Recent Developments in MARTINI

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Probing Quark-Gluon Matter with Jets
BNL
July, 25-th, 2018

McGill Team

- Charles Gale
- Sangyong Jeon
- *Li Yan* ⟹ Fudan Univ, China
- *Alina Czajka* ⟹ NCNR, Warsaw
- *Dani Pablos* (JETSCAPE)
- *Shuzhe Shi* (Joining in September)

- Chanwook Park : *Did most of the work presented here*
- Mayank Singh
- Scott McDonald
- Siggi Hauksson
- Igor Kozlov
- Rouzbeh Modarresi-Yazdi
- Jessica Churchil
- Matthew Heffernan
- Melissa Mendes (Joining in September)

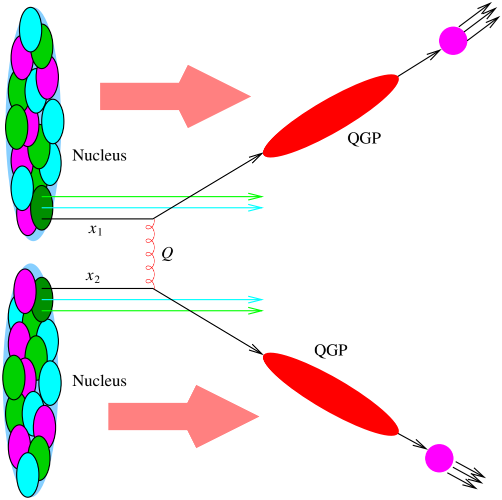
Part of JETSCAPE and BEST. Many alumni.

Talk about MARTINI

Talk about MARTINI

- Brief review of MARTINI
- New features
- Results

Schematic Understanding of Jet Quenching



HIC Jet production scheme:

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'}$$
$$\times f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f)$$
$$\times \frac{d\sigma_{ab\to cd}}{dt}$$
$$\times \mathcal{P}(x_c \to x'_c | T, u^\mu)$$
$$\times D(z'_c, Q)$$

$\mathcal{P}(x_c \to x'_c | T, u^\mu)$: Medium modification of high energy parton property

MARTINI

MARTINI

[Schenke, Jeon & Gale, Phys.Rev. C80 (2009) 054913,
Young, Schenke, Jeon & Gale, Phys.Rev. C86 (2012) 034905]

- Modular Alogorithm for Relativistic Treatment of Heavy IoN Interactions
- Hybrid approach
  - Calculate Hydrodynamic evolution of the soft mode (MUSIC)
  - Propagate jets in the evolving medium according to the McGill-AMY radiation rates[1] & the leading order elastic scattering rates[2]
- A part of JETSCAPE v1.0 release (https://github.com/JETSCAPE)

JETSCAPE

[http://jetscape.wayne.edu/]

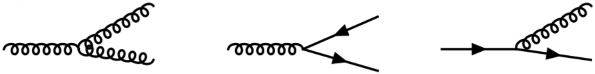
[1] Arnold, Moore & Yaffe, JHEP 0206:030,2002,
Jeon & Moore, Phys.Rev. C71 (2005) 034901.

[2] Qin, Ruppert, Gale, Jeon, Moore & Mustafa, Phys.Rev.Lett. 100 (2008) 072301.

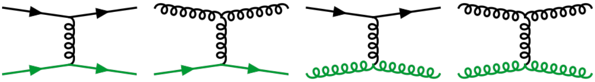
Parton propagation

Process included in MARTINI (all of them can be switched on & off):

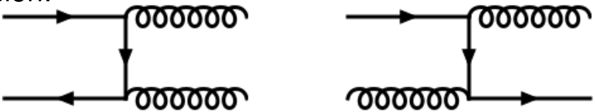
- Inelastic (AMY):



- Elastic:



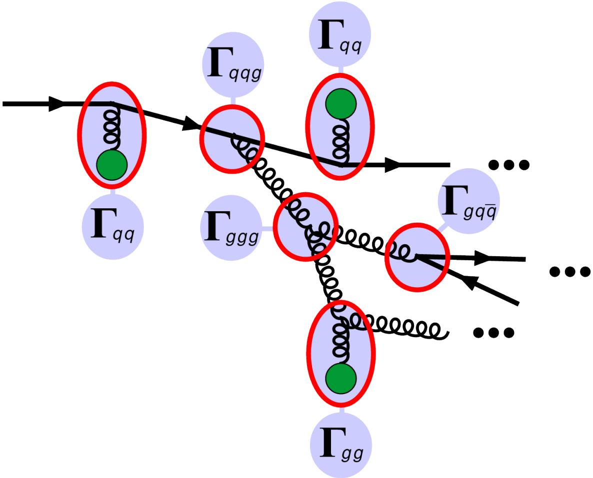
- Conversion:



- Photon: emission & conversion

[Figures by B. Schenke]

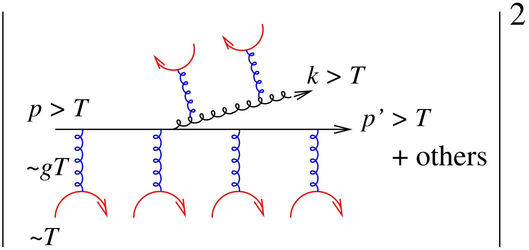
Parton propagation



An example path in MARTINI

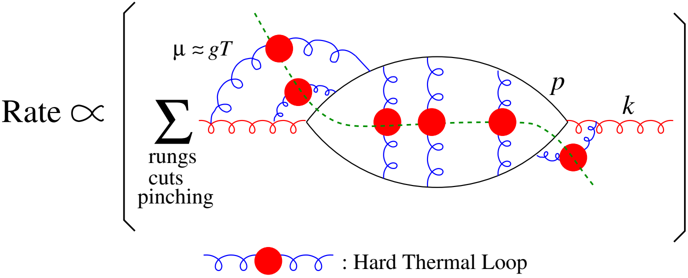
[Figure by B. Schenke]

McGill-AMY radiation rate



- Medium is weakly coupled QGP with thermal quarks and gluons
- Requires $g \ll 1$, $p > T$, $k > T$
- Sum all interactions with the medium

McGill-AMY radiation rate



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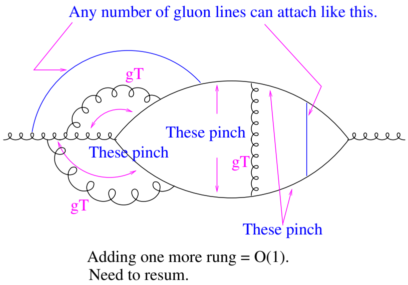
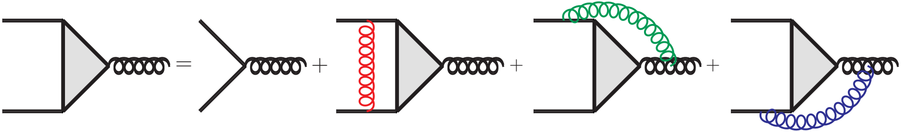
McGill-AMY radiation rate



- Medium is weakly coupled QGP with thermal quarks and gluons
- Requires $g \ll 1$, $p > T$, $k > T$
- Sum all interactions with the medium
- Leading order: 3 different kinds of collinear pinching poles

McGill-AMY radiation rate

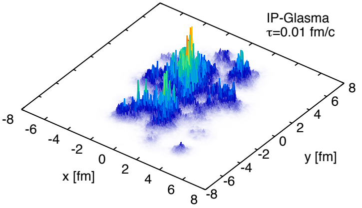
- SD-Eq for the vertex **F**:

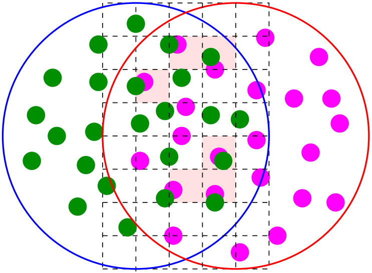


[Figure from G. Qin]

- Close the other end to get $d\Gamma(p,k)/dk$
- $d\Gamma(p,k)/dk$ Tabulated and interpolated in MARTINI simulations

MARTINI Step 1 - Jet positions





[Figures by B. Schenke]

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} \cdots$$

- Start with an initial condition. Use the positions of the nucleons to determine the binary collision sites.
- Embed PYTHIA jets at the binary collision sites.
- Weight the event with the jet cross-section.

MARTINI Step 2 - Initial jet spectrum

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^\mu) D(z'_c, Q)$$

- PYTHIA 8.2 generates the hard collision
- PDF selection using LHAPDF
- Shadowing through EKS98 or EPS09 (Default)
- Isospin effect for neutrons taken into account
- Initial & final state PYTHIA showers are fully done before entering the medium

MARTINI Step 3 - Medium evolution

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^\mu) D(z'_c, Q)$$

- First load the hydro evolution history
- All partons start at $\tau = 0$ i.e. $z = 0, t = 0$
- Move the position until $\tau = \tau_0$

MARTINI Step 3 - Medium evolution

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^\mu) D(z'_c, Q)$$

- *Go to the rest frame of cell* where the jet parton is currently at
- According to the local conditions, calculate the total interaction probability within Δt_{local}

$$P = \sum_{i=el,rad} \Delta t_{local} \int dk \frac{d\Gamma_i}{dk}$$

- If $x_{\text{random}} < P$, then do interactions
- If so, decide which process to do

MARTINI Step 3 - Medium evolution

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^{\mu}) D(z'_c, Q)$$

- This procedure solves the rate equation

$$\frac{dP(p)}{dt} = \int_k P(p+k) \frac{d\Gamma(p+k,k)}{dk} - P(p) \int_k \frac{d\Gamma(p,k)}{dk}$$

- Includes both radiations and elastic scatterings
- Includes recoil partons – Details later

MARTINI Step 3 - Medium evolution

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^\mu) D(z'_c, Q)$$

- If the energy of the emitted parton is above $E_{\rm cut}$ $(\sim 4T)$, add it to the hard parton list
- If a recoiled tharmal parton has $E \geq E_{\rm cut}$, add it to the hard parton list (optional)
- Lorentz transform back to the lab frame and move the final state particles to the next position
- Repeat

MARTINI Step 4 - Hadronization

$$\frac{d\sigma_{AB}}{dt} = \int_{\text{geometry}} \int_{abcdc'} f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \frac{d\sigma_{ab\to cd}}{dt} \times \mathcal{P}(x_c \to x'_c | T, u^\mu) D(z'_c, Q)$$

- Evolution ends when
  - $E_{jet} < E_{\rm cut}$ in the cell rest frame or
  - The jet enters the hadronic phase
- Hadronization through PYTHIA's lund fragmenation model
  - Keeping track of colors
  - Recoil partons connect only to themselves
- Optional: Particles from Hydro (via Cooper-Frye) + Jets both go into UrQMD
- FASTJET for Jet construction

New Features

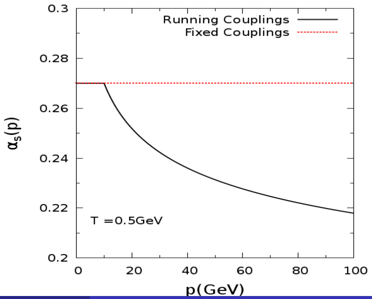
Feature 1 – Running Coupling

- Where the couplings appear:
  Rate for $p > T, k > T$ (valid for $p \gg T$ and $k \gg T$ as well)

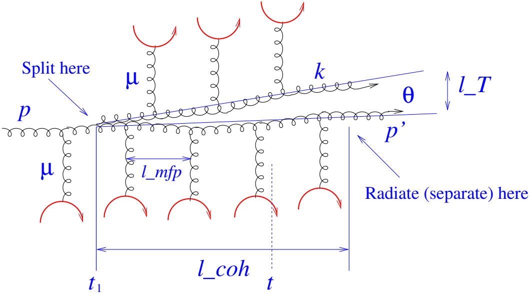
$$\frac{dN_g(p,k)}{dkdt} = \frac{g_s^2}{16\pi p^7}\frac{1}{1 \pm e^{-k/T}}\frac{1}{1 \pm e^{-(p-k)/T}} \times$$
$$\times \left\{ \begin{array}{ll} C_f \frac{1+(1-x)^2}{x^3(1-x)^2} & q \to qg \\ 2N_f T_f \frac{x^2+(1-x)^2}{x^2(1-x)^2} & g \to q\bar{q} \\ C_a \frac{1+x^4+(1-x)^4}{x^3(1-x)^3} & g \to gg \end{array} \right\} \times \int \frac{d^2\mathbf{h}}{(2\pi)^2} 2\mathbf{h} \cdot \mathrm{Re}\, \mathbf{F}(\mathbf{h}, p, k),$$

where $x = k/p$

- The first g_s^2: Splitting vertex
  Scale: $Q^2 \sim \hat{q}\ell_{\rm coh} \sim \sqrt{k\hat{q}}$
- Second g_s^2: In the soft part. Fixed.



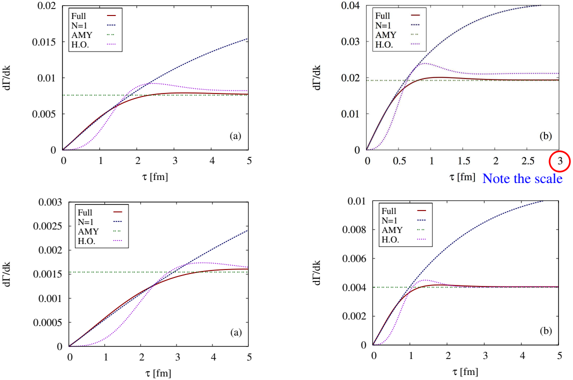
Feature 2 – Finite-size in the LPM effect



- A typical radiation[3] process leading to the QCD LPM effect (AMY)
- Original AMY provides the radiation rate $d\Gamma(p,k)/dk$ in the large distance limit
- Coherence effect: Second radiation is suppressed during $\ell_{\rm coh}$
- Whole process can be formulated as a time-dependent 2-D Schrödinger equation

[3] Radiation here means full separation from the original parton.

[Caron-Huot & Gale PRC 82 064902 (2010). Also see Zakharov JETP 65, 615 (1997).]



3GeV daughter from 16 GeV parent, $T = 200, 400\,\text{MeV}$

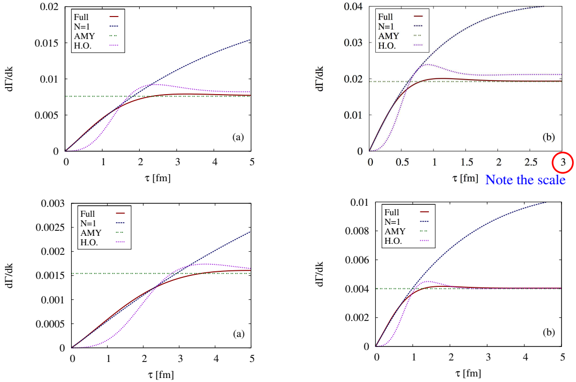
8GeV daughter from 16 GeV parent, $T = 200, 400\,\text{MeV}$

$N = 1$: Opacity expansion, H.O.: BDMPS SHO

- Interpret: Rate for the *next* radiation after *radiating* at $\tau = 0$
- At $\tau \gtrsim \ell_{\rm coh}$, the AMY rate is recoverd
- Slope near 0 *stiffer* when T is larger or when k is smaller

Finite size effect by Caron-Huot & Gale (Static medium)

[Caron-Huot & Gale PRC 82 064902 (2010). Also see Zakharov JETP 65, 615 (1997).]



3GeV daughter from 16 GeV parent, $T = 200, 400\,\text{MeV}$

8GeV daughter from 16 GeV parent, $T = 200, 400\,\text{MeV}$

$N = 1$: Opacity expansion, H.O.: BDMPS SHO

- Full solution: Interpolates between the 1-st order opacity expansion (OE1 $\simeq$ GLV) and AMY:

$$\frac{d\Gamma}{dk} \approx \theta(\ell_{\rm coh} - \tau)\frac{d\Gamma_{OE1}}{dk} + \theta(\tau - \ell_{\rm coh})\frac{d\Gamma_{AMY}}{dk}$$

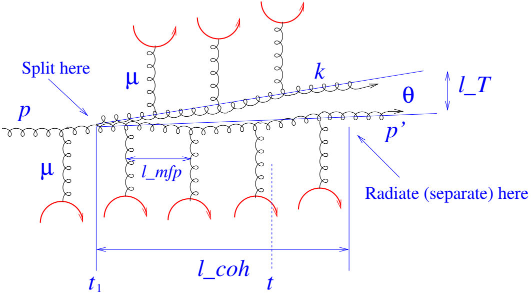
Implementation?

- In-medium radiation rate at $t < \infty$
  [Caron-Huot & Gale PRC 82 064902 (2010)]

$$\frac{d\Gamma^a_{bc}}{dk} = \frac{P^{a(0)}_{bc}(x)}{\pi p} \mathrm{Im}\left(\int_0^t dt' \int_{\mathbf{q}_\perp, \mathbf{p}_\perp} \frac{\mathbf{q}_\perp \cdot \mathbf{p}_\perp}{\delta E(\mathbf{q}_\perp)} \mathcal{C}(t)\, K(t, \mathbf{q}_\perp; t', \mathbf{p}_\perp) \right)$$

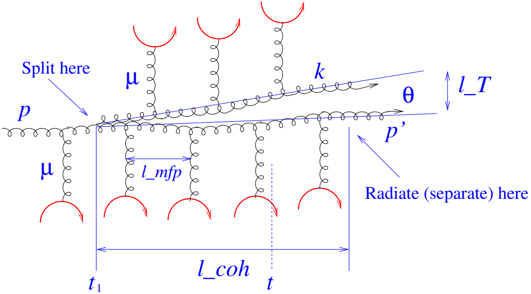
- 0 is when the parton last radiated
- t' is when the daughter is split
- t is when the daughter is finally radiated (separated)
- K: Propagator for the 2-D Schrödinger equation
- This is the right thing to do, but not easy to implement in an evolving medium

Practical implementaton

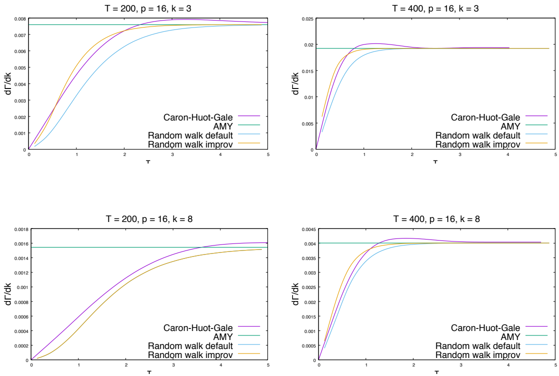


- Implement the *approximate shape*
- Follow the physical idea laid out by C-H&G (also by BDMPS-Z)
  - $p, k, p' \gg T$
  - $\theta \ll 1$
  - Elastic collsions happen in much shorter time/length scale
  - Transverse separation $\ll 1/\mu$ near t_1

Practical implementaton



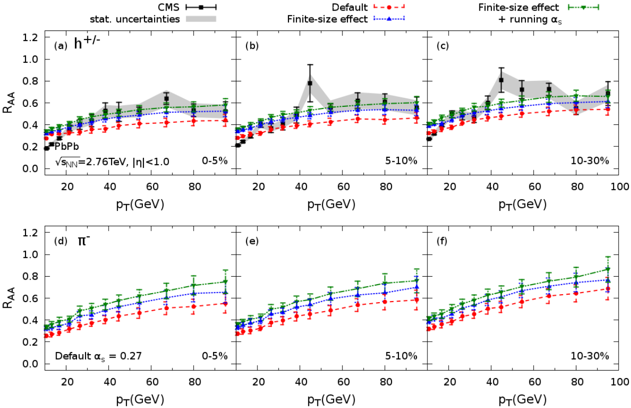
- Daughter created with the AMY rate
- Both the mother and the daughter random-walk with the step-size distribution given by $d\sigma_{\rm el}/d^2\mathbf{q} \propto 1/\mathbf{q}^2(\mathbf{q}^2+\mu^2)$
- Do not allow another split until $\ell_T \approx \rho_T/|\mathbf{k}_T - \mathbf{p}_T|$,
- $\rho_T \lesssim 1/2$ is a parameter
- Allows *local* evolution $\Longrightarrow$ Can deal with dynamic medium



- Approach to AMY approximately reproduced
- The coherence length (time) $\ell_{\rm coh}$ approximately reproduced
- Default $\rho_T = 0.5$

- "Improv" here means slightly modified elastic scattering spectrum. (Better to have a variable $\rho_T(p, T)$ between 0.15 to 0.5.)

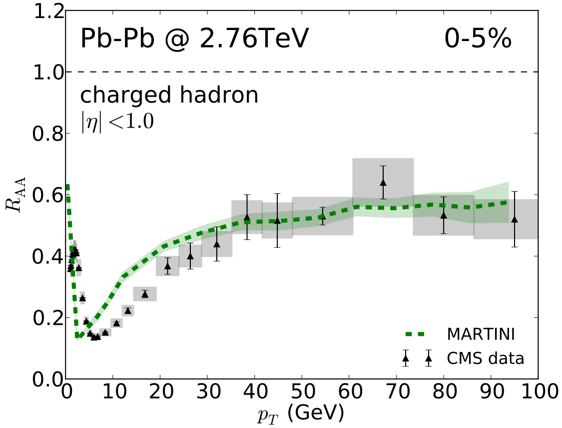
Effects of FSE + RC



- With $\rho_T = 0.5$
- Both effects *reduce* the jet energy loss

[Chanwook Park's Master's Thesis] (2015)

Effects of FSE + RC – Latest



[arXiv:1807.06550, C. Park, S. Jeon & C. Gale]

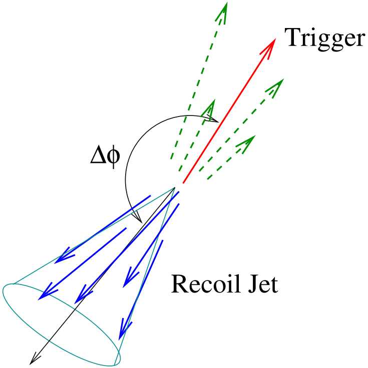
- With $\rho_T = 0.5$
- Low p_T part should improve with $\rho_T(p, T)$

Hadron-Jet Correlation

[ALICE, JHEP 09 (2015) 170]

- Semi-Inclusive recoil jet distribution

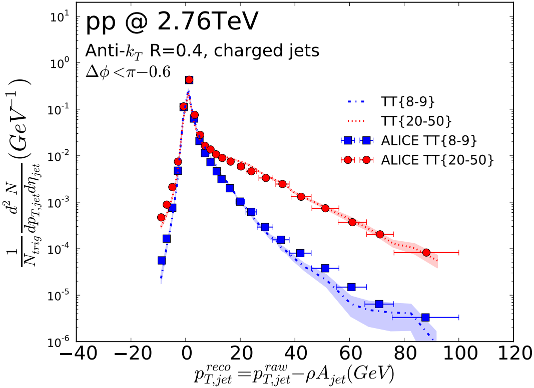
$$H_{TT}(p_{T,\mathrm{jet}}, \eta_{\mathrm{jet}}) \equiv \frac{1}{N^{AA}_{\mathrm{trig}}} \frac{d^2 N^{AA}_{\mathrm{jet}}}{dp^{\mathrm{ch}}_{T,\mathrm{jet}} d\eta_{\mathrm{jet}}}\bigg|_{p_{T,\mathrm{trig}} \in \mathrm{TT}}, \qquad V_{TT}(\Delta\phi) \equiv \frac{1}{N^{AA}_{\mathrm{trig}}} \frac{d^2 N^{AA}_{\mathrm{jet}}}{dp^{\mathrm{ch}}_{T,\mathrm{jet}} d\Delta\phi}\bigg|_{p_{T,\mathrm{trig}} \in \mathrm{TT}}$$



- Spectrum of recoil jets provided that a hard hadron is found in TT (Trigger Tracks). Includes no-jet cases.
- TT represents the trigger range. For example, $H_{8,9}(p_T, \eta)$ represents the jet spectrum with the trigger hadron within $(8\,\mathrm{GeV}, 9\,\mathrm{GeV})$

Hadron-Jet Correlation

[Preliminary]

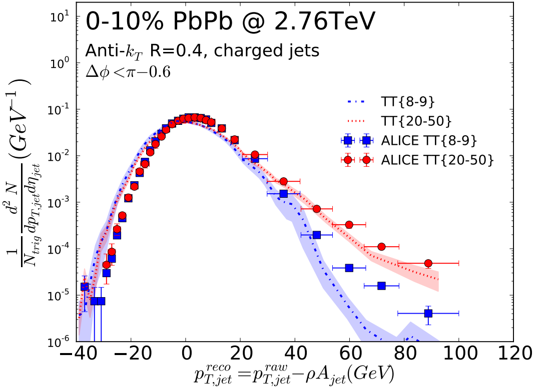


[Calculation by C. Park]

- 2.76 TeV pp reference.
- Trigger track (TT) for 8 – 9 GeV (the reference TT class) and TT for 20 – 50GeV (signal TT class).
- $\rho = \text{median}\left\{\frac{p^{i,\text{raw}}_{T,\text{jet}}}{A^i_{\text{jet}}}\right\}$ with the highest two jet p_T's excluded.

Hadron-Jet Correlation

[Preliminary]

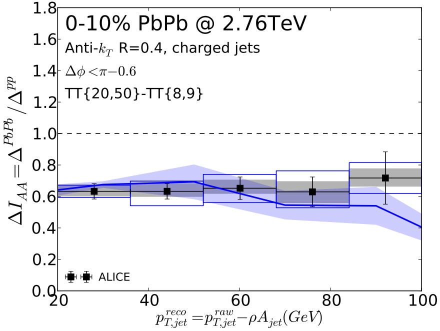


[Calculation by C. Park]

- Same as before but for 0-10 % PbPb collisions.
- The slight shift to the left is due to slight centrality mis-match between experimental and theoretical results
- Large $p_{T,jet}^{reco}$ part needs more statistics.

[Preliminary]

$$\Delta = H_{20,50}(p_T,\eta) - H_{8,9}(p_T,\eta)$$



- Experimental feature roughly reproduced
- High p_T: Need more statistics

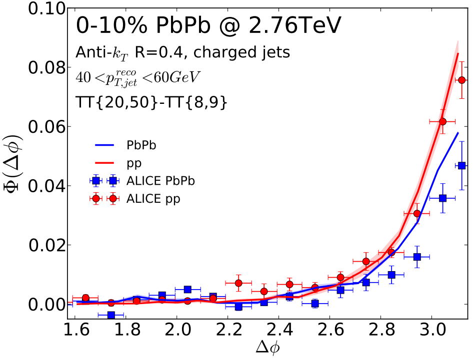
[Calculation by C. Park]

$$H_{TT}(p_{T,\mathrm{jet}}, \eta_{\mathrm{jet}}) \equiv \frac{1}{N_{\mathrm{trig}}^{AA}} \frac{d^2 N_{\mathrm{jet}}^{AA}}{dp_{T,\mathrm{jet}}^{\mathrm{ch}} d\eta_{\mathrm{jet}}}\bigg|_{p_{T,\mathrm{trig}} \in \mathrm{TT}},$$

Hadron-Jet Correlation

[Preliminary]

$$\Phi(\Delta\phi) = V_{20,50}(\Delta\phi) - V_{8,9}(\Delta\phi)$$



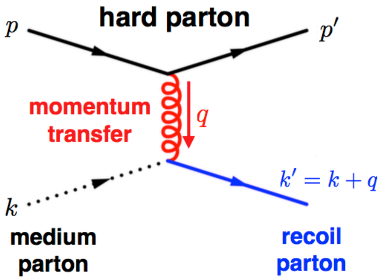
[Calculation by C. Park.

Recoil partons and medium response to be added.]

- High trigger (HT): Trigger p_T and the recoil jet direction tends to align
- Low trigger (LT): Trigger direction and the recoil jet direction are less correlated
- (HT) − (LT) still retains $\Delta\phi = \pi$ peak
- Medium interaction deflects jets: The trigger-jet correlation is degraded

$$V_{TT}(\Delta\phi) \equiv \left(1/N^{AA}_{\rm trig}\right) d^2 N^{AA}_{\rm jet} / dp^{\rm ch}_{T,\rm jet} d\Delta\phi \Big|_{p_{T,\rm trig} \in \rm TT}$$

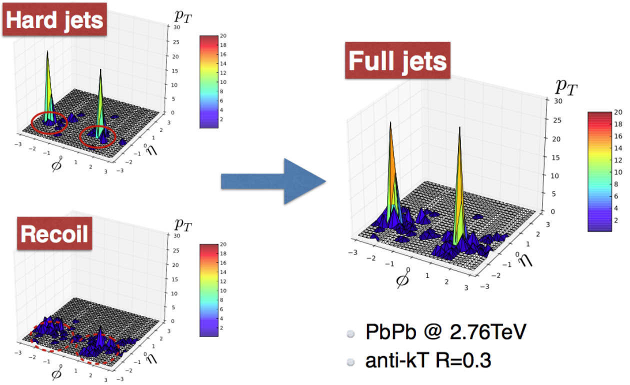
Feature 3 – Recoil Partons



[Figure by C. Park]

- Sample p' according to the in-medium elastic cross-section
- $q = p - p'$
- Sample k from the thermal medium under the constraint that $k'^2 = (k+q)^2 = 0$
- If $|\mathbf{k}'| \geq p_{\rm cut}$, k' behaves like a "jet parton"

What the recoil partons do



[Figure by C. Park]

- Spread jet energy-momentum to larger angles
- Affect jet shapes and jet masses

Recoil in MARTINI

- Kinematical cut: $p_{\rm cut} \simeq 4T$
  $\Longrightarrow$ The same as the AMY cut-off.
- Recoil partons with $|\mathbf{k}'| < p_{\rm cut}$:
  $\Longrightarrow$ $p - p' = q$ should be treated the source term for the medium response
- Radiated partons with $|\mathbf{p}'| \simeq 4T$ are sources that contribute p' to the medium

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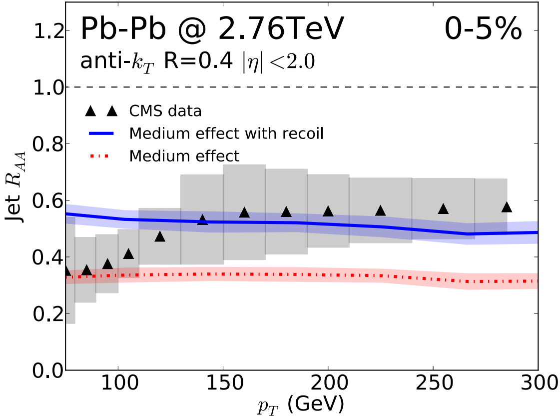
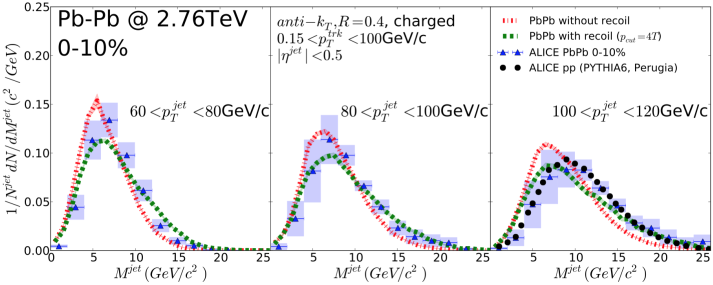
M. Singh, C. Park & S. McDonald are working on medium response (coming soon).

[Preliminary]



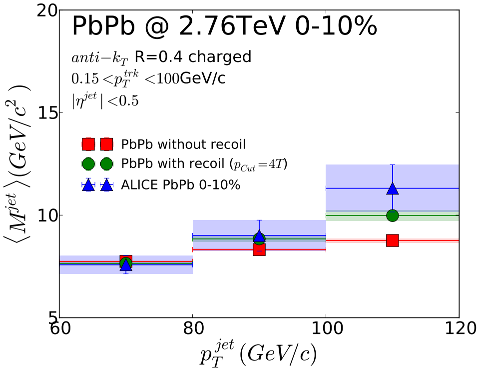
- Recoil crucial for the high p_T part
- Low p_T part still under study

Jet Invariant Mass w/ Recoil



[arXiv:1807.06550, C. Park, S. Jeon & C. Gale]

- Jet broadening *decreases* $M_{\rm jet}$
- Energy absorbed into the medium *decreases* $M_{\rm jet}$
- Recoils *increase* $M_{\rm jet}$ by adding thermal energy to $P_{\rm tot}$
- In *AA* collisions, these effects seem to largely cancel each other out



[arXiv:1807.06550, C. Park, S. Jeon & C. Gale]

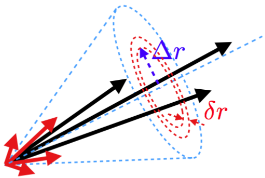
- But not entirely!
- Recoil effect less visible for low p_T
- Recoil crucial in explaining higher jet mass for high p_T
- Medium response should help

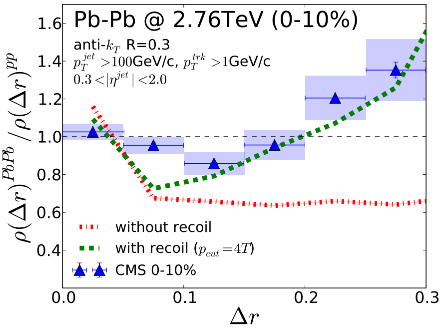
Jet-Shape Function w/ Recoil

Jet-shape function (here $\Delta r = R$)

$$\rho(\Delta r) = \frac{1}{\delta r}\frac{1}{N_{\text{jet}}}\sum_{\text{jets}} \frac{\sum_{\text{tracks}\in[r_a,r_b)} p_T^{\text{track}}}{p_T^{\text{jet}}}$$

with $r_a = \Delta r - \delta r/2$ and $r_b = \Delta r + \delta r/2$

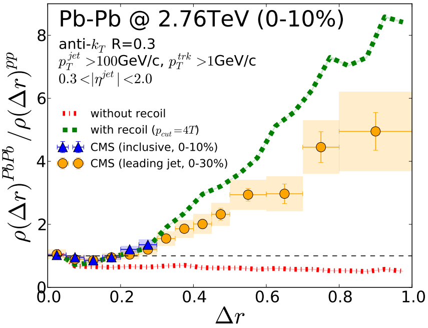




- Recoils spread energy and momentum $\Longrightarrow$ Important at large angles

[arXiv:1807.06550, C. Park, S. Jeon & C. Gale]

Jet-Shape Function w/ Recoil



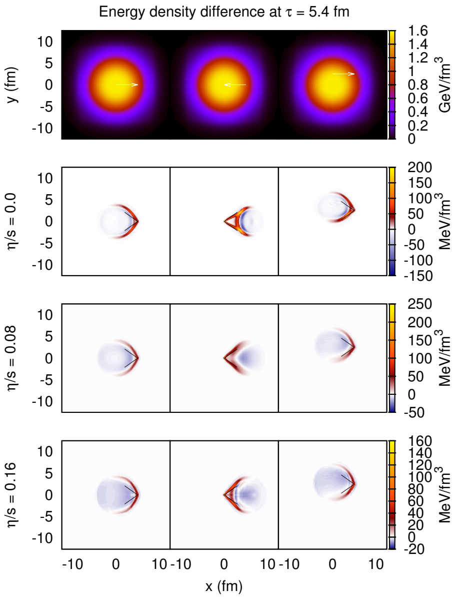
[arXiv:1807.06550, C. Park, S. Jeon & C. Gale]

- Combined plot of the inclusive jet shape function in $0 - 10\,\%$ and the leading jet shape function in $0 - 30\,\%$
- Importance of recoils is clearly visible although quantitative comparison is not $100\,\%$ meaningful

Summary

- MARTINI with a set of new features
  - Running coupling
  - Finite time AMY-McGill radiation rate
  - Recoil partons
- Goal: Consistent description of high p_T hadrons and jets
- Medium response to be added
- Multiple papers in preps

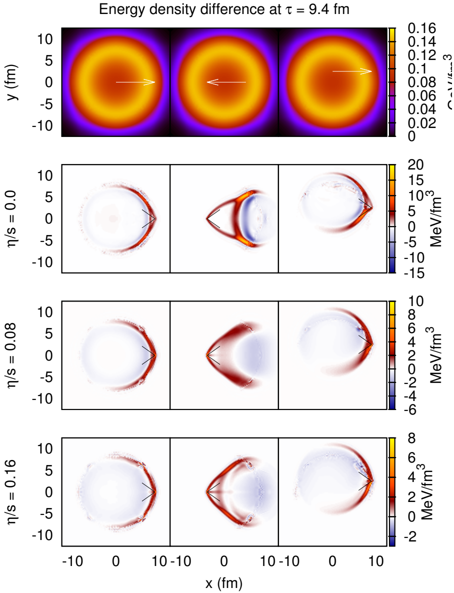
Medium response



- At $\tau = 5.4\,\mathrm{fm}$
- $\delta\epsilon/\epsilon \sim 10\,\%$
- Diffusion wake clearly visible – The higher η/s, the stronger the wake.
- The strength and the angle of the shock depends on η/s – Note that $\eta/s = 1/4\pi$ has *higher* temperature – Reheating

[Mayank Singh & Chanwook Park]

Medium response

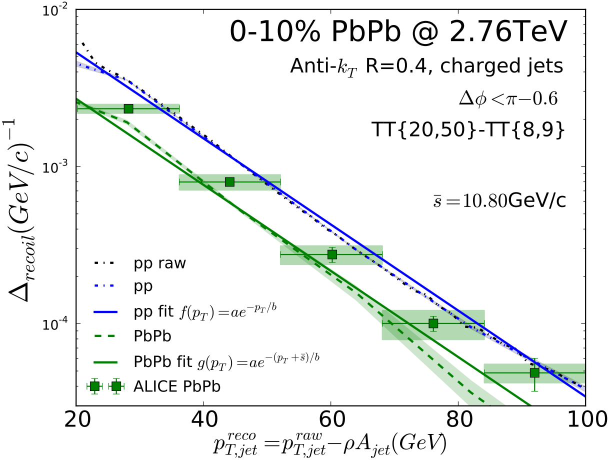


- Later time at $\tau = 9.4\,\mathrm{fm}$
- $\delta\epsilon/\epsilon \sim 10\,\%$
- The strength and the angle of the shock depends on η/s – The higher η/s, the weaker the shock wave – Dissipation wins

[Mayank Singh & Chanwook Park]

Backup Slides

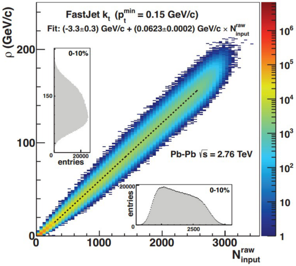
[Preliminary]

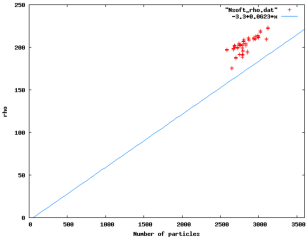


[Calculation by C. Park]

Hadron-Jet Correlation

[Preliminary]





[ALICE, JHEP 1203 (2012) 053]

The slight shift to the left is because of centrality mis-match. The background density ρ is higher at given number of charged particles $\Longrightarrow$ $(-\rho A_{jet})$ is large $\Longrightarrow$ The left shift of the curves.

[Calculation by C. Park]

Local or Holistic – Finite time effects

- In-medium radiation rate

$$\frac{d\Gamma^a_{bc}}{dk} = \frac{P^{a(0)}_{bc}(x)}{\pi p}\mathrm{Im}\left(\int_0^t dt' \int_{\mathbf{q}_\perp,\mathbf{p}_\perp} \frac{\mathbf{q}_\perp\cdot\mathbf{p}_\perp}{\delta E(\mathbf{q}_\perp)}\mathcal{C}(t)\, K(t,\mathbf{q}_\perp;t',\mathbf{p}_\perp)\right)$$

- $K(t,\mathbf{q}_\perp;t',\mathbf{p}_\perp)$: Retarded propagator of parton a from t' to t with the Hamiltonian $H = \delta E - i\mathcal{C}$

  where

$$\delta E = \frac{p\mathbf{p}_\perp^2}{k(p-k)} + \frac{m_b^2}{2k} + \frac{m_c^2}{2(p-k)} - \frac{m_a^2}{2p}$$

$$\mathcal{C} = \frac{C_b + C_c - C_a}{2} v(\mathbf{x}_\perp) + \frac{C_a + C_c - C_b}{2} v(\frac{k}{p}\mathbf{x}_\perp) + \frac{C_a + C_b - C_c}{2} v(\frac{p-k}{p}\mathbf{x}_\perp)$$

  with

$$v(\mathbf{x}_\perp) = \int_{\mathbf{q}_\perp} C(\mathbf{q}_\perp)(1 - e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}) \quad \text{and } C(\mathbf{q}_\perp) = \frac{g_s^2 m_D^2 T}{\mathbf{q}_\perp^2(\mathbf{q}_\perp^2 + m_D^2)}$$

  The same as the AMY kernel, but for finite t.

The rate equation to solve

- Rate equation to solve

$$
\begin{aligned}
\frac{dP_{q,\bar{q}}(p)}{dt} &= \int_k P_{q,\bar{q}}(p+k)\frac{d\Gamma^q_{qg}(p+k,k)}{dkdt} - \int_k P_{q,\bar{q}}(p)\frac{d\Gamma^q_{qg}(p,k)}{dkdt} \\
&\quad +2\int_k P_g(p+k)\frac{d\Gamma^g_{q\bar{q}}(p+k,k)}{dkdt}\,, \\
\frac{dP_g(p)}{dt} &= \int_k P_{q,\bar{q}}(p+k)\frac{d\Gamma^q_{qg}(p+k,p)}{dkdt} + \int_k P_g(p+k)\frac{d\Gamma^g_{gg}(p+k,k)}{dkdt} \\
&\quad - \int_k P_g(p)\left(\frac{d\Gamma^g_{q\bar{q}}(p,k)}{dkdt} + \frac{d\Gamma^g_{gg}(p,k)}{dkdt}\Theta(k-p/2)\right)
\end{aligned}
$$

- Actual solution by Monte-Carlo