Disentangle covariant Wigner functions for chiral fermions

Qun Wang

University of Science and Technology of China

Open problems and opportunities in chiral fluids
(July 17-19, 2018, Santa Fe)

Outline

- Introduction
- Wigner functions of chiral fermions in external electromagnetic field
  [J. Gao, et al., 1203.0725; J. Gao, QW, 1504.07334]
- Disentangle covariant Wigner functions for chiral fermions
  [J. Gao, Z.T. Liang, QW, X.N. Wang, 1802.06216]

Introduction

- High energy HIC with $\sqrt{s}$:

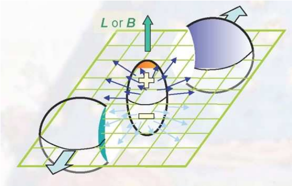
$$v \sim 1 - \frac{m_p^2}{2s}, \quad \gamma \sim \frac{\sqrt{s}}{m_p}$$

- Electric field in the rest frame of one nucleus

$$\mathbf{E} = \frac{Ze}{R^2}\hat{\mathbf{r}}$$

Boost to Lab frame ($v_z \approx 0.999989$c for 200 GeV, MeV$^2 \approx 1.44 \times 10^{13}$ Gs)

$$\begin{aligned} \mathbf{B} &= -\gamma \mathbf{v}_z \times \mathbf{E} = B\mathbf{e}_\phi \\ eB &\sim 2\gamma v_z \frac{Ze^2}{R^2} \sim 10 m_\pi^2 \sim 3 \times 10^{18}\mathrm{Gs} \end{aligned}$$



Fukushima, Kharzeev, Warringa (2008)

Skokov (2009), Deng, Huang (2012)

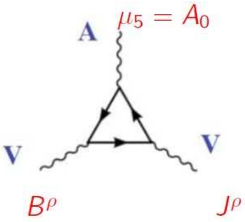
many others ...

Introduction

- Anomalies: the classical symmetry of the Lagrangian broken by quantum effects:
  (a) chiral symmetry by axial anomaly
  (b) scale symmetry by scale anomaly
- Chiral anomaly implies VVA coupling: if A^μ ($\mu_5 = A_0$) and V^μ (Magnetic field B^μ) are background field, then V^μ (vector current J^μ) is induced.
- Chiral Magnetic effect:

$$J^\rho = \hbar \frac{Q^2}{2\pi^2} \mu_5 B^\rho$$

It's a quantum and topological effect (no higher order corrections).



Kharzeev, McLerran, Warringa (2008)
Fukushima, Kharzeev, Warringa (2008)

Recent reviews, e.g.:
Kharzeev, Landsteiner, Schmitt, Yee (2013)
Kharzeev, Liao, Voloshin, Wang (2015)

Classical transport equation

- Distribution function in phase space $f(t, \mathbf{x}, \mathbf{p})$
- Classical Boltzmann equation with background EM fields

$$p^{\mu}(\partial_{\mu} - QF_{\mu\nu}\partial_{p}^{\nu})f(t, \mathbf{x}, \mathbf{p}) = C[f] \tag{1}$$

- Conserved current and energy-momentum tensor

$$\begin{aligned} j^{\mu}(x) &= \int \frac{d^3p}{(2\pi)^3 E_p} p^{\mu} f(t, \mathbf{x}, \mathbf{p}) \\ T^{\mu\nu}(x) &= \int \frac{d^3p}{(2\pi)^3 E_p} p^{\mu} p^{\nu} f(t, \mathbf{x}, \mathbf{p}) \end{aligned} \tag{2}$$

- Classical transport theory: one $f(t, \mathbf{x}, \mathbf{p})$ and one equation!

Quantum transport in Wigner function

- The Wigner function for fermions in EM field

$$W_{\alpha\beta}(X,p) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot y/\hbar} \left\langle \bar{\psi}_\beta(x_+)\, U(x_+,x_-)\, \psi_\alpha(x_-)\right\rangle$$

$[\hat{X},\hat{p}]=0$ Gauge link

where $x_\pm = X \pm y/2$.

- $W(X,p)$ (4×4 complex matrix)$\Rightarrow$ 32 variables $\Rightarrow W^\dagger = \gamma_0 W \gamma_0$
$\Rightarrow$ 16 variables (Scalar, Pseudoscalar, Vector, Axial vector, Tensor)
$\Rightarrow$ 8 variables for chiral fermions (Vector, Axial vector)
$\Rightarrow$ 4 for RH and 4 for LH.

Some early works on Wigner functions of QED/QCD:

Heinz (1983); Elze, Gyulassy, Vasak (1986); Vasak, Gyulassy, Elze (1987);

Zhuang, Heinz (1996); Blaizot, Iancu (2002); QW, Redlich, Stoecker, Greiner (2002)

Wigner function

- The WF can be decomposed in 16 independent generators of Clifford algebra,

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu \mathscr{V}_\mu + \gamma^5\gamma^\mu \mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

whose coefficients $\mathscr{F}(1)$, $\mathscr{P}(1)$, $\mathscr{V}_\mu(4)$, $\mathscr{A}_\mu(4)$ and $\mathscr{S}_{\mu\nu}(6)$ are the scalar, pseudo-scalar, vector, axial-vector and tensor components.

- For chiral fermions, $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ are decoupled from $\mathscr{F}$, $\mathscr{P}$ and $\mathscr{S}_{\mu\nu}$.

Some early works on Wigner functions of QED/QCD:

Heinz (1983); Elze, Gyulassy, Vasak (1986); Vasak, Gyulassy, Elze (1987);

Zhuang, Heinz (1996); Blaizot, Iancu (2002); QW, Redlich, Stoecker, Greiner (2002)

WF-EOM for chiral fermions in constant field

- The vector component $\mathscr{J}^{s}_{\mu}(x,p)$ for chiral fermions (with chirality $s=\pm$) is defined as

$$\mathscr{J}^{s}_{\mu}(x,p) = \frac{1}{2}[\mathscr{V}_{\mu}(x,p) + s\mathscr{A}_{\mu}(x,p)] \tag{3}$$

- The EOM for $\mathscr{J}^{\mu}_{s}(x,p)$ in constant field read

$\nabla_{\mu} = \partial^{X}_{\mu} - QF_{\mu\nu}\partial^{\nu}_{p}$

3 evo eqs for $\mathscr{J}^{i}$ ($\mu\nu = 0i$)

3 constr eqs for $\mathscr{J}^{i}$ and $\mathscr{J}_{0}$ ($\mu\nu = ij$)

$$\begin{aligned} p^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 && \longrightarrow \text{Mass-shell condition} \\ \nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 && \longrightarrow \text{Evolution equation for } \mathscr{J}_{0} \\ 2s(p^{\lambda}\mathscr{J}^{\rho}_{s} - p^{\rho}\mathscr{J}^{\lambda}_{s}) &= -\hbar\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}^{s}_{\nu} \end{aligned} \tag{4}$$

- There are 8 equations for 4 variables in $\mathscr{J}^{\mu}_{s}(x,p)$ for a given s.

Vasak, Gyulassy, Elze (1987); Gao, Liang, Pu, QW, Wang (2012); Chen, Pu, QW, Wang (2013);

Gao, QW (2015); Gao, Pu, QW (2017)

WF-EOM in non-constant field

- The set of (eight) equations for the vector component $\mathscr{J}_s^{\mu}(x,p)=(\mathscr{J}_0,\mathscr{J}^i)$ for chiral fermions (with chirality s)

Equation 3-8:
3 evolution eqs for $\mathscr{J}^i$ $(\mu\nu=0i)$
3 constraint eqs for $\mathscr{J}^i$ and $\mathscr{J}_0$ $(\mu\nu=ij)$

$$\begin{aligned}\Pi^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \longrightarrow \text{Equation 1: mass-shell condition}\\ G^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \longrightarrow \text{Equation 2: evolution equation for } \mathscr{J}_0\\ 2s(\Pi^{\lambda}\mathscr{J}_s^{\rho}-\Pi^{\rho}\mathscr{J}_s^{\lambda}) &= -\hbar\epsilon^{\mu\nu\lambda\rho}G_{\mu}\mathscr{J}_{\nu}^{s} \qquad (5)\end{aligned}$$

- where

$j_1(z)=(\sin z - z\cos z)/z^2$, $j_1(0)=0$

$$\begin{aligned}\Pi^{\mu} &= p^{\mu}-\hbar\frac{1}{2}j_1\left(\frac{\hbar}{2}\partial_x\cdot\partial_p\right)QF^{\mu\nu}\partial_{\nu}^{p}\\ G^{\mu} &= \partial_x^{\mu}-j_0\left(\frac{\hbar}{2}\partial_x\cdot\partial_p\right)QF^{\mu\nu}\partial_{\nu}^{p} \qquad (6)\end{aligned}$$

∂_x only acts on $F^{\mu\nu}$

$j_0(z)=\sin z/z$, $j_0(0)=1$

Vasak, Gyulassy, Elze (1987)

Question to be answered

- The question has long been asked: to what extent a chiral fermion quantum system in electromagnetic fields can be described by the quasi-classical distribution function.
- The answer is yes. The rigorous proof is based on the semi-classical expansion in $\hbar$ for the covariant Wigner functions.
- This remarkable property of chiral fermions will significantly simplify the kinetic simulation of chiral effects in heavy ion collisions and Dirac/Weyl semimetals.

Semiclassical expansion in $\hbar$

- Semiclassical expansion in powers of $\hbar$

$$
\begin{aligned}
\mathscr{J}_\mu &= \sum_{n=0}^{\infty} \hbar^n \mathscr{J}_\mu^{(n)} \\
\Pi^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} \Pi^\mu_{(2n)} \\
G^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} G^\mu_{(2n)}
\end{aligned}
\tag{7}
$$

- where

∂_x acts only on $F^{\mu\nu}$

$$
\begin{aligned}
G^\mu_{(2n)} &= \frac{(-1)^{n+1}}{2^{2n}(2n+1)!} (\partial_x \cdot \partial_p)^{2n} F^{\mu\nu} \partial^p_\nu \\
\Pi^\mu_{(2n)} &= \frac{(-1)^n n}{2^{2n-1}(2n+1)!} (\partial_x \cdot \partial_p)^{2n-1} F^{\mu\nu} \partial^p_\nu
\end{aligned}
\tag{8}
$$

∂_x acts only on $F^{\mu\nu}$

The 0-th and 2nd order operators

- 0-th order operators (in comoving frame)

$$\begin{aligned} G^{\mu}_{(0)} &= \partial^{\mu}_{x} - QF^{\mu\nu}\partial^{p}_{\nu} = (G^{(0)}_{0}, -\mathbf{G}^{(0)}) \\ G^{(0)}_{0} &= \partial_{t} + Q\mathbf{E}\cdot\nabla_{p} \\ \mathbf{G}^{(0)} &= \nabla_{x} + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_{p} \\ \Pi^{\mu}_{(0)} &= (p_0, \mathbf{p}) \end{aligned} \tag{9}$$

Only $G^{(0)}_{0}$ contains time derivative ∂_t on the Wigner function

- 2nd order operators

$$\begin{aligned} G^{\mu}_{(2)} &= \frac{Q^2}{24}(\partial_x\cdot\partial_p)^2 F^{\mu\nu}\partial^{p}_{\nu} \\ \Pi^{\mu}_{(2)} &= -\frac{Q^2}{12}(\partial_x\cdot\partial_p)F^{\mu\nu}\partial^{p}_{\nu} \end{aligned} \tag{10}$$

Convention

- In comoving frame with $u^\mu = (1, \mathbf{0})$, any vector X^μ

$$X^\mu = (X_0, \mathbf{X})$$

For general u^μ, we have

$$X^\mu = (u \cdot X, \Delta^\mu_\nu X^\nu)$$

- The wigner functions have the same form

$$\begin{aligned}\mathscr{J}_s^\mu(x,p) &= (\mathscr{J}_0, \boldsymbol{\mathscr{J}}) \\ &\to (u \cdot \mathscr{J}, \Delta^\mu_\nu \mathscr{J}^\nu)\end{aligned} \tag{11}$$

EOM for Wigner function

- Let us write EOM explicitly in time and spatial components

G_0 contains 0-th order:
$G_0^{(0)} = \partial_t + Q\mathbf{E} \cdot \nabla_p$

$$
\begin{aligned}
G_0 \mathscr{J}_0 + \mathbf{G} \cdot \boldsymbol{\mathscr{J}} &= 0 & (12) \\
\hbar \left[G_0 \boldsymbol{\mathscr{J}} + \mathbf{G} \mathscr{J}_0 \right] &= 2s(\boldsymbol{\Pi} \times \boldsymbol{\mathscr{J}}) & (13) \\
\Pi_0 \mathscr{J}_0 - \boldsymbol{\Pi} \cdot \boldsymbol{\mathscr{J}} &= 0 & (14) \\
-\hbar \mathbf{G} \times \boldsymbol{\mathscr{J}} &= 2s\left(\boldsymbol{\Pi} \mathscr{J}_0 - \Pi_0 \boldsymbol{\mathscr{J}} \right) & (15)
\end{aligned}
$$

The 0-th order equations

- The evolution equations at $O(\hbar^0)$

$G_0^{(0)} = \partial_t + Q\mathbf{E}\cdot\nabla_p$ $\quad\quad$ $\mathbf{G}^{(0)} = \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_p$

$\hbar G_0^{(0)} \mathscr{J}^{(0)}$ is absent at $O(\hbar)$ $\longrightarrow$

$$\begin{aligned} G_0^{(0)} \mathscr{J}_0^{(0)} + \mathbf{G}^{(0)} \cdot \boldsymbol{\mathscr{J}}^{(0)} &= 0 &(16) \\ 2s(\mathbf{p} \times \boldsymbol{\mathscr{J}}^{(0)}) &= 0 &(17) \end{aligned}$$

- Constraint equations at $O(\hbar^0)$

$$\begin{aligned} p_0 \mathscr{J}_0^{(0)} - \mathbf{p} \cdot \boldsymbol{\mathscr{J}}^{(0)} &= 0 &(18) \\ 2s(p_0 \boldsymbol{\mathscr{J}}^{(0)} - \mathbf{p} \mathscr{J}_0^{(0)}) &= 0 &(19) \end{aligned}$$

The 0-th solution

- From Eqs. (19,18) we obtain

$$\begin{aligned}\mathscr{J}^{(0)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)} \\ (p_0^2-\mathbf{p}^2)\mathscr{J}_0^{(0)} &= 0 \\ \Rightarrow \mathscr{J}_0^{(0)} &= p_0 f^{(0)}(x,p)\delta(p^2) \end{aligned} \tag{20}$$

- The evolution equation (17) is satisfied automatically. The evolution for $f^{(0)}$ is described by Eq. (16),

$$\begin{aligned}(\partial_t + Q\mathbf{E}\cdot\nabla_p)\mathscr{J}_0^{(0)} & \\ +(\nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_p)\cdot\left(\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}\right) &= 0 \end{aligned} \tag{21}$$

The 1-st order equations

- The evolution equations at $O(\hbar)$

Contains $\partial_t \mathscr{J}_0^{(1)}$
$$G_0^{(0)} \mathscr{J}_0^{(1)} + \mathbf{G}^{(0)} \cdot \boldsymbol{\mathscr{J}}^{(1)} = 0 \quad (22)$$

Contains $\partial_t \boldsymbol{\mathscr{J}}^{(0)}$
$$G_0^{(0)} \boldsymbol{\mathscr{J}}^{(0)} + \mathbf{G}^{(0)} \mathscr{J}_0^{(0)} = 2s(\mathbf{p} \times \boldsymbol{\mathscr{J}}^{(1)}) \quad (23)$$

- The constraint equations at $O(\hbar)$

Mass-shell condition
$$p_0 \mathscr{J}_0^{(1)} - \mathbf{p} \cdot \boldsymbol{\mathscr{J}}^{(1)} = 0 \quad (24)$$

Constraint equation to relate $\boldsymbol{\mathscr{J}}$ in terms of $\mathscr{J}_0$
$$\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(0)} = 2s(p_0 \boldsymbol{\mathscr{J}}^{(1)} - \mathbf{p} \mathscr{J}_0^{(1)}) \quad (25)$$

The 1-st order solution

- From Eq. (25) we obtain

$$\mathscr{J}^{(1)} = \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(1)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \mathscr{J}^{(0)} \tag{26}$$

and insert it into Eq. (24) we have

$\mathbf{G}^{(0)} = \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B} \times \nabla_p$

$$(p_0^2 - \mathbf{p}^2)\mathscr{J}_0^{(1)} \quad = \quad \frac{s}{2}\mathbf{p} \cdot \left(\mathbf{G}^{(0)} \times \mathscr{J}^{(0)}\right) \tag{27}$$

$\mathscr{J}^{(0)} = \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}$

So we obtain the solution

$$\mathscr{J}_0^{(1)} = p_0 f^{(1)}\delta(p^2) + sQ(\mathbf{p} \cdot \mathbf{B})f^{(0)}\delta'(p^2) \tag{28}$$

$\delta'(y) \equiv \frac{d\delta(y)}{dy} = -\frac{1}{y}\delta(y)$

- The evolution $\mathscr{J}_0^{(1)}$ or $f^{(1)}$ is described by Eq. (22).

Two evolution equations for $\mathscr{J}_0^{(0)}$

- Insert Eq. (26) into Eq. (23), the first order contribution from $\mathbf{p} \times \boldsymbol{\mathscr{J}}^{(0)} \sim \mathbf{p} \times \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(1)} = 0$ is vanishing, so we obtain another evolution equation for $\mathscr{J}_0^{(0)}$ through $\boldsymbol{\mathscr{J}}^{(0)}$

$\mathbf{G}^{(0)} = \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B} \times \nabla_p$

$\boldsymbol{\mathscr{J}}^{(0)} = \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(0)}$

$G_0^{(0)} = \partial_t + Q\mathbf{E} \cdot \nabla_p$

$$G_0^{(0)} \boldsymbol{\mathscr{J}}^{(0)} + \mathbf{G}^{(0)} \mathscr{J}_0^{(0)} = \frac{1}{p_0}\mathbf{p} \times \left(\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(0)}\right) \tag{29}$$

$\boldsymbol{\mathscr{J}}^{(0)} = \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(0)}$

Remember that the original evolution equation (16) for $\mathscr{J}_0^{(0)}$

$$G_0^{(0)} \mathscr{J}_0^{(0)} + \mathbf{G}^{(0)} \cdot \boldsymbol{\mathscr{J}}^{(0)} = 0 \tag{30}$$

- These two evolution equations can be shown to be consistent with each other!

The n-th order equations

- [EVO1] Evolution equation for $\mathscr{J}_0^{(n)}$ (with $\partial_t \mathscr{J}_0^{(n)}$)
- [EVO2] Evolution equation for $\boldsymbol{\mathscr{J}}^{(n-1)}$ (with $\partial_t \boldsymbol{\mathscr{J}}^{(n-1)}$ and a term $\mathbf{p}\times \boldsymbol{\mathscr{J}}^{(n)}$)
- [MASS-SH] One constraint equation for mass-shell for $\mathscr{J}_0^{(n)}$
- [CONSTRAINT] One Constraint equation to relate $\boldsymbol{\mathscr{J}}^{(n)}$ to $\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \cdots, \mathscr{J}_0^{(0)}$
- (1) From Eq. [CONSTRAINT], solve $\boldsymbol{\mathscr{J}}^{(n)}$ in the form $\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n)} + F[\mathscr{J}_0^{(n-1)}, \cdots, \mathscr{J}_0^{(0)}]$
- (2) Inserting $\boldsymbol{\mathscr{J}}^{(n)}$ into Eq. [EVO2] and we obtain another evolutiuon equation for $\partial_t \mathscr{J}_0^{(n-1)}$ invloving $\mathscr{J}_0^{(n-1)}, \cdots, \mathscr{J}_0^{(0)}$ other than Eq. [EVO1] at (n-1)-th order. Note that the n-th order term in $\mathbf{p}\times \boldsymbol{\mathscr{J}}^{(n)}$ is vanishing due to $\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n)}$ in $\boldsymbol{\mathscr{J}}^{(n)}$.

Two evolution equations for $\mathscr{J}_0^{(n)}$

- There are two evolution equations for $\mathscr{J}_0^{(n)}$: the original one [EVO1] at the n-th order and the one converted from that for $\mathscr{J}^{(n+1)}$ in [EVO2] at (n+1)-th order.
- These two evolution equations have to be consistent to each other.
- We can prove that to any order of $\hbar$ the evolution equation derived from $\mathscr{J}^{(n+1)}$ is automatically satisfied once the original evolution equation for $\mathscr{J}_0^{(n)}$ and the mass-shell equation are satisfied.
- This means: the time component $\mathscr{J}_0^{(n)}$ or equivalently distribution function $f^{(n)}$ is sufficient to describe the kinetics of chiral fermions in the EM field.

Reduced equations

- In conclusion, the system of equations is reduced to two equations at the n-th order:
- One evolution equation and one mass-shell constraint equation for $\mathscr{J}_0^{(n)}$ once lower order components $\mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots, \mathscr{J}_0^{(0)}$ are known.
- The end of the proof. [Gao, Liang, QW, Wang, 1802.06216]

Second order results

- Solving mass shell equations we obtain $\mathscr{J}_0^{(0,1,2)}$ up to $O(\hbar^2)$

$$\delta'(y) \equiv \frac{d\delta(y)}{dy} = -\frac{1}{y}\delta(y)$$

$$\delta''(y) = \frac{2}{y^2}\delta(y)$$

$$\begin{aligned}
\mathscr{J}_0^{(0)} &= p_0 f^{(0)}\delta(p^2) \\
\mathscr{J}_0^{(1)} &= p_0 f^{(1)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(0)}\delta'(p^2) \\
\mathscr{J}_0^{(2)} &= p_0 f^{(2)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(1)}\delta'(p^2) + Q^2\frac{(\mathbf{p}\cdot\mathbf{B})^2}{2p_0} f^{(0)}\delta''(p^2) \\
&\quad + \frac{1}{4p^2}\mathbf{p}\cdot\left\{\mathbf{G}^{(0)}\times\left[\frac{1}{p_0}\mathbf{G}^{(0)}\times\left(\mathbf{p}f^{(0)}\delta(p^2)\right)\right]\right\} \\
&\quad - \frac{p_0}{p^2}\Pi_\mu^{(2)}p^\mu f^{(0)}\delta(p^2) \\
&\quad + \frac{1}{p^2}\mathbf{p}\cdot\left(\mathbf{\Pi}^{(2)}p_0 - \Pi_0^{(2)}\mathbf{p}\right) f^{(0)}\delta(p^2)
\end{aligned} \tag{31}$$

$f^{(0,1,2)}$ determined by evolution equation at each order

Second order results

- Spatial components $\mathscr{J}^{(n)}$ are given by $\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n)}$ and a function of lower order time-components $\mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots \mathscr{J}_0^{(0)}$:

$$\begin{aligned}
\boldsymbol{\mathscr{J}}^{(0)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)} \\
\boldsymbol{\mathscr{J}}^{(1)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(1)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(0)} \\
\boldsymbol{\mathscr{J}}^{(2)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(2)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(1)} - \frac{1}{p_0}\Pi_0^{(2)}\boldsymbol{\mathscr{J}}^{(0)} \\
&\quad + \frac{1}{p_0}\mathbf{\Pi}^{(2)}\mathscr{J}_0^{(0)}
\end{aligned} \tag{32}$$

Mass-shell condition up to $O(\hbar)$

- We collect first three lines of Eq. (31), $\mathscr{J}_0^{(0)} + \hbar \mathscr{J}_0^{(1)} + \hbar^2 \mathscr{J}_0^{(2)}$, to obtain

$$\mathscr{J}_0 \approx p_0 f(x,p)\delta(\tilde{p}^2) \tag{33}$$

where

Quantum effect

$$\begin{aligned} \tilde{p}^2 &\equiv p^2 + \hbar s Q \frac{\mathbf{p}\cdot\mathbf{B}}{p_0} \\ f(x,p) &\equiv f^{(0)} + \hbar f^{(1)} + \hbar^2 f^{(2)} \end{aligned} \tag{34}$$

- The mass-shell condition $\delta(\tilde{p}^2)$ gives

$\mathbf{\Omega}_p \equiv \frac{\mathbf{p}}{2|\mathbf{p}|^3}$

$$E_p^{(\pm)} = \pm E_p (1 \mp \hbar s Q \mathbf{B}\cdot\mathbf{\Omega}_p) \tag{35}$$

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Gao, QW (2015); Hidaka, Yang, Pu (2017); Huang, Shi, Jiang, Liao, Zhuang (2018); Gao, Liang, QW, Wang (2018)

CKE in three-momentum

- For non-zero energy, $|\mathbf{p}| \neq 0$, we obtain the previous result

$\mathbf{v} \equiv \nabla_p E_p^{(+)}$

$\tilde{\mathbf{E}} \equiv \mathbf{E} - Q^{-1}\nabla_x E_p^{(+)}$

$\boldsymbol{\Omega}_p \equiv \frac{\mathbf{p}}{2|\mathbf{p}|^3}$

$$\begin{aligned} & (1+\hbar sQ\boldsymbol{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p}) \\ + & \left[\mathbf{v}+\hbar sQ(\mathbf{E}\times\boldsymbol{\Omega}_p)+\hbar sQ\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\nabla_x f(x,E_p,\mathbf{p}) \\ + & \left[Q\tilde{\mathbf{E}}+Q\mathbf{v}\times\mathbf{B}+\hbar sQ^2(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_p\right]\cdot\nabla_p f(x,E_p,\mathbf{p}) = 0 \end{aligned} \tag{36}$$

- At $|\mathbf{p}| = 0$, there are two additional terms in the above CKE which are singular but were previously neglected,

$$\begin{aligned} & \hbar s\,(\mathbf{E}\cdot\mathbf{B})\,(\nabla_p\cdot\boldsymbol{\Omega}_p)\,f(x,E_p,\mathbf{p}) \\ & -\lim_{\Lambda\to 0}\frac{2\hbar s}{\Lambda}\,(\mathbf{E}\cdot\mathbf{p})\,(\mathbf{B}\cdot\mathbf{p})\,\delta'(\Lambda^2-\mathbf{p}^2)f(x,\Lambda,\mathbf{p}) \end{aligned} \tag{37}$$

Derivation of Eq. (41):

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Hidaka, Yang, Pu (2017);

Gao, Liang, QW, Wang (2018); Huang, Shi, Jiang, Liao, Zhuang (2018)

New source of chiral anomaly

- These two terms come from total derivatives in p_0 and are relevant to the anomalous conservation equation

$$
\begin{aligned}
\partial_t j_0 + \boldsymbol{\nabla}_x \cdot \mathbf{j} &= -\frac{\hbar s Q^2}{2} \int d^3\mathbf{p} \Big[\overbrace{(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_p \cdot \boldsymbol{\nabla}_p f}^{\text{Previous terms } (I_1)} \\
\text{New terms } I_2) \quad & + (\mathbf{E}\cdot\mathbf{B})\left(\boldsymbol{\nabla}_p \cdot \boldsymbol{\Omega}_p\right) f \\
\text{New terms } (I_3) \quad & - \lim_{\Lambda\to 0} \frac{2}{\Lambda} (\mathbf{E}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \delta'(\Lambda^2 - \mathbf{p}^2) f \Big], \qquad (38)
\end{aligned}
$$

- Since

$$
\begin{aligned}
I_1 + I_2 &\sim \int d^3\mathbf{p} \nabla_p (\boldsymbol{\Omega}_p f) = 0 \\
I_1 &= I_3 = \hbar \frac{sQ^2}{4\pi^2} (\mathbf{E}\cdot\mathbf{B}) \qquad (39)
\end{aligned}
$$

so the last term contributes to the anomaly.

Related works: Landsteiner, Rebhan (2011); Mueller, Venugopalan (2017, 2018); Fujikawa (2018)

Wigner functions in a general Lorentz frame

- With an auxiliary velocity u^{μ}, we can decompose a vector X^{μ}

$$X^{\mu}=(X\cdot u)u^{\mu}+\bar{X}^{\mu} \tag{40}$$

Eq. (32) becomes

$$\begin{aligned}
\bar{\mathscr{J}}_{\mu}^{(0)} &= \bar{p}_{\mu}\frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p}\\
\bar{\mathscr{J}}_{\mu}^{(1)} &= \bar{p}_{\mu}\frac{u\cdot\mathscr{J}^{(1)}}{u\cdot p}-\frac{s}{2(u\cdot p)}\epsilon^{\mu\nu\rho\sigma}u_{\nu}\bar{\nabla}_{\sigma}\bar{\mathscr{J}}_{\rho}^{(0)}
\end{aligned} \tag{41}$$

Eq. (31) becomes

$$\begin{aligned}
\frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p} &= f^{(0)}\delta\left(p^{2}\right)\\
\frac{u\cdot\mathscr{J}^{(1)}}{u\cdot p} &= f^{(1)}\delta\left(p^{2}\right)-sQ\frac{B\cdot p}{u\cdot p}f^{(0)}\delta'\left(p^{2}\right)
\end{aligned} \tag{42}$$

Lorentz covariance and side-jump

- We can combine Eq. (41) and (42) as

$$\mathscr{J}_{\mu}^{(0)} = p_{\mu}\frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p}$$

$$\mathscr{J}_{\mu}^{(1)} = p_{\mu}\frac{u\cdot\mathscr{J}^{(1)}}{u\cdot p} - \frac{s}{2(u\cdot p)}\epsilon^{\mu\nu\rho\sigma}u_{\nu}\nabla_{\sigma}\mathscr{J}_{\rho}^{(0)} \tag{43}$$

- We can also choose any other velocity u'_{μ} to make the decomposition. Then we have

$$\mathscr{J}_{(0)}^{\prime\mu} = p^{\mu}\frac{u'\cdot\mathscr{J}^{(0)}}{u'\cdot p}$$

$$\mathscr{J}_{(1)}^{\prime\mu} = p^{\mu}\frac{u'\cdot\mathscr{J}^{(1)}}{u'\cdot p} - \frac{s}{2u'\cdot p}\epsilon^{\mu\nu\rho\sigma}u'_{\nu}\nabla_{\sigma}\mathscr{J}_{\rho}^{(0)} \tag{44}$$

Lorentz covariance and side-jump

- We can easily check $\mathscr{J}'^{\mu}_{(0)} = \mathscr{J}^{\mu}_{(0)}$ as

$$\begin{aligned} \delta \mathscr{J}^{\mu}_{(0)} &= \mathscr{J}'^{\mu}_{(0)} - \mathscr{J}^{\mu}_{(0)} \\ &= p^{\mu} \frac{(u \cdot p)\left(u' \cdot \mathscr{J}_{(0)}\right) - (u' \cdot p)\left(u \cdot \mathscr{J}_{(0)}\right)}{(u' \cdot p)(u \cdot p)} \\ &= 0 \end{aligned} \tag{45}$$

  where we have used $\mathscr{J}^{\mu}_{(0)} \propto p^{\mu}$ Then we see $\mathscr{J}^{(0)}_{\mu}$ is independent of the choice of u^{μ}.

- We can also verify $\mathscr{J}'^{\mu}_{(1)} = \mathscr{J}^{\mu}_{(1)}$ (more complicated but straightforward).

Lorentz covariance and side-jump

- We look at the difference of $\frac{u\cdot\mathscr{J}_{(1)}}{u\cdot p}$ (distribution function) from the change of the observer's velocity

$$\begin{aligned}
\delta\left(\frac{u\cdot\mathscr{J}_{(1)}}{u\cdot p}\right) &= p^{\mu}\frac{(u\cdot p)\left(u'\cdot\mathscr{J}_{(1)}\right)-(u'\cdot p)\left(u\cdot\mathscr{J}_{(1)}\right)}{(u'\cdot p)(u\cdot p)} \\
&= p^{\mu}\frac{u^{\rho}u'^{\sigma}\left(p_{\rho}\mathscr{J}_{\sigma}^{(1)}-p_{\sigma}\mathscr{J}_{\rho}^{(1)}\right)}{(u'\cdot p)(u\cdot p)} \\
&= -\hbar s\epsilon_{\mu\nu\rho\sigma}\frac{u^{\rho}u'^{\sigma}\nabla^{\mu}\mathscr{J}_{(0)}^{\nu}}{2(u'\cdot p)(u\cdot p)} \qquad (46)
\end{aligned}$$

where we have used $2s(p_{\rho}\mathscr{J}_{\sigma}^{(1)}-p_{\sigma}\mathscr{J}_{\rho}^{(1)})=-\hbar\epsilon_{\mu\nu\rho\sigma}\nabla^{\mu}\mathscr{J}_{(0)}^{\nu}$.

Chen, Son, Stephanov, Yee, Yin (2014); Hidaka, Pu, Yang (2017);

Gao, Liang, QW, Wang (2018); Huang, Shi, Jiang, Liao, Zhuang (2018)

Summary of main results

- A semiclassical expansion of covariant WF in $\hbar$ for chiral fermions in a background EM field.
- Non-equilibrium formalism: without assumptions of equilibrium conditions as in our previous works
- Proof: to any order of $\hbar$ that only the time-component of the WF is independent while other components are explicit derivatives.
- Proof: to any order of $\hbar$ that a system of kinetic equations for multiple-components of WF can be reduced to one CKE involving only the single-component distribution function.

Summary of main results

- Remarkable properties for chiral fermions: will significantly simplify the description and simulation of chiral effects in HIC and Dirac/Weyl semimetals.
- We present the CKE in four-momentum up to $O(\hbar^2)$.
- We find that the chiral anomaly may arise from a new source in CKE, in contrast to the well-known scenario of the Berry phase term.
- Our results can be re-written in a general Lorentz frame with a reference 4-velocity and we show that the non-trivial transformation of the distribution function in different frames (side-jump) emerges in a natural way.

Backup slides

The n-th order evolution equations

- The evolution equations at $O(\hbar^n)$ read

Contains $\partial_t \mathscr{I}_0^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \mathscr{I}_0^{(n-2i)} + \mathbf{G}^{(2i)} \cdot \boldsymbol{\mathscr{I}}^{(n-2i)} \right] = 0 \tag{47}$$

Contains $\partial_t \boldsymbol{\mathscr{I}}^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \boldsymbol{\mathscr{I}}^{(n-2i)} + \mathbf{G}^{(2i)} \mathscr{I}_0^{(n-2i)} \right] = 2s \sum_{i=0}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \boldsymbol{\mathscr{I}}^{(n-2i+1)} \tag{48}$$

The n-th order constraint equations

- The constraint equations at $O(\hbar^n)$ read

Mass-shell condition

$$\sum_{i=0}^{[n/2]} \left[\Pi_0^{(2i)} \mathscr{J}_0^{(n-2i)} - \mathbf{\Pi}^{(2i)} \cdot \boldsymbol{\mathscr{J}}^{(n-2i)}\right] = 0 \tag{49}$$

$$2s \sum_{i=0}^{[(n+1)/2]} \left[\mathbf{\Pi}^{(2i)} \mathscr{J}_0^{(n-2i+1)} - \Pi_0^{(2i)} \boldsymbol{\mathscr{J}}^{(n-2i+1)}\right]$$

Can solve $\boldsymbol{\mathscr{J}}^{(n+1)}$

Constraint equation to relate $\boldsymbol{\mathscr{J}}$

in terms of $\mathscr{J}_0$

$$= -\sum_{i=0}^{[n/2]} \mathbf{G}^{(2i)} \times \boldsymbol{\mathscr{J}}^{(n-2i)} \tag{50}$$

Solve $\boldsymbol{\mathscr{J}}^{(n+1)}$ as function of $\mathscr{J}_0^{(0)}, \cdots \mathscr{J}_0^{(n+1)}$

- From Eq. (50) we can solve

$$\begin{aligned}\boldsymbol{\mathscr{J}}^{(n+1)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n+1)} + \frac{s}{2p_0}\sum_{i=0}^{[n/2]}\mathbf{G}^{(2i)} \times \underbrace{\boldsymbol{\mathscr{J}}^{(n-2i)}}_{\boldsymbol{\mathscr{J}}^{(n)},\, \boldsymbol{\mathscr{J}}^{(n-2)}\ldots} \\ &+\frac{1}{p_0}\sum_{i=1}^{[(n+1)/2]}\left[\mathbf{\Pi}^{(2i)}\underbrace{\mathscr{J}_0^{(n-2i+1)}}_{\mathscr{J}_0^{(n-1)},\, \mathscr{J}_0^{(n-3)}\ldots} - \Pi_0^{(2i)}\underbrace{\boldsymbol{\mathscr{J}}^{(n-2i+1)}}_{\boldsymbol{\mathscr{J}}^{(n-1)},\, \boldsymbol{\mathscr{J}}^{(n-3)}\ldots}\right] \quad (51)\end{aligned}$$

- By recursively using the above for $\mathbf{G}^{(2i)} \times \boldsymbol{\mathscr{J}}^{(n-2i)}$ and $\Pi_0^{(2i)}\boldsymbol{\mathscr{J}}^{(n-2i+1)}$ one can finally express $\boldsymbol{\mathscr{J}}^{(n+1)}\left[\mathscr{J}_0^{(0)}, \cdots, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n+1)}\right]$.

- Evolution equation for $\partial_t \mathscr{J}^{(n)}$ in (48) becomes

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots]$

$= 2s\,(\mathbf{p} \times \mathscr{J}^{(n+1)})$

$$\sum_{i=0}^{[n/2]} \Big[G_0^{(2i)} \underbrace{\mathscr{J}^{(n-2i)}}_{\mathscr{J}^{(n)},\ \mathscr{J}^{(n-2)} \cdots} + \mathbf{G}^{(2i)} \underbrace{\mathscr{J}_0^{(n-2i)}}_{\mathscr{J}_0^{(n)},\ \mathscr{J}_0^{(n-2)} \cdots} \Big]$$

$$= 2s\,\mathbf{p} \times \mathscr{J}^{(n+1)} + 2s \sum_{i=1}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \underbrace{\mathscr{J}^{(n-2i+1)}}_{\mathscr{J}^{(n-1)},\ \mathscr{J}^{(n-3)} \cdots} \tag{52}$$

- We use Eq. (51)

$$\begin{aligned} \mathscr{J}^{(n+1)} &= \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(n+1)} \\ &\quad + \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots , \right] \end{aligned} \tag{53}$$

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots]$

$= 2s\,\mathbf{p} \times \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots \right]$

Evolution equation for $\mathscr{J}^{(n)}$

- The evolution equation for $\partial_t \mathscr{J}^{(n)}$ in (48) is now converted to another evolution for $\partial_t \mathscr{J}_0^{(n)}$

$$\mathbf{F}\left[\frac{\mathbf{p}}{p_0}\partial_t \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \cdots\right] = 2s\,\mathbf{p}\times\mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots\right] \tag{54}$$

- The original evolution equation (47)

$$\begin{aligned}&\left(\partial_t + Q\mathbf{E}\cdot\nabla_p\right)\mathscr{J}_0^{(n)} + \mathbf{G}^{(0)}\cdot\mathscr{J}^{(n)} \\ &\sum_{i=1}^{[n/2]}\left[G_0^{(2i)}\mathscr{J}_0^{(n-2i)} + \mathbf{G}^{(2i)}\cdot\mathscr{J}^{(n-2i)}\right] = 0\end{aligned} \tag{55}$$