World-line approach to chiral kinetic theory

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Problems and opportunities in chiral fluids
July 17-19, Santa Fe

with Raju Venugopalan and Yi Yin

arXiv:1701.03331, 1702.01233, 1712.04057

Motivations to understand chiral transport

Macroscopic manifestations of chiral anomalies

- **condensed matter**
  chiral anomaly in Weyl semimetals, transport of energy and information at room temperature

- **astrophysics**
  helical perturbations in large scale electro-magnetic fields, magnetic fields in stars, novel instability mechanisms (=transport)

- **cosmology**
  baryogenesis, CP violation and the structure of non-Abelian gauge theories

- **nuclear and high energy physics**
  chiral fluids in ultra-relativistic heavy ion collisions, probing the topological structure of Quantum Chromodynamics

Chiral fluids and topological properties of Quantum Chromodynamics

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- **Anomalies test P and CP odd field configurations**

$$\partial_\mu j_5^\mu = -\frac{g^2 N_f}{16\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$$

Chiral fluids and topological properties of Quantum Chromodynamics

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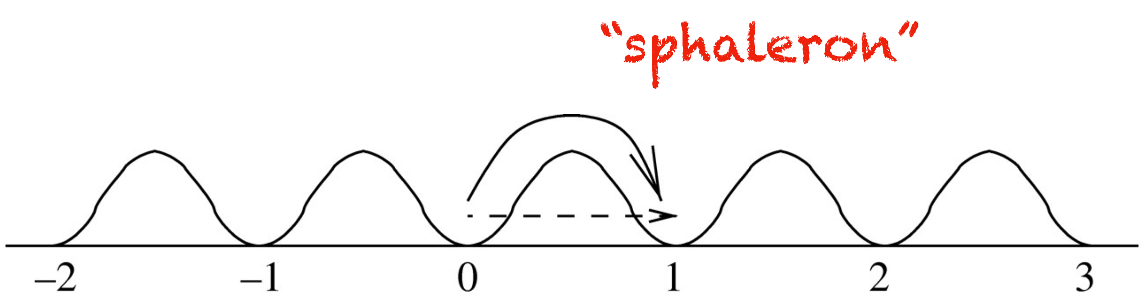
- **P and CP *odd* field configurations connected to the topological structure of QCD**



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Chiral fluids and topological properties of Quantum Chromodynamics

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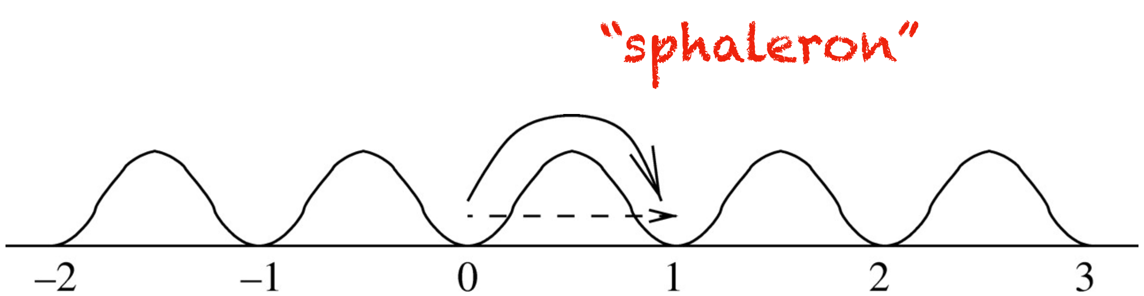

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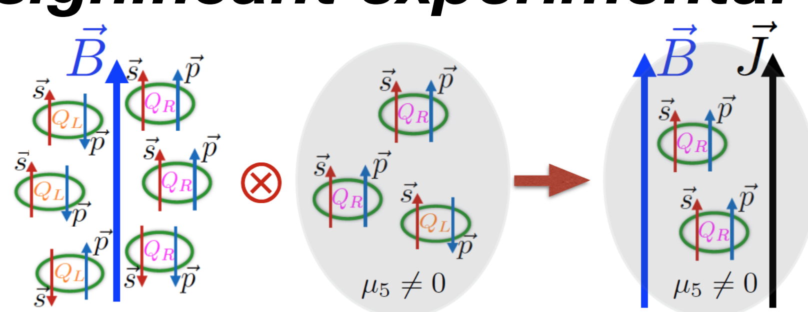
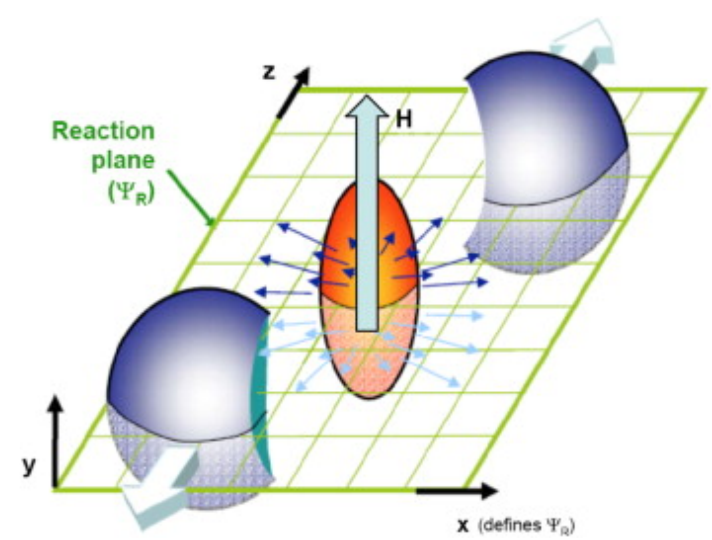
- **Ultra-relativistic Heavy Ion Collisions:** ***significant experimental effort***



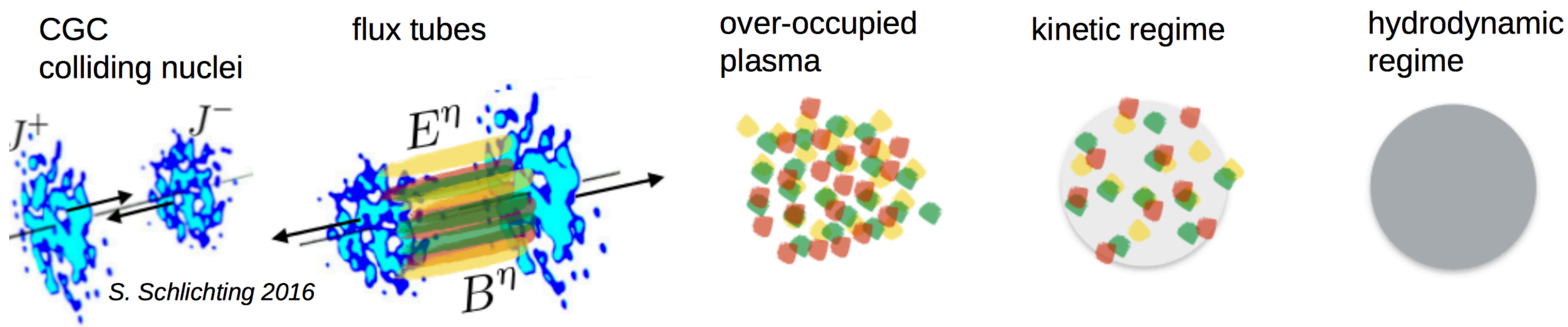
"Magnetic Field related"

"Vortical Effects"

"Polarization of Hyperons"



The life of a chiral fluid



Out of equilibrium → **Equilibrium?**

"classical statistical simulations"

"spin/chiral hydrodynamics"

"kinetic theory and chirality"

"lattice simulations"

"holography"

"EFT"

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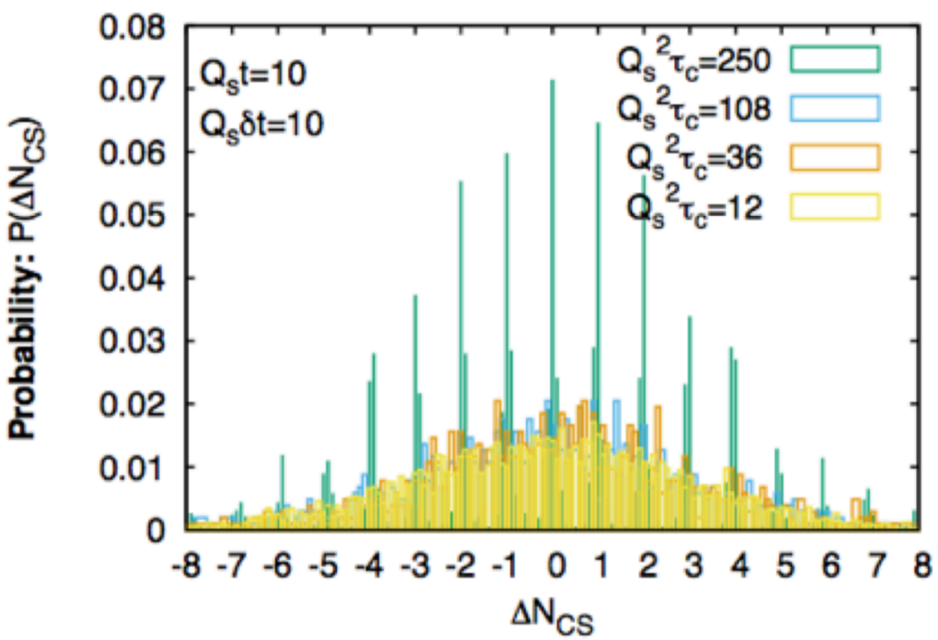

Real-time lattice simulations
Mace, Schlichting, Venugopalan
PRD93 (2016) no.7, 074036

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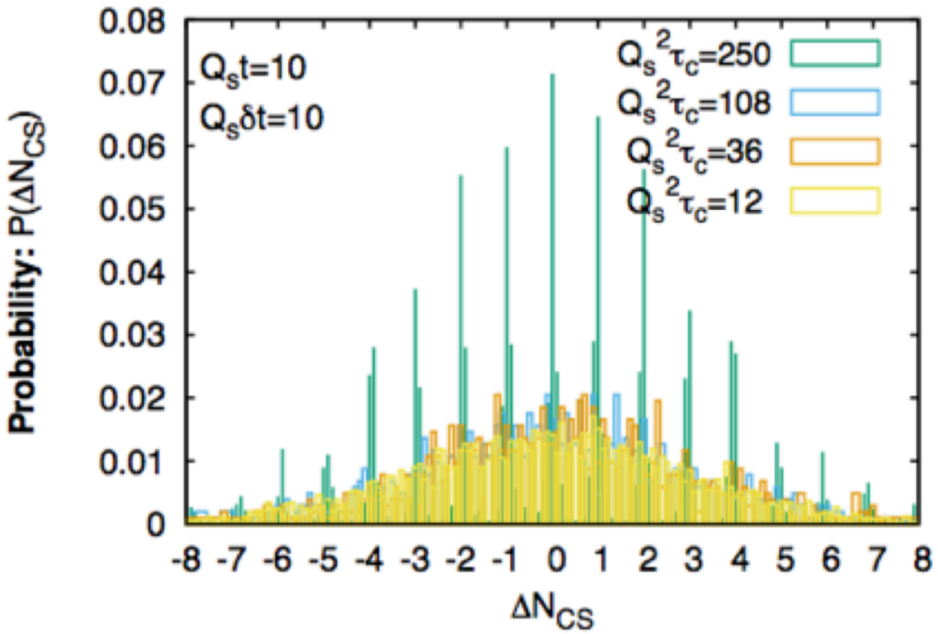

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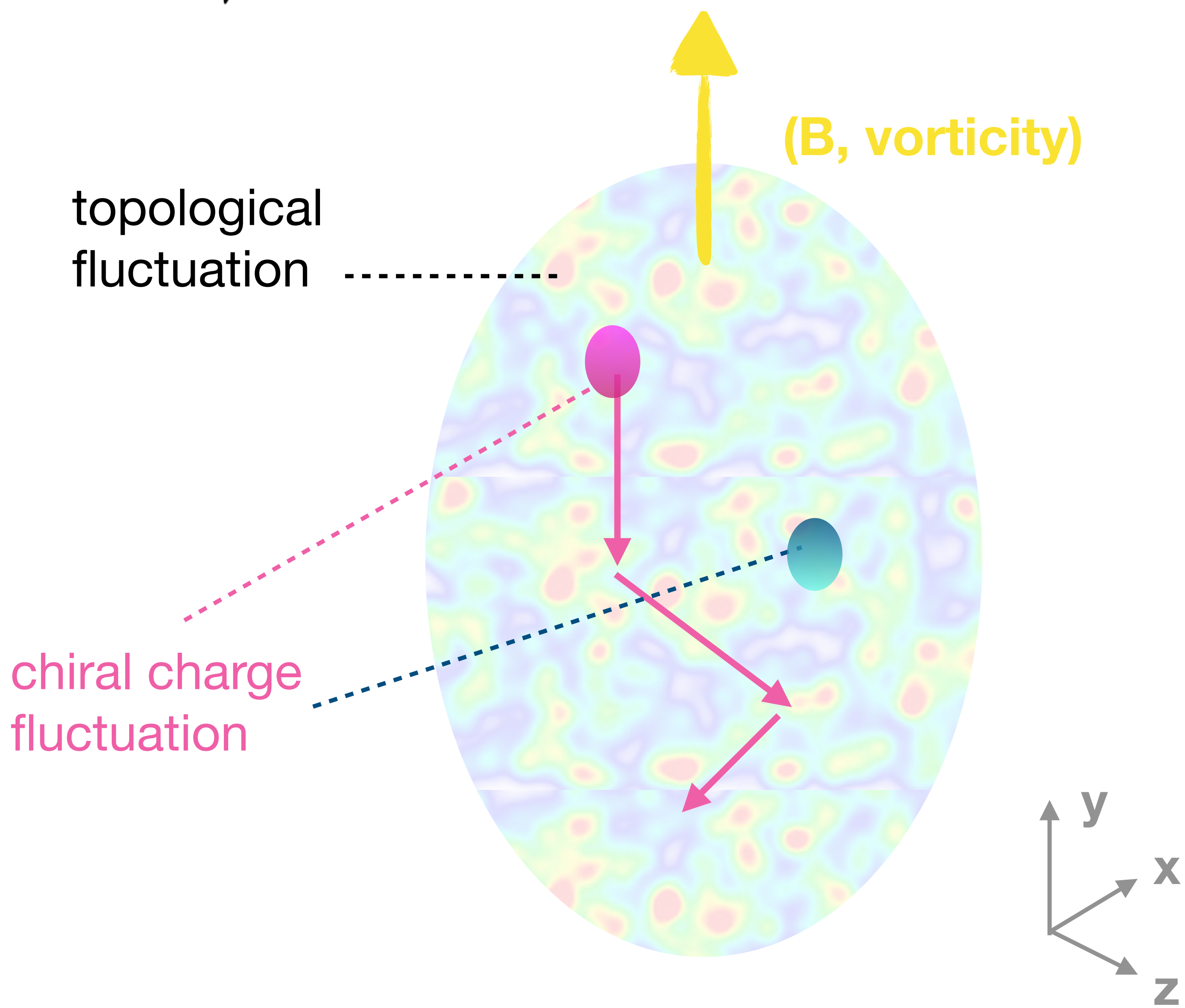

Towards chiral transport

- **Aim for relativistic Boltzmann equation for chiral fermions in weak-coupling, dilute limit**
- **Chirality, spin and interactions with topological background in quasi particle picture?**
- **Relevant scales and mechanisms?**

World-line approach

1. World-line approach to Quantum Field theory

- **One-loop effective action** *Polyakov; Bern, Kosower; Strassler*

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right) \qquad \mathcal{L} = \Phi^\dagger D^2 \Phi$$

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- **Integral representation of log (heat-kernel)**

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t}\left(e^{-\sigma t} - e^{-t}\right)$$

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- **Effective action: QM path integral of particle on circle (Strassler, 1992)**

$$\Gamma[A] \;=\; \int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\,\mathcal{P}\exp\left[-\int_0^T d\tau\,\left(\frac{1}{2\varepsilon}\dot{x}^2 + igA[x(\tau)]\cdot\dot{x}\right)\right]$$

relativistic point-particle action

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- **Effective action for fermions (D'Hoker and Gagne)**

$$S[A,B] = \int d^4x\, \bar{\psi}\left(i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/\right)\psi \longrightarrow W[A,B] = W_{\mathbb{R}}[A,B] + iW_{\mathbb{I}}[A,B]$$

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$$W_{\mathbb{R}} = -\frac{1}{8} \log \det(\tilde{\Sigma}^2) = -\frac{1}{8} \mathrm{Tr} \log(\tilde{\Sigma}^2)$$

$$\tilde{\Sigma}^2 = (p - \mathcal{A})^2 \, \mathbb{I}_8 + \frac{i}{2} \Gamma_\mu \Gamma_\nu F_{\mu\nu}[\mathcal{A}]$$

$$\mathcal{A} = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$$

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$$\mathcal{L} = \frac{\dot{x}^2}{2\mathcal{E}} + \dot{x}_\mu A^\mu(x) + \frac{i}{2}\psi^\mu\dot{\psi}_\mu - \frac{i\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu + \ldots$$

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- **'SUSY spinning particle models' via anti-commuting variables**

Berezin & Marinov, Barducci, Balachandran, Casalbuoni, Brink, Howe, DiVecchia (70s-80s)

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$$W_{\mathbb{I}} = \frac{i\mathcal{E}}{64} \int_{-1}^{1} d\alpha \int_{0}^{\infty} dT \; \mathrm{Tr} \left\{ \hat{M} e^{-\frac{\mathcal{E}}{2} T \tilde{\Sigma}^2_{(\alpha)}} \right\}$$

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- **Anomaly related to Grassmann zero modes on the world line** (see also Polyakov's book 80's)

$$\partial_\mu \langle j^5_\mu(y) \rangle \equiv \partial_\mu \frac{i\delta W_{\mathbb{I}}}{\delta B_\mu(y)} \Big|_{B=0} = -\frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(y) F_{\rho\sigma}(y)$$

Berry's phase and the adiabatic limit

- **In non-relativistic, adiabatic limit**

$$H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S} \cdot \left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right] \times \boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B} \cdot \boldsymbol{S}}{m}$$

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- **real part of world-line effective action contains a Berry phase**

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- **while the anomaly is related to the imaginary part…**

c.f. talk by K. Fujikawa,
see e.g. Phys.Rev. D97 (2018) no.1, 016018

Towards chiral kinetic theory

2. Towards chiral kinetic theory

- **World-line approach to Schwinger-Keldysh (SK) path integral**

$$Z = \int [d\xi]\ \exp\left(-G[\xi]\right) \int_{\mathcal{C}} [dA]\ \exp\left(iS_{\rm eff}\right)$$

$$S_{\rm eff}[A,\xi] \ = \ -\tfrac{1}{4}\int_{\mathcal{C}} d^4x\, F_{\mu\nu}F^{\mu\nu} + W[A,\xi]$$

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- **Chiral Fermion Hamiltonian**

$$H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_- \qquad c_\pm \;\equiv\; \tfrac{1}{2}\left(\pm P_\mu\psi^\mu + \tfrac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right)$$

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- **chiral fermions and local topological fluctuations**

$$C(t,\delta t) = \frac{1}{V}\Big\langle\big(N_{CS}(t+\delta t) - N_{CS}(t)\big)^2\Big\rangle$$

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  (not necessarily thermal)

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- **What is the dynamics of** $\delta f(x,P,Q,\psi)$ **?**

- **What are the relevant scales for** $F^{\mu\nu}(x)$ **?**

Relevant work

- **Arnold, Son, Yaffe,** Thermal non-abelian plasmas, sphaleron transitions, Phys.Rev. D55 (1997) 6264-6273

$$R \sim (g^2 T)^{-1}, \qquad t_{sph} \sim (g^4 T)^{-1}$$

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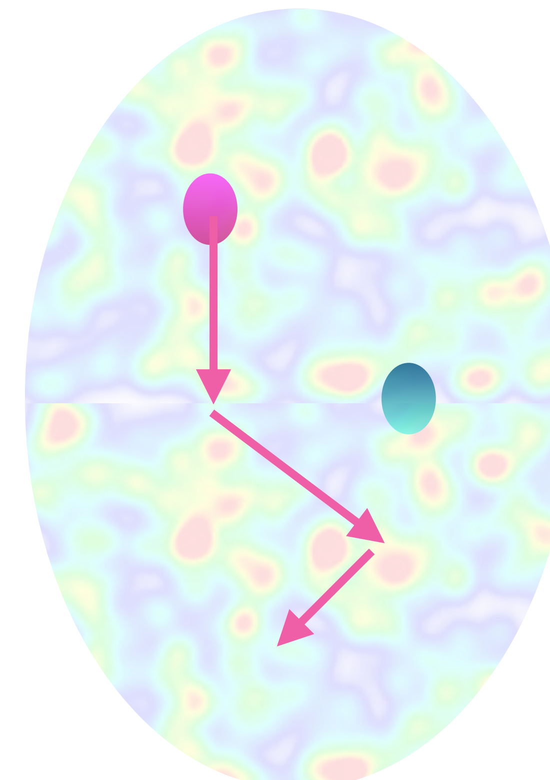

- **Bodeker,** real-time dynamics for topological transitions, via stochastic Boltzmann-Vlasov equations, Phys.Lett. B426 (1998) 351-360

- **Kelly, Liu, Litim, Manuel,** kinetic theory with internal degrees of freedom color, Phys.Rev.Lett. 72 (1994) 3461-3463, Phys.Rev.Lett. 82 (1999) 4981-4984

- **Jalilian-Marian, Jeon, Venugopalan, Wirstam,** kinetic theory from world line approach to QFT, colored-scalar particles

2. Towards chiral kinetic theory

- **Equations of motion from world-line Hamiltonian**

$$\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{\epsilon^{\mu\nu\alpha\beta}}{6}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]\Big)$$

$$+ \frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta + \frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] = C[\delta f, \delta F]$$

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$$C[\delta f, \delta F] \equiv -\varepsilon\langle\delta f\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\delta F^{\mu\nu}\rangle\psi_\nu + \frac{i\varepsilon}{2}\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\partial^\mu\delta F_{\alpha\beta}\rangle\psi^\alpha\psi^\beta$$

$$-\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\delta F^{\mu\alpha}\rangle\big(\varepsilon P_\alpha + \frac{i}{2}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\big)$$

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- **Collision terms from fluctuations**

$$C[\delta f, \delta F] \equiv -\varepsilon\langle\delta f\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\delta F^{\mu\nu}\rangle\psi_\nu + \frac{i\varepsilon}{2}\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\partial^\mu\delta F_{\alpha\beta}\rangle\psi^\alpha\psi^\beta$$

$$-\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\delta F^{\mu\alpha}\rangle\big(\varepsilon P_\alpha + \frac{i}{2}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\big)$$

- **Using Maxwell / Yang-Mills equations —> hierarchy of equations**

$$\langle\delta f\,\delta f\rangle,\quad \langle\delta F^{\alpha\beta}\,\delta F^{\mu\nu}\rangle,\quad \langle\delta F\,\delta F\,\delta F\,\delta F\rangle,\ldots$$

$$C(t,\delta t) = \frac{1}{V}\Big\langle\big(N_{CS}(t+\delta t) - N_{CS}(t)\big)^2\Big\rangle$$

c.f. BBGKY hierarchy, Landau Lifshitz, "Statistical Physics" and "Physical Kinetics"

Summary

- Consistent Chiral Kinetic Theory using the world-line approach to QFT
- Berry phase and anomaly
- Issues of Lorentz covariance and chirality/spin/helicity
- Non-equilibrium many-body (SK) formulation:
  quantum kinetic theory in saddle point limit
- Fluctuations and scales matter

Back-up:
Origin of the anomaly and Berry phase

The non-relativistic, $H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S}\cdot\left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right] \times \boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B}\cdot\boldsymbol{S}}{m}$ $\quad S^i = -\frac{i}{2}\epsilon^{ijk}\psi^j\psi^k$

and adiabatic limit of the world-line representation contains a Berry phase

$$W_{\mathbb{R}} = \int \mathscr{D}x\mathscr{D}p \, \exp\left(i \int dt \left[\dot{\mathbf{x}}\cdot\mathbf{p} - \tilde{H}\right]\right)$$

$$\tilde{H} = mc^2 + \frac{(\boldsymbol{p} - \mathbf{A}/c)^2}{2m} + A^0(x) - \dot{\mathbf{p}}\cdot\mathscr{A}(\boldsymbol{p}) \qquad \mathscr{A}(\boldsymbol{p}) \equiv -i\langle\psi^+(\boldsymbol{p})|\boldsymbol{\nabla}_p|\psi^+(\boldsymbol{p})\rangle$$

Interesting related development: K. Fujikawa 2005 and 2017

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Origin of the anomaly and Berry phase

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In the adiabatic limit, for large chemical potential and zero mass we reproduce the results of Son & Yamamoto, Stephanov and Yin …

… from the real part, which we have shown to be independent of the anomaly

Interesting related development: K. Fujikawa 2005 and 2017

Fujikawa 2005

The notion of Berry's phase is known to be useful in various physical contexts [17]-[18], and the topological considerations are often crucial to obtain a qualitative understanding of what is going on. Our analysis however shows that the topological interpretation of Berry's phase associated with level crossing generally fails in practical physical settings with any finite T. The notion of "approximate topology" has no rigorous meaning, and it is important to keep this approximate topological property of geometric phases associated with level crossing in mind when one applies the notion of geometric phases to concrete physical processes. This approximate topological property is in sharp contrast to the Aharonov-Bohm phase [8] which is induced by the time-independent gauge potential and topologically exact for any finite time interval T. The similarity and difference between the geometric phase and the Aharonov-Bohm phase have been recognized in the early literature [1, 8], but our second quantized formulation, in which the analysis of the geometric phase is reduced to a diagonalization of the effective Hamiltonian, allowed us to analyze the topological properties precisely in the infinitesimal neighborhood of level crossing.

What we have shown in the present paper is that this expectation is not realized, and the similarity between the two is superficial.

7

3. Bonus: Side-jumps

An issue of Lorentz covariance and spin: side jumps

Lorentz Invariance in Chiral Kinetic Theory
Jing-Yuan Chen, Dam T. Son (Chicago U.), Mikhail A. Stephanov (Illinois U., Chicago & Chicago U.), Ho-Ung Yee (Illinois U., Chicago & RIKEN BNL), Yi Yin (Illinois U., Chicago).
Published in **Phys.Rev.Lett. 113 (2014) no.18, 182302**

Collisions in Chiral Kinetic Theory
Jing-Yuan Chen, Dam T. Son (Chicago U.), Mikhail A. Stephanov (Illinois U., Chicago). Feb 24, 2015. 5 pp.
Published in **Phys.Rev.Lett. 115 (2015) no.2, 021601**

Seems puzzling, but is not

$$H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_- \qquad c_\pm \equiv \frac{1}{2}\left(\pm P_\mu\psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right)$$

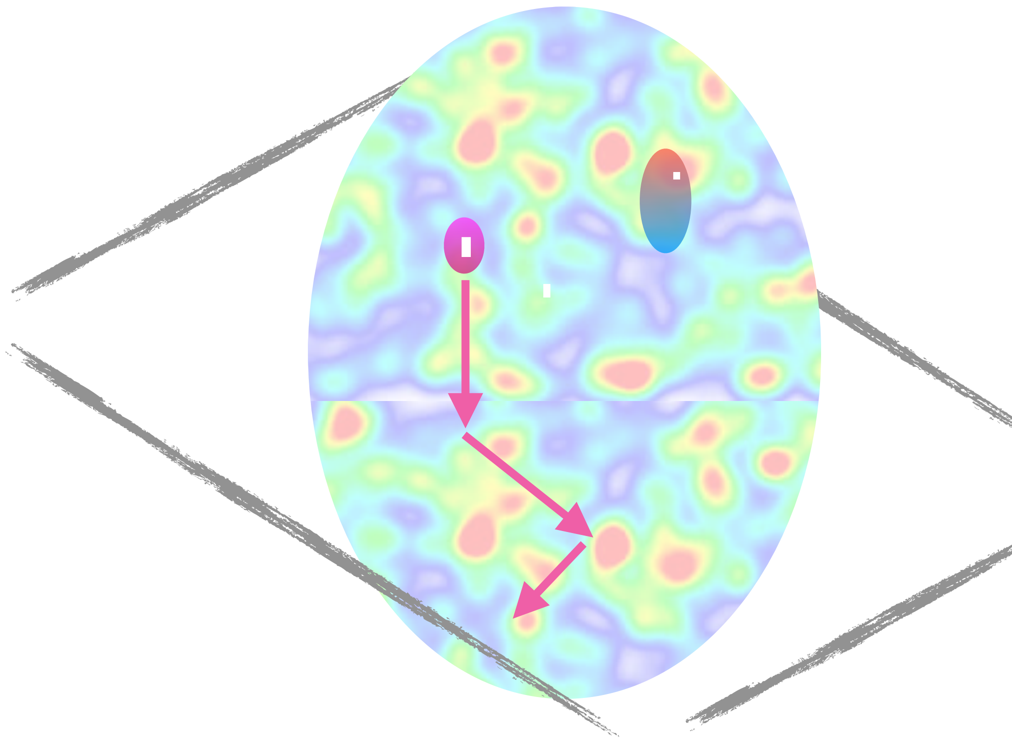
mass-shell constraint

helicity constraint

gauge parameters (Lagrange multiplier), 1st class constraints

- **'Position' $x^\mu(\tau)$ at given 'time' is not gauge invariant, because τ is not (reparametrization invariance of world line of relativistic particle).**
- **Similarly defining the local spin frame is somewhat ambiguous.**

Backup: Chiral Kinetic Theory



- **Collision terms generated by (topological) fluctuations in the medium**
- **Similar equations exist for the fluctuations, which in the dilute limit can be solved in closed form (c.f. BBGKY hierarchy)**

$$\delta f\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu+\frac{i}{2}\psi^\mu\bar{\chi}-\frac{1}{6}\epsilon^{\mu\nu\alpha\beta}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big]+\frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha$$
$$-\frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta+\frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi}-\frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big]$$
$$+\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha+\frac{P^\mu}{2}\bar{\chi}+\frac{i\epsilon^{\mu\nu\alpha\beta}}{4}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]=K[\delta F]\,,$$

$$K[\delta F]\equiv-\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\delta F^{\mu\alpha}P_\alpha-\frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\delta F_{\alpha\beta}\psi^\beta$$
$$+\frac{i}{2}\delta F^{\mu\alpha}\psi_\alpha\bar{\chi}-\frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\delta F^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big]+\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\delta F^{\mu\alpha}\psi_\alpha\Big]\Big)$$

Backup: Grassmann extended phase space

Poisson/Dirac Brakets

$$\{A,B\}_{\mathrm{D}} = A\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\vec{\partial}}{\partial p^\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\vec{\partial}}{\partial x^\mu} + \frac{1}{2}\Big[\frac{\overleftarrow{\partial}}{\partial \psi^\mu}\frac{\vec{\partial}}{\partial p^\mu_\psi} + \frac{\overleftarrow{\partial}}{\partial p^\mu_\psi}\frac{\vec{\partial}}{\partial \psi^\mu}\Big] + \frac{1}{2}\left[\frac{\overleftarrow{\partial}}{\partial \psi_5}\frac{\vec{\partial}}{\partial p_5} + \frac{\overleftarrow{\partial}}{\partial p_5}\frac{\vec{\partial}}{\partial \psi_5}\right]\Big)B\,. \tag{6}$$

The Dirac brackets between any two elements of the extended phase space are given by

$$\{x^\mu, p^\nu\} = \{x^\mu, P^\nu\} = g^{\mu\nu}, \tag{7}$$
$$\{P^\mu, P^\nu\} = F^{\mu\nu}\,, \tag{8}$$
$$\{P^\alpha, F^{\mu\nu}\} = -\partial^\alpha F^{\mu\nu}\,, \tag{9}$$
$$\{p_{\psi,\mu}, \psi_\nu\} = \frac{g_{\mu\nu}}{2}\,, \tag{10}$$
$$\{p_5, \psi_5\} = \frac{1}{2}\,, \tag{11}$$

while all other brackets vanish. Further, using Eq.(4) and Eq.(5), we obtain

$$\{\psi_\mu, \psi_\nu\} = -i g_{\mu\nu}\,, \tag{12}$$
$$\{\psi_5, \psi_5\} = -i\,. \tag{13}$$

The conjugate variables for x^μ are

$$p^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = P^\mu + A^\mu\,,$$

where

$$P^\mu \equiv \frac{\dot{x}^\mu}{\varepsilon} - \frac{i}{4\varepsilon}\dot{x}_\mu\left(\psi^\mu + \frac{i\epsilon^{\mu\nu\alpha\beta}}{3}\psi_\nu\psi_\alpha\beta\right)\chi_+ + \frac{i}{4\varepsilon}\dot{x}_\mu\left(\psi^\mu - \frac{i\epsilon^{\mu\nu\alpha\beta}}{3}\psi_\nu\psi_\alpha\beta\right)\chi_-\,.$$

Backup: Chiral phase space

Weyl equation $\frac{1}{2}(\gamma \cdot p)(1 \pm \gamma^5)\Psi = 0\,.$

Weyl Hamiltonian $H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_-\,,$

$$c_\pm \equiv \tfrac{1}{2}\left(\pm P_\mu \psi^\mu + \tfrac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right).$$

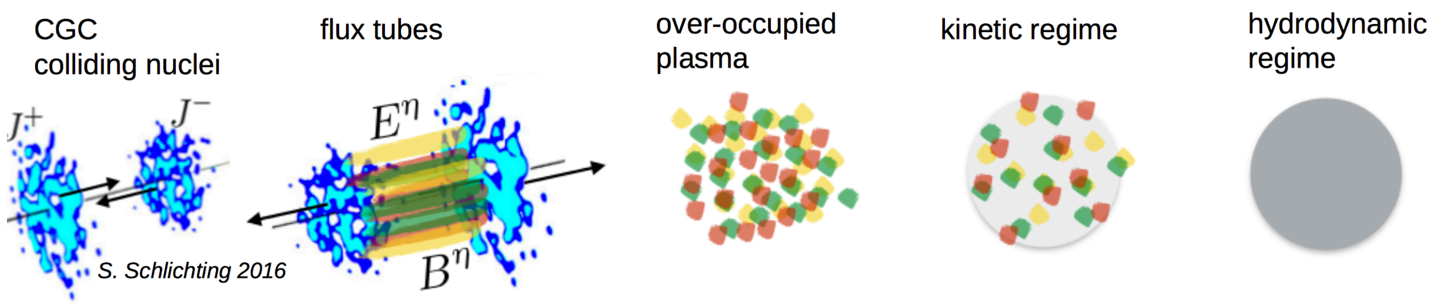
Phase space measure $d^4\psi = (-i/(\sqrt{2})^4)d\psi^3 d\psi^2 d\psi^1 d\psi^0$

$$\varepsilon \tilde{f}_\pm = 2i(\pm P_\mu \psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta)\epsilon^{ijk}\psi^i\psi^j\psi^k\,. \tag{22}$$

The above expression can be quantized by identifying $\psi^\mu \to \gamma^5\gamma^\mu/\sqrt{2}$. This gives

$$\varepsilon \tilde{f}_\pm \to \rho_\pm = \frac{1}{2}(\gamma \cdot P)(1 \pm \gamma^5)\gamma^0\,, \tag{23}$$

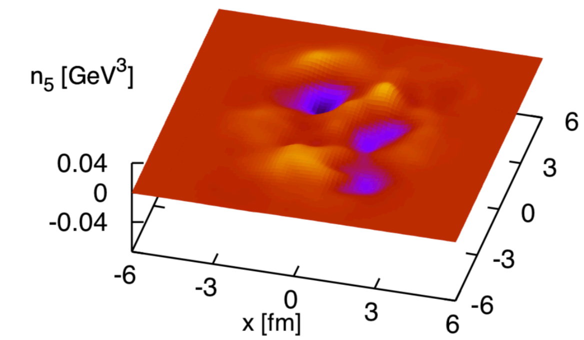
Back-up: Overview and Literature
Pre-equilibrium dynamics of the CME



Large Magnetic Fields

Fermion production | 'Anomalous transport'

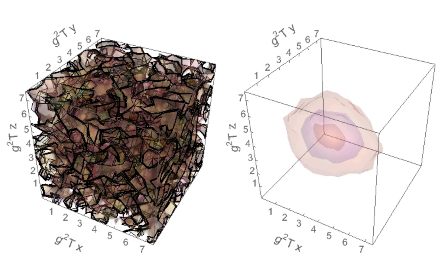
Anomalous Hydro

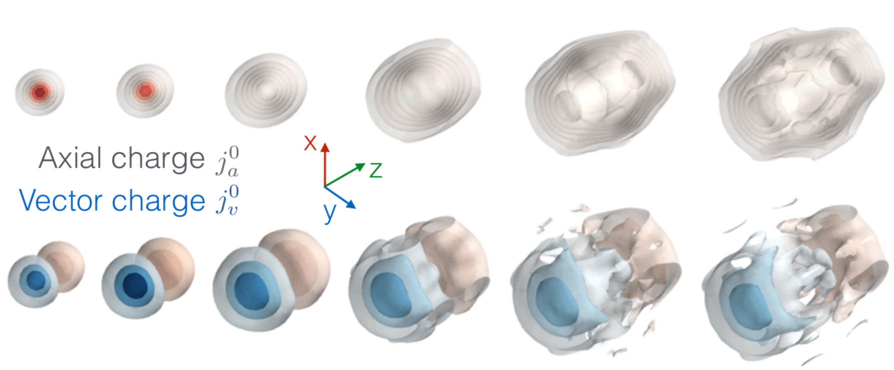


Son, Surowka 2009;
Nair, Ray, Roy 2012
Hongo, Hirnono, Hirano 2013,
Hirono, Hirano, Kharzeev 2014,
Florkowski, Friman, Jaiswal, Speranza 2017

See talk by
W. Florkowski

Real-time lattice simulations





M. Mace, S. Schlichting, R. Venugopalan, PRD 93, 074036
M. Mace, NM, S. Schlichting, S. Sharma: PRL 117, 061601, PRD 95, 036023, NPA 967, 752
N. Tanji, NM, J. Berges: PRD 93, 074507
N. Tanji, J. Berges, PRD 97, 034013

See talk by N. Tanji

Chiral Kinetic Theory

?

Wigner functions, Gao, Pu, Wang, Huang, Shi, Jiang, Liao, Zhuang, Yang, …

Berry phases, Son & Yamamoto, Stephanov & Yi, Stone,
Manuel, Torres-Rincon, Hidaka, Pu, Yang, Sun, Ko, Li, ….

See talks in this session