Entanglement negativity: from fermionic systems to tensor network

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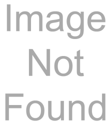
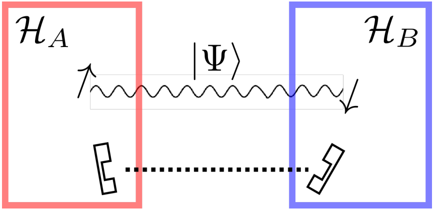


- *I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.* [– Erwin Schrödinger –]
- *Quantum entanglement has emerged as a universal phenomenon underlying the behavior of strongly interacting systems across vastly different scales. The workshop will address the use of methods based on quantum entanglement to address the hadron structure and thermalization in high energy collisions. We plan to bring together the experts working on the theory and applications of quantum entanglement in high energy, nuclear, condensed matter, and cold atom physics with the goal of finding new approaches to the long-standing problems of quark confinement and hadron structure.* [– Quantum Entanglement at Collider Energies –]

Quantum entanglement; basic setup



- Local quantum operations and classical communications (LOCC):

$$\rho \longrightarrow (A \otimes B)\rho(A \otimes B)^\dagger$$

$$\text{but not} \quad \rho \longrightarrow K_{AB}\,\rho\,K^\dagger_{AB}.$$

- "Quantum entanglement" = What cannot be generated by LOCC.

Quantum entanglement; how to quantify it?

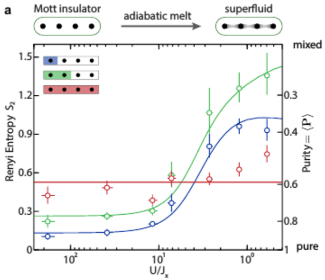
- von-Neumann entanglement entropy:

$$S_A := -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

  where ρ_A is the reduced density matrix.

$$\rho_A := \mathrm{Tr}_B \rho_{A\cup B}$$

- S_A for pure state $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$ decreases monotonically under LOCC.
- How to measure it experimentally? [R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. N. Rispoli, M. Greiner, Nature (2015)]



Entanglement in mixed states?

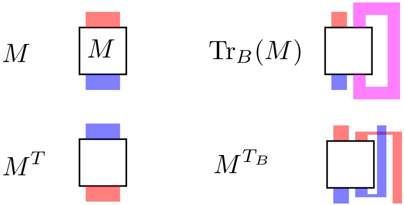
- How to quantify quantum entanglement between A and B when $\rho_{A\cup B}$ is *mixed* ? E.g., finite temperature, A, B is a part of bigger system.
- The entanglement entropy is an entanglement measure only for pure states. It is not monotone under LOCC.

Partial transpose (bosonic case)

- Definition: for an operator M, its partial transpose M^{T_B} is

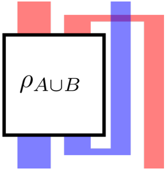
$$\langle e_i^{(A)} e_j^{(B)} | M^{T_B} | e_k^{(A)} e_l^{(B)} \rangle := \langle e_i^{(A)} e_l^{(B)} | M | e_k^{(A)} e_j^{(B)} \rangle$$

where $|e_i^{(A,B)}\rangle$ is the basis of $\mathcal{H}_{A,B}$.



Partial transpose and entanglement

$$\rho_{A\cup B}^{T_B}$$



Partial transpose and quantum entanglement

- Bell pair: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|]$$

- Partial transpose:

$$\rho^{T_2} = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - \underline{|00\rangle\langle 11|} - \underline{|11\rangle\langle 00|}]$$

- Entangled states are badly affected by partial transpose: Negative eigenvalues: $Spec(\rho^{T_2}) = \{1/2, 1/2, 1/2, -1/2\}$.
- C.f. For a classical state:

$$\rho = \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11|] = \rho^{T_2}$$

Partial transpose and Entanglement negativity

- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose*,

$$\mathcal{N}(\rho) := \frac{1}{2}\left(||\rho^{T_B}||_1 - 1\right), \quad \mathcal{E}(\rho) := \log ||\rho^{T_B}||_1,$$

  [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

- For mixed states, Negativity can extract quantum correlations only.
- The logarithmic negativity is not convex but an entanglement monotone. [Plenio (2005)]

Outline

1. Introduction: Partial transpose and negativity
2. Part I: Partial transpose and negativity for fermionic systems
3. Part II: Negativity in holographic models

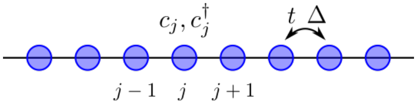
Part I: Partial transpose and negativity in fermionic systems

- Partial transpose is useful to detect entanglement in many-body states.
- How about fermion systems? E.g., the Kitaev chain
- Based on:
  - "Partial time-reversal transformation and entanglement negativity in fermionic systems", arXiv:1611.07536
  - "Entanglement negativity of fermions: monotonicity, separability criterion and classification of few-mode states ", arXiv:1804.08637
  - "Finite-temperature entanglement negativity of Fermi surface", arXiv:1807.09808

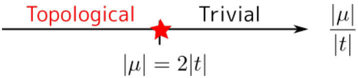
The Kitaev chain

- The Kitaev chain

$$H = \sum_j \left[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \right] - \mu \sum_j c_j^\dagger c_j$$



- Phase diagram: there are only two phases:

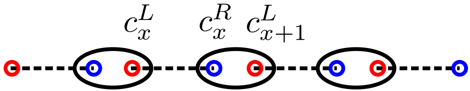


- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

Ground state; Majorana dimers

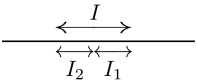
- Fractionalizing an electron into two Majoranas:

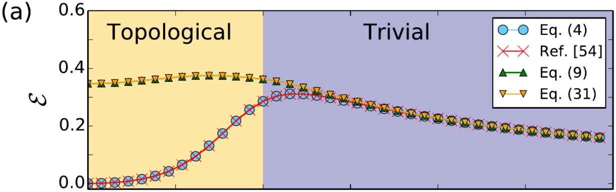
$$c_x = c_x^L + i c_x^R, \quad c_x^\dagger = c_x^L - i c_x^R.$$



Issues in fermionic systems (1)

- Consider log negativity $\mathcal{E}$ for two adjacent intervals of equal length. $(L = 4\ell = 8)$





- Vertical axis: μ/t ranging from 0 to 6.
- (Blue circles and Red corsses) is computed by Jordan-Wigner + bosonic partial transpose
- Log negativity fails to capture Majorana dimers.

Issues in fermionic systems (2)

- Partial transpose of bosonic Gaussian states is still Gaussian; easy to compute by using the correlation matrix
- Partial transpose of fermionic Gaussian states are not Gaussian
  - ρ^{T_1} can be written in terms of two Gaussian operators $O_\pm$:

$$\rho^{T_1} = \frac{1-i}{2} O_+ + \frac{1+i}{2} O_-$$

  - Negativity estimators/bounds using $\mathrm{Tr}\,[\sqrt{O_+ O_-}]$ [Herzog-Y. Wang (16), Eisert-Eisler-Zimborás (16)]
  - Spin structures: [Coser-Tonni-Calabrese, Herzog-Wang]

Partial transpose for fermions – our definition

[Shiozaki-Shapourian-SR (16)]

- Fermion operator algebra does not trivially factorize for $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$.
- Expand the density matrix in terms of Majorana fermions:

$$\rho_A = const. + \sum_{p_{1,2}} \rho_{p_1 p_2} c_{p_1} c_{p_2} + \sum_{p_{1,\dots,4}} \rho_{p_1 p_2 p_3 p_4} c_{p_1} c_{p_2} c_{p_3} c_{p_4} + \cdots$$

- Group them in terms of subregions:

$$\rho_A = \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} \underbrace{c_{p_1}^{A_1} \cdots c_{p_m}^{A_1}}_{\in A_1} \underbrace{c_{q_1}^{A_2} \cdots c_{q_n}^{A_2}}_{\in A_2}$$

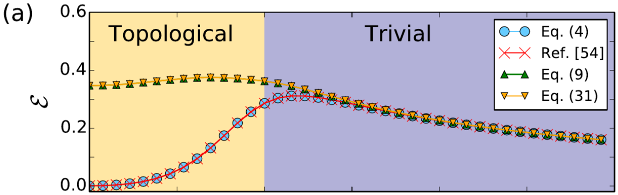
- Define partial transpose by $\rho_{p,q} \to \rho_{p,q} i^m$:

$$\rho_A^{T_1} = \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} i^m c_{p_1}^{A_1} \cdots c_{p_m}^{A_1} c_{q_1}^{A_2} \cdots c_{q_n}^{A_2}$$

- C.f. fermionic matrix product states perspective [Bultinck et al]
- Gaussian states stay Gaussian under our partial transpose

Comparison with previous definitions

[Shiozaki-Shapourian-SR (16)]



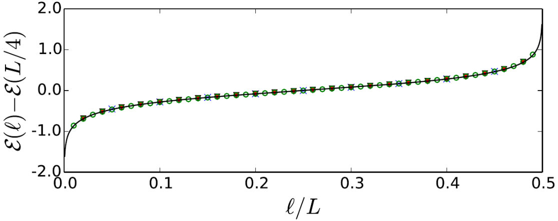
- (Blue circles and Red crosses): Old (bosonic) definition
- (Green triangles and Orange triangles) Our definition;
- At critical point: agrees with CFT prediction by Calabrese-Cardy-Tonni.

Motivation behind the construction

- Partial transpose can change the topology of spacetime: quantum field theory on an unoriented spacetime [Pollmann-Turner, Calabrese-Cardy-Tonni, Shiozaki-SR]
- In the topological phase, the path integral on an unoriented spacetime can be computed using topological quantum field theory (TQFT).
- The relevant TQFT are invertible, fermionic and defined on unoriented spacetime ("Pin" TQFT) [Kapustin, Hsieh-Cho-Sule-SR-Leigh, Kapustin-Thorngren-Turzillo-Wang, Hsieh-Cho-SR, Witten, Freed-Hopkins, Metlitski, Barkeshli-Bonderson-Jian-Cheng-Walker, Yonekura-Tachikawa, and many others]
- We use TQFT as a guide to search for a proper definition of partial transpose for fermions.

Critical point

- The logarithmic negativity for two adjacent intervals of equal length ℓ at the critical point (the SSH model).

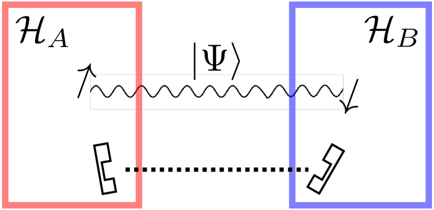


- The numerical result using the free fermion formula (points) with $L = 40$-400 agrees with the CFT result (solid line). [Calabrese-Cardy-Tonni]

$$\mathcal{E} = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$$

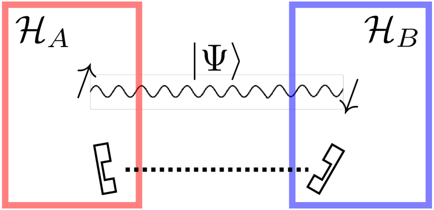
- Analytical derivation by using the replica method + Fisher-Hartwig.

Monotoncity under LOCC



- For bosonic systems, negativity is LOCC monotone
- I.e., what cannot be generated by LOCC = "quantum entanglement".
- von-Neumann entanglement entropy decreases monotonically at $T = 0$, but *not* at $T > 0$.

Monotoncity under LOCC



- We have introduced fermionic version of partial transpose, and negativity, but is it a good entanglement measure? Is it monotone under LOCC?
- In [Shapourian-SR (18)], we proved that if LOCC are taken to be fermion number parity preserving. fermionic entanglement negativity is monotone; a proper entanglement measure.

Application: Fermi surface at finite T

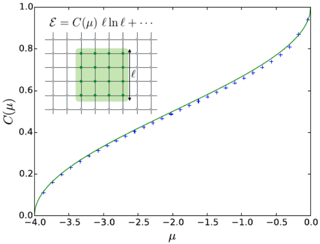
- Renyi entanglement entropy:

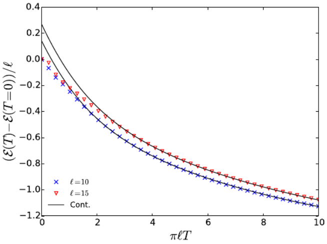
$$S_n = \frac{n+1}{6n} C_2 \cdot \ell \ln \left| \frac{\beta}{\pi a_0} \sinh \frac{\pi \ell}{\beta} \right|$$

$$\text{where} \quad C_2 = \frac{1}{8\pi} \int_{\partial\Omega} \int_{\partial\Gamma} dS_k dS_x |n_x \cdot n_k|$$

- Negativity:

$$\mathcal{E} = C_2 \cdot \frac{\ell}{2} \left[\ln \left(\frac{\beta}{\pi a_0} \sinh \frac{\pi \ell}{\beta} \right) - \frac{\pi \ell}{\beta} \right]$$

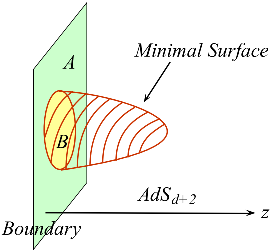




- No sudden death

Part II: Negativity in holographic models

- Questions: Is there a geometric/holographic interpretation of negativity?
- C.f. Holographic entanglement entropy formula

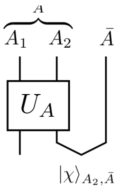


- Based on:
  - " Entanglement negativity and minimal entanglement wedge cross sections in holographic theories" arXiv:1808.00446

Holographic code

- A toy model of holography using quantum error correcting code; [Almheiri-Dong-Harlow (15), Harlow (17)]

$$|\tilde{i}\rangle = U_A(|i\rangle_{A_1} \otimes |\chi\rangle_{A_2,\bar{A}}), \quad |\chi\rangle_{A_2,\bar{A}} \in \mathscr{H}_{A_2,\bar{A}}.$$



- This code can correct for the erasure of, e.g., the 3rd "qutrit",

$$U_A^\dagger |\tilde{i}\rangle = |i\rangle_{A_1} |\chi\rangle_{A_2\bar{A}} ,$$

- Captures many aspects of holography; black holes, bulk reconstruction, subregion duality, holographic entanglement entropy, etc.

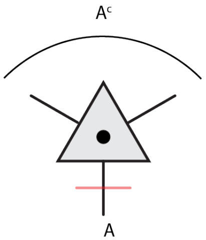
- E.g., 3-qutrit code: where

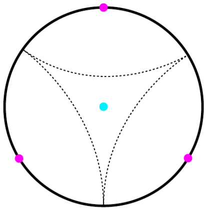
$$|\tilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle),$$

$$|\tilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle),$$

$$|\tilde{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle),$$

$$|\chi\rangle \equiv \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle).$$





Holographic entanglement entropy

- Encode input state ρ on $\mathcal{H}_{A_1}$

$$\tilde{\rho} = U_A(\rho_{A_1} \otimes |\chi\rangle \langle\chi|_{A_2,\bar{A}})U_A^\dagger.$$

- Entanglement entropies for $\tilde{\rho}_A = \mathrm{Tr}_{\bar{A}}\, \tilde{\rho}$ and $\tilde{\rho}_{\bar{A}} = \mathrm{Tr}_A\, \tilde{\rho}$:

$$S(\tilde{\rho}_A) = S(\chi_{A_2}) + S(\tilde{\rho}), \quad S(\tilde{\rho}_{\bar{A}}) = S(\chi_{A_2}).$$

  (where $\chi_{A_2} \equiv \mathrm{Tr}_{\bar{A}}\ |\chi\rangle \langle\chi|_{A_2,\bar{A}}$)

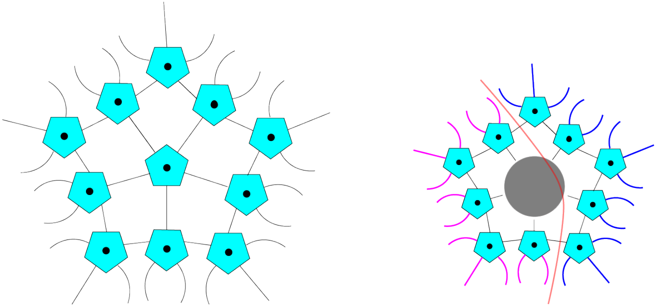
- By identifying $S(\chi_{A_2})I_{code}$ as the "area operator", $\mathscr{L}$,

$$\langle\mathscr{L}\rangle = S(\chi_{A_2}) = -\sum_a p_a \log p_a,$$

  an "holographic formula" for error-correcting codes is obtained.

Perfect tensor network code

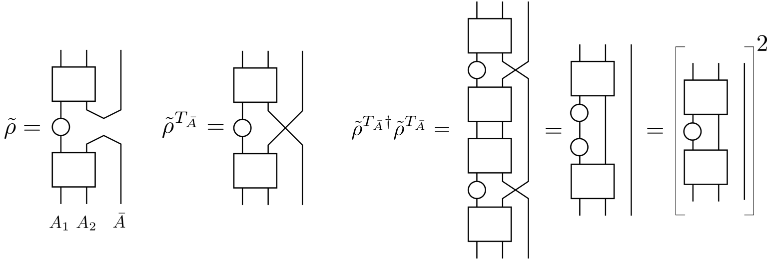
- Error correcting code encoding multiple "bulk" logical qubits into multiple "boundary" physical qubits [Pastawski-Yoshida-Harlow-Preskill(15)]



- Consisting of perfect tensors.

Negativity in holographic error correcting code

- Diagramatic computation:



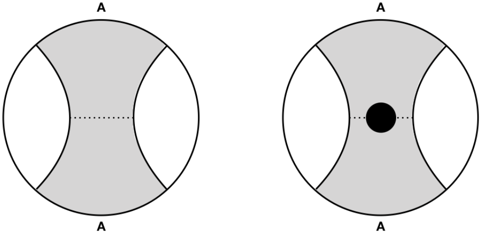
$$\mathcal{N}(\tilde{\rho}) = \frac{\left(\sum_a \sqrt{p_a}\right)^2 - 1}{2}, \quad \mathscr{E}(\tilde{\rho}) = \log\left(\sum_a \sqrt{p_a}\right)^2.$$

- The negativity is equal to $\langle \mathscr{L} \rangle$ when χ_{A_2} is maximally mixed:

$$\mathcal{E}(\tilde{\rho}) = \langle \mathscr{L} \rangle = \log(|\tilde{A}|).$$

Entanglement wedge

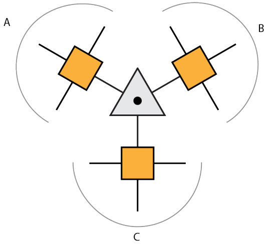
- Negativity is captured by the so-called entanglement wedge [Headrick et al (14), Jafferis-Suh (14), Jafferis-Lewkowycz-Maldacena-Suh (15), ...] (minimal entanglement wedge cross section).

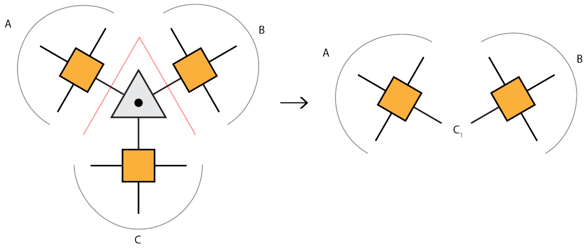


- Previous work: Entanglement of purification [Takayanagi-Umemoto(17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]

9-qutrit model

- We have tested out entanglement wedge formula for 9-qutrit model.





AdS_3/CFT_2

- How about negativity in the full fledged AdS/CFT? No time to discuss ... but rather interesting.
- In holographic code models; many quantities are "degenerate"; mutual information, negativity, entanglement of purification.
- Back reaction is expected; since, e.g., in certain case, negativity is Renyi entropy at $n = 1/2$ [Dong(16)]
- See our paper for more detailed comparisons.

Summary

- Based on the topological field theory intuition, we introduced partial transpose for fermionic systems.
- The (log) negativity using the fermionic partial transpose can capture the formation of Majorana dimers in the Kitaev chain.
- Partial transpose of fermionic Gaussian states are Gaussian, and hence easy to compute.
- Entanglement negativity and entanglement wedge cross section