

Relativistic hydrodynamics with spin

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W. Florkowski, B. Friman, A. Jaiswal, ES, PRC **97**, 041901 (2018)

W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, ES, PRD **97**, 116017 (2018)

W. Florkowski, ES, F. Becattini, arXiv:1803.11098



Open Problems and Opportunities in Chiral Fluids

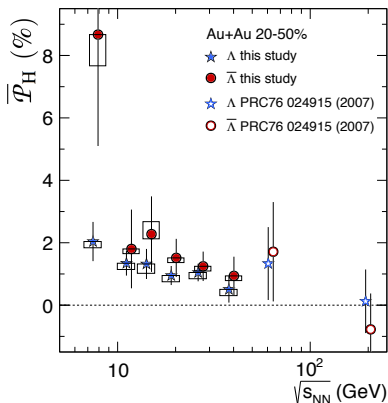
Santa Fe, July 16-19, 2018

Outline

- ▶ Why we need spin degrees of freedom in hydrodynamics
- ▶ Equations for perfect-fluid hydrodynamics with spin
- ▶ Analytic and numerical results

Motivation

- ▶ Non-central nuclear collisions $\Rightarrow$ Large global angular momentum
 $\Rightarrow$ May generate spin polarization of hot and dense matter
- ▶ Connection between spin polarization and vorticity
- ▶ Measurement of Λ hyperon polarization: "Most vortical fluid"



L. Adamczyk et al. (STAR), Nature 548 62-65

Basic conservation laws

Poincare symmetry leads to basic conservation laws

- ▶ Energy-momentum tensor $\hat{T}^{\mu\nu}$

Conservation of energy and momentum:

$$\partial_\mu \hat{T}^{\mu\nu} = 0$$

- ▶ Total angular momentum tensor ("orbital" + "spin")

$$\hat{J}^{\lambda,\mu\nu} = x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu} + \hat{S}^{\lambda,\mu\nu}$$

Conservation of total angular momentum:

$$\partial_\lambda \hat{J}^{\lambda,\mu\nu} = 0 \implies \partial_\lambda \hat{S}^{\lambda,\mu\nu} = \hat{T}^{\nu\mu} - \hat{T}^{\mu\nu}$$

Spin tensor $\hat{S}^{\lambda,\mu\nu}$ is in general **not** conserved

Pseudo-gauge transformations

- ▶ Total energy-momentum and angular momentum must be fixed

$$\hat{P}^\mu = \int d^3\Sigma_\lambda \hat{T}^{\lambda\mu} \quad \hat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \hat{J}^{\lambda, \mu\nu}$$

- ▶ Densities are defined up to divergences

⇒ **Pseudo-gauge transformations:**

(F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976))

$$\begin{aligned}\hat{T}'^{\mu\nu} &= \hat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda(\hat{\Phi}^{\lambda, \mu\nu} + \hat{\Phi}^{\mu, \nu\lambda} + \hat{\Phi}^{\nu, \mu\lambda}) \\ \hat{S}'^{\lambda, \mu\nu} &= \hat{S}^{\lambda, \mu\nu} - \hat{\Phi}^{\lambda, \mu\nu}\end{aligned}$$

Leave $\hat{P}^\mu$ and $\hat{J}^{\mu\nu}$ invariant

- ▶ Belinfante's case

$$\begin{aligned}\hat{T}_{\text{Bel}}^{\mu\nu} &= \hat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda(\hat{S}^{\lambda, \mu\nu} + \hat{S}^{\mu, \nu\lambda} + \hat{S}^{\nu, \mu\lambda}) \\ \hat{S}_{\text{Bel}}^{\lambda, \mu\nu} &= 0\end{aligned}$$

Density operators for local equilibrium

F. Becattini, W. Florkowski, ES (in preparation)

- **Belinfante's case:** Constraints on energy and momentum ($S_{\text{Bel}}^{\lambda, \mu\nu} = 0$)

$$\hat{\rho}_{\text{Bel}} = \exp \left[- \int d^3 \Sigma_{\mu}(x) \hat{T}_{\text{Bel}}^{\mu\nu}(x) \beta_{\nu} \right]$$

- **Canonical case:** constraints on energy, momentum and angular momentum

$$\hat{\rho}_{\text{Can}} = \exp \left[- \int d^3 \Sigma_{\mu}(x) \left(\hat{T}_{\text{Can}}^{\mu\nu}(x) \beta_{\nu} - \frac{1}{2} \hat{S}_{\text{Can}}^{\mu, \lambda\nu}(x) \omega_{\lambda\nu} \right) \right]$$

After pseudo gauge transformation (considering Dirac theory):

$$\hat{\rho}_{\text{Can}} = \exp \left[- \int d^3 \Sigma_{\mu}(x) \left(\hat{T}_{\text{Bel}}^{\mu\nu}(x) \beta_{\nu} - \frac{1}{2} \left(\omega_{\lambda\nu} + \frac{1}{2} (\partial_{\lambda} \beta_{\nu} - \partial_{\nu} \beta_{\lambda}) \right) \hat{S}_{\text{Can}}^{\mu, \lambda\nu}(x) \right) \right]$$

$$\hat{\rho}_{\text{Bel}} \neq \hat{\rho}_{\text{Can}} \text{ unless } \omega_{\mu\nu} = -\frac{1}{2} (\partial_{\mu} \beta_{\nu} - \partial_{\nu} \beta_{\mu})$$

Measurable quantities related to $W(p)_{\alpha\omega} = \text{tr}(\hat{\rho} a^{\dagger}(p)_{\alpha} a(p)_{\omega})$.

In general $W_{\text{Bel}}(p)_{\alpha\omega} \neq W_{\text{Can}}(p)_{\alpha\omega}$

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Global thermodynamic equilibrium

- Density operator for quantum mechanical system

$$\hat{\rho} = \exp \left[- \int d^3 \Sigma_{\mu}(x) \left(\hat{T}^{\mu\nu}(x) \beta_{\nu} - \frac{1}{2} \hat{S}^{\mu, \alpha\beta}(x) \omega_{\alpha\beta} \right) \right]$$

Global thermodynamic equilibrium



$\hat{\rho}$ independent of time



$$\partial_{\mu} \beta_{\nu} + \partial_{\nu} \beta_{\mu} = 0,$$

$$\omega_{\mu\nu} = -\frac{1}{2} (\partial_{\mu} \beta_{\nu} - \partial_{\nu} \beta_{\mu}) = \text{const} \quad (\text{thermal vorticity})$$

Global vs. local equilibrium

Present phenomenology prescription used to describe the data:

- 1) Run any type of hydro (perfect or viscous)
- 2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
- 3) Calculate thermal vorticity
- 4) Identify thermal vorticity with the spin-polarization tensor $\omega_{\mu\nu}$
- 5) Make predictions about spin polarization

This talk:

In **local equilibrium** thermal vorticity and $\omega_{\mu\nu}$ are in general different

$$\omega_{\mu\nu} \neq -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

$\beta_\mu(x)$ and $\omega_{\mu\nu}(x)$ are independent quantities

- ▶ Eventually, β_μ and $\omega_{\mu\nu}$ become related at global equilibrium
- ▶ Spin polarization may be early-stage effect that survives the whole evolution

Local distribution functions for spin-1/2 particles

- ▶ Starting point (Becattini et al., Annals. Phys. 338 32):

$$f_{rs}^+(x, p) = \frac{1}{2m} \bar{u}_r(p) X^+ u_s(p), \quad f_{rs}^-(x, p) = -\frac{1}{2m} \bar{v}_s(p) X^- v_r(p)$$

$$X^\pm = \exp[\pm \xi(x) - \beta_\mu(x) p^\mu] M^\pm$$

$$M^\pm = \exp\left[\pm \frac{1}{2} \omega_{\mu\nu}(x) \Sigma^{\mu\nu}\right]$$

with $\beta^\mu = u^\mu/T$, $\xi = \mu/T$, $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$

- ▶ $\omega_{\mu\nu}$ analogue to EM field-strength tensor $F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta B^\gamma$

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma$$

- ▶ Assumptions: $k \cdot \omega = k \cdot u = \omega \cdot u = 0$, and $k \cdot k - \omega \cdot \omega \geq 0$ (ζ is real)

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega_{\mu\nu} \Sigma^{\mu\nu}, \quad \zeta \equiv \frac{1}{2} \sqrt{k \cdot k - \omega \cdot \omega}$$

- ▶ ζ imaginary – W. Florkowski, ES, F. Becattini, arXiv:1803.11098

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Charge current and energy-momentum tensor

- **Charge current** (de Groot, van Leeuwen, van Weert)

$$N^\mu = \int \frac{d^3 p}{2(2\pi)^3 E_p} p^\mu [\text{tr}_4(X^+) - \text{tr}_4(X^-)] = n u^\mu$$

$$n = 4 \cosh(\zeta) \sinh(\xi) n_{(0)}(T) = \underbrace{(e^\zeta + e^{-\zeta})}_{\text{spin-up} + \text{spin-down}} \overbrace{(e^\xi - e^{-\xi})}^{\text{particles} - \text{antiparticles}} n_{(0)}(T)$$

Boltzmann average

$$n_{(0)}(T) \equiv \int \frac{d^3 p}{(2\pi)^3 E_p} (u \cdot p) e^{-\beta \cdot p}$$

- **Energy-momentum tensor** (de Groot, van Leeuwen, van Weert)

$$T^{\mu\nu} = \int \frac{d^3 p}{2(2\pi)^3 E_p} p^\mu p^\nu [\text{tr}_4(X^+) + \text{tr}_4(X^-)] = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu},$$

$$\varepsilon = 4 \cosh(\zeta) \cosh(\xi) \varepsilon_{(0)}(T)$$

$$P = 4 \cosh(\zeta) \cosh(\xi) P_{(0)}(T),$$

Entropy current

Generalization of the Boltzmann expression

$$S^\mu = - \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left(\text{tr}_4 [X^+ (\ln X^+ - 1)] + \text{tr}_4 [X^- (\ln X^- - 1)] \right)$$

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu n - \Omega w}{T}$$

Ω defined through the relation $\zeta = \Omega/T$

$$w = 4 \sinh(\zeta) \cosh(\xi) n_{(0)}$$

New thermodynamic variable Ω – "Spin chemical potential"

$$s = \left. \frac{\partial P(T, \mu, \Omega)}{\partial T} \right|_{\mu, \Omega}, \quad n = \left. \frac{\partial P(T, \mu, \Omega)}{\partial \mu} \right|_{T, \Omega}, \quad w = \left. \frac{\partial P(T, \mu, \Omega)}{\partial \Omega} \right|_{T, \mu}$$

Hydrodynamic spin background equations

- ▶ Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu} = 0 \implies \partial_\mu [(\varepsilon + P)u^\mu] = u^\mu \partial_\mu P \equiv \frac{dP}{d\tau}$$

Evaluating the derivatives

$$T \partial_\mu (su^\mu) + \mu \partial_\mu (nu^\mu) + \Omega \partial_\mu (wu^\mu) = 0.$$

Charge conservation:

$$\partial_\mu (nu^\mu) = 0.$$

- ▶ Perfect fluid $\Rightarrow$ entropy conservation $\Rightarrow$ we demand:

Additional conservation law

$$\partial_\mu (wu^\mu) = 0$$

Self-consistent entropy conservation $\partial_\mu (su^\mu) = 0$

Spin dynamics

Symmetric $T^{\mu\nu}$



$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

Form of the spin tensor (Becattini, Tinti, Annals Phys. 325, 1566)

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\lambda \text{tr}_4 [(X^+ - X^-) \Sigma^{\mu\nu}] = \frac{wu^\lambda}{4\zeta} \omega^{\mu\nu}$$

Rescaled spin-polarization tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$

$$u^\lambda \partial_\lambda \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0$$

- ▶ Parallel transport of the spin polarization direction along the fluid stream lines
- ▶ Non-trivial spin dynamics

Global equilibrium with rotation

Stationary vortex (I)

- Hydrodynamic flow $u^\mu = \gamma(1, \mathbf{v})$ (rigid rotor):

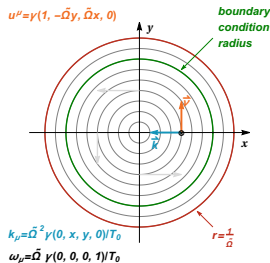
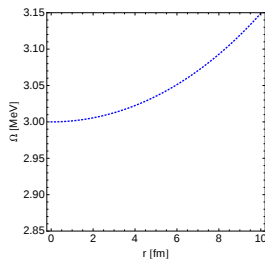
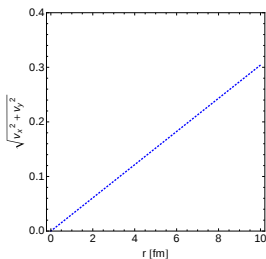
$$u^0 = \gamma, \quad u^1 = -\gamma \tilde{\Omega} y, \quad u^2 = \gamma \tilde{\Omega} x, \quad u^3 = 0,$$

$\tilde{\Omega}$ is a constant, $\gamma = 1/\sqrt{1 - \tilde{\Omega}^2 r^2}$, $r^2 = x^2 + y^2$

Due to limiting light speed, $0 \leq r \leq R < 1/\tilde{\Omega}$

- Solution for hydrodynamic spin background:

$$T = T_0 \gamma, \quad \mu = \mu_0 \gamma, \quad \Omega = \Omega_0 \gamma$$



Global equilibrium with rotation

Stationary vortex (II)

- ▶ Unpolarized vortex: $\omega_{\mu\nu} = 0$ and $\Omega_0 = 0$
- ▶ **Polarized** vortex (**global equilibrium**):

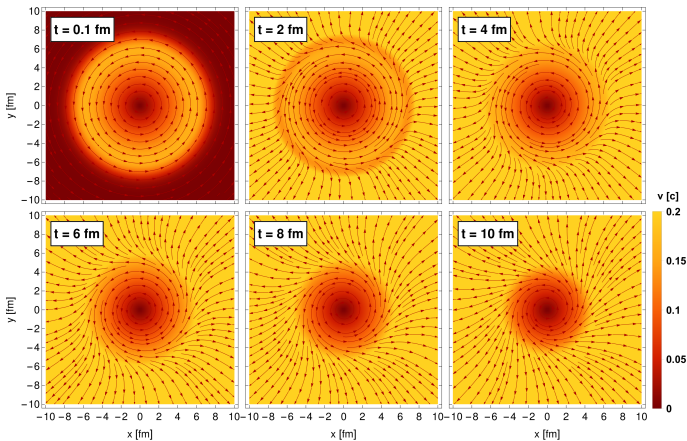
$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Omega}/T_0 & 0 \\ 0 & -\tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\Omega} = 2\Omega_0.$$

Spin-polarization tensor = thermal vorticity:

$$\omega_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

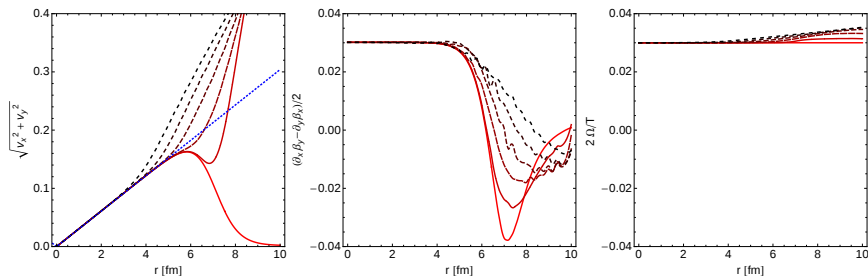
Isolated vortex (I)

- ▶ External boundary is removed $\Rightarrow$ Expansion into external vacuum
- ▶ Time dependent problem solved numerically



Color gradient: strength of fluid velocity

Isolated vortex (II)



Time increases by 2 fm: red $\rightarrow$ black lines

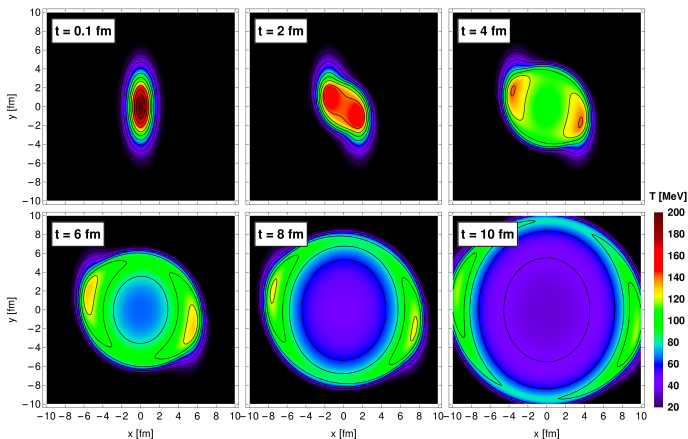
- Spin-polarization tensor $\neq$ thermal vorticity

Quasi-realistic model for low energy collisions

Initial temperature and flow profile

$$T_i = T_0 \exp \left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2} \right), \quad \tilde{\Omega} = \frac{1}{r} \tanh \left(\frac{r}{r_0} \right)$$

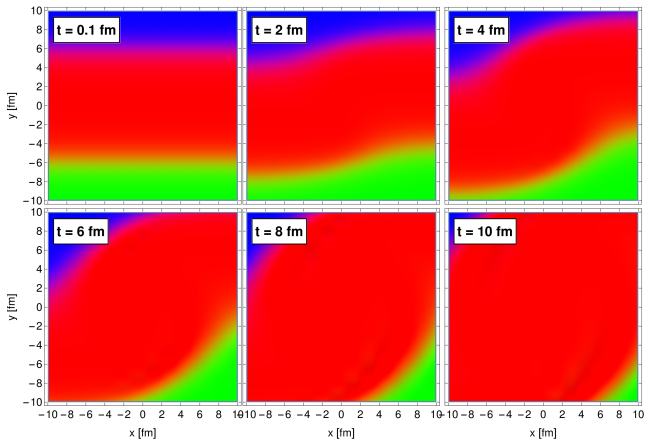
$$\sigma_x = 1, \sigma_y = 2.6, \sigma_z = 2, r_0 = 1$$



Polarization evolution

$$\bar{\omega}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\omega}_{xy} & \bar{\omega}_{xz} \\ 0 & -\bar{\omega}_{xy} & 0 & \bar{\omega}_{yz} \\ 0 & -\bar{\omega}_{xz} & -\bar{\omega}_{yz} & 0 \end{bmatrix}$$

■ $\bar{\omega}_{xy}$ ■ $\bar{\omega}_{xz}$ ■ $\bar{\omega}_{yz}$



Summary

- ▶ Hydrodynamic framework which includes evolution of spin density in a consistent fashion
- ▶ Minimal extension of well-established perfect-fluid picture
- ▶ Polarization evolution in heavy-ion collisions
- ▶ Advantage to study dynamics of systems in **local equilibrium**, compared to studies where global equilibrium was assumed

Outlook

- ▶ Spin-orbit interaction
- ▶ Dissipative effects
- ▶ ...

BACKUP

Local distribution functions for spin-1/2 particles

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with $\beta^\mu = u^\mu/T$, $\xi = \mu/T$, $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$

- ▶ $\omega_{\mu\nu}$ analogue to EM field-strength tensor $F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta B^\gamma$

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma$$

- ▶ Polarization tensor expressed like EM field-strength tensor

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

Spin-density matrix

- ▶ **Pure state:** $|\psi\rangle = \sum_{\lambda} c_{\lambda} |\lambda\rangle$
Expectation value of an operator $\langle O \rangle = \langle \psi | O | \psi \rangle$
- ▶ **Mixed state:** incoherent mixture of $|\psi_i\rangle$ with statistical weight a_i

$$f = \sum_i a_i |\psi_i\rangle \langle \psi_i| = \sum_{\lambda, \lambda'} f_{\lambda \lambda'} |\lambda\rangle \langle \lambda'|$$

$$f_{\lambda \lambda'} = \sum_i a_i c_{\lambda}^{(i)} c_{\lambda'}^{(i)*}. \text{ Expectation value: } \langle O \rangle = \text{Tr}(f O)$$

Spin-1/2 particle (2×2 hermitian matrix):

$$f = \frac{1}{2}(1 + \mathcal{P} \cdot \sigma)$$

- ▶ **Polarization 3-vector:** $\mathcal{P} = \langle \sigma \rangle = \text{Tr}(f \sigma)$

$$\begin{aligned} |\mathcal{P}| &= 1 && \text{Pure state} \\ 0 < |\mathcal{P}| &< 1 && \text{Mixed state} \\ |\mathcal{P}| &= 0 && \text{Completely unpolarized mixed state} \end{aligned}$$

Polarization 3-vector $\mathcal{P}$

- Expansion in terms of Pauli matrices

$$f^{\pm}(x, p) = e^{\pm\xi - p \cdot \beta} \left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta} \mathbf{P} \cdot \boldsymbol{\sigma} \right]$$

$$\mathbf{P} = \frac{1}{m} \left[E_p \mathbf{b} - \mathbf{p} \times \mathbf{e} - \frac{\mathbf{p} \cdot \mathbf{b}}{E_p + m} \mathbf{p} \right] = \mathbf{b}_*$$

* denotes the PARTICLE REST FRAME

- Average polarization vector

$$\mathcal{P} = \frac{1}{2} \frac{\text{tr}_2 [(f^+ + f^-) \boldsymbol{\sigma}]}{\text{tr}_2 [f^+ + f^-]} = -\frac{1}{2} \tanh(\zeta) \frac{\mathbf{P}}{2\zeta}$$

$$\mathcal{P} = -\frac{1}{2} \tanh \left[\frac{1}{2} \sqrt{\mathbf{b}_* \cdot \mathbf{b}_* - \mathbf{e}_* \cdot \mathbf{e}_*} \right] \frac{\mathbf{b}_*}{\sqrt{\mathbf{b}_* \cdot \mathbf{b}_* - \mathbf{e}_* \cdot \mathbf{e}_*}}$$

- **CAUTION:** Valid for small $\omega_{\mu\nu}$ and particle momenta p

Spin tensor $S^{\lambda, \mu\nu}$

- ▶ Total energy-momentum and angular momentum must be fixed

$$P^\mu = \int d^3\Sigma_\lambda T^{\lambda\mu} \quad J^{\mu\nu} = \int d^3\Sigma_\lambda J^{\lambda, \mu\nu}$$

- ▶ Densities are defined up to divergences
 $\Rightarrow$ **Pseudo-gauge transformations:**

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi^{\lambda, \mu\nu} + \Phi^{\mu, \nu\lambda} + \Phi^{\nu, \mu\lambda})$$
$$S'^{\lambda, \mu\nu} = S^{\lambda, \mu\nu} - \Phi^{\lambda, \mu\nu} + \partial_\alpha Z^{\alpha\lambda, \mu\nu}$$

Leave P^μ and $J^{\mu\nu}$ invariant

Pauli-Lubański four-vector (I)

- ▶ Phase-space density of total angular momentum of particle with momentum p

$$E_p \frac{dJ^{\lambda, \mu\nu}(x, p)}{d^3 p}$$

- ▶ Pauli-Lubański four-vector

$$E_p \frac{d\Delta\Pi_\mu(x, p)}{d^3 p} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Delta\Sigma_\lambda(x) E_p \frac{dJ^{\lambda, \nu\alpha}(x, p)}{d^3 p} \frac{p^\beta}{m}$$

- ▶ Which spin tensor do we use?

$$S^{\lambda, \mu\nu} = \kappa \int \frac{d^3 p}{2E_p} p^\lambda \text{tr}_4 [(X^+ - X^-) \Sigma^{\mu\nu}] = \frac{wu^\lambda}{4\zeta} \omega^{\mu\nu}$$

- ▶ Total angular momentum density becomes

$$E_p \frac{dJ^{\lambda, \nu\alpha}(x, p)}{d^3 p} = \frac{\kappa}{2} p^\lambda (x^\nu p^\alpha - x^\alpha p^\nu) \text{tr}_4 (X^+ + X^-) + \frac{\kappa}{2} p^\lambda \text{tr}_4 [(X^+ - X^-) \Sigma^{\nu\alpha}]$$

Pauli-Lubański four-vector (II)

- ▶ Particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2} \Delta\Sigma \cdot p \operatorname{tr}_4 (X^+ + X^-)$$

- ▶ PL per particle

$$\pi_\mu(x, p) = \frac{\Delta\Pi_\mu(x, p)}{\Delta\mathcal{N}(x, p)}$$

- ▶ PL four-vector in the PRF agrees with the Polarization vector (!)

$$\pi_*^0 = 0, \quad \pi_* = \mathcal{P}$$

- ▶ What about other forms for spin tensor?

Different form for spin tensor

- ▶ PL four-vector

$$E_p \frac{d\Delta\Pi_\mu(x, p)}{d^3p} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Delta\Sigma_\lambda(x) E_p \frac{dJ^{\lambda,\nu\alpha}(x, p)}{d^3p} \frac{p^\beta}{m}$$

- ▶ Canonical spin tensor

$$S_{\text{can}}^{\lambda,\mu\nu} = \kappa \int \frac{d^3p}{2E_p} (p^\lambda \text{tr}_4 [(X^+ - X^-) \Sigma^{\mu\nu}] \\ - p^\mu \text{tr}_4 [(X^+ - X^-) \Sigma^{\lambda\nu}] + p^\nu \text{tr}_4 [(X^+ - X^-) \Sigma^{\lambda\mu}])$$

- ▶ de Groot, van Leeuwen, van Weert spin tensor

$$S_{\text{GLW}}^{\lambda,\mu\nu} = \kappa \int \frac{d^3p}{2E_p} p^\lambda \left(\text{tr}_4 [(X^+ - X^-) \Sigma^{\mu\nu}] \right. \\ \left. + \frac{i}{2m^2} \text{tr}_4 [(X^+ - X^-) p_\alpha \gamma^\alpha (\gamma^\mu p^\nu - \gamma^\nu p^\mu)] \right)$$

PL is identical for all forms of spin tensor considered!