How Entangled Spin Chain Can Be?

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Entropy of a Subsystem

$$\mathbf{S}_{\text{class}} = -\sum_n p_n \log p_n$$

Here p_n are probabilities. In classical case if the entropy is zero than all $p_n = 0$ exept of one, which is equal to 1. If the total entropy is 0 than there is no entropy in any subsystem.
Not so in quantum case. There can be entropy in a subsystem, also the total entropy is 0. More entropy in a subsystem [keeping total entropy equal to 0] stronger quantum fluctuations are.

Growth

Infinite chain with local interaction and unique ground state.
How fast the entropy of a block of spins can grow with the length of the block?
Gapped models: area law Srednicki 1993.

Many gapp-less models: logarithmic growth
C.Holzhey, F.Larsen, F.Wilczek , 1994
consistent with CFT

Can entanglement entropy grow faster for a Hamiltonian with local interaction?

Overview of Fredkin model

- A chain of interacting half integer spins: $(k - 1/2)$. Integer k will appear as number of colors.
- Locality: spins in three nearest lattice sites interact.
- Density of the Hamiltonian minimizes at each lattice site. Frustration free.
- The ground state of the model is described by random walk on upper half plane of a square lattice.

Notations

- For a chain of even length N, the Hilbert total space is a tensor product $W \otimes W \otimes W \otimes .... \otimes W$ of the local Hilbert spaces at the different sites. Simplest case $k = 1$ (spin $1/2$) $\dim W = 2$.
- We will write the Hamiltonian as a sum of local operators H_j [density of the Hamiltonian]. The subscript indicates a lattice site, where the H_j actes non-trivially:
$1 \otimes 1... \otimes 1 \otimes \tilde{H} \otimes 1 \otimes ... \otimes 1$: we have (j-1) factors of 1 before $\tilde{H}$.
- The Pauli matrices are given by:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z = \frac{1}{2}(1 + \Pi_{i,i+1})$$

Hamiltonian for spins 1/2

$$H_{bulk} = \sum_{j=1}^{N} (1+\sigma_j^z)(1-\vec{\sigma}_{j+1}\cdot\vec{\sigma}_{j+2}) + \left(1-\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}\right)(1-\sigma_{j+2}^z\right)$$

- Can be rewritten in terms of *Fredkin gate* F_{ijk}: it permutes sites **j** and **k** if site **i** is in state $\left|\uparrow\right\rangle_i$. Nothing happen to the sites **j** and **k** if site **i** is in state $\left|\downarrow\right\rangle_i$.

$$H_{bulk} = \sum_{j=1}^{N} 1 - F_{j,j+1,j+2} + 1 - \sigma_{j+2}^x \left(F_{j+2,j+1,j}\right) \sigma_{j+2}^x$$

- We add local magnetic field at the boundary

$$H_\partial = \left|\downarrow_1\right\rangle\left\langle\downarrow_1\right| + \left|\uparrow_N\right\rangle\left\langle\uparrow_N\right|$$

It make the ground state unique.

Historical remark

About 50 years ago computer scientists discussed: how much energy is necessary for computation? Rolf Landauer discovered that the energy has to be spent only for cleaning the database: $kT \log 2$ per bit.
Tommasso Toffoli and Edward Fredkin suggested reversible computation based of 3-bit operations. Fredkin gate was introduced in 1982. See
Richard Feynman: Lectures on Computation
Wikipedia

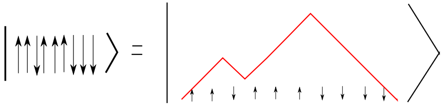
Properties of the Hamiltonian

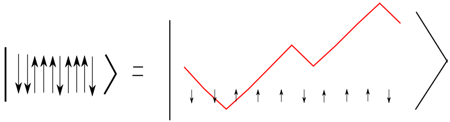
$$H_{bulk} = \sum_{j=1}^{N} (1+\sigma_j^z)(1-\vec{\sigma}_{j+1}\cdot\vec{\sigma}_{j+2}) + (1-\vec{\sigma}_j\cdot\vec{\sigma}_{j+1})(1-\sigma_{j+2}^z)$$

- The Hamiltonian is unfrustrated: each local term vanishes (minimizes) at the ground state.
- The Hamiltonian is invariant under joint reflection: $j \to N-j$ and $S_z \to -S_z$
- The Hamiltonian commutes with third component of total spin: $\sum_j \sigma_z$
- Each term in the Hamiltonian is positive semidefinite.

Spin states as paths

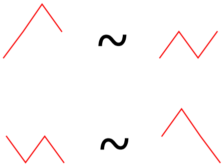
- Identify basis states with paths on an integer lattice assigning spin up to a step up and spin down to a step down, $|\uparrow\downarrow\uparrow\rangle = |/\backslash/\rangle, |\downarrow\uparrow\downarrow\rangle = |\backslash/\backslash\rangle$.
- The map defines a path uniquely up to a constant shifting of the height axis, we set the height of the lowest point of the path to zero.





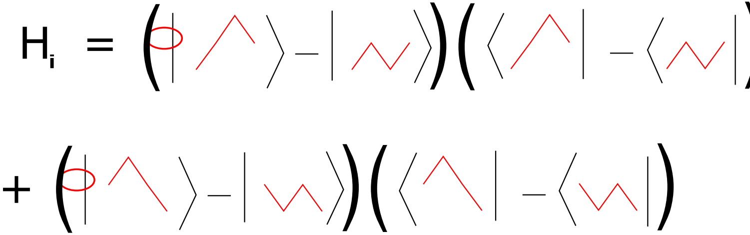
Defining an equivalence relation on paths

- Let us define a local equivalence relation on paths. We say that two paths are equivalent if they are related by a sequence of the Fredkin moves below.
- The equivalence relation allows us to move a $/\backslash$ peak to any point in the path.
- The moves conserve endpoints of the path.
- The ground state will be invariant under Fredkin moves.



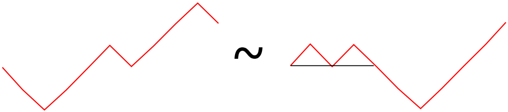
Rewrite the Hamiltonian

- We rewrite the Hamiltonian in terms of projectors



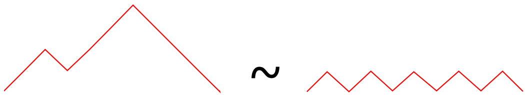
Equivalence relation on paths

- We can move a $/\backslash$ peak to any point in the path.
- The moves preserves the endpoints of the path.
- If we repeatedly move the last peak of the path to the beginning, the process will converge to a sequence of $/\backslash$ peaks followed by a single dip with no peaks.
- Any two paths with the same endpoints will be mapped to the same path after N/2 steps.
- The equivalence classes are fully characterized by the heights of the path endpoints. We call them $C_{a,b}(N)$, where **a** and **b** are the heights of the first and last endpoints.



Choosing the ground state

- For equivalence class of paths $C_{a,b}(N)$, we define the Dyck state $|C_{a,b}(N)\rangle$ as the symmetric sum [equal coeficients] of all basis states corresponding to the paths in $C_{a,b}(N)$.
- The states $|C_{a,b}(N)\rangle$ are ground states of all terms in the bulk Hamiltonian.
- The state $|C_{0,0}(N)\rangle = |C(N)\rangle$ which we call *the* Dyck state is of particular interest: it is the only bulk ground state which is also a ground state of the boundary part of teh Hamiltonian: $H_\partial = |{\downarrow_1}\rangle\langle{\downarrow_1}| + |{\uparrow_N}\rangle\langle{\uparrow_N}|$.
- $|C(N)\rangle$ is the superposition of all Dyck paths (paths which never go below of their endpoints). These are counted by the famous Catalan numbers $C_N = \prod_{l=2}^{N} \frac{N+l}{l}$
- The ground state is invariant under Fredkin moves.



The Schmidt coefficient= entanglement spectrum

- The ground state $|C(N)\rangle$ has a Schmidt decomposition $\sum_m \sqrt{p_m}\,|C_{0,m}(L)\rangle \otimes |C_{m,0}(N-L)\rangle$ for a block of lenght L.
- The Schmidt coefficients in our case are given by $p_m = \frac{|C_{0,m}(L)||C_{0,m}(N-L)|}{C(N)}$.
- Here
$$|C_{a,b}(L)| = \binom{L}{\frac{L+a+b}{2}} - \binom{L}{\frac{L+a+b}{2}+1}$$
- The Schmidt rank, is the number of nonzero Schmidt coefficients: in our case is $\left\lfloor \frac{L}{2} \right\rfloor$.
- The entanglement entropy is $S = -\sum_m p_m Log(p_m)$.

Entanglement entropy for spin 1/2 is boring

- Using Stirling's we get for N = 2n, L = 2l, m=2h

$$p_m \approx \frac{h^2}{Z} \exp\left(-h^2\left[\frac{1}{l} + \frac{1}{n-l}\right]\right), \qquad \sum_m p_m = 1$$

- Entanglement entropy is:

$$S \approx \frac{1}{2} Log(L) + O(C)$$

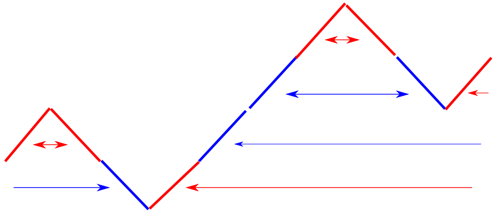
Constant term is around 0.437

Generalization to higher spins: colored Dyck walks

- For spin $3/2$, we have to consider colored paths: red, blue.
- For $3/2$, we can identify $m = 3/2$ with a red up step, $m = 1/2$ with a blue up step, $m = -1/2$ with a blue down step, and $m = -3/2$ with a red down step.
- To analyze colored paths, we need to mention matched steps.

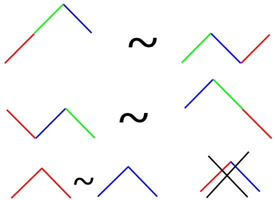
Spin 3/2 and up: colorings and matchings

- Up step and a down step are *matched* if the the up step is of the form $(i, j) \to (i+1, j+1)$ and the down step is the first down step occurring after our up step which is of the form $(i', j+1) \to (i'+1, j)$ [**same height**]. Equivalently, two steps are matched if the subpath between them is a Dyck path.
- *Matched steps should have the same color.*



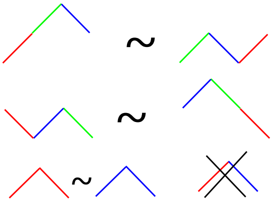
Fredkin moves, coloring rules

- Fredkin moves will move peaks along with their colors.
- We introduce the coloring rules, which allow us to recolor matched peaks and forbid matched pairs from having different colors.
- The ground state will be invariant under Fredkin moves.



Moving colores

- The colored Fredkin moves allow us to reduce any path with colored steps to one where all matched steps are adjacent to their match. The coloring rules can then be applied to recolor matched pairs or to exclude invalid path colorings.
- This allows us to define equivalence classes of colored paths which are defined only by their endpoints, and the colors of the unmatched steps to their left and right.



Hamiltonian for spin 3/2 and higher

- These rules can be implemented by the following SU(k)-invariant Hamiltonian of the form $H = H_F + H_X + H_\partial$.
- H_F implements Fredkin moves. It can be expressed in terms of operators P_j^+, P_j^- which project onto up/down steps disregarding the colors, and the cyclic permutation operators $C_{i,j,k}$:

$$
\begin{aligned}
H_F = \sum_{j=1}^{N-2} P_j^+ P_{j+1}^+ P_{j+2}^- + P_j^+ P_{j+1}^- P_{j+2}^+ - \\
- P_j^+ P_{j+1}^+ P_{j+2}^- C_{j,j+1,j+2} - C^\dagger_{j,j+1,j+2} P_j^+ P_{j+1}^+ P_{j+2}^- + \\
+ P_j^+ P_{j+1}^- P_{j+2}^- + P_j^- P_{j+1}^+ P_{j+2}^- - \\
- P_j^+ P_{j+1}^- P_{j+2}^- C^\dagger_{j,j+1,j+2} - C_{j,j+1,j+2} P_j^+ P_{j+1}^- P_{j+2}^-
\end{aligned}
$$

The Hamiltonian

- The matching term H_X, is defined using local SU(k) generators T_j^a acting on the color space. Up steps lie in the fundamental representation of SU(k), while down step colors lie in its conjugate repesentation.
- The color matching then simply corresponds to projecting the colors of matched pairs onto SU(k) singlets.
- The Hamiltonian is symmetric under SU(k), it commutes with generators $T^a = \sum_j T_j^a$

$$
\begin{aligned}
H_X =& \sum_{j=1}^{N-1} P_j^+ P_{j+1}^- \left[\sum_a (T_j^a + T_{j+1}^a)^2 \right] \\
H_\partial =& P_1^- + P_N^+
\end{aligned}
$$

Ground state for spin 3/2 and higher

- The ground state is the symmetric sum [equal coeficients] of all the proper colorings of all the Dyck paths

Schmidt decomposition

- The Schmidt coefficients in colored case are closely related to the spin $1/2$ case for the block of size L.
- It looks like $\sum_{m,c} \sqrt{q_{m,c}}\, |C_{0,\varnothing,m,c}(L)\rangle \otimes |C_{m,c,0,\varnothing}(N-L)\rangle$ where c is a string of k-bits (k is the number of colors) representing all possible colorings of the unmatched steps in the left and right blocks.
- we have $q_{m,c} = k^m p_m$ which is independent of c.

Spin 3/2 and up: Entanglement entropy

- The entropy can simply be written as $-\sum_m p_m \log(k^m p_m) = -\sum_m p_m[\log(p_m) + m\log(k)]$
- The first term is just the spin 1/2 entropy, while the second is $\log(k)$ times the expectation value of the path height, which scales as $O(\sqrt{N})$
- The leading term $L \to \infty$ of entanglement entropy is:

$$\frac{2}{\sqrt{\pi}}\log(k)\sqrt{2\frac{L(N-L)}{N}}$$

It increase as a square root of the size of the block of spins L. Faster than usual $\log L$.

q-deformation is similar to replacement of XXX by XXZ

$$H_q = H_F^q + H_X + H_\partial$$

$H_F^q = \sum_{j=1}^{N-2} \sum_{c_1,c_2,c_3=1}^{k} [\Big(|\uparrow_j^{c_1}\uparrow_{j+1}^{c_2}\downarrow_{j+2}^{c_3}> -q^2|\uparrow_j^{c_2}\uparrow_{j+1}^{c_3}\downarrow_{j+2}^{c_1}>\Big)$
$\Big(<\uparrow_j^{c_1}\uparrow_{j+1}^{c_2}\downarrow_{j+2}^{c_3}| - q^2 <\uparrow_j^{c_2}\uparrow_{j+1}^{c_3}\downarrow_{j+2}^{c_1}|\Big) +$
$\Big(q^2|\downarrow_j^{c_1}\uparrow_{j+1}^{c_2}\downarrow_{j+2}^{c_3}> -|\uparrow_j^{c_2}\uparrow_{j+1}^{c_3}\downarrow_{j+2}^{c_1}>\Big)$
$\Big(q^2 <\downarrow_j^{c_1}\uparrow_{j+1}^{c_2}\downarrow_{j+2}^{c_3}| - <\uparrow_j^{c_2}\uparrow_{j+1}^{c_3}\downarrow_{j+2}^{c_1}|\Big)]\frac{1}{1+q^4}$

The ground state:

$$|GS>= \sum_{\text{colored Dyck walks}} q^{A(w)}|w>$$

$A(w)$ is area under Dyck walk. For $q>1$ and $k>1$ entanglement entropy is linear in the size of the block: $L \log k$.