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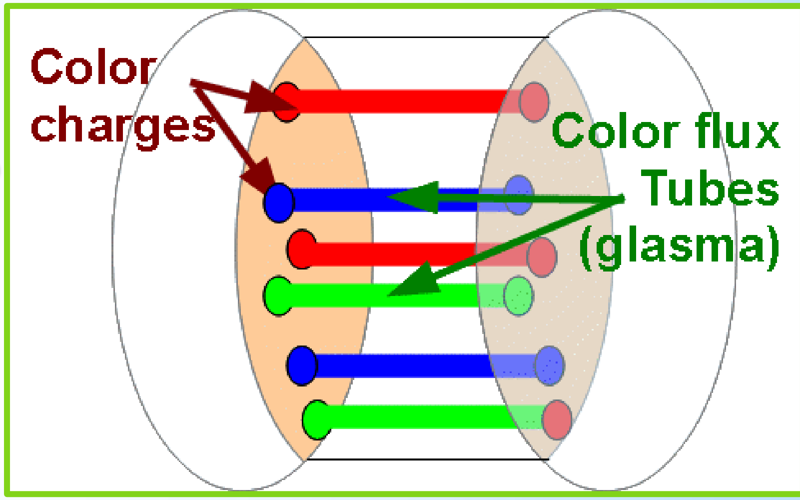
Quantum chaos and entanglement generation in matrix quantum mechanics

Motivation

Initially pure states of colliding nuclei
Overpopulated gluon states
Almost “classical” gauge fields



Chaotic Classical Dynamics [Saviddy,Susskind…]

- Gauge fields forget initial conditions

…but is it enough for **Thermalization?**

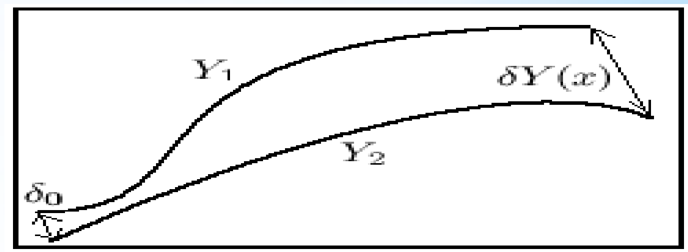
How a nearly thermal state is generated out of pure states? Relevant timescales?

Motivation

Thermalization for quantum systems?

- Quantum extension of Lyapunov exponents - OTOCs $<[P(0),X(t)]^2>$
- Generation of entanglement between subsystems

$$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$



- Quasinormal ringing

Timescales: quantum vs classical?

☹ QFT tools extremely limited…

☺ …Holography provides intuition

In this talk:

Numerical attempt to look at the real-time dynamics of gauge-like models beyond classical limit

Of course, not an exact simulation, but should be good at early times

Approximating all states by Gaussians, study entanglement generation

Our system:

Pure Yang-Mills theory dimensionally reduced from *D=d+1* down to *D=0+1*

$$L = \frac{1}{2g}\left[\mathrm{tr}\,\dot{X}^i\dot{X}^i - \frac{1}{2}\mathrm{tr}\,\left[X^i, X^j\right]^2\right]$$

- X^i is the ex-gauge field
- Classically chaotic dynamics
- For reasons to be explained below, we use *d=9* $\Rightarrow$ *1/d* expansion/mean field is a good description

BFSS Model: Classically chaotic system with a holographic dual

N=1 Supersymmetric Yang-Mills in D=1+9:

Reduce to a single point = BFSS matrix model [Banks, Fischler, Shenker, Susskind'1997]

$$L = \frac{1}{2g}\left[\mathrm{tr}\dot{X}^i\dot{X}^i + 2\theta^T\dot{\theta} - \frac{1}{2}\mathrm{tr}[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right]$$

N x N hermitian matrices

Majorana-Weyl fermions, N x N hermitian

System of *N* D0 branes joined by open strings [Witten'96]:

- X^{ii}_{μ} = D0 brane positions
- X^{ij}_{μ} = open string excitations

Classical chaos and BH physics

Stringy interpretation:

- Dynamics of gravitating D0 branes
- Thermalized state = black hole
- Classical chaos = info scrambling

[Sekino,Susskind]

Expected to be "maximally chaotic" at low temperatures!

Bounds on chaos

Reasonable physical assumptions
Analyticity of OTOCs ⟶

$$\boxed{\lambda_L < 2\pi T}$$

(QGP ~0.1 fm/c)

[Maldacena Shenker Stanford'15]

- **Holographic models with black holes saturate the bound(e.g. SYK)**
- **In contrast, for classical YM** $\boxed{\lambda_L \sim T^{1/4}}$

What happens at low T ???

Gaussian state approximation

Simple example:
Double-well potential

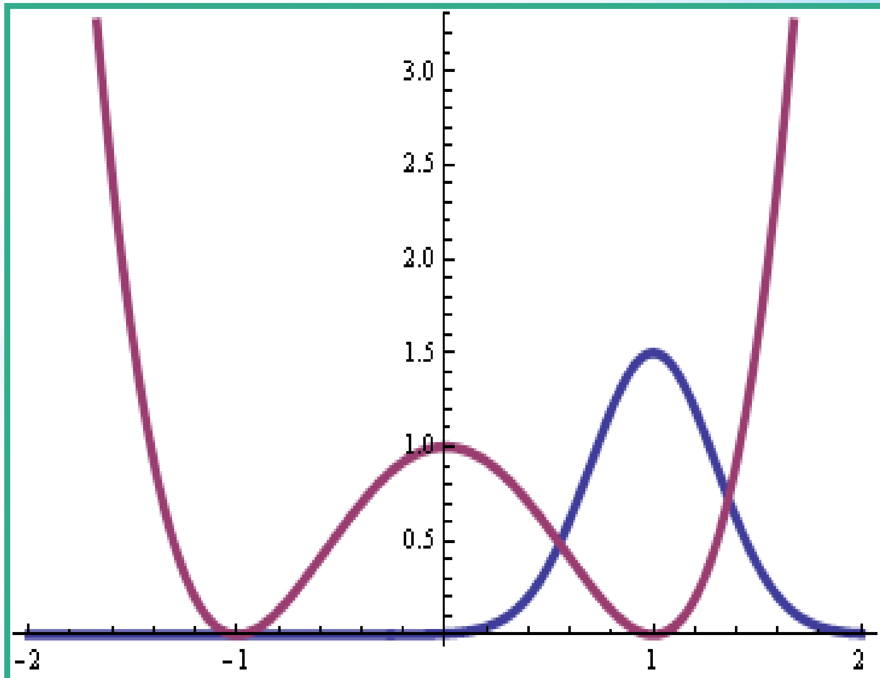
$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{a\hat{x}^2}{2} + \frac{b\hat{x}^3}{3} + \frac{c\hat{x}^4}{4}$$

Heisenberg equations of motion

$$\partial_t \hat{x} = \hat{p},$$
$$\partial_t \hat{p} = -a\hat{x} - b\hat{x}^2 - c\hat{x}^3$$

Also, for example

$$\partial_t \left(\hat{x}\hat{x}\right) = \hat{x}\hat{p} + \hat{p}\hat{x}$$



Next step: Gaussian Wigner function

Assume Gaussian wave function at any t

Simpler: Gaussian Wigner function

$$\langle \hat{x}^2 \rangle = x^2 + \sigma_{xx},$$
$$\langle \hat{p}^2 \rangle = p^2 + \sigma_{pp},$$
$$\langle \frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2} \rangle = xp + \sigma_{xp}$$

For other correlators: use Wick theorem!

$$\langle \hat{x}^4 \rangle = x^4 + 6x^2\sigma_{xx} + 3\sigma_x x^2,$$
$$\langle \hat{x}^2\hat{p} \rangle = x^2 p + 2x\sigma_{xp} + p\sigma_{xx}$$

Derive closed equations for

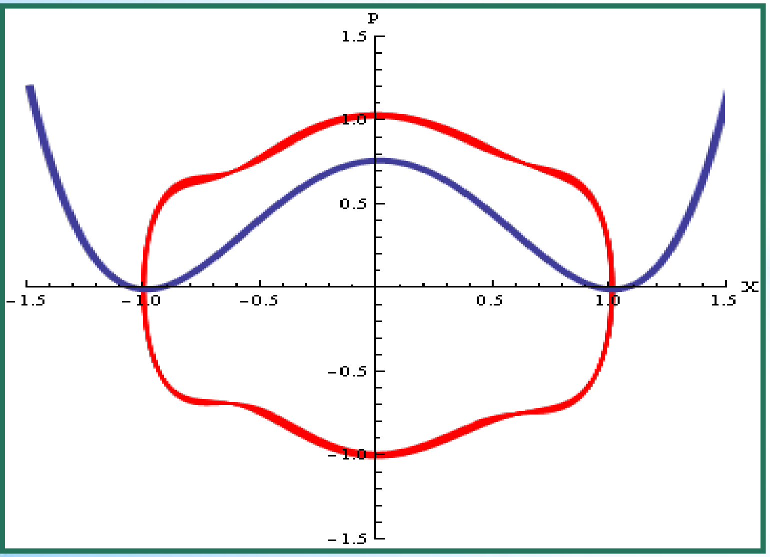
***x*, *p*, σ_{xx} , σ_{xp} , σ_{pp}**

Origin of tunnelling

$$\partial_t p = -ax - bx^2 - cx^3 - b\sigma_{xx} - 3cx\sigma_{xx},$$
$$\partial_t x = p$$

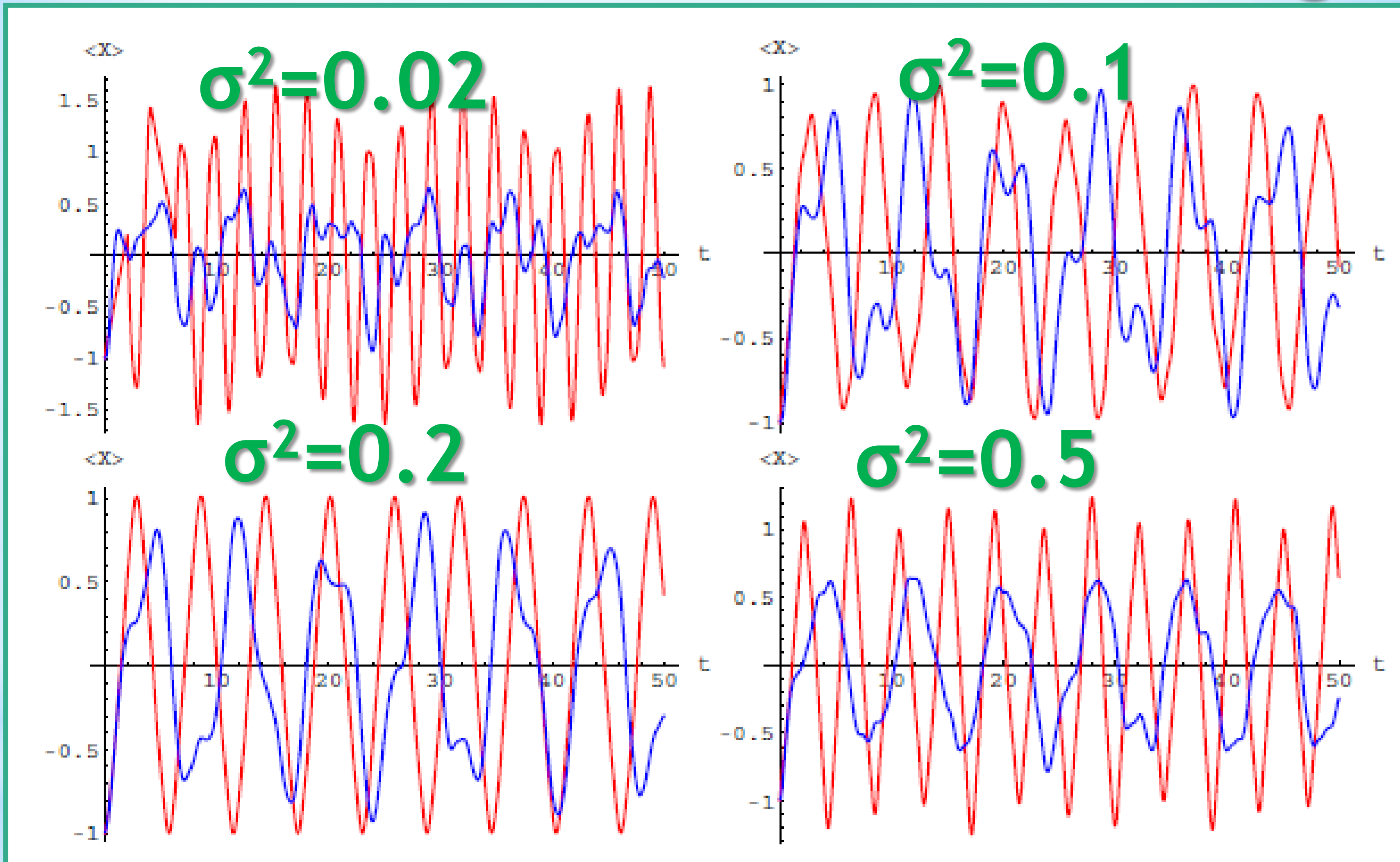
Positive force even at *x=0* (classical minimum)

$$\partial_t \sigma_{xx} = 2\sigma_{xp},$$
$$\partial_t \sigma_{xp} = \sigma_{pp} - a\sigma_{xx} - 2bx\sigma_{xx} - 3cx^2\sigma_{xx} - 3c\sigma_{xx}^2,$$
$$\partial_t \sigma_{pp} = -2\left(a\sigma_{xp} + 2bx\sigma_{xp} + 3cx^2\sigma_{xp} + 3c\sigma_{xx}\sigma_{xp}\right)$$



Quantum force causes classical trajectory to leave classical minimum

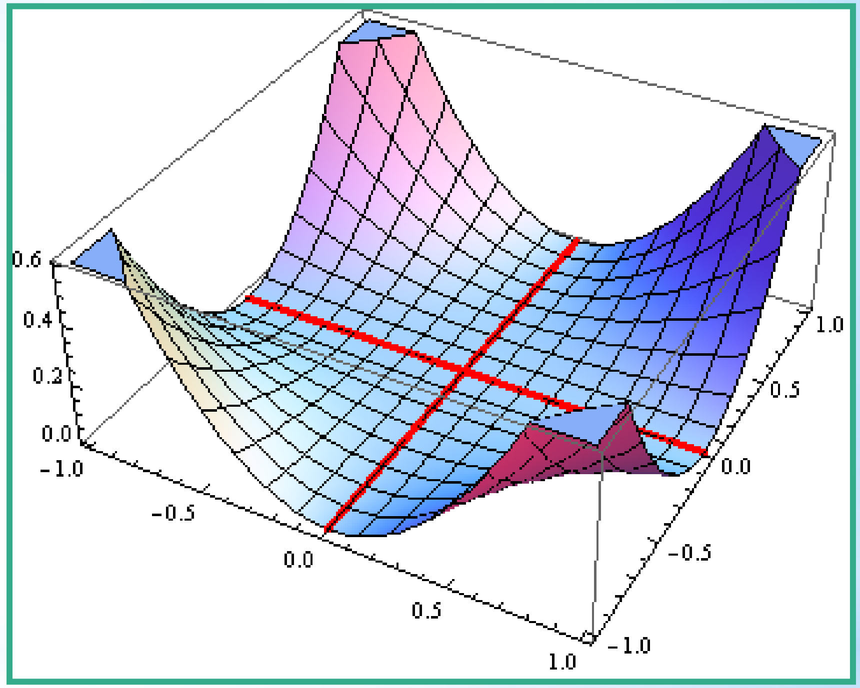
Gaussian state vs exact Schrödinger



- Early-time evolution OK
- Tunnelling period qualitatively OK

2D potential with flat directions
(closer to BFSS model)
[SUSY QM, Lüscher, de Wit, Nicolai]

$$\hat{H} = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\kappa}{2}\hat{x}^2\hat{y}^2$$

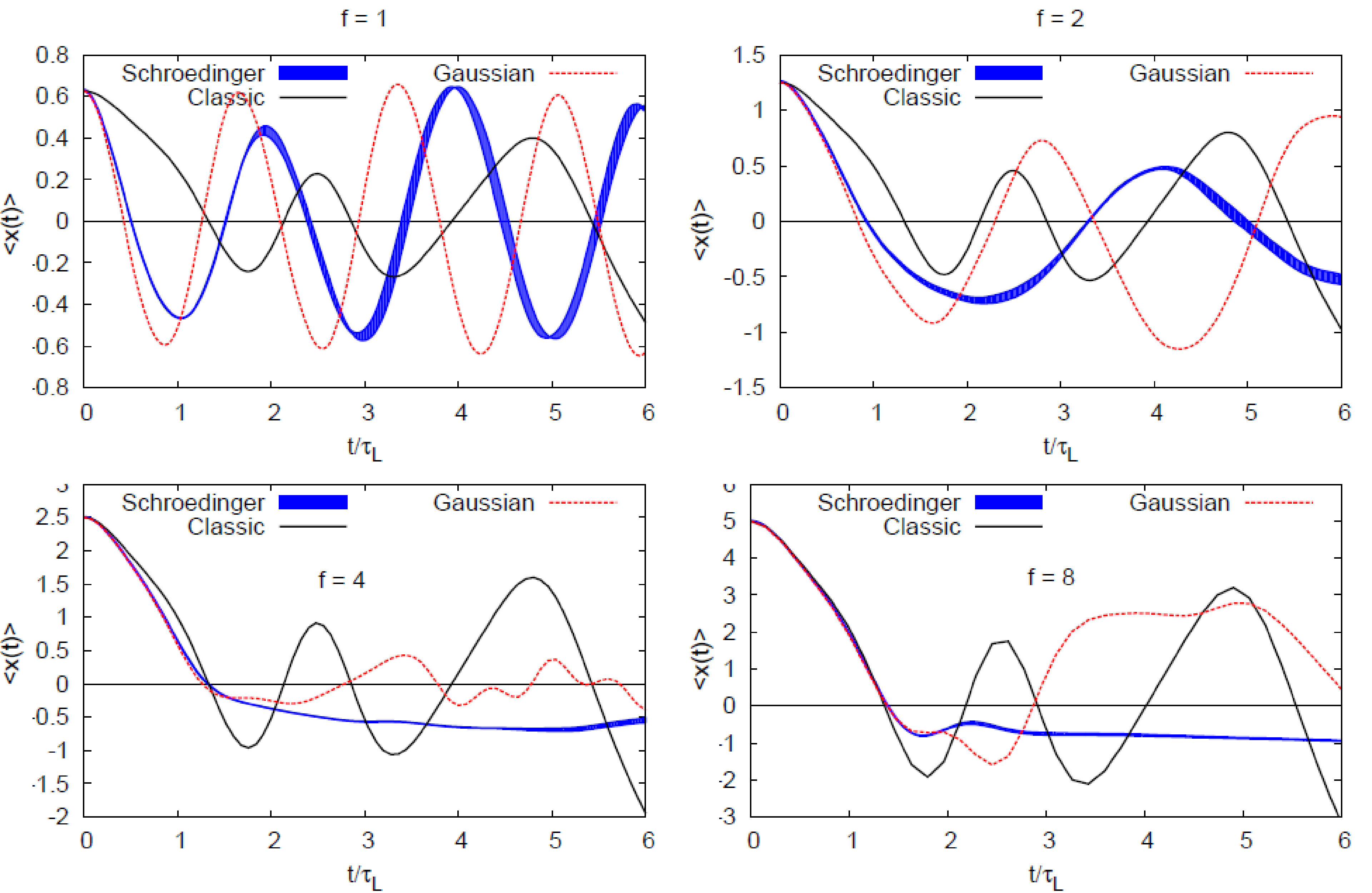


Classic runaway along *x=0 or y=0*

Classically chaotic!

We start with a Gaussian wave packet at distance *f* from the origin (away from flat directions)

Gaussian state vs exact Schrödinger



Gaussian state approximation

- ✓ Is good for at least two classical Lyapunov times
- ✓ Maps pure states to pure states [discussion follows below]
- ✓ Allows to study entanglement
- ✓ Closely related to semiclassics
- ✓ Is better for chaotic than for regular systems [nlin/0406054]
- ✓ Is likely safe in the large-N limit
- X Is not a unitary evolution

Matrix QM: Hamiltonian formulation

$$\hat{H} = \frac{1}{2}\hat{P}_i^a \hat{P}_i^b + \frac{1}{4} C_{abc} C_{ade} \hat{X}_i^b \hat{X}_j^c \hat{X}_i^d \hat{X}_j^e + \frac{i}{2} C_{abc} \hat{\psi}_\alpha^a \left[\sigma_i\right]_{\alpha\beta} \hat{X}_i^b \hat{\psi}_\beta^c,$$

a,b,c – su(N) Lie algebra indices

Heisenberg equations of motion

$$\boxed{\partial_t \hat{X}_i^a = \hat{P}_i^a}$$

$$\boxed{\partial_t \hat{P}_i^a = -C_{abc} C_{cde} \hat{X}_j^b \hat{X}_i^d \hat{X}_j^e - \frac{i}{2} C_{bac} \sigma^i_{\alpha\beta} \hat{\psi}_\alpha^b \hat{\psi}_\beta^c,}$$

$$\boxed{\partial_t \hat{\psi}_\alpha^a = C_{abc} X_i^b \sigma^i_{\alpha\beta} \hat{\psi}_\beta^c}$$

GS approximatio for matrix QM

$$\partial_t P^a_i = -C_{abc}C_{cde}X^b_j X^d_i X^e_j - \frac{i}{2}C_{bac}\sigma^i_{\alpha\beta}\langle\psi^b_\alpha\psi^c_\beta\rangle -$$
$$- C_{abc}C_{cde}X^b_j[XX]^{de}_{ij} - C_{abc}C_{cde}[XX]^{be}_{jj}X^d_i - C_{abc}C_{cde}[XX]^{bd}_{ji}X^e_j$$

$$\partial_t [XX]^{ab}_{ij} = [XP]^{ab}_{ij} + [XP]^{ba}_{ji},$$
$$\partial_t [XP]^{af}_{ik} = [PP]^{af}_{ik} - C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XX]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XX]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XX]^{ef}_{jk},$$
$$\partial_t [PP]^{af}_{ik} = -C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XP]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XP]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XP]^{ef}_{jk} + (\{a,i\} \leftrightarrow \{f,k\})$$

- **CPU time ~ *N^5* (double commutators)**
- **RAM memory ~ *N^4***
- ***SUSY broken, unfortunately …***

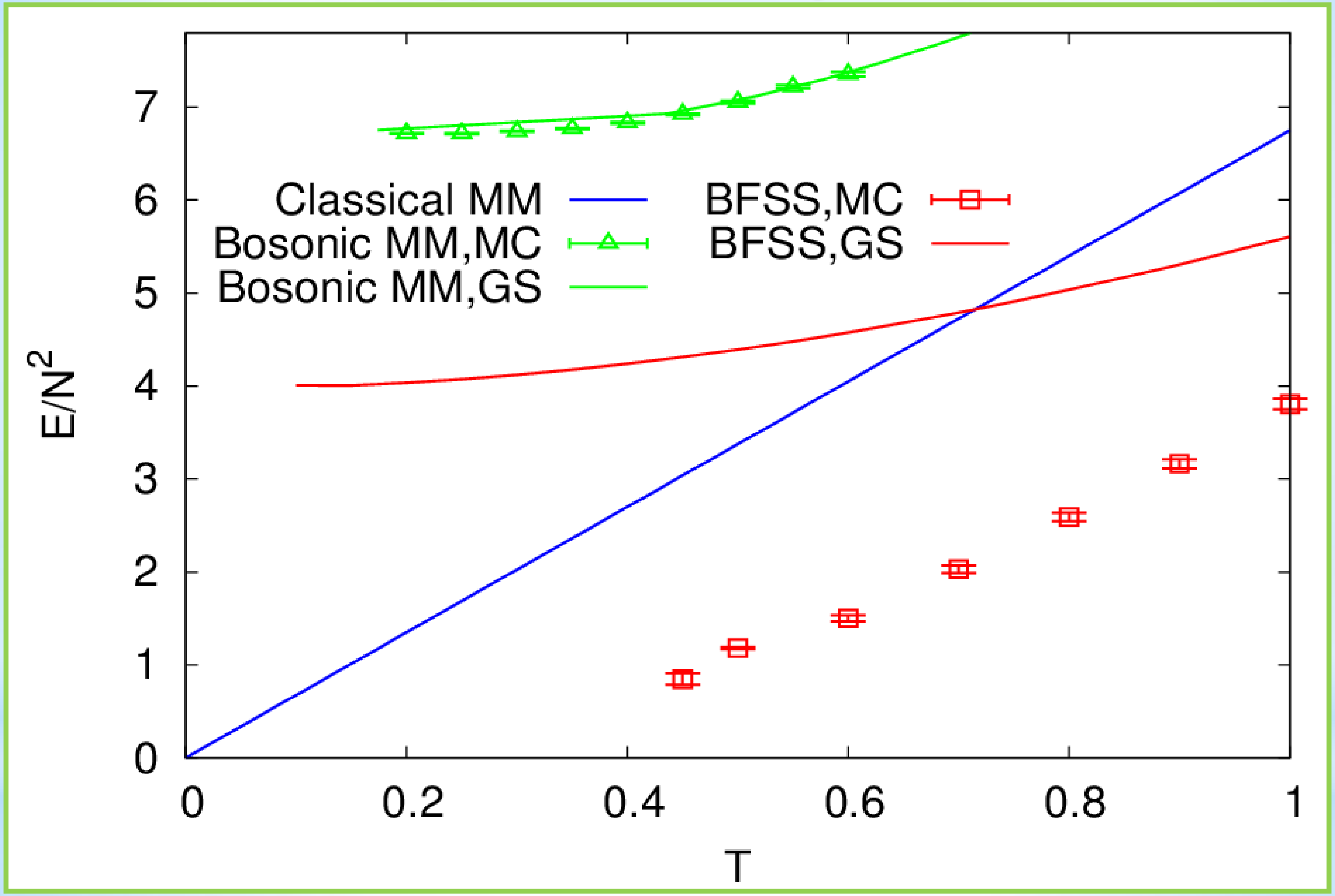
Equation of state and temperature

- Consider mixed Gaussian states with fixed energy $E = \langle H \rangle$
- *Maximize entropy w.r.t.* $\langle xx \rangle, \langle pp \rangle$
- Calculate temperature using

$$\boxed{T^{-1} = \frac{\partial S}{\partial E}}$$

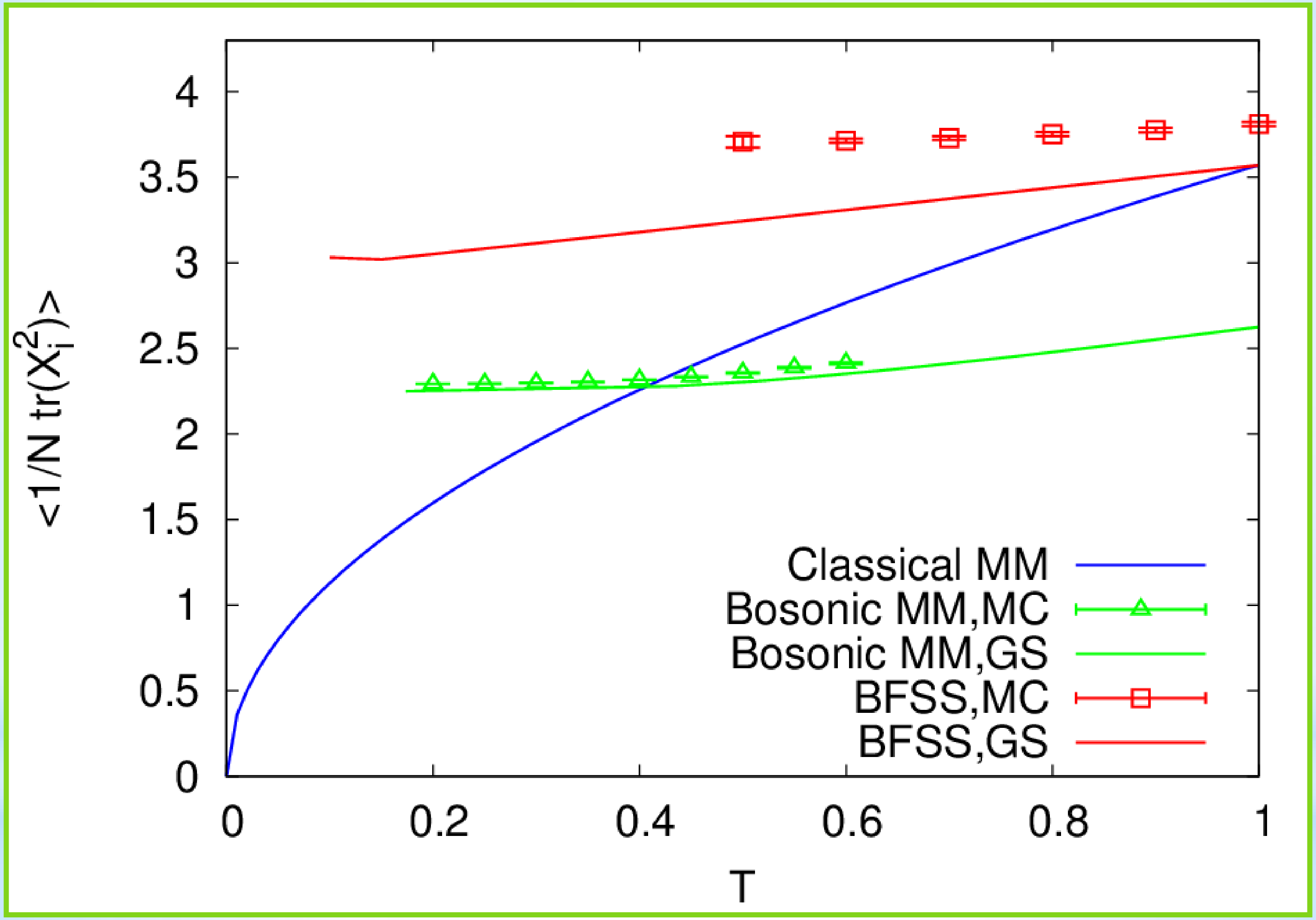
- Can be done analytically using rotational and SU(N) symmetries

Energy vs temperature



MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], **good agreemend for large d**

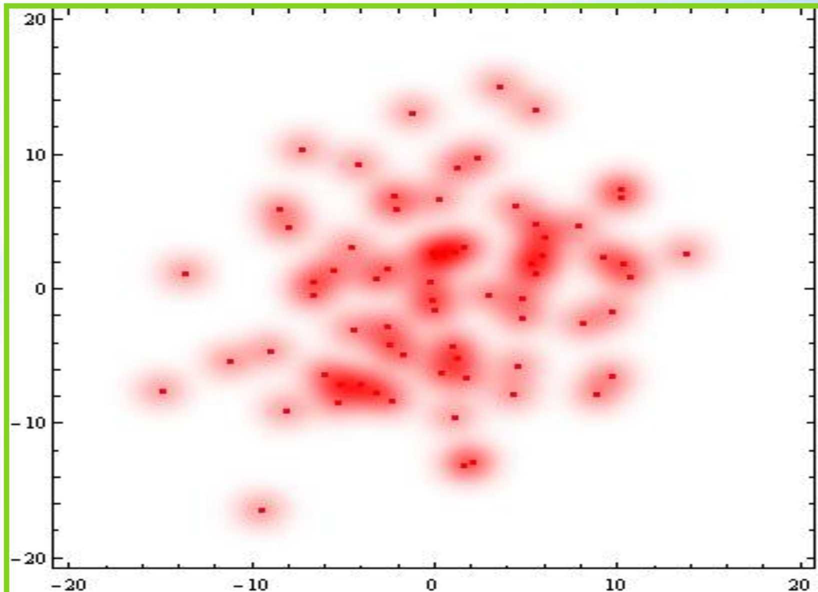
$<1/N\, Tr(X_i^2)>$ vs temperature



MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], good agreement for pure gauge

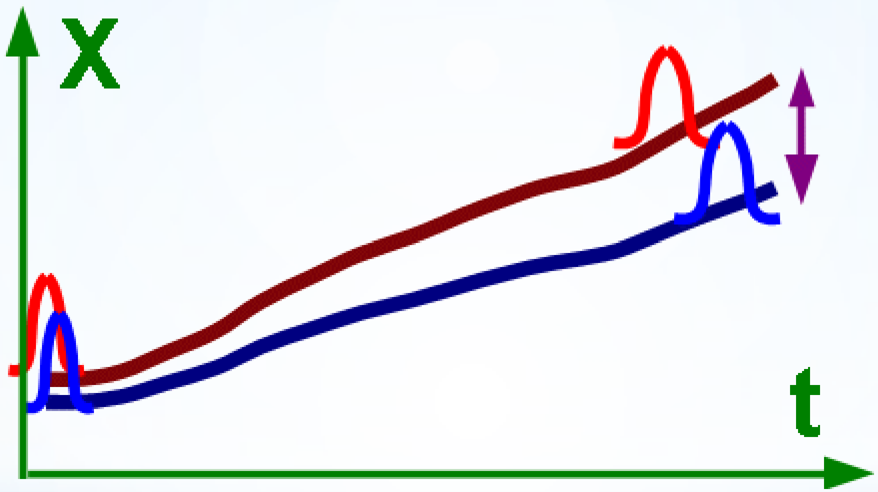
“Thermal” initial conditions

- At $T=0$ pure “ground” state with minimal $<pp>$,$<xx>$
- At $T>0$ mixed states, interpret as mixture of pure states, shifted by “classical” coordinates with dispersion $<xx>-<xx>_0$
- Makes difference for non-unitary evolution
- Fermions in ground state at fixed classical coordinates



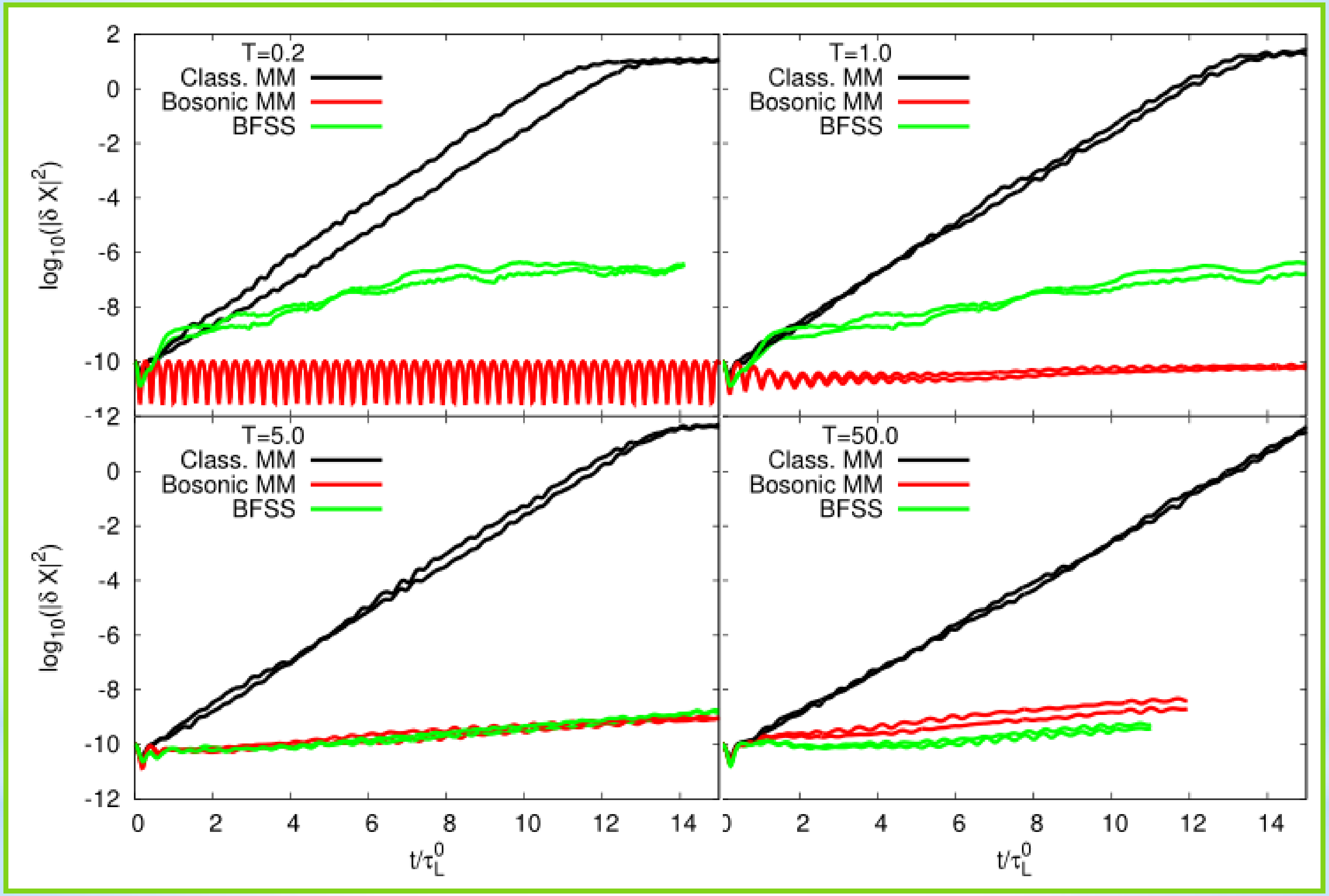
OTOCs and Lyapunov distances

Our approximation for OTOC $[X(t),P(0)]^2$: Distance between centers of slightly shifted wave packets



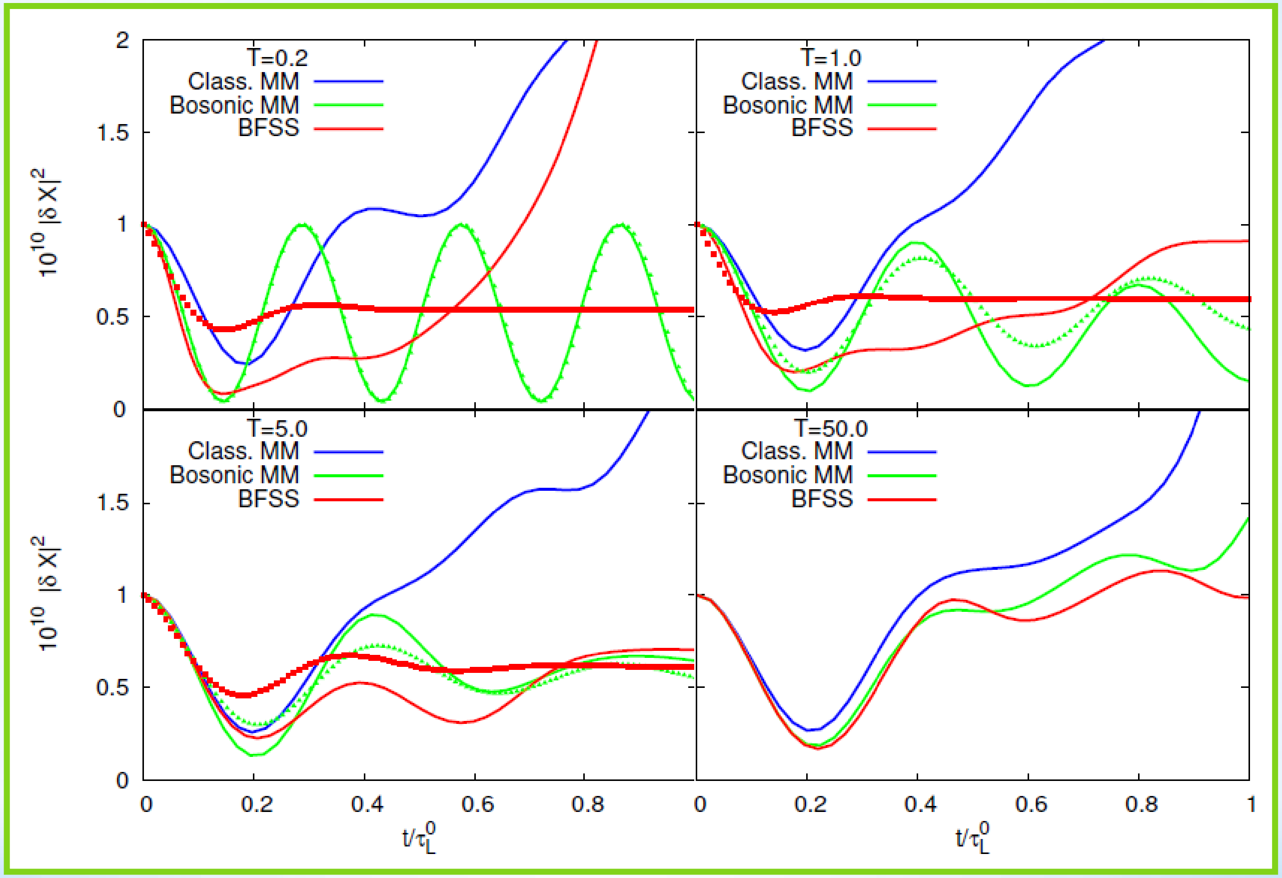
Difference between X^a_i coordinates
Two initial conditions with X^a_i shifted by random
$|\varepsilon^a_i| \sim 10^{-5}$

Lyapunov distances



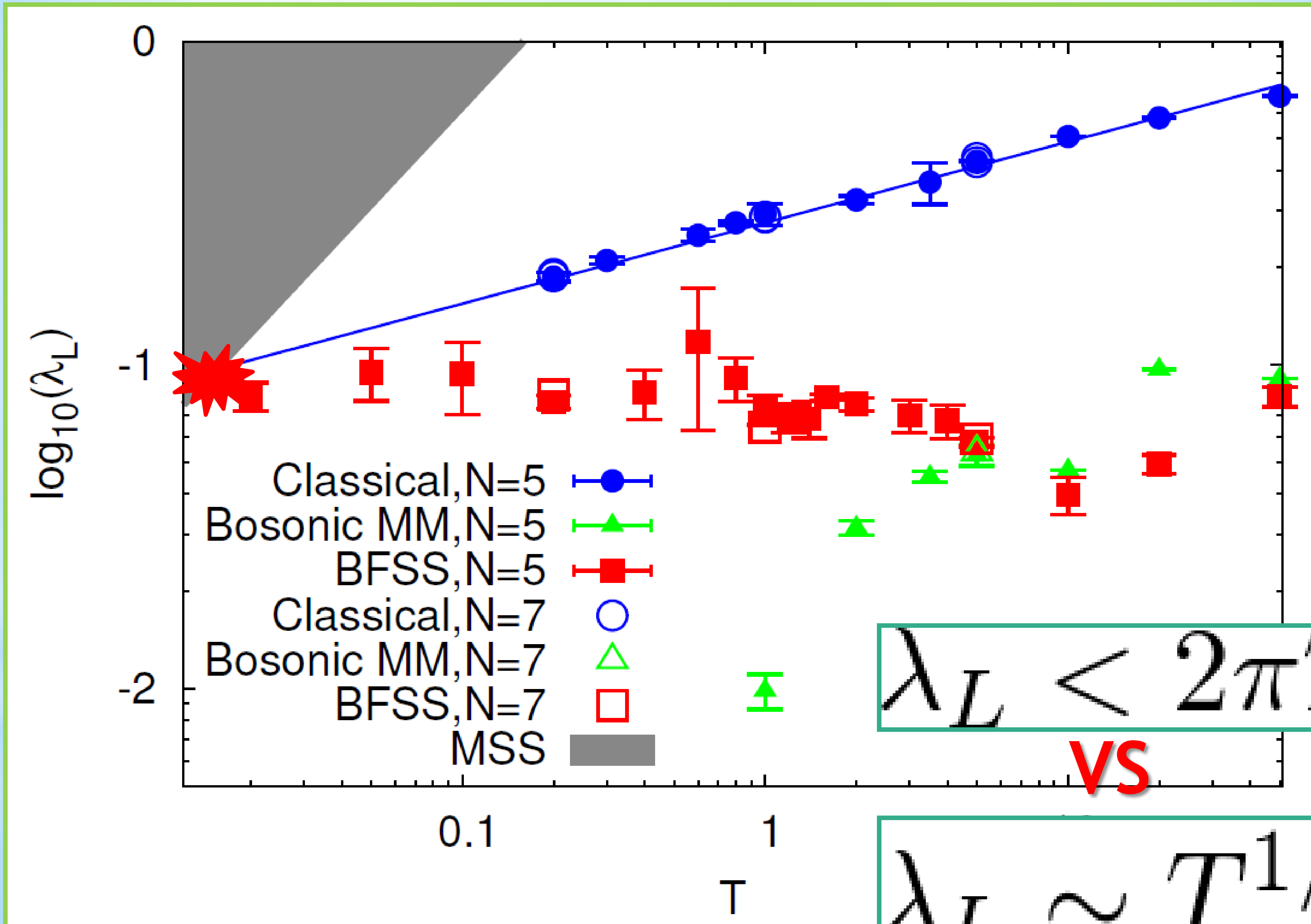
Early times: Very similar to classical dynamics
Late times: significantly slower growth

Dissipation and quasinormal ringing



- Quasinormal ringing at early times
- Exponential growth at late times
- Lyapunov time happens to be larger than dissipation time

Lyapunov exponents and MSS bound



$$\lambda_L < 2\pi T$$

VS

$$\lambda_L \sim T^{1/4}$$

Entanglement entropy: Gaussian states

$$\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|$$

- **Reduced density matrix is also Gaussian!**
- **Entropy of (mixed) Gaussian state:**

$$S = -\mathrm{tr}\,(\hat{\rho}\log\hat{\rho})$$

$$S = \sum_k \left(f_k + \frac{1}{2}\right)\ln\left(f_k + \frac{1}{2}\right) - \\ -\sum_k \left(f_k - \frac{1}{2}\right)\ln\left(f_k - \frac{1}{2}\right).$$

- **f_k are symplectic eigenvalues of the block matrix**

$$\Delta = \begin{pmatrix} \langle\langle\, \hat{X}_i^a \hat{X}_j^b \,\rangle\rangle & \langle\langle\, \hat{X}_i^a \hat{P}_j^b \,\rangle\rangle \\ \langle\langle\, \hat{X}_j^b \hat{P}_i^a \,\rangle\rangle & \langle\langle\, \hat{P}_i^a \hat{P}_j^b \,\rangle\rangle \end{pmatrix}$$

- **Uncertainty principle: $f_k \geq 1/2$**

Gaussian states: symplectic structure

- Symplectic eigenvalues of *(2 N)x(2 N)* real, symmetric, positive-definite matrix *A*:

  Eigenvalues of ΩA, Pairs $\pm i f_k$

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- For the correlator block matrix of entire system, our evolution equations can be written as $\partial_t (\Delta\Omega) = \Upsilon (\Delta\Omega) - (\Delta\Omega)\Upsilon.$

$$\Upsilon = \begin{pmatrix} 0 & I \\ -V & 0 \end{pmatrix}$$

$$V_{AB} = 3V_{ABCD}\langle \hat{X}_C \hat{X}_D \rangle$$

$$\hat{H} = \hat{P}_A^2/2 + V_{ABCD}\hat{X}_A\hat{X}_B\hat{X}_C\hat{X}_D/4$$

- Symplectic eigenvalues are conserved
- Pure states are mapped to pure states

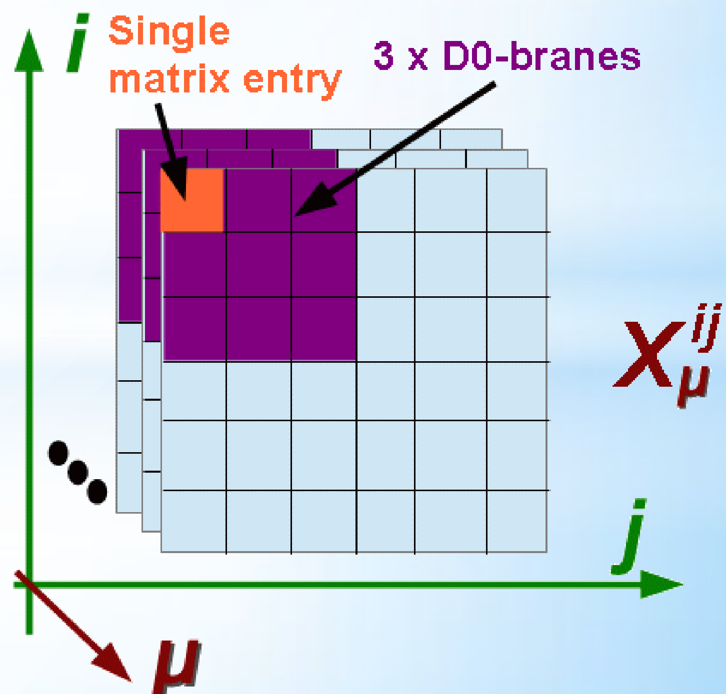
Entanglement entropy

$$\boxed{\rho_A = \mathrm{Tr}_{\,B}\, |\Psi\rangle\langle\Psi|}$$

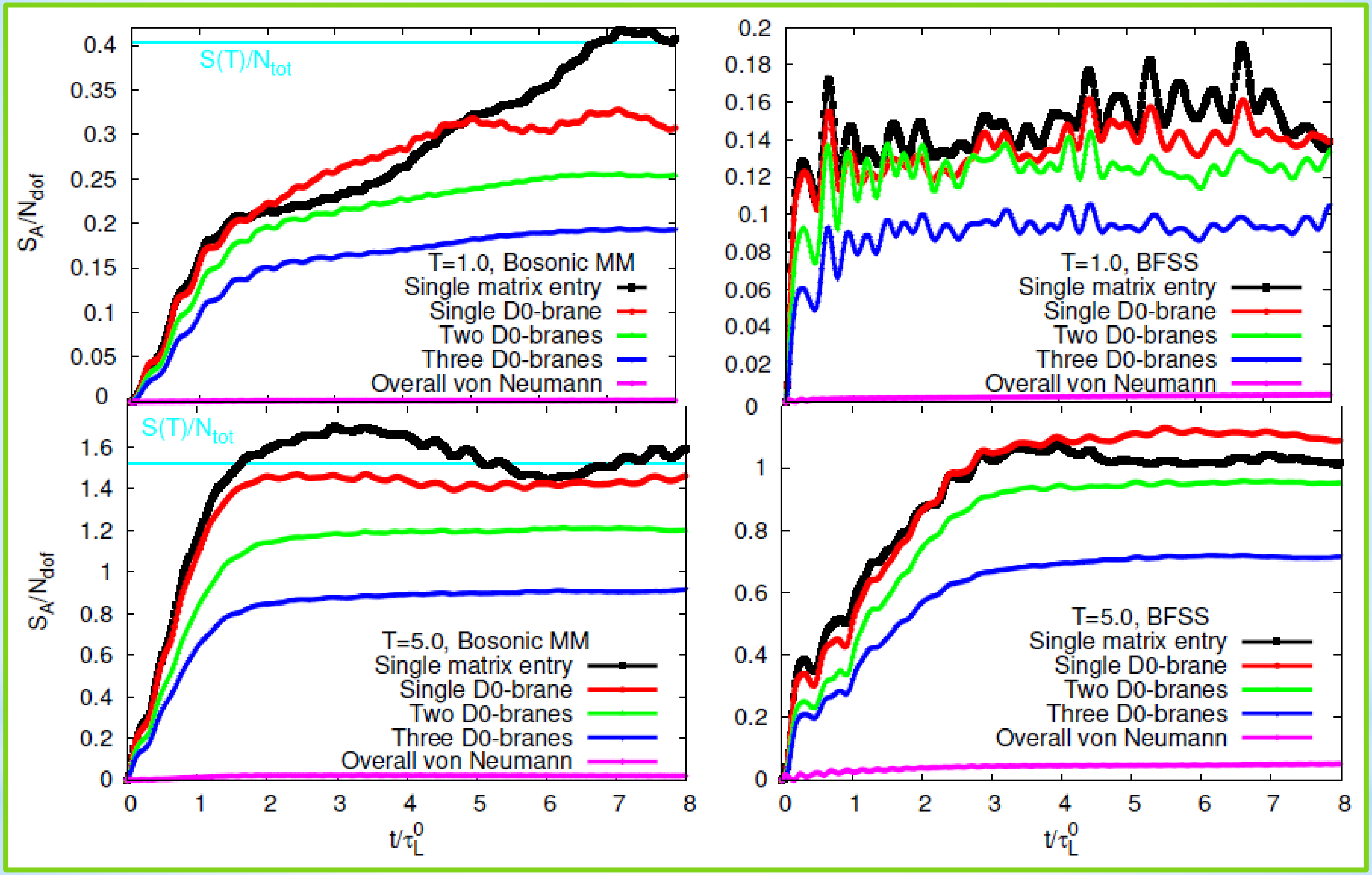
$$\boxed{S_A = -\mathrm{Tr}\left(\rho_A \log\left(\rho_A\right)\right)}$$

- Chaotic systems are expected to entangle A and B ➡ Entanglement entropy saturates

- Subsystem A is a matrix block
- We start with a product state for each matrix component



Entanglement vs time



Late-time saturation = information scrambling

#d.o.f's = {1, 9, 36, 81}

Micro- vs. Macro-canonical ensemble

- For pure gauge BFSS,
- For sufficiently small subsystem $N_{dof} << N_{tot}$

Saturation value of entanglement entropy is:

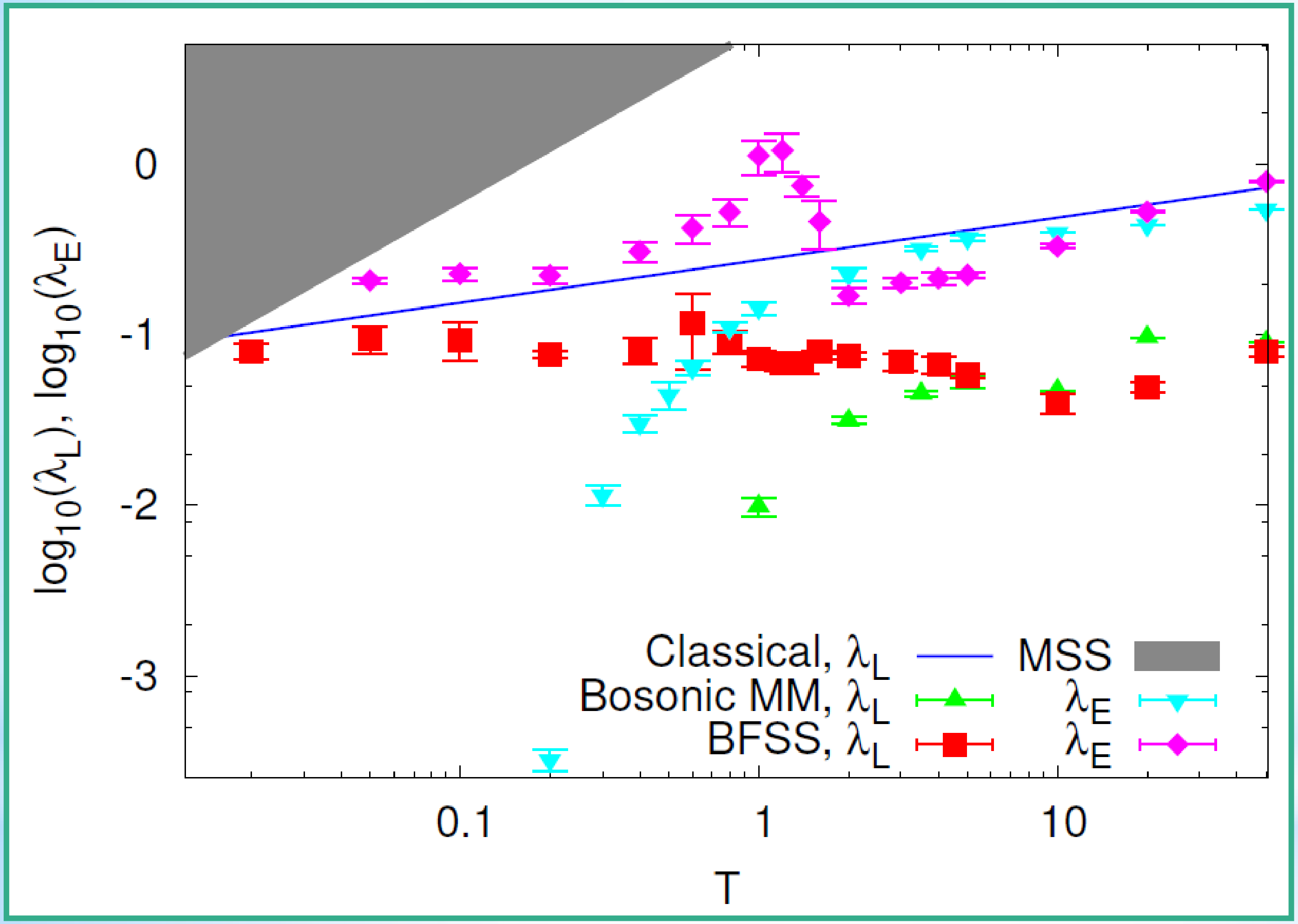
$$\boxed{S_{max} \approx S\left(T\right) N_{dof} / N_{tot}}$$

Entanglement entropy of a pure state

Von Neumann entropy of a thermal state, defines EoS and T

- Entanglement entropy is locally indistinguishable from thermal entropy
- Real-time thermalization of microcanonical ensemble

Entanglement saturation time
(vs Lyapunov exponents)



Fit $S_E(t)$ as

$A\ Tanh(t/\tau_E)$

Entanglement saturates much faster than Lyapunov time, at high T – classical Lyapunov

Summary: Lyapunov exponents

- Longer quantum Lyapunov times vs. classical, important for MSS bound
- “Confining” regime non-chaotic
- Full BFSS model chaotic at all T
- Lyapunov time longer than dissipation time
- Potential bias, since Lyapunov growth at late times, approximation might fail

Summary: Entanglement

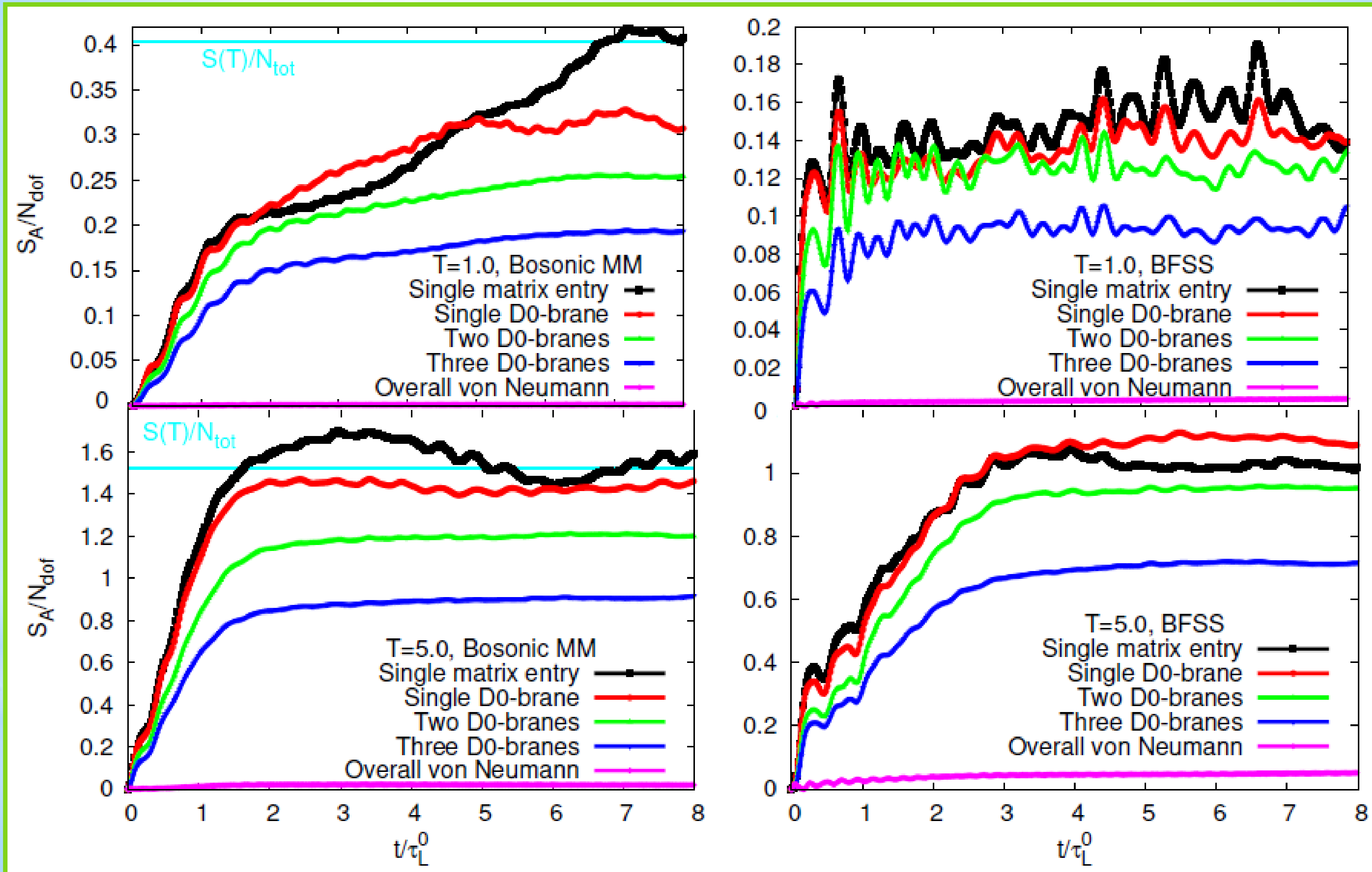
- “Scrambling” behavior for entanglement entropy
- Entanglement saturation timescale is the shortest
- Saturation value given by thermal entropy, Evidence for real-time thermalization!
- At high T governed by classic, rather than quantum Lyapunov
- Entanglement entropy is the best short-time probe of thermalization in our simulations

Summary

- Gaussian state approximation: $\sim V^2$ scaling of CPU time for QCD/ Yang-Mills
- Feasible on moderately large lattices
- Quantum effects on thermalization?
- Topological transitions in real time?

Backup slides

Micro- vs. Macro-canonical ensemble



Late-time saturation value?

OTOCs and Lyapunov distances

OTOC in an overfull basis of Gaussian states:

$$\mathrm{Tr}\left(\hat{\rho}\left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right]^2\right) =$$
$$\int dX'\,dP' \left\langle \langle X, P| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X', P'\rangle \times \right.$$
$$\left. \times \langle X', P'| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X, P\rangle \right\rangle_c .$$

Saturated by saddle point at *X=X', P=P'* !!!

OTOC in terms of infinitesimal shift:

$$\langle X, P| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X', P'\rangle =$$
$$= -i\frac{\partial}{\partial \epsilon_j^b} \langle X, P| e^{i\epsilon \hat{P}(0)} \hat{X}_i^a(t) e^{-i\epsilon \hat{P}(0)} |X', P'\rangle,$$

Very similar to classical Lyapunov distance!!!

Fate of supersymmetry

16 supercharges in BFSS model:

$$\hat{Q}_\alpha = \hat{P}_i^a\,[\sigma_i]_{\alpha\beta}\,\hat{\psi}_\beta^a - \frac{1}{4}C_{abc}\hat{X}_i^b\hat{X}_j^c\,[\sigma_{ij}]_{\alpha\beta}\,\hat{\psi}_\beta^a$$

$$\sigma_{ij} \equiv \sigma_i\sigma_j - \sigma_j\sigma_i$$

$$\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = 2\delta_{\alpha\beta}\hat{H} - 2\left(\sigma_i\right)_{\alpha\beta}\hat{X}_i^a\hat{J}^a$$

$$\left[\hat{H}, \hat{Q}_\gamma\right] = -i\hat{\psi}_\gamma^a\hat{J}^a$$

Gauge transformations

$$\hat{J}^a = C_{abc}\hat{X}_i^b\hat{P}_i^c - \frac{i}{2}C_{abc}\hat{\psi}_\alpha^b\hat{\psi}_\alpha^c$$

Fate of supersymmetry

In full quantum theory

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) = 0$$

Fierz identity (cyclic shift of indices):

$$[\sigma_i]_{\alpha\beta} [\sigma_i]_{\gamma\delta} + [\sigma_i]_{\alpha\gamma} [\sigma_i]_{\beta\delta} + [\sigma_i]_{\alpha\delta} [\sigma_i]_{\gamma\beta} = \\ = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}$$

In CSFT approximation

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \rangle \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) \neq 0$$

Fermionic 3pt function seems necessary!

Ungauging the BFSS model

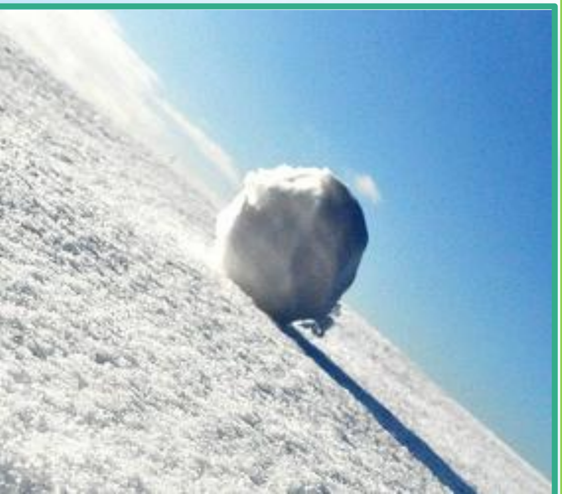
- Gauge constraints

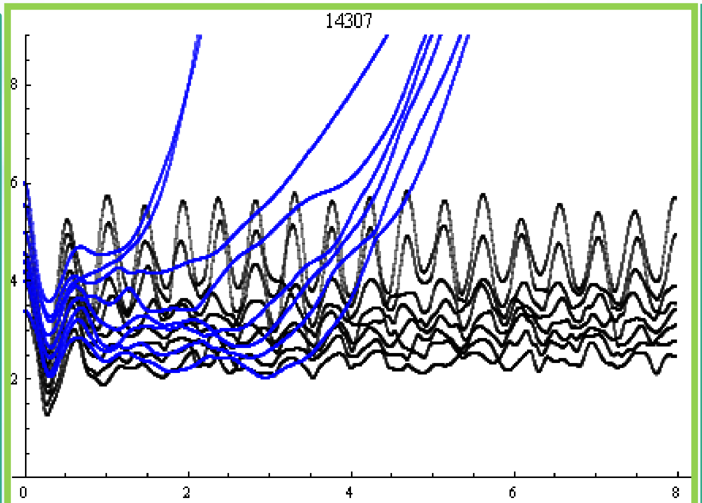
$$\hat{J}_a = C_{abc}\hat{X}_i^b\hat{P}_i^c - \frac{i}{2}C_{abc}\hat{\psi}_\alpha^b\hat{\psi}_\alpha^c \qquad \hat{J}_a\left|\psi\right\rangle = 0$$

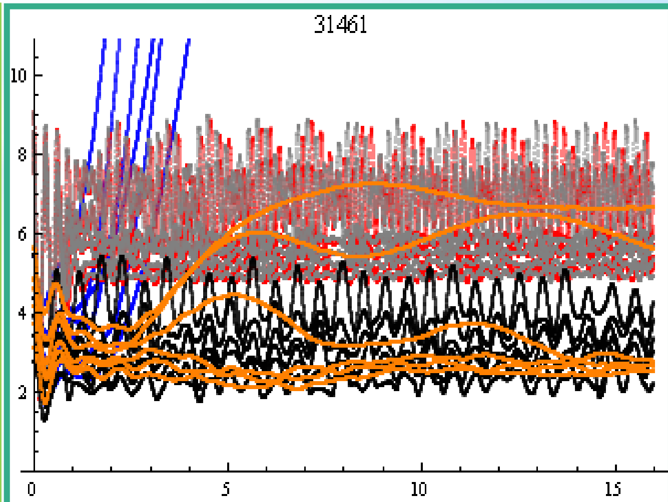
- For Gaussian states we can only have a weaker constraint $\left\langle\psi\right|\hat{J}_a\left|\psi\right\rangle = 0$
- We work with ungauged model [Maldacena,Milekhin'1802.00428] (e.g. LGT with unit Polyakov loops)
- Ungauging preserves most of the features of the original model [Berkowitz,Hanada, Rinaldi, Vranas 1802.02985]

Summary: Outlook

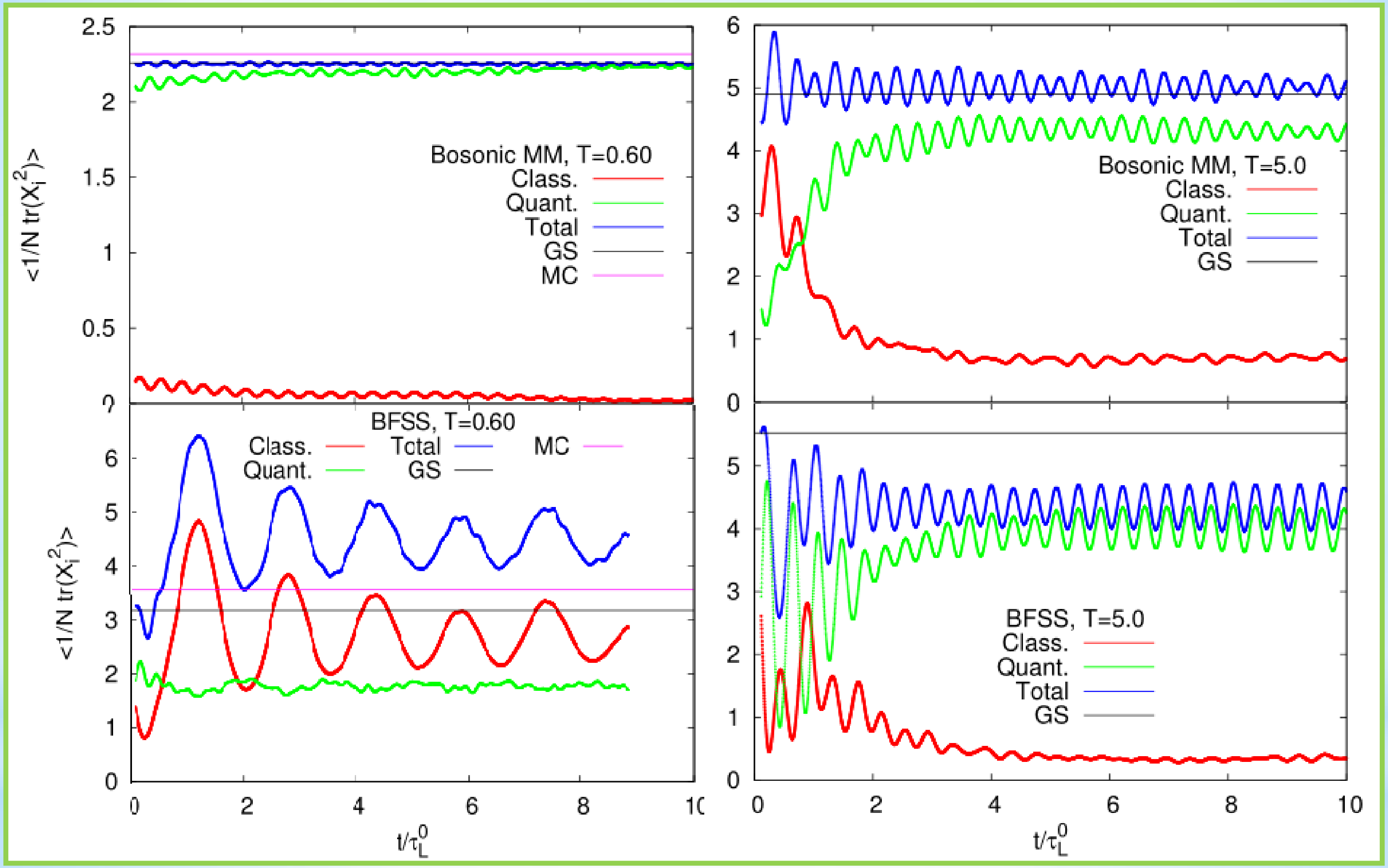
- **Hawking radiation of D0 branes conjectured**
- **We do see it if quantum bosonic corrections are omitted**
- **Bosonic quantum corrections remove the instability imperfect cancellation because of broken SUSY?**







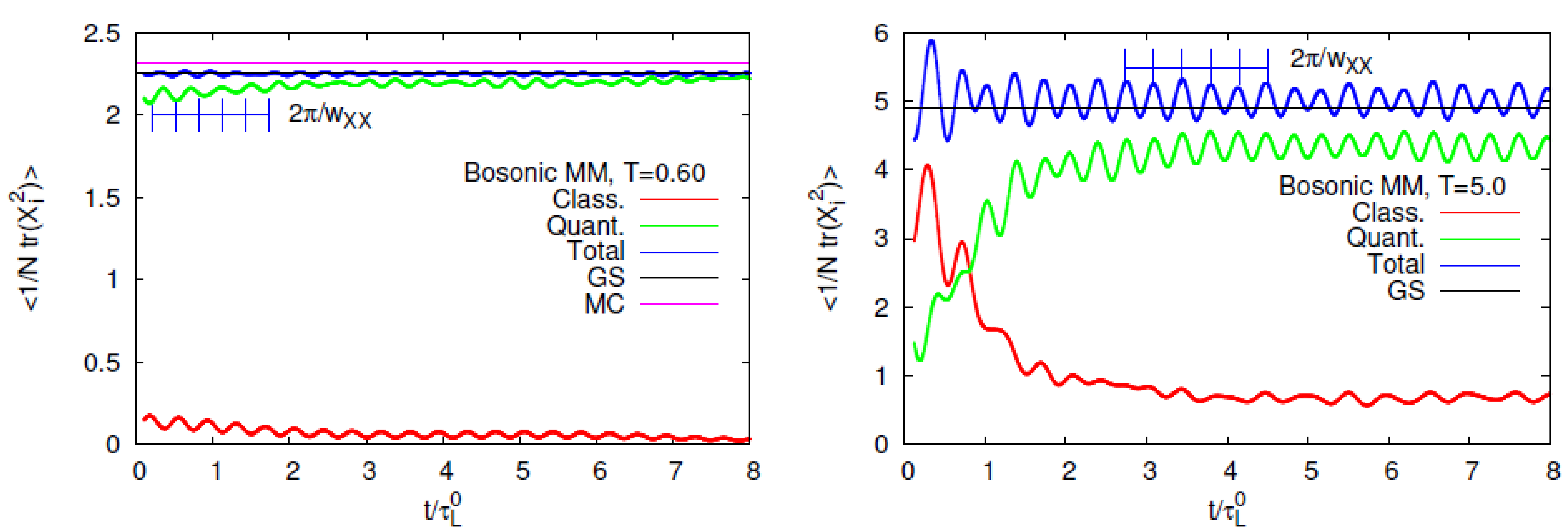
Real-time evolution: $\langle 1/N\, Tr(X_i^2)\rangle$



Wavepacket spread vs classical shrinking

For BFSS $\langle 1/N\, Tr(X_i^2)\rangle$ grows, instability?

Quasinormal ringing I

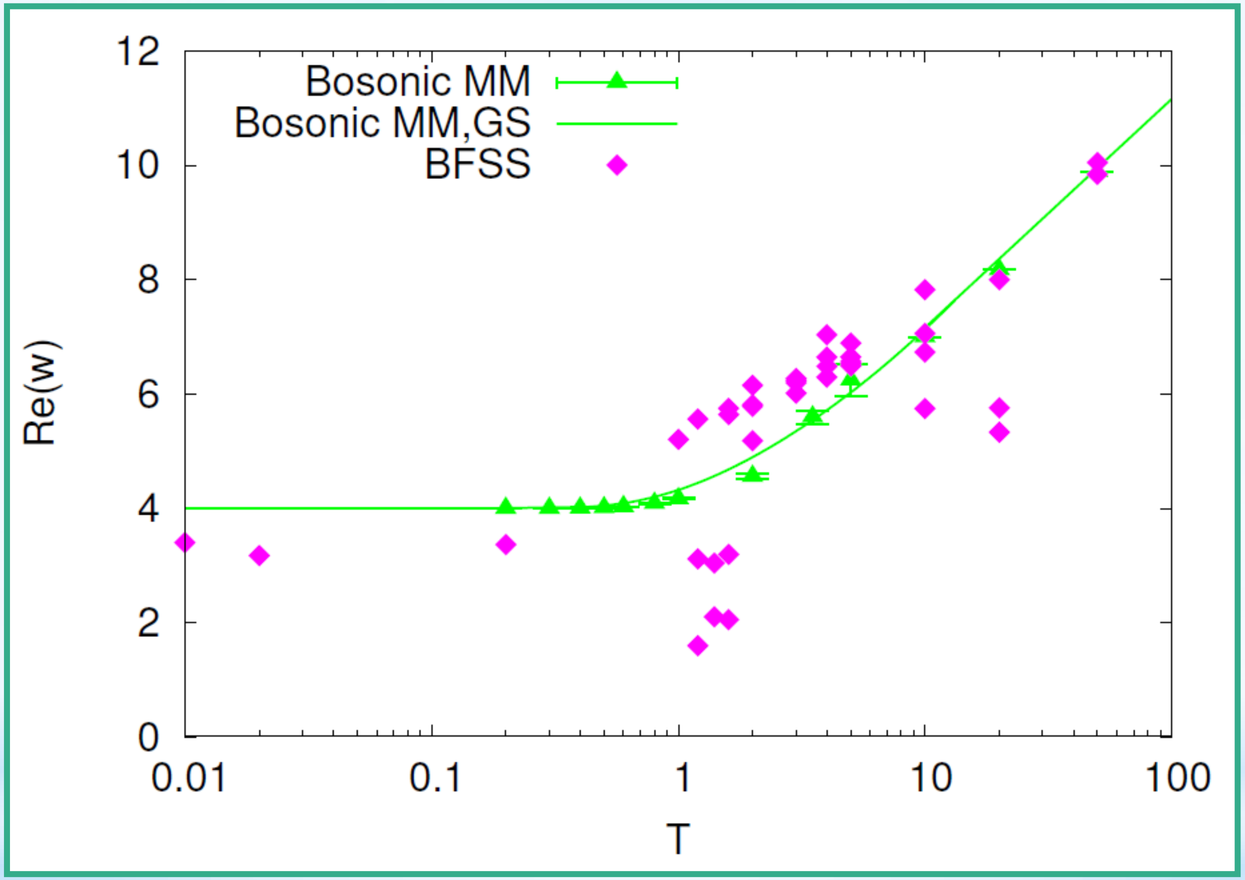


Linearizing equations of motion around thermal equilibrium, we get oscillations with frequencies:

- $w_X = (2d-2)/d <1/N\, tr(X_i^2)>$ **(X and P)**
- $w_{XX} = 6\, w_X^2$ **(XX, XP and PP)**

To-be quasinormal modes!

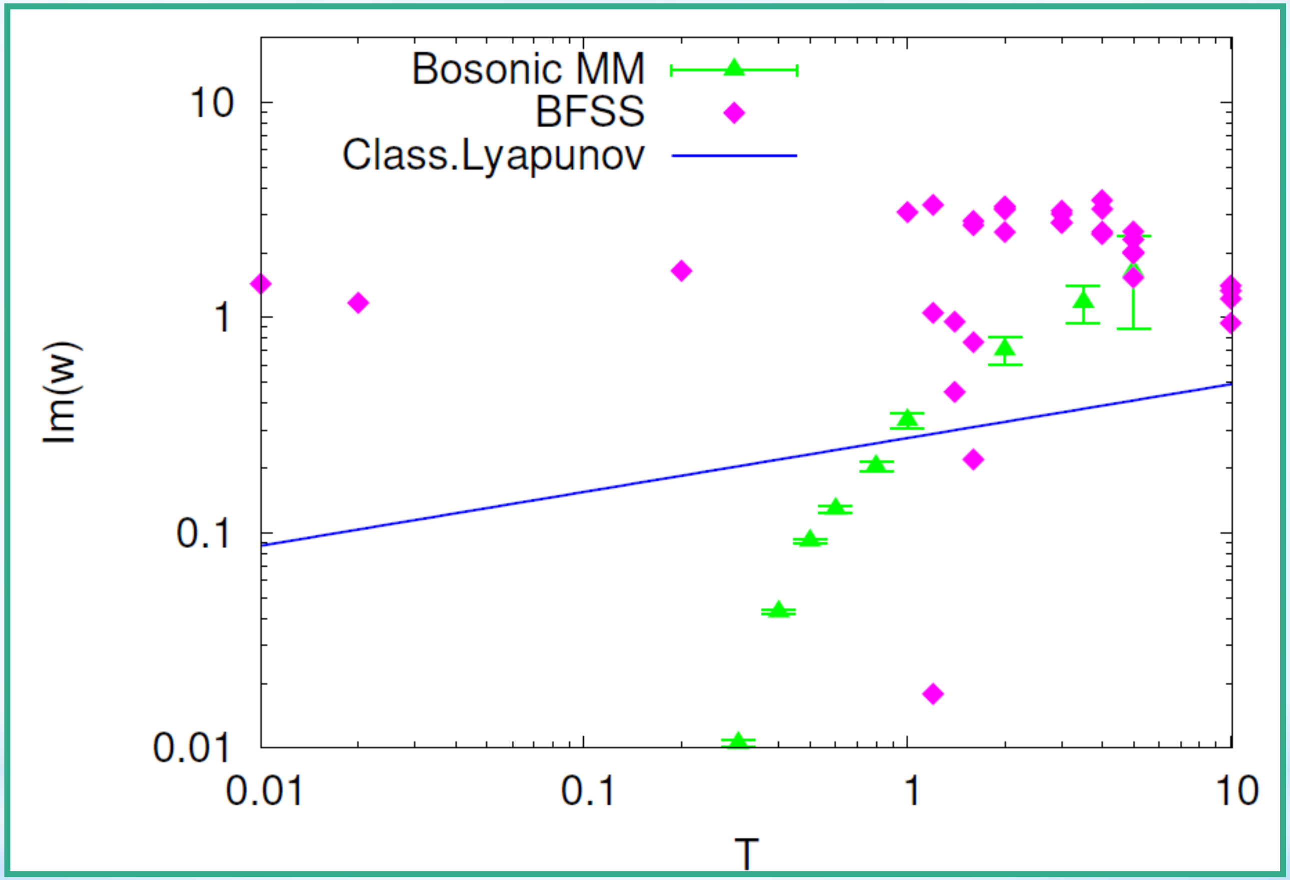
Quasinormal ringing
Re(w) vs Temperature



High-T scaling: $w_{xx} = 4.89\ T^{1/4}$

vs. $w_{xx} = 5.15\ T^{1/4}$ [Romatschke, Hanada]

Quasinormal ringing
Im(w) vs Temperature



Dissipation rate vanishes in the confinement regime, in contrast to BFSS

Real-time evolution

- Thermal initial conditions
- Randomly shifted Gaussian wave functions
- Only a few instances of random initial conditions
- Good self-averaging at sufficiently large N
- Numerically solving the evolution equations for X, P, $\langle XX \rangle$, $\langle XP \rangle$, $\langle PP \rangle$, $\langle \psi\psi \rangle$
- We use $N=5$ and $N=7$ (remember N^5 scaling of CPU time)