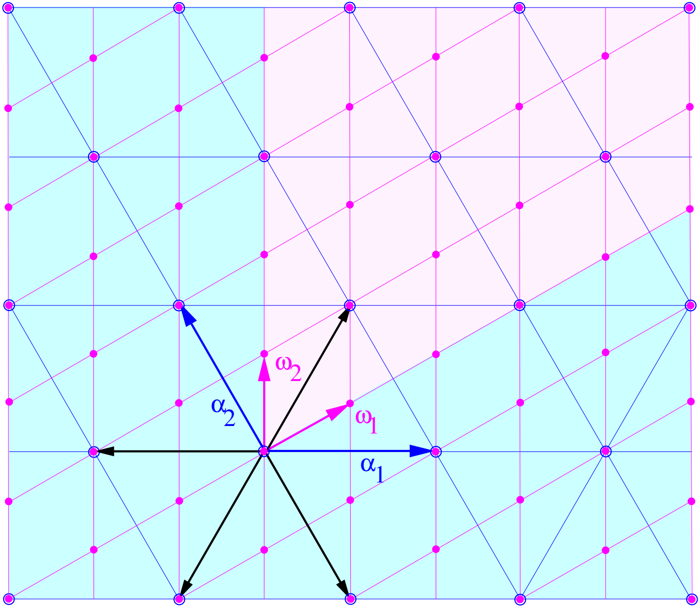
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Chiral currents from Anomalies

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Santa-Fe July 18 2018

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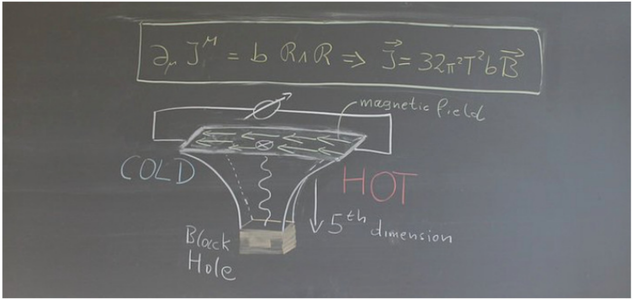
SCIENCE

An Experiment in Zurich Brings Us Nearer to a Black Hole's Mysteries

By KENNETH CHANG JULY 19, 2017



A chalkboard illustration of the string theory calculation that shows how the axial gravitational anomaly produces current. Karl Landsteiner

LETTER

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Experimental signatures of the mixed axial–gravitational anomaly in the Weyl semimetal NbP

Johannes Gooth[1,2], Anna C. Niemann[1,3], Tobias Meng[4], Adolfo G. Grushin[5], Karl Landsteiner[6], Bernd Gotsmann[2], Fabian Menges[2], Marcus Schmidt[7], Chandra Shekhar[7], Vicky Süß[7], Ruben Hühne[3], Bernd Rellinghaus[3], Claudia Felser[7], Binghai Yan[7,8] & Kornelius Nielsch[1,3]

Experiment aims to verify:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= F^{\mu\nu} J_\mu - \frac{k}{384\pi^2}\frac{\epsilon^{\rho\sigma\alpha\beta}}{\sqrt{-g}} \nabla_\mu [F_{\rho\sigma} R^{\nu\mu}{}_{\alpha\beta}], \\ \nabla_\mu J^\mu &= -\frac{k}{32\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\mu\nu}F_{\rho\sigma} - \frac{k}{768\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma}, \end{aligned}$$

Here k is number of Weyl fermions, or the Berry flux for a single Weyl node.

What the experiment measures

Contribution to energy current for 3d Weyl fermion with $\hat{H} = \boldsymbol{\sigma} \cdot \mathbf{k}$

$$\mathbf{J}_\epsilon = \mathbf{B}\left[\frac{\mu^2}{8\pi^2} + \frac{1}{24}T^2\right]$$

Simple explanation:

- The $\mathbf{B}$ field makes $B/2\pi$ one-dimensional chiral fermions per unit area.
- These have $\epsilon = +k$
- Energy current/density from each one-dimensional chiral fermion:

$$J_\epsilon = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\epsilon\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}} - \theta(-\epsilon)\right\} = 2\pi\left(\frac{\mu^2}{8\pi^2} + \frac{1}{24}T^2\right)$$

- Could even have been worked out by Sommerfeld in 1928!

Are we really exploring anomaly physics?

Yes!

Energy-Momentum Anomaly

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu - \frac{k}{384\pi^2} \frac{\epsilon^{\rho\sigma\alpha\beta}}{\sqrt{-g}} \nabla_\mu [F_{\rho\sigma} R^{\nu\mu}{}_{\alpha\beta}],$$

Origin of Gravitational Anomaly in 2 dimensions

- Set $z = x + iy$ and use conformal coordinates

$$ds^2 = \exp\{\phi(z,\bar{z})\} d\bar{z} \otimes dz$$

- Example: non-chiral scalar field $\hat{\varphi}$ has central charge $c = 1$ and energy-momentum operator is $\hat{T}(z) =: \partial_z \hat{\varphi} \partial_z \hat{\varphi}:$
- Actual energy-momentum tensor is

$$\begin{aligned} T_{zz} &= \hat{T}(z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \tfrac{1}{2}(\partial_z\phi)^2\right) \\ T_{\bar{z}\bar{z}} &= \hat{T}(\bar{z}) + \frac{c}{24\pi}\left(\partial^2_{\bar{z}\bar{z}}\phi - \tfrac{1}{2}(\partial_{\bar{z}}\phi)^2\right) \\ T_{\bar{z}z} &= -\frac{c}{24\pi}\partial^2_{\bar{z}z}\phi \end{aligned}$$

- Now $\Gamma^z_{zz} = \partial_z\phi$, $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}}\phi$, all others zero. So find

$$\nabla^z T_{zz} + \nabla^{\bar{z}} T_{\bar{z}z} = 0$$

Chiral field

- Energy-momentum tensor for chiral field

$$
\begin{aligned}
T_{\bar{z}\bar{z}} &= 0 \\
T_{zz} &= \hat{T}(z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \tfrac{1}{2}(\partial_z\phi)^2\right) \\
T_{\bar{z}z} &= -\frac{c}{48\pi}\partial^2_{\bar{z}z}\phi
\end{aligned}
$$

- In these coordinates Ricci scalar $R = R^{ij}{}_{ij}$ is given by.

$$R = -4e^{-\phi}\partial^2_{\bar{z}z}\phi$$

- Now we find

$$\boxed{\nabla^z T_{zz} + \nabla^{\bar{z}} T_{\bar{z}z} = -\frac{c}{96\pi}\partial_z R}$$

1+1 d Gravitational Energy-Momentum Anomaly

Gravitational Anomaly in 1+1 dimensions

- 1+1 Chiral Fermion Anomaly in Minkowski signature coordinates:

$$\nabla_\mu T^{\mu\nu} = -\frac{1}{96\pi}\frac{\epsilon^{\nu\sigma}}{\sqrt{-g}}\partial_\sigma R$$

- Apply to r, t Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2$$

where $f(r) \to 1$ at large r, and vanishes linearly as $r \to r_{H+}$.

- Ricci Scalar $R = -f''$.
- Timelike Killing vector η^ν gives genuine (non)- conservation

$$\nabla_\mu (T^{\mu\nu}\eta_\nu) \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\sqrt{-g}T^r{}_t = \frac{1}{96\pi}\frac{\epsilon^{\nu\sigma}\eta_\nu}{\sqrt{-g}}\partial_\sigma R,$$

Hawking radiation in 1+1 dimensions

(Robinson, Wilczek 2005; Banerjee 2008;...)

- Have

$$\frac{\partial}{\partial r}(\sqrt{-g}T^r{}_t) = \frac{1}{96\pi} f\partial_r f'' = \frac{1}{96\pi}\frac{\partial}{\partial r}\left(ff'' - \frac{1}{2}(f')^2\right),$$

- Integrate

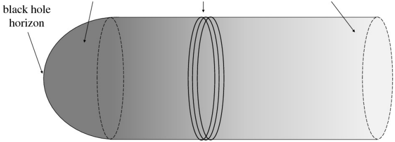
$$\sqrt{-g}T^r{}_t\Big|_{r_H}^{\infty} = \frac{1}{96\pi}\left(ff'' - \frac{1}{2}(f')^2\right)\Bigg|_{r_H}^{\infty}.$$

- Nothing is coming out of the black hole: the LHS is zero at r_H.
- The RHS is zero at infinity, and equal to $(1/96\pi)(f')^2/2$ at the horizon.
- Thus

$$T^r{}_t(r\to\infty) \equiv J_\epsilon = \frac{1}{48\pi}\kappa^2 = \frac{1}{12}\pi T^2_{\text{Hawking}}, \quad \checkmark$$

where $\kappa = f'(r_H)/2$ is the surface gravity and $T_{\text{Hawking}} = \kappa/2\pi$.

Hawking radiation



- Euclidean Schwarzschild manifold is skin of a salami sausage with circumference $\beta = 2\pi/\kappa = 1/T_{\text{Hawking}}$.(Gibbons, Perry 1976)
- Recall that Sommerfeld gives

$$\begin{aligned} J_\epsilon &= \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\epsilon\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}} - \theta(-\epsilon)\right\} = 2\pi\left(\frac{\mu^2}{8\pi^2} + \frac{1}{24}T^2\right) \\ &\to \frac{1}{12}\pi T^2, \quad \text{when } \mu = 0. \end{aligned}$$

- Energy-momentum anomaly $\Rightarrow$ Sommerfeld.

Aside: Hawking radiation in the Laboratory?

M. Stone, Class. Quant. Gravity 30 085003 (2013)

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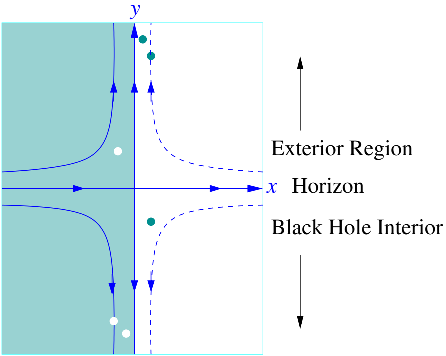

- Hall bar with electrodes to anti-confine.
- Inflowing current divides at "event horizon"
- Solve lowest Landau-level quantum scattering problem (parabolic-cylinder functions)
- Find that chiral fermion edge is in Hawking-temperature thermal state

Gravitational Axial Anomaly

$$\nabla_\mu J^\mu = -\frac{k}{32\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}F_{\mu\nu}F_{\rho\sigma} - \frac{k}{768\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma}$$

Chiral Vortical Effect

- In a (Born) frame rotating with angular velocity Ω, have on-axis current (Vilenkin 1979, Landsteiner, Megias, Peña-Benitez 2011):

$$\begin{aligned} J_z(r=0) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\epsilon^2\,d\epsilon\left(\frac{1}{1+e^{\beta(\epsilon-(\mu+\Omega/2))}}-\frac{1}{1+e^{\beta(\epsilon-(\mu-\Omega/2))}}\right)\\ &= \frac{\mu^2\Omega}{4\pi^2}+\frac{\Omega^3}{48\pi^2}+\frac{1}{12}\Omega T^2.\\ &= \Omega\left(\frac{\mu^2}{4\pi^2}+\frac{1}{12}T^2\right)+O(\Omega^3). \end{aligned}$$

- CME with $B\to 2\Omega$ Coriolis force? (Stephanov, Son *et al.*)
— but no $4\to 2$ dimensional reduction, and no $T^2/12$ this way
- Pontryagin (R^2) term in axial anomaly? (Landsteiner *et al.*) Yes!

Axial-anomaly origin of CVE

- Obtain CVE from the mixed axial anomaly by using the 4-d metric

$$ds^2 = -f(z)\frac{\left(dt - \Omega\, r^2 d\phi\right)^2}{(1-\Omega^2 r^2)} + \frac{1}{f(z)}dz^2 + dr^2 + \frac{r^2\,(d\phi - \Omega\, dt)^2}{(1-\Omega^2 r^2)}.$$

- Looks complicated, but $ds^2 \to -dt^2 + dz^2 + dr^2 + r^2 d\phi^2$ as $f(z) \to 1$.
- Horizon at $f(z_H) = 0$ rotates with angular velocity Ω
- Again $T_{\text{Hawking}} = f'(z_H)/4\pi$
- Pontryagin density:

$$\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} R^a{}_{b\mu\nu} R^b{}_{a\rho\sigma} = \Omega\,\frac{\partial}{\partial z}(r[f'(z)]^2) + O[\Omega^3]$$

Axial-anomaly origin of CVE

- Assume no current at horizon, and integrate up

$$\nabla_\mu J^\mu = -\frac{k}{768\pi^2}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma},$$

  from horizon to infinity.
- Again, by a seeeming miracle, result depends only on value of $f'(z_H) = 4\pi T_{\rm Hawking}$.
- Find contribution to current is

$$J_z = \frac{\Omega}{12} T^2 + O(\Omega^3).$$

$\Rightarrow$ CVE consequence of gravitational axial anomaly

General anomaly-related currents

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu - p g^{\mu\nu} + \xi_{TB}(B^\mu u^\nu + B^\nu u^\mu) + \xi_{T\omega}(\omega^\mu u^\nu + \omega^\nu u^\mu) \\ J_n^\mu &= n u^\mu + \xi_{JB} B^\mu + \xi_{J\omega}\omega^\mu \quad \text{number current} \\ J_S^\mu &= s u^\mu + \xi_{SB} B^\mu + \xi_{S\omega}\omega^\mu \quad \text{entropy current} \end{aligned}$$

No-entropy-production imposes relationships (Stephanov, Yee 2015)

$$\begin{aligned} \xi_{JB} &= C\mu, \qquad \xi_{J\omega} = C\mu^2 + X_B T^2 \\ \xi_{SB} &= X_B T, \quad \xi_{S\omega} = 2\mu T X_B + X_\omega T^2 \\ \xi_{TB} &= \frac{1}{2}\left(C\mu^2 + X_B T^2\right), \\ \xi_{T\omega} &= \frac{2}{3}\left(C\mu^3 + 3X_B \mu T^2 + X_\omega T^3\right) \end{aligned}$$

For the ideal Weyl gas

$$C = \frac{1}{4\pi^2}, \quad X_B = \frac{1}{12}, \quad X_\omega = 0.$$

We have confirmed that X_B is gravitational anomaly

Curved space Dirac/Weyl equation

Massless Dirac equation in curved space:

$$\not{D}\Psi \equiv \gamma^a e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu}\right)\Psi = 0$$

- $\mathbf{e}_a = e_a^\mu \partial_\mu$ is a Minkowski-orthonormal vierbein
- $\omega_{ab\mu} dx^\mu$ is spin connection 1-form
- $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b], \quad \Gamma_\mu = \frac{1}{2}\sigma^{ab}\omega_{ab\mu}.$

Decompose

$$\not{D} = \begin{bmatrix} 0 & \not{D}_+ \\ \not{D}_- & 0 \end{bmatrix}$$

Gravitational Anomaly from Dirac

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- Work in frame rotating with horizon to order Ω
- Find

$$
\begin{aligned}
0 \;=\; & \left[-\frac{1}{\sqrt{f[z]}}\frac{\partial}{\partial t}+\sqrt{f[z]}\sigma_3\left(\frac{\partial}{\partial z}+\frac{f'}{4f}-\frac{i\Omega}{2}\right)\right.\\
& \left.+\sigma_1\left(\frac{\partial}{\partial r}+\frac{1}{2r}\right)+\sigma_2\left(\frac{1}{r}\frac{\partial}{\partial \phi}+r\Omega\frac{\partial}{\partial t}\right)\right]\Psi
\end{aligned}
$$

- Absorb the $1/2r$ and $f'/4f$ by setting

$$
\Psi=\frac{1}{r^{1/2}f^{1/4}}\tilde{\Psi}
$$

- Introduce tortoise coordinate z_* by $dz_*=dz/f(z)$.

Gravitational Anomaly from Dirac

- Find

$$0 = \left[\frac{\partial}{\partial t} + \sigma_3\left(\frac{\partial}{\partial z_*} - \frac{i}{2}f(z)\Omega\right) + \sqrt{f(z)}\left\{\sigma_1\frac{\partial}{\partial r} + \sigma_2\left(\frac{1}{r}\frac{\partial}{\partial \phi} + r\Omega\frac{\partial}{\partial t}\right)\right\}\right]\Psi$$

- Could absorb a z_* dependent phase into $\tilde{\Psi}$ to remove the $if(z)\Omega/2$.
- Bad idea! — this is an axial transformation.
- Origin of spectral flow and anomaly

My Conclusions:

Experiment does not directly involve gravity yet it highlights the intimate and fascinating connection between temperature and gravity

...first seen by