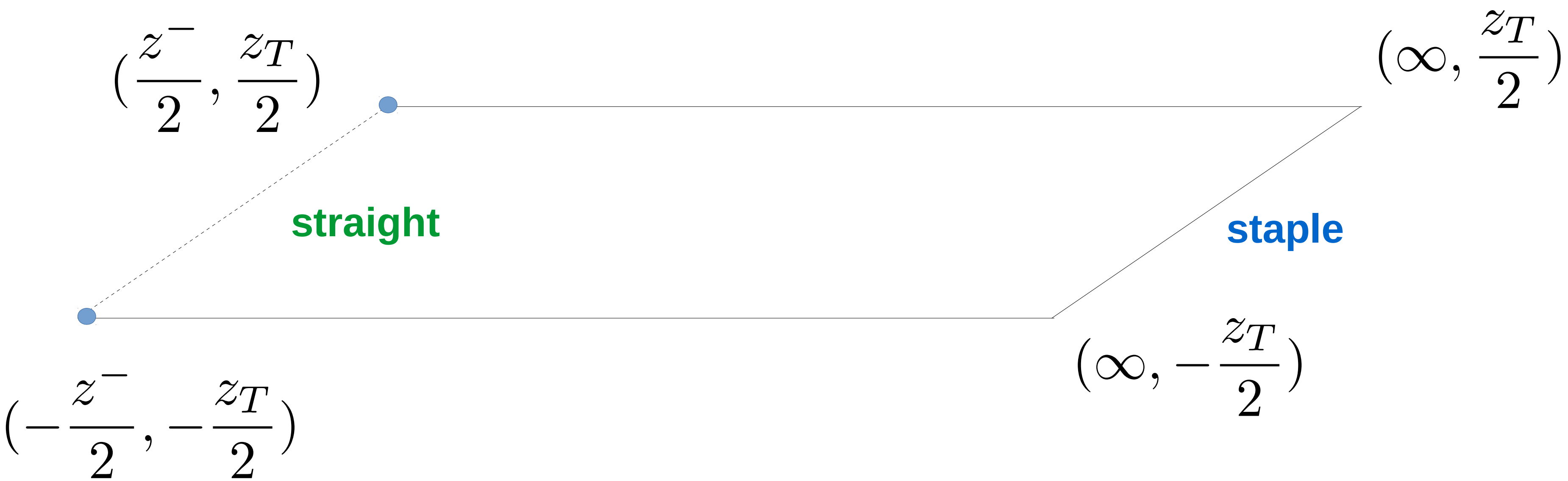


Moments of twist three GPDs -Quark gluon structure



$$
\begin{aligned}
\int dx \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \left(1 + \frac{\Delta_T^2}{4M^2}\right) \int dx \widetilde{H} \quad \stackrel{\Delta_T \to 0}{\Longrightarrow} \int dx \left(H'_{2T} - \widetilde{H} \right) \\
&\equiv \int dx \, g_2 = 0 \\
\int dx \, x \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \frac{1}{2}\left(1 + \frac{\Delta_T^2}{4M^2}\right) \int dx x \widetilde{H} + \frac{\Delta_T^2}{8M^2} \int dx (H + E) \\
&+ \frac{m}{2M} \int dx \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) \\
\int dx \, x^2 \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \frac{1}{3}\left(1 + \frac{\Delta_T^2}{4M^2}\right) \int dx x^2 \widetilde{H} + \frac{\Delta_T^2}{6M^2} \int dx x (H + E) \\
&+ \frac{2m}{3M} \int dx x \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) + \frac{2}{3} \int dx \, x \, \mathcal{M}_{G_{12}} \Big|_{v=0} \, .
\end{aligned}
$$

Quark gluon quark contributions



$$(\overrightarrow{\partial} - igA)\mathcal{U}\Big|_{-z/2} =$$

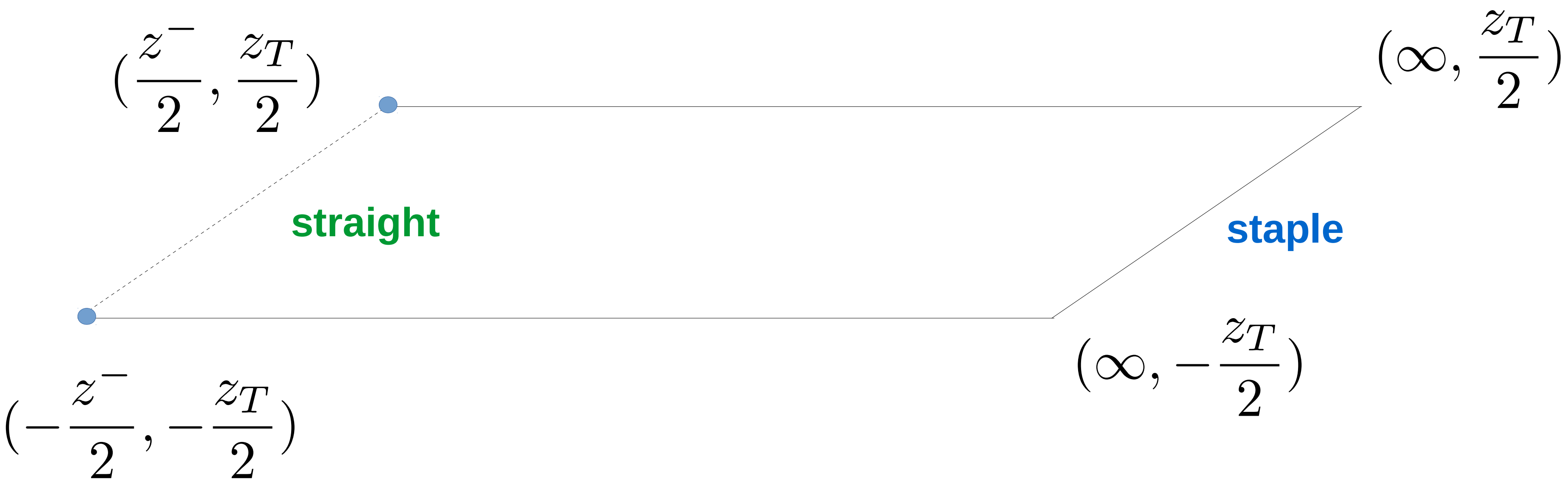
$$igz^- \int_0^1 ds\,(1-s)$$

$$\cdot\, U(-z/2, -z/2+v+sz)\gamma_\mu F^{+\mu}(-z/2+v+sz)U(-z/2+v+sz, z/2)$$

$$+igv^- \int_0^1 ds\, U(-z/2, -z/2+sv)\gamma_\mu F^{+\mu}(-z/2+sv)U(-z/2+sv, z/2)$$

staple arm

Quark gluon quark contributions



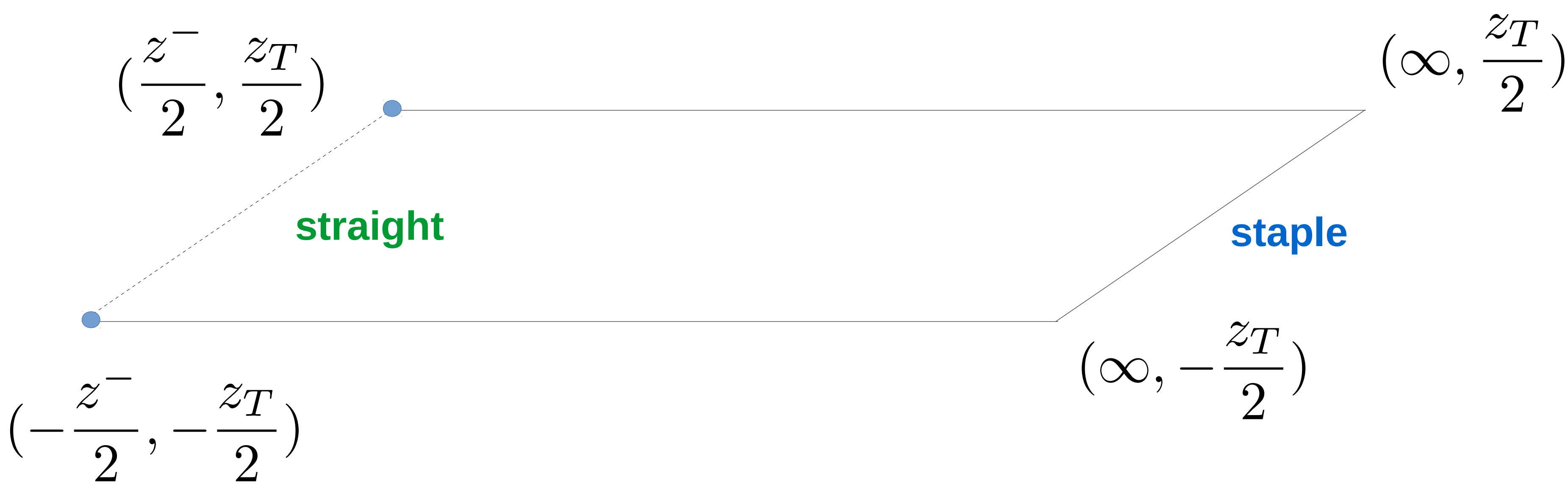
$$\int dx \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} \quad =$$

$$i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0)\psi(0)|p, \Lambda\rangle$$

$$\int dx \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} \quad =$$

$$-g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 U(0, sv) F^{+i}(sv) U(sv, 0)\psi(0)|p, \Lambda\rangle$$

Quark gluon quark contributions



$$\int dx \int d^2k_T \, \mathcal{M}^{i,S}_{\Lambda'\Lambda} \quad = $$

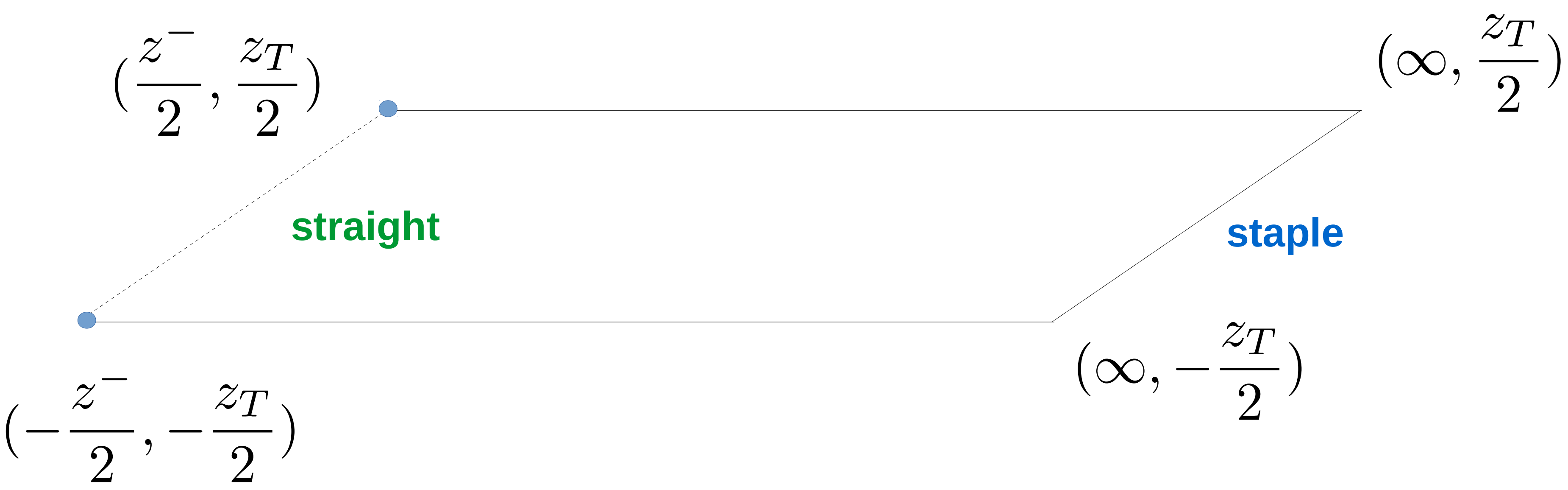
$$i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0)\psi(0)|p, \Lambda\rangle$$

Zero for straight gauge link

$$\int dx \int d^2k_T \, \mathcal{M}^{i,A}_{\Lambda'\Lambda} \quad = $$

$$-g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 U(0, sv) F^{+i}(sv) U(sv, 0)\psi(0)|p, \Lambda\rangle$$

Quark gluon quark contributions



$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = \frac{ig}{4(P^+)^2} \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 F^{+i}(0)\psi(0)|p,\Lambda\rangle$$

$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \frac{g}{4(P^+)^2} \epsilon^{ij} \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ F^{+j}(0)\psi(0)|p,\Lambda\rangle$$

Equations of Motion Relations

Starting with the equation of motion and its conjugate we arrive at the following

$$-\frac{\Delta^+}{2} W_{\Lambda\Lambda'}^{[\gamma^i\gamma^5]} + ik^+\epsilon^{ij} W_{\Lambda\Lambda'}^{[\gamma^j]} = -\frac{\Delta^i}{2} W_{\Lambda\Lambda'}^{[\gamma^+\gamma^5]} + i\epsilon^{ij} k_T^j W_{\Lambda\Lambda'}^{[\gamma^+]} - \mathcal{M}_{\Lambda\Lambda'}^{i,S}$$

$$-k^+ W_{\Lambda\Lambda'}^{[\gamma^i\gamma^5]} + \frac{i\Delta^+}{2}\epsilon^{ij} W_{\Lambda\Lambda'}^{[\gamma^j]} + k^i W_{\Lambda\Lambda'}^{[\gamma^+\gamma^5]} = i\epsilon^{ij}\frac{\Delta^j}{2} W_{\Lambda\Lambda'}^{[\gamma^+]} - mW_{\Lambda\Lambda'}^{[i\sigma^{i+}\gamma^5]} - i\mathcal{M}_{\Lambda\Lambda'}^{i,A}$$

- Each W is a correlator that can be parameterized using GTMDs / GPDs.

$$W_{\Lambda\Lambda'}^{\Gamma} = \int \frac{dz^- \, d^2\mathbf{z}_T}{(2\pi)^3} e^{ixP^+z^- - i\bar{\mathbf{k}}_T\cdot\mathbf{z}_T} \langle p', \Lambda' \mid \bar{\psi}\left(-\frac{z}{2}\right)\mathcal{U}\Gamma\psi\left(\frac{z}{2}\right) \mid p, \Lambda\rangle\Big|_{z^+=0}$$

Generalized Lorentz Invariance Relations

- Parametrization of the quark quark correlator at different levels
- LIRs occur because the number of GPCFs is less than the number of GTMDs.

$$\begin{aligned} \mathcal{W}_{\Lambda\Lambda'}^{[\gamma^\mu]} &= \frac{\bar{U}U}{M}(P^\mu A_1^F + k^\mu A_2^F + \Delta^\mu A_3^F) + i\frac{\bar{U}\sigma^{\mu k}U}{M}A_5^F + i\frac{\bar{U}\sigma^{\mu\Delta}U}{M}A_6^F \\ &+ i\frac{\bar{U}\sigma^{k\Delta}U}{M^3}(P^\mu A_8^F + k^\mu A_9^F + \Delta^\mu A_{17}^F) \end{aligned}$$

Explicit k_T coefficient

$$W_{\Lambda,\Lambda'}^{[\gamma^+]} = \frac{1}{2M}\bar{U}(p',\Lambda')[F_{11} + \frac{i\sigma^{i+}k_T^i}{\bar{p}_+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{\bar{p}_+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{M^2}F_{14}]U(p,\Lambda)$$

$$F_{\Lambda,\Lambda'}^{[\gamma^i]} = \frac{1}{2(P^+)^2}\bar{U}\left[i\sigma^{+i}H_{2T} + \frac{\gamma^+\Delta_T^i}{2M}E_{2T} + \frac{P^+\Delta_T^i}{M^2}\tilde{H}_{2T} - \frac{P^+\gamma^i}{M}\tilde{E}_{2T}\right]U$$

Generalized Lorentz Invariance Relations

- The As are a function of the following scalar variables :

$$\sigma \equiv \frac{2k.P}{M^2}, \qquad \tau \equiv \frac{k^2}{M^2}, \qquad \sigma' \equiv \frac{k.\Delta}{\Delta^2} = \frac{k_T.\Delta_T}{\Delta_T^2}$$

For $\Delta^+ = 0$

$$\begin{aligned}\int dk^- A(k^2, k.P, k.\Delta \ldots) \quad &\to \quad \frac{M^2}{2P^+}\int d\sigma A \\ &\to \quad \frac{M^2}{2P^+}\int d\sigma' d\sigma d\tau \delta\left(\frac{k_T^2}{M^2} - x\sigma + \tau + \frac{x^2P^2}{M^2}\right)\delta\left(\sigma' - \frac{k_T.\Delta_T}{\Delta_T^2}\right)A(\sigma,\tau,\sigma')\end{aligned}$$

Generalized Lorentz Invariance Relations

$$F_{14}^{(1)} = \int d\sigma d\sigma' d\tau \frac{M^3}{2} J \left[A_8^F + x A_9^F\right] \qquad J = \sqrt{x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}}$$

$$\tilde{E}_{2T} = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \left[\left(x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}\right) A_9^F - \sigma' A_5^F - A_6^F\right]$$

$$H + E = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \sigma' A_5^F + A_6^F + \left(\frac{\sigma}{2} - \frac{x P^2}{M^2}\right) \left(A_8^F + x A_9^F\right)$$

$$\begin{aligned} -\frac{dF_{14}^{(1)}}{dx} &= \tilde{E}_{2T} + H + E \\ F_{14}^{(1)}(x) &= \int_x^1 dy \left(\tilde{E}_{2T}(y) + H(y) + E(y)\right) \end{aligned}$$

Distribution of OAM in x !

k_T^2 moment of a twist two function

Twist three function

EoM relations for Orbital Angular Momentum

$$x\tilde{E}_{2T} = -\tilde{H} + 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} F_{14} + \frac{\Delta^i}{\Delta_T^2}\int d^2k_T (\mathcal{M}^{i,S}_{++} - \mathcal{M}^{i,S}_{--})$$

$$x\left(E'_{2T} + 2\tilde{H}'_{2T}\right) = -H + \frac{m}{M}(E_T + 2\tilde{H}_T) - 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} G_{11} - \mathcal{M}_{G_{11}}$$

$$\mathcal{M}_{G_{11}} = \frac{2i\epsilon^{im}\Delta^m}{\Delta_T^2}(\mathcal{M}^{i,A}_{++} + \mathcal{M}^{i,A}_{--})$$

EoM relations for Transversely Polarized Proton

$$
\begin{aligned}
0 \quad = \quad & \frac{\Delta_T^2}{4M^2}E + \frac{1}{2}G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2}G_{11}^{(1)} - x\left(H'_{2T} + \frac{\Delta_T^2}{2M^2}\widetilde{H}'_{2T}\right) + \frac{m}{M}\left(H_T + \frac{\Delta_T^2}{2M^2}\widetilde{H}_T\right) \\
& - \frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\int d^2k_T \left((\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{+-} + (-\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{-+}\right)
\end{aligned}
$$

Axial vector

$$
\begin{aligned}
- \quad & x\left(F_{23} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{24}\right) + \frac{1}{2M^2}\left(\Delta_T^2 G_{13} + k_T\cdot\Delta_T G_{12}\right) \\
+ \quad & \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{12} + \frac{\Delta^i}{2M\Delta_T^2}\left((\Delta^1 - i\Delta^2)\mathcal{M}^{i,S}_{-+} + (\Delta^1 + i\Delta^2)\mathcal{M}^{i,S}_{+-}\right) = 0.
\end{aligned}
$$

Vector

EoM relations for Transversely Polarized Proton

Twist 3

$$0 = \frac{\Delta_T^2}{4M^2} E + \frac{1}{2} G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2} G_{11}^{(1)} \boxed{- x \left(H'_{2T} + \frac{\Delta_T^2}{2M^2} \widetilde{H}'_{2T} \right)} + \frac{m}{M} \left(H_T + \frac{\Delta_T^2}{2M^2} \widetilde{H}_T \right)$$
$$- \frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2} \int d^2k_T \left((\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,A} + (-\Delta^1 + i\Delta^2)\mathcal{M}_{-+}^{i,A} \right)$$

d_2 (in the forward limit)

Axial vector

H_{2T}

Twist 3

$$\boxed{- \quad x \left(F_{23} + \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2 \Delta_T^2} F_{24} \right)} + \frac{1}{2M^2} \left(\Delta_T^2 G_{13} + k_T \cdot \Delta_T G_{12} \right)$$
$$+ \quad \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2 \Delta_T^2} F_{12} + \frac{\Delta^i}{2M\Delta_T^2} \left((\Delta^1 - i\Delta^2)\mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,S} \right) = 0.$$

Vector

EoM relations for Transversely Polarized Proton

Twist 3

$$0 = \frac{\Delta_T^2}{4M^2}E + \frac{1}{2}G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2}G_{11}^{(1)} \boxed{- x\left(H'_{2T} + \frac{\Delta_T^2}{2M^2}\widetilde{H}'_{2T}\right)} + \frac{m}{M}\left(H_T + \frac{\Delta_T^2}{2M^2}\widetilde{H}_T\right)$$

$$-\frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\int d^2k_T\left((\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,A} + (-\Delta^1 + i\Delta^2)\mathcal{M}_{-+}^{i,A}\right)$$

d_2 (in the forward limit)

Axial vector

H_{2T}

Twist 3

$$\boxed{- \quad x\left(F_{23} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{24}\right)} + \frac{1}{2M^2}\left(\Delta_T^2 G_{13} + k_T\cdot\Delta_T G_{12}\right)$$

$$+ \quad \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{12} + \frac{\Delta^i}{2M\Delta_T^2}\left((\Delta^1 - i\Delta^2)\mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,S}\right) = 0.$$

$$\boxed{f_{1T}^{\perp(1)} = -F_{12}^{o(1)} = \mathcal{M}_{F_{12}}|_{\Delta_T=0}}$$

Vector

LIR violating term

$$
\begin{aligned}
\mathcal{A}_{F_{14}}(x) &\equiv v^- \frac{(2P^+)^2}{M^2} \int d^2 k_T \int dk^- \left[\frac{k_T \cdot \Delta_T}{\Delta_T^2} (A_{11} + x A_{12}) + A_{14} \right. \\
&+ \left. \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \left(\frac{\partial A_8}{\partial (k \cdot v)} + x \frac{\partial A_9}{\partial (k \cdot v)} \right) \right] \\
&= \frac{dF_{14}^{(1)}}{dx} - \left. \frac{dF_{14}^{(1)}}{dx} \right|_{v=0} \qquad (1)
\end{aligned}
$$

$$
F_{14}^{(1)} - \left. F_{14}^{(1)} \right|_{v=0} = \mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \, (1)
$$

$$
\mathcal{A}_{F_{14}}(x) = \frac{d}{dx} \left(\mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \right) (1)
$$

$$
\begin{aligned}
- \int dx \left. \left(F_{14}^{(1)} - \left. F_{14}^{(1)} \right|_{v=0} \right) \right|_{\Delta_T = 0} &= \qquad (1) \\
- \frac{\partial}{\partial \Delta^i} i \epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', + | \bar{\psi}(0) \gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0) \psi(0) | p, + \rangle \Big|_{\Delta_T = 0} ,
\end{aligned}
$$

Wandzura Wilczek Relations

$$2\tilde{H}'_{2T} + E'_{2T} = -\int_x^1 \frac{dy}{y}\tilde{H} + \left[\frac{H}{x} - \int_x^1 \frac{dy}{y^2} H\right]$$

$$+ \quad \frac{m}{M}\left[\frac{1}{x}(2\tilde{H}_T + E_T) - \int_x^1 \frac{dy}{y^2}(2\tilde{H}_T + E_T)\right] + \frac{\mathcal{M}_{G_{11}}}{x} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{G_{11}}$$

Wandzura Wilczek Relations

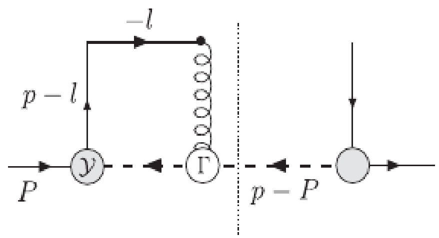
$$
\begin{aligned}
H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \quad = \quad & \left(1 + \frac{\Delta_T^2}{4M^2}\right) \int_x^1 \frac{dy}{y} \widetilde{H} + \frac{m}{M} \left[\frac{1}{x}\left(H_T - \frac{\Delta_T^2}{4M^2} E_T\right)\right. \\
& \left. - \int_x^1 \frac{dy}{y^2}\left(H_T - \frac{\Delta_T^2}{4M^2} E_T\right)\right] + \frac{\Delta_T^2}{4M^2}\left[\frac{1}{x}(H+E) - \int_x^1 \frac{dy}{y^2}(H+E)\right] \\
& + \left[\frac{\mathcal{M}_{G_{12}}}{x} - \int_x^1 \frac{dy}{y^2} \mathcal{M}_{G_{12}}\right] - \int_x^1 \frac{dy}{y} \mathcal{A}_{G_{12}}. \qquad (1)
\end{aligned}
$$

Including Final State Interactions

Brandon Kriesten

- Ji → Straight Gauge link

- Jaffe Manohar → Staple Link



- The difference is the torque

$$L_q^{JM} - L_q^{Ji} = \int \frac{d^2 z_T dz^-}{(2\pi)^3} \langle P', \Lambda' | \bar{\psi}(z) \gamma^+ (-g) \int_{z^-}^{\infty} dy^- U \left[z_1 G^{+1}(y^-) - z_2 G^{+2}(y^-) \right] U \psi(z) | P, \Lambda \rangle \Big|_{z^+=0}$$

Burkardt (2013)