TMD evolution as a double-scale evolution

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based on [1803.11089]

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May 2018

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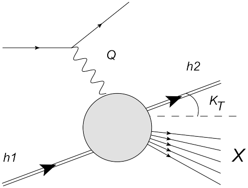
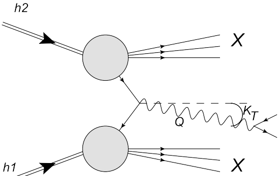
Brief introduction

Transverse momentum dependent = TMD

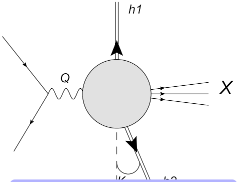
TMD factorization describes p_T-spectrum of the "double-inclusive processes" at small-p_T, where the transverse momentum is dominantly generated by the orbital motion of partons.



Drell-Yan



SIDIS



$e^+e^- \to h_1 + h_2 + X$

q is momentum of initiating EW-boson
$q^2 = \pm Q^2$
q_T^μ transverse component

$$\begin{cases} Q^2 \gg \Lambda^2_{QCD} \\ Q^2 \gg q_T^2 \end{cases}$$

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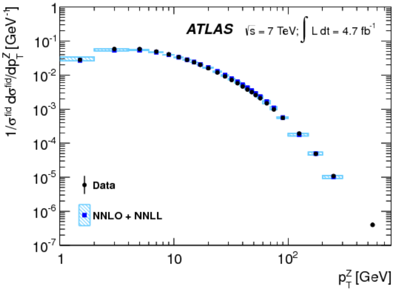
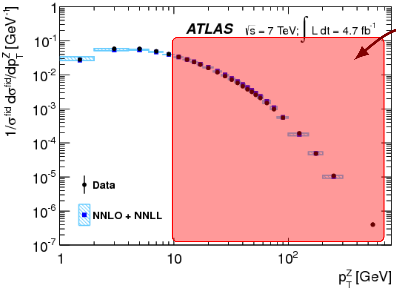
Example: pT-spectum of Z-boson



Different ranges of pT-spectrum
are dominated
by different physics

Example: pT-spectum of Z-boson



Pertubative (large) transverse momentum

Collinear factorization

Monte-Carlo event generators

Different ranges of pT-spectrum are dominated by different physics

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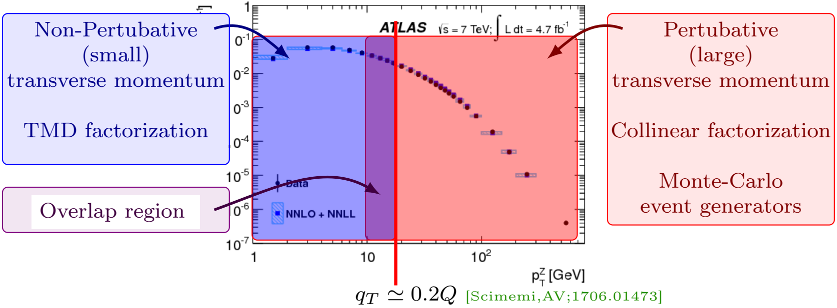
Example: pT-spectum of Z-boson



Transverse momentum dependent factorization
and
collinear factorization
independent and complimentary

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Current status

Experimental data

unpolarized

- Large amount of Drell-Yan data: from 5GeV to 120 GeV
- Some data is extremely precise (ATLAS Z-boson measurements)
- Large amount of SIDIS data (low energy only)
- e^+e^--annihilation data (not well investigated yet)

Theory

- TMD factorization is proved
  - Factorization of collinear part [Collins,Becher,Neubert,Scimemi,...; 2010-2012]
  - Factorization of rapidity divergences [AV;1707.07606]
- Perturbative parts are known up high orders
  - Hard part: 3-loops
  - Evolution: 3-loops
  - Matching: 2-loop (1-loop polarized)

Current status

TMD phenomenology

- Many separate fits of subsets of data.
- (practically) All studies are done at LO (often without evolution).
- There is the first global fit of DY+SIDIS [Bacchetta, et al; 1703.10157].
- There is a single example of higher perturbative order fit (NLO, NNLO) [Scimemi,AV; 1706.01473].
- There is only as single attempt to estimate theory uncertainty band [Scimemi,AV; 1706.01473].

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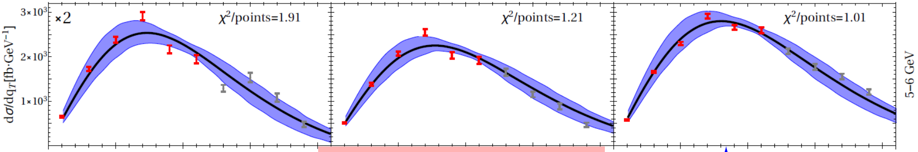
The first NNLO fit and extraction of (unpol.) TMDPDF [Scimemi,AV;1706.01473]

- The largest number of data point (DY)
- The largest energy separation
- Consideration of various orders (NLO,NNLL,NNLO)
- Studies of theory error-bands

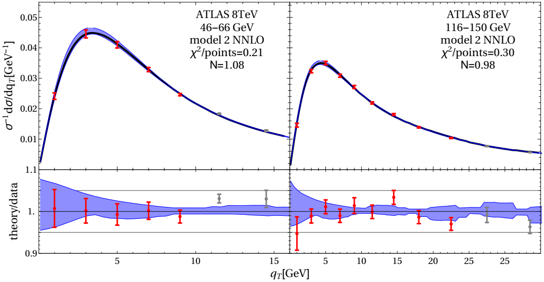
Included data (at $q_T < 0.2Q$)

	reaction	$\sqrt{s}$	Q	comment	points
E288	$p + Cu \to \gamma^* \to \mu\mu$	19.4 GeV	4-9 GeV	norm=0.8	35
E288	$p + Cu \to \gamma^* \to \mu\mu$	23.8 GeV	4-9 GeV	norm=0.8	45
E288	$p + Cu \to \gamma^* \to \mu\mu$	27.4 GeV	4-9 & 11-14 GeV	norm=0.8	66
CDF+D0	$p + \bar{p} \to Z \to ee$	1.8 TeV	66-116 GeV		44
CDF+D0	$p + \bar{p} \to Z \to ee$	1.96 TeV	66-116 GeV		43
ATLAS	$p + p \to Z \to \mu\mu$	7 & 8 TeV	66-116 GeV	tiny errors!	18
CMS	$p + p \to Z \to \mu\mu$	7 & 8 TeV	60-120 GeV		14
LHCb	$p + p \to Z \to \mu\mu$	7 & 8 & 13 TeV	60-120 GeV		30
ATLAS	$p + p \to Z/\gamma^* \to \mu\mu$	8 TeV	46-66 GeV		5
ATLAS	$p + p \to Z/\gamma^* \to \mu\mu$	8 TeV	116-150 GeV		9
				Total	309



Drell-Yan at $Q = 5 - 6\text{GeV}$

Drell-Yan at $Q = 116 - 150\text{GeV}$



Evolution is a key element

Here:

- 3-loop evolution
- 2-loop coefficient function
- 2-loop matching

plots from [1706.01473]

- The main difficulty was to make all ingredients work together.

Next goal is to join SIDIS and DY, and to make a global fit.

Questions of internal consistency are ultimately important.

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Questions of internal consistency are ultimately important.

TMD evolution is the central element of the factorization. Precise knowledge and understanding of it is required to make a consistent description of modern data (2GeV $\leftrightarrow$ 150GeV).

- There are problems in it
- Generally, the "traditional" formulation is overcomplicated

Outline

Outline

- TMD evolution in a nutshell
  - Equations, solutions, etc.
  - TMD evolution field and its structure
- Effects of truncation perturbation theory
  - Violation of integrability condition, and solution-dependence of TMD evolution
  - Methods to fix the ambiguity.
- ζ-prescription
  - Physical meaning of ζ-prescription
  - Optimal TMD distribution.
- TMD cross-section and perturbative uncertainties.

Evolution of transverse momentum dependent (TMD) distributions
=
TMD evolution

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TMD evolution is used for two practical purposes

- Compare different experiments
- Modeling TMD distribution

$$\frac{d\sigma}{dX} \sim \int d^2b\, e^{i(bq_T)} H_{ff'}(Q,\mu) F_{f\leftarrow h}(x_1,b;\mu,\zeta_1) F_{f'\leftarrow h}(x_2,b;\mu,\zeta_2)$$

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Minimize $\ln(Q/\mu)$
$\mu = Q$

$\zeta_1\zeta_2 = Q^4$
or
$\zeta_1 = \zeta_2 = Q^2$

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Minimize $\mathbf{L}_\mu$, $\mathbf{L}_{\sqrt{\zeta}}$

$\mu \sim \sqrt{\zeta} \sim b^{-1}$

$$F(x,b;\mu,\zeta) \sim C(x,b;\mu,\zeta) \otimes \mathrm{PDF}(x,\mu)$$

Typical model for TMD includes matching

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$$F(x,b;\mu_f,\zeta_f) = R[b,(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,b;\mu_i,\zeta_i)$$

Final scale ⟼ TMD evolution factor ⟼ Initial scale

Minimize $\mathbf{L}_\mu$, $\mathbf{L}_{\sqrt{\zeta}}$
$\mu \sim \sqrt{\zeta} \sim b^{-1}$

$$F(x,b;\mu,\zeta) \sim C(x,b;\mu,\zeta) \otimes \mathrm{PDF}(x,\mu)$$

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TMD evolution: theory

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (2)$$

- γ_F – TMD anomalous dimension
- $\mathcal{D}$ – rapidity anomalous dimension ($= -\frac{\tilde{K}}{2}$[Collins' book], $= K$[Bacchetta, at al,1703.10157])
- Anomalous dimensions are *universal*, i.e. independent on hadron, polarization, PDF/FF(see proof [AV;1707.07606]).
- Anomalous dimension depend only on flavor (gluon/quark). Skip index f in the following.

TMD evolution: theory

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

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Solution: $F(x,\mathbf{b};\mu_f,\zeta_f) = R[\mathbf{b};(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]F(x,\mathbf{b};\mu_i,\zeta_i)$

Expression for R is known as "Sudakov exponent"

$$\times \exp\left\{ \ln\frac{\sqrt{\zeta_A}}{\mu_b}\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_D(g(\mu');1) - \ln\frac{\sqrt{\zeta_A}}{\mu'}\gamma_K(g(\mu'))\right]\right\}. \quad (13.70)$$

This is probably the best formula for calculating and fitting TMD fragmentation functions;

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TMD evolution: theory

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta), \quad (1)$$

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- There are theoretical traps in TMD evolution.
- They became evident at high-perturbative orders.
- Each problem is small, but there are many of them.

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Problem 1: Violation of transitivity

$$R[\mathbf{b}; X \to Y]\; R[\mathbf{b}; Y \to X] = 1$$

$C_2 Q \to \mu_0$ by [Collins' book]

$$\times \exp\left\{ -\int_{\mu_0}^{C_2 Q} \frac{d\mu}{\mu} \left[\ln \frac{C_2 Q}{\mu} \gamma_K(g(\mu)) - 2\gamma_A\left(1/C_2^2, g(\mu)\right) \right] \right\}$$
$$\times \exp\left[\frac{1}{2}(y_{p_A} - y_{p_B}) K\left(m_g, m, \mu_0, g(\mu_0)\right) \right], \qquad (10.131)$$

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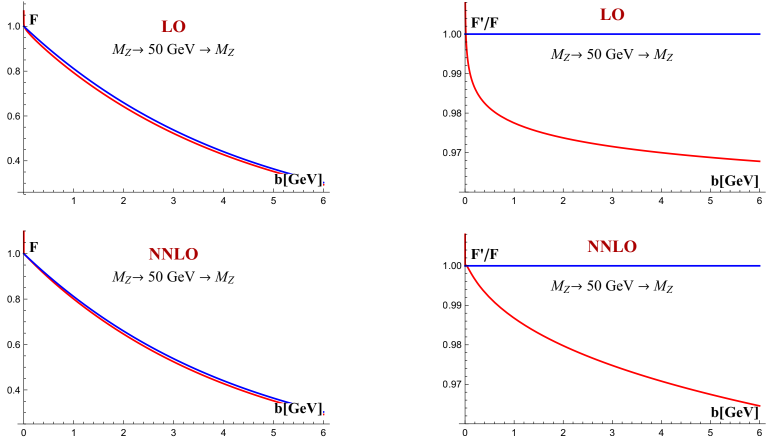

There is a violation of transitivity $\sim 2\ \%$ which seems better at NNLO

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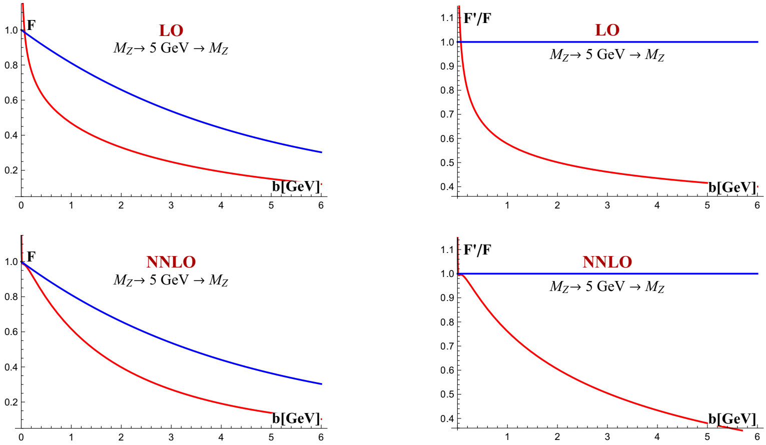
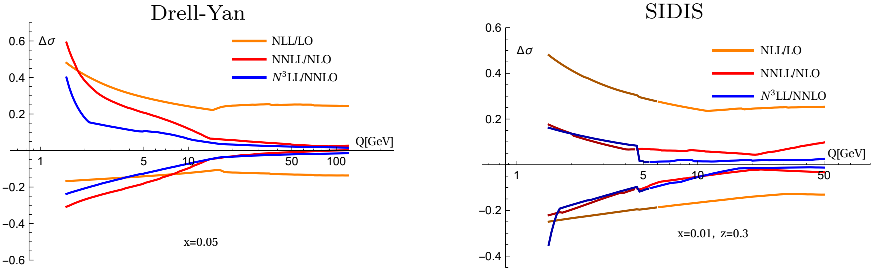


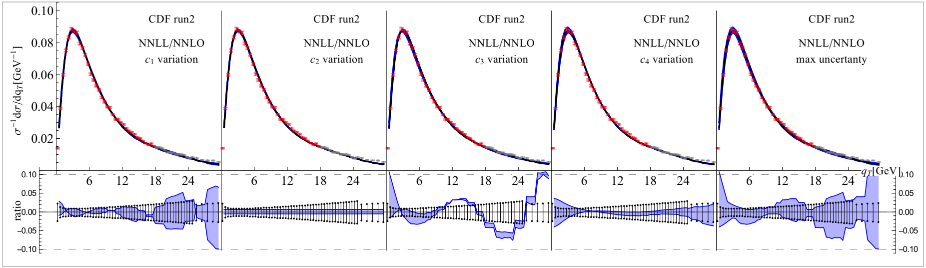
There is a VERY strong violation of transitivity, which seems worse at NNLO

Problem 2: Anomalous behavior of variations bands



- The variations of constants does not decrease at large-Q.
- Opposite it start to increase at large-Q.
- NNLO band seems larger then NLO

Problem 3: Anomalous behavior of variations



In [Scimemi,AV; 1706.01473] there was a study of a perturbative stability. With the help of variation of scales.

- The variations of constants c_1 and c_3 are the largest despite these are 3-loop series (compare to c_2 and c_4 which are 2-loop)
- The variation of c_1 and c_3 are numerically unstable (see artifacts)

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Problem 4: Strong dependence on μ

- It seems that TMD fits are seriously dependent on the values of μ (μ_b, μ^*, etc)
- Often the parameter μ is used as a subject of fit. E.g. $b_{\max}$ parameter.
- Is it evidence of perturbative instability? Difficult to answer, since there is no dedicated study on it.

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In fact, these are consequences of a larger problem:

> not self-consistency of TMD evolution in the "naive" form
> within perturbation theory.

Under "naive" I refer to, say formulas given in [Collins textbook], [Aybat,Rogers,1101.5057],[Echevarria,et al,1208.1281],...

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Let us examine the TMD evolution equation again

$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x,b;\mu,\zeta) = \frac{\gamma_F^f(\mu,\zeta)}{2} F_{f\leftarrow h}(x,b;\mu,\zeta),$$
$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x,b;\mu,\zeta) = -\mathcal{D}^f(\mu,b) F_{f\leftarrow h}(x,b;\mu,\zeta),$$

The solution of TMD evolution equation (i.e. R) exists (in the mathematical sense) only if

$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} = -\mu^2 \frac{d}{d\mu^2} \mathcal{D}(\mu,b)$$

integrability condition

Integrability condition is satisfied due to the collinear overlap of divergences

$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} = -\Gamma_{\text{cusp}}(\mu)$$
$$\mu^2 \frac{d}{d\mu^2} \mathcal{D}(\mu,b) = \Gamma_{\text{cusp}}(\mu)$$

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integrability condition

Solution is

$$R[b;(\mu_f,\zeta_f) \to (\mu_i,\zeta_i)] = \exp\left[\int_P \left(\gamma_F(\mu,\zeta)\frac{d\mu}{\mu} - \mathcal{D}(\mu,b)\frac{d\zeta}{\zeta}\right)\right]$$

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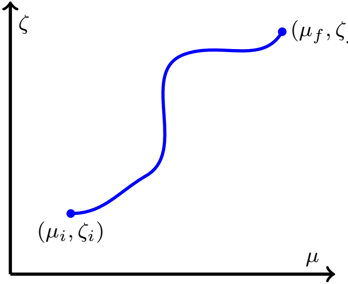
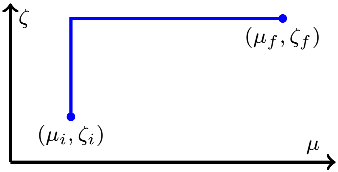


The solution is independent on the path of the integration due to *integrability condition*

Examples



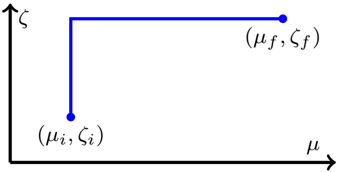
Solution 1

$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - \mathcal{D}(\mu_i, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$

[Collins' textbook],[Aybat,Rogers,1101.5057],...

99% popular

Examples



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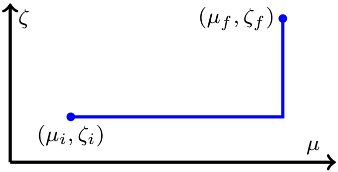
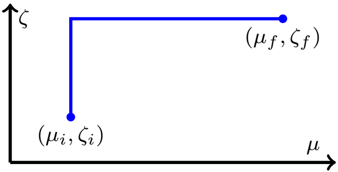


Solution 2

$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - \mathcal{D}(\mu_f, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$

Examples



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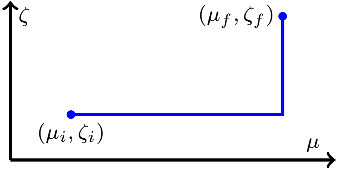
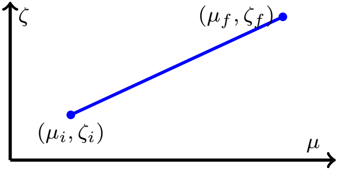


Solution 2

$$\ln R=\int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\gamma_F(\mu,\zeta_i)-\mathcal{D}(\mu_f,b)\ln\left(\frac{\zeta_f}{\zeta_i}\right)$$



Solution 3

$$\ln R=\int_0^1\Big(\gamma_F(\mu(t),\zeta(t))\frac{\mu_f-\mu_i}{(\mu_f-\mu_i)t+\mu_i}-\mathcal{D}(\mu(t),b)\frac{\zeta_f-\zeta_i}{(\zeta_f-\zeta_i)t+\zeta_i}\Big)dt$$

TMD evolution is essentially 2D task.
Let me introduce convenient notation.

Evolution scales	Anomalous dimensions
$$\boldsymbol{\nu} = (\ln\left(\frac{\mu^2}{1\ \text{GeV}^2}\right), \ln\left(\frac{\zeta}{1\ \text{GeV}^2}\right)).$$	$$\mathbf{E}(\boldsymbol{\nu}, b) = (\frac{\gamma_F(\boldsymbol{\nu})}{2}, -\mathcal{D}(\boldsymbol{\nu}, b)).$$
2d vector	vector field

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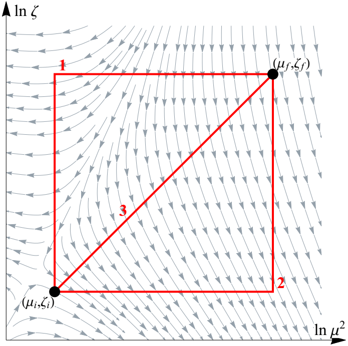

Evolution equation

$$\boldsymbol{\nabla} F(x, b; \boldsymbol{\nu}) = \mathbf{E}(\boldsymbol{\nu}, b) F(x, b; \boldsymbol{\nu})$$

Solution

$$\ln R[b, \boldsymbol{\nu}_f \to \boldsymbol{\nu}_i] = \int_P \mathbf{E} \cdot d\boldsymbol{\nu}$$

Scalar potential

The integrability condition is the condition that evolution field $\mathbf{E}$ is *irrotational* (*conservative*)

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

Thus, it is determined by a *scalar potential*

$$\mathbf{E}(\boldsymbol{\nu}, b) = \boldsymbol{\nabla} U(\boldsymbol{\nu}, b)$$

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Evolution is the difference between potentials

$$\ln R[b; \boldsymbol{\nu}_f \to \boldsymbol{\nu}_i] = U(\boldsymbol{\nu}_f, b) - U(\boldsymbol{\nu}_i, b).$$

Scalar potential can be easily found

$$U(\boldsymbol{\nu}, b) = \int^{\nu_1} \frac{\Gamma(s)s - \gamma_V(s)}{2} ds - \mathcal{D}(\boldsymbol{\nu}, b)\nu_2 + \text{const}(b),$$

TMD evolution in the perturbation theory

In the real live we can operate only with the first several terms of perturbation theory. Therefore, the *integrability* condition is violated

$$\mu \frac{d\mathcal{D}(\mu)}{d\mu} \neq \Gamma(\mu)$$

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Simple example at 1-loop

$$\mathcal{D} = a_s(\mu)\frac{\Gamma_0}{2}\mathbf{L}_\mu$$

$$\begin{aligned}\mu\frac{d\mathcal{D}}{d\mu} &= a_s(\mu)\frac{\Gamma_0}{2}\left(\mu\frac{d}{d\mu}\mathbf{L}_\mu\right) + \left(\mu\frac{da_s(\mu)}{d\mu}\right)\frac{\Gamma_0}{2}\mathbf{L}_\mu \\ &= a_s(\mu)\Gamma_0 - \beta_0 a_s^2(\mu)\Gamma_0\mathbf{L}_\mu \neq a_s(\mu)\Gamma_0\end{aligned}$$

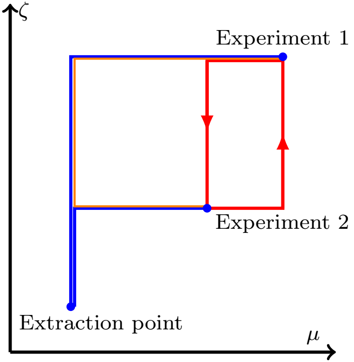
At N'th order of perturbation theory $\Gamma - d\mathcal{D} \sim a_s^{N+1}\mathbf{L}_\mu^N$

- Since $a_s \sim \ln^{-1}\mu$ there is always (at any finite N) value of b(fixed) then $\delta\Gamma \gg 1$
- The value of μ does not play a role
- In fact, this term is ALWAYS NLO, in the standard resummation counting ($a_s L \sim 1$).
- The NP models for $\mathcal{D}$ only enforce the problem.

In PT the TMD evolution dependents on the path

Transitivity

$$R[b;(\mu_1,\zeta_1)\to(\mu_2,\zeta_2)] = R[b;(\mu_1,\zeta_1)\to(\mu_3,\zeta_3)]R[b;(\mu_3,\zeta_3)\to(\mu_2,\zeta_2)]$$

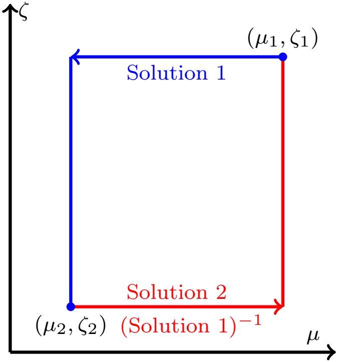


In the fitting procedure
different experiments (different Q)
define the same point (same b)

But (generally) different Q's
are unrelated

Inversion

$$R[b;\{\mu_1,\zeta_1\}\to\{\mu_2,\zeta_2\}]=R^{-1}[b;\{\mu_2,\zeta_2\}\to\{\mu_1,\zeta_1\}]$$



$$R[b;\{\mu_1,\zeta_1\}\xrightarrow{1}\{\mu_2,\zeta_2\}]=R^{-1}[b;\{\mu_2,\zeta_2\}\xrightarrow{2}\{\mu_1,\zeta_1\}]$$
$$\neq R^{-1}[b;\{\mu_2,\zeta_2\}\xrightarrow{1}\{\mu_1,\zeta_1\}]$$

There is no simple way to compare different fits!

Reverse ingeneering for each fit!

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Helmeholz decomposition

$$\mathbf{E} = \tilde{\mathbf{E}} + \mathbf{\Theta}$$

$\tilde{\mathbf{E}}$	conservative (*irrotational*) component	$\text{curl}\tilde{\mathbf{E}} = 0$
$\mathbf{\Theta}$	*divergence-free* component	$\boldsymbol{\nabla} \cdot \mathbf{\Theta} = 0$

$$\tilde{\mathbf{E}} \cdot \mathbf{\Theta} = 0$$

$$\text{curl}\mathbf{E} = \text{curl}\mathbf{\Theta} = \frac{\delta\Gamma}{2}$$

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$$\tilde{\mathbf{E}} \cdot \mathbf{\Theta} = 0$$

$$\text{curl}\mathbf{E} = \text{curl}\mathbf{\Theta} = \frac{\delta\Gamma}{2}$$

Ambiguous scalar potential

The *divergence-free* component is an artifact of truncated PT. It prevents the definition of scalar potential

$$\boldsymbol{\nabla}\tilde{U} = \tilde{\mathbf{E}}, \qquad \mathbf{curl}V = \mathbf{\Theta}$$

$$\nabla^2\tilde{U} = \frac{d\gamma_F}{d\ln\mu} \qquad \text{vs.} \qquad \boldsymbol{\nabla}U = \mathbf{E}$$

Poisson equation solution is defined up to $\nabla^2 f = 0$.

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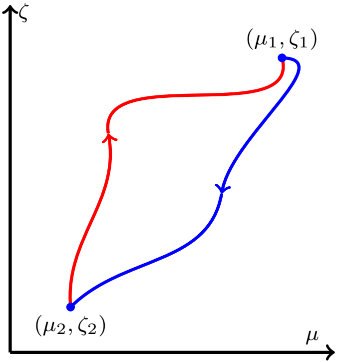
Non-conservative evolution



$$\oint_C \mathbf{E}\cdot d\nu = \int_\Omega d^2\nu\,\mathrm{curl}\boldsymbol{\Theta} = \frac{1}{2}\int_\Omega d^2\nu\,\delta\Gamma(\nu,b)$$

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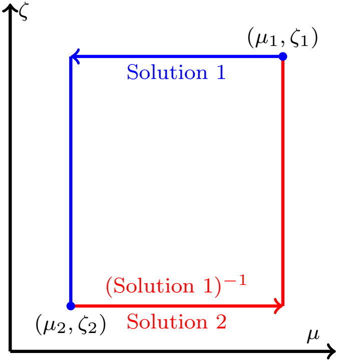
Non-conservative evolution



$$\oint_C \mathbf{E}\cdot d\nu = \int_\Omega d^2\nu\,\text{curl}\boldsymbol{\Theta} = \frac{1}{2}\int_\Omega d^2\nu\,\delta\Gamma(\nu,b)$$

$$\ln\frac{\text{solution 1}}{\text{solution 2}} = \ln\left(\frac{\zeta_f}{\zeta_i}\right)\int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\delta\Gamma(\mu,b)$$

The "longer" evolution – the bigger error

That is why for $Z-$boson

error is larger

How to fix it?

There is no unique way to fix this ambiguity, in the absence of extra all-order/non-perturbative statement on TMD anomalous dimensions.

Some possibilities

- Lets use a single evolution line $\mu^2 = \zeta$, and the solution 3
  - \+ Restore self-consistency and inversion
  - \- - Everyone stick to a single line. No freedom for modeling.
  - \- Numerically more expensive
- Lets set $\boldsymbol{\Theta} = 0$, and use only $\tilde{\mathbf{E}}$
  - \+ + Ideal solution which does not restrict anything
  - \- The procedure is not unique, we need to set boundary conditions
- Lets repair the integrability condition by adding terms beyond PT
  - \+ + Very simple
  - \- The procedure is not unique
  - Equivalent to some boundary condition (do not know which)

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In PT the integrability condition is violated

We can repair it by accounting "higher-then-allowed" terms of perturbation theory

$$\mu\frac{d\mathcal{D}(\mu,b)}{d\mu} \neq -\zeta\frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta)=-\Gamma(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{D}(\mu,b)\neq\Gamma(\mu)$$

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Improved $\mathcal{D}$ scenario "CSS-like"

$$\mu\frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta\frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu)$$

$$\mathcal{D}(\mu,b) = \int_{\mu_0}^{\mu}\frac{d\mu}{\mu}\Gamma(\mu) + \mathcal{D}(\mu_0,b) \qquad \text{[CS,1981]}$$

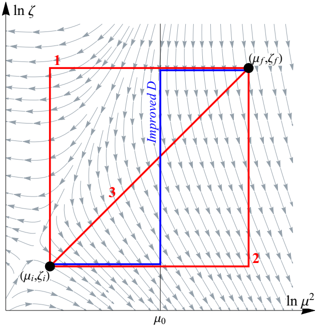
$$\begin{aligned}\ln R[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i);\mu_0] \quad &= \quad \int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\left(\Gamma(\mu)\ln\left(\frac{\mu^2}{\zeta_f}\right)-\gamma_V(\mu)\right)\\ &\quad -\int_{\mu_0}^{\mu_i}\frac{d\mu}{\mu}\Gamma(\mu)\ln\left(\frac{\zeta_f}{\zeta_i}\right)-\mathcal{D}(\mu_0,b)\ln\left(\frac{\zeta_f}{\zeta_i}\right).\end{aligned}$$

- μ_0 is some new scale where "perturbation theory works".
- In fact it is the composition of solution 1 and 2

Improved $\mathcal{D}$ scenario "CSS-like"

$$\mu \frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta \frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\zeta \frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu), \qquad \mu \frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu)$$



- Transitivity and inversion hold
  If μ_0 is kept explicit (not $\mu_0 = \mu_i$ as typically used)
- If different μ_0 are used, the problem of comparison returns
- If different non-perturbative models are used, the problem also returns
- The evolution (quite strongly) depends on μ_0 (c_1 variation band)

Improved γ scenario
Use integrability condition as the definition

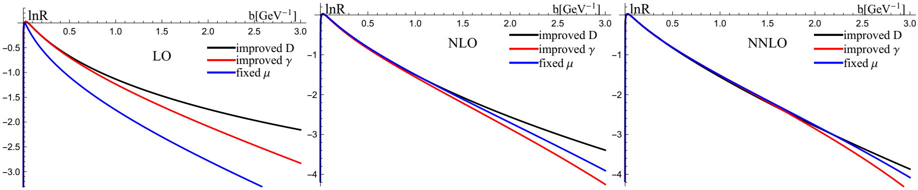
$$\mu \frac{d\mathcal{D}(\mu,b)}{d\mu} = -\zeta \frac{d\gamma_F(\mu,\zeta)}{d\zeta}$$

$$\gamma_F(\mu,\zeta) \to \gamma_M(\mu,\zeta,b) = -\mu \frac{d}{d\mu}\mathcal{D}(\mu,b) \ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_V(\mu)$$

$$\begin{aligned}\ln R[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)] &= -\int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu}\left(2\mathcal{D}(\mu,b)+\gamma_V(\mu)\right) \\ &+\mathcal{D}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right) - \mathcal{D}(\mu_i,b)\ln\left(\frac{\mu_i^2}{\zeta_i}\right).\end{aligned}$$

- Explicitly transitive, and inverse.
- Simple non-perturbative generalization ($\mathcal{D} \to \mathcal{D}_{NP}$)
- No extra scales. The evolution field is explicitly conservative.

The difference between solutions is $\underbrace{\sim a^{N+1}\mathbf{L}_\mu}_{\text{main b}}$ or $\underbrace{\sim a^{N+1}\mathbf{L}_\mu\mathbf{L}_{\mu_0}^N}_{\text{large b}}$



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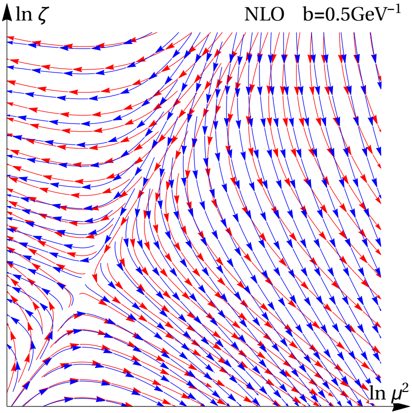
How strong is modification of the field?



Part 2:
ζ-prescription

The final scales (μ_f, ζ_f) are fixed by process kinematics $\sim (Q, Q^2)$.
The initial scale are fixed only by model of TMD distribution.

Small-b matching

At small-b one can match TMD to collinear distribution by OPE

$$\text{TMD}(x, b; \mu_i, \zeta_i) = C(x, \mathbf{L}_\mu, \mathbf{L}_{\sqrt{\zeta}}, \mu) \otimes \text{PDF}(x, \mu)$$

- It is often used as an zero-level input to the model of TMD.
- It guaranties agreement with high energy experiments.
- It also requires the evolution from $(Q, Q^2) \to (\mu_i, \zeta_i)$, which are typically selected as

$$\mu_i^2 = \zeta_i \sim \frac{1}{b^2}$$

$$F(\underbrace{x, b}_{\text{params.}}; \underbrace{\mu, \zeta}_{\text{scales}})$$

TMD case

$$d\sigma \sim \int d^2 b e^{iqb} H(Q) \{R\Big(Q \to \frac{1}{b}\Big)\}^2 F_1(x_1, b; b^{-1}, b^{-2}) F_2(x_2, b; b^{-1}, b^{-2})$$

This is the standard approach that is used in majority of applications.

$$F_1(x_1, b; b^{-1}, b^{-2}) \to \text{phenomenological parametrization}$$

Analogy in DIS

$$d\sigma \sim C(Q, x) \otimes R\Big(Q \to 1/x\Big) \otimes f(x, 1/x)$$

$$f(x, 1/x) \to \text{phenomenological parametrization}$$

$$F(\underbrace{x\ ,\ b}_{\text{params.}}\ ;\ \underbrace{\mu\ ,\ \zeta}_{\text{scales}})$$

TMD case

$$d\sigma \sim \int d^2b e^{iqb} H(Q) \{R\Big(Q \to \frac{1}{b}\Big)\}^2 F_1(x_1, b; b^{-1}, b^{-2}) F_2(x_2, b; b^{-1}, b^{-2})$$

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Analogy in DIS

$$d\sigma \sim C(Q, x) \otimes R\Big(Q \to 1/x\Big) \otimes f(x, 1/x)$$

$$f(x, 1/x) \to \text{phenomenological parametrization}$$

It is non-sense!

$$F(\underbrace{x\,,\,b}_{\text{params.}}\;;\;\underbrace{\mu\,,\,\zeta}_{\text{scales}})$$

TMD case

$$d\sigma \sim \int d^2b e^{iqb} H(Q)\{R\Big(Q\to\frac{1}{b}\Big)\}^2 F_1(x_1,b;b^{-1},b^{-2})F_2(x_2,b;b^{-1},b^{-2})$$

This is the standard approach that is used in majority of applications.

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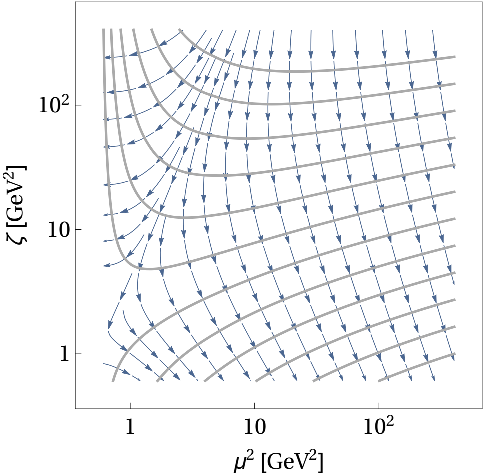
Analogy in DIS

$$\require{cancel}\cancel{d\sigma\sim C(Q,x)\otimes R\Big(x;Q\to 1/x\Big)\otimes f(x,1/x)}$$

$$d\sigma\sim C(Q,x)\otimes R\Big(x;Q\to 1\text{GeV}\Big)\otimes f(x,1\text{GeV})$$

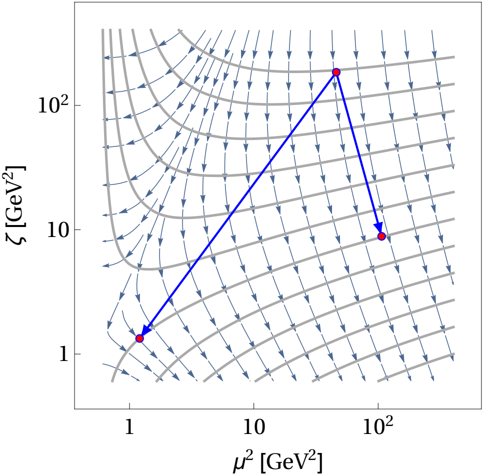
$$f(x,1\text{GeV}) \to \text{phenomenological parametrization}$$

In the TMD case there is no notion of a scale, because it is defined on a plane



The scaling is defined by
~~a difference between scales~~
a difference between potentials

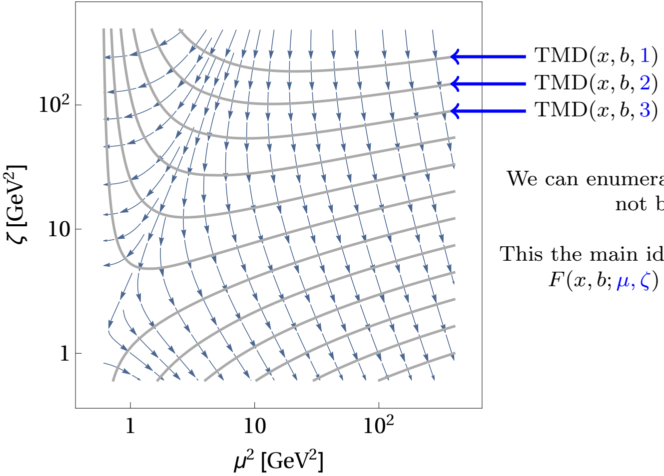
In the TMD case there is no notion of a scale, because it is defined on a plane



The scaling is defined by
~~a difference between scales~~
a difference between potentials

Evolution factor to both points
is the same
although the scales are
different by 10^2GeV^2

TMD distributions on the same equipotential line are equivalent.



We can enumerate them by a lines not by (μ, ζ)

This the main idea of ζ-prescription

$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$

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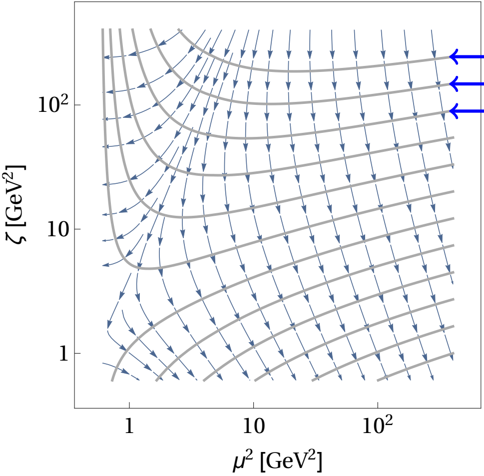
TMD distributions on the same equipotential line are equivalent.



TMD$(x, b, 1)$
TMD$(x, b, 2)$
TMD$(x, b, 3)$

We can enumerate them by a lines not by (μ, ζ)

This the main idea of ζ-prescription
$F(x, b; \mu, \zeta) \to F(z, b; \text{line})$

Intresting to know:
"ζ-prescription" is an idiotic term.
Refered to the initial "naive" version [Scimemi,AV,1706.01473].

In ζ-prescription we set
$$\zeta \to \zeta_\mu(\boldsymbol{\nu})$$

- TMDs are "enumerated" by $\boldsymbol{\nu}$ (the number of line)
- TMDs are "naive" scale-independent

$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0 \qquad\qquad \Rightarrow \text{No double-logs in the matching.}$$

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- TMDs are "naive" scale-independent

$$\mu \frac{d}{d\mu} F(x, b; \mu, \zeta_\mu) = 0 \qquad \Rightarrow \text{No double-logs in the matching.}$$

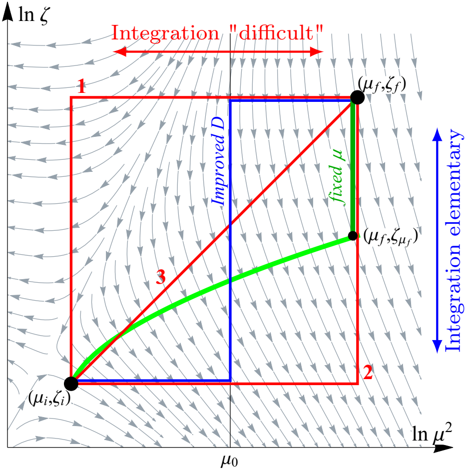
TMD distribution depends only on the "number" of equipotential line

$$F(x, \mathbf{b}; \mu, \zeta) \to F(x, \mathbf{b}; \nu)$$

$$\frac{dF(x, \mathbf{b}; \nu)}{d\nu} = \frac{dU(\mathbf{b}; \nu)}{d\nu} F(x, \mathbf{b}; \nu)$$

$$\Updownarrow$$

$$F(x, \mathbf{b}; \nu) = e^{U(\mathbf{b};\nu) - U(\mathbf{b};\nu_0)} F(x, \mathbf{b}; \nu_0)$$



$$R = \left(\frac{\zeta_f}{\zeta_{\mu_f}}\right)^{-\mathcal{D}(\mu_f,b)}$$

- Numerically simple (and fast)
- $\mu_f = Q$ thus a_s is small
- Alternative form of Sudakov exponent

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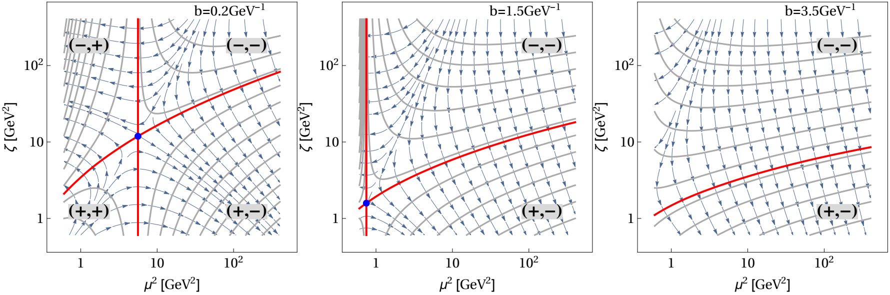
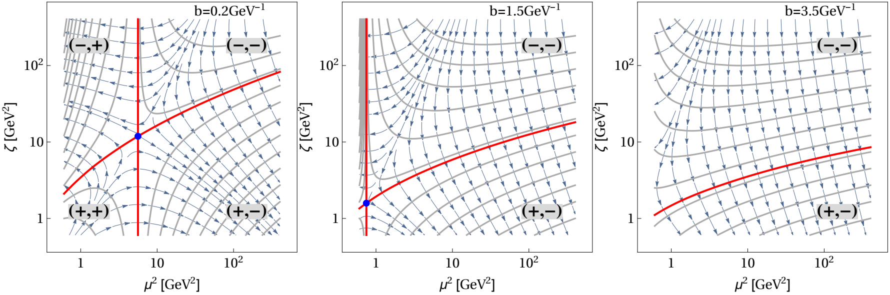
Which line is the best?



- Some non-interesting singularities at $\mu, \zeta \to \infty$
- Landau pole at $\mu = \Lambda$
- Saddle point (blue dot)

$$\mathcal{D}(\mu_{\text{saddle}}, b) = 0, \qquad \gamma_M(\mu_{\text{saddle}}, \zeta_{\text{saddle}}, b) = 0$$

Which line is the best?



- Due to presence of saddle point the set of uquipotential lines is split into subsets with restricted domains
- **Subset 1:** $\mu > \mu_{\text{saddle}}$
- **Subset 2:** $\mu < \mu_{\text{saddle}}$
- **Special line:** The one which passes though the saddle point (μ is unrestricted)
- Special lines dissect the evolution planes into quadratures of the "same evolution sign".

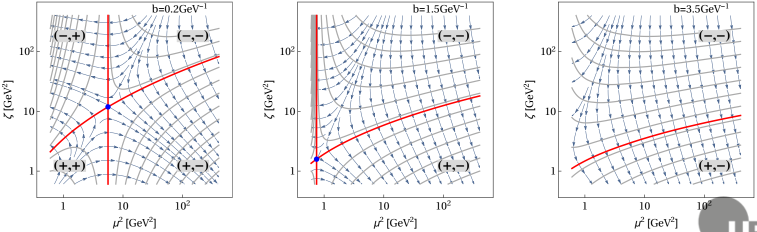
Universal scale-independent TMD

There is a unique line which passes though all μ's

The optimal TMD distribution

$$F(x,b) = F(x,b;\mu,\zeta_\mu)$$

where ζ_μ is the special line.



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TMD cross-section

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q)\{\tilde{R}^f[b;Q]\}^2 \tilde{F}_{f\leftarrow h}(x_1,b)\tilde{F}_{f'\leftarrow h}(x_2,b),$$

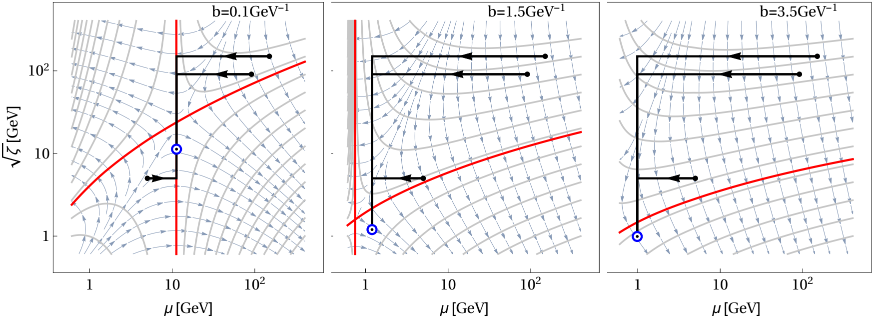
with $\zeta_f = \mu_f^2 = Q^2$

$$\tilde{R}^f[b;Q] \quad = \quad (Qb)^{-\mathcal{D}^f_{\text{NP}}(Q,b)} \exp\{-\mathcal{D}^f_{\text{NP}}(Q,b) v^f(Q,b)\}$$

- v is given perturbative series, $v = \frac{3}{2} + a_s...$
- $\tilde{F}$ is TMD in the "naive" ζ-prescription

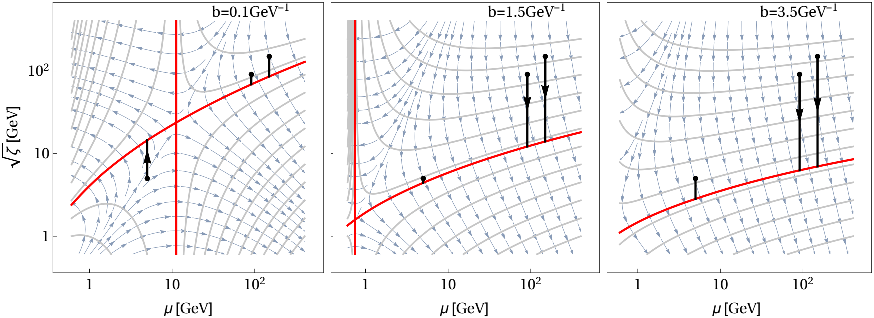
- There are no approximations (*ala* high energy expansion of integrals).
- There are only 2 (μ_f, ζ_f) scales and no solution dependence.
- Clear separation of TMD evolution from the TMD distribution.

CSS version
$(Q, Q^2) \to (\mu_b, \mu_b^2)$



Here $\mu_b = \dfrac{C_0}{b^*}$ with $b_{\max} = 1.2\text{GeV}^{-1}$

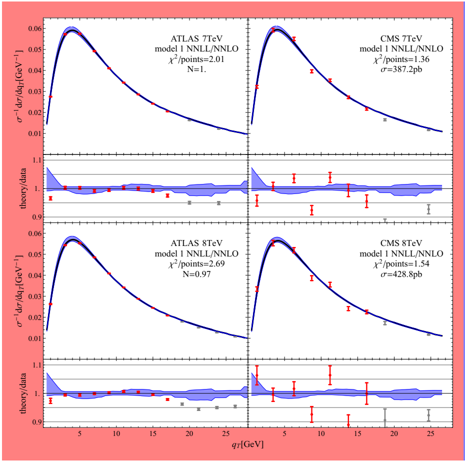
Optimal version
$(Q, Q^2) \to (Q, \zeta_Q)$



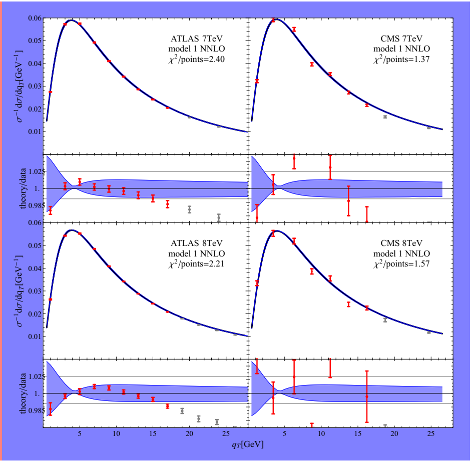
Despite it looks very different
it does just the same job as the Sudakov exponent
but faster, numerically more accurate and without extra intermidiate scales

Uncertainties of TMD cross-section (1)

CSS-like definition



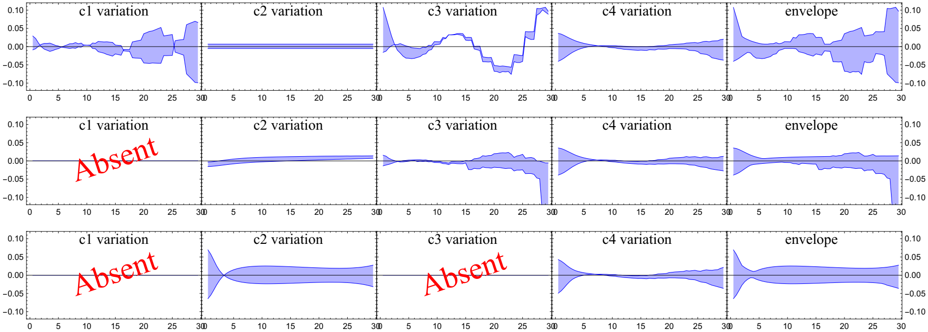
Optimal definition



Update of the NNLO DY fit,
χ^2-values practically the same (a bit better), parameters within (previous) error-bars
significant reduction of theory uncertainties.

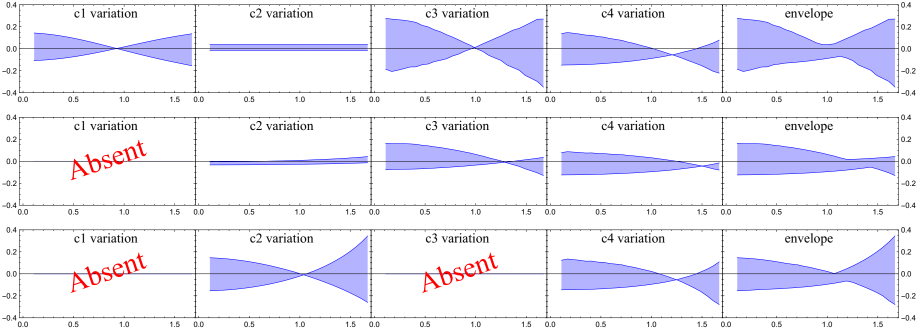
Uncertainties of TMD cross-section (1)

Z-boson production at CDF run 2

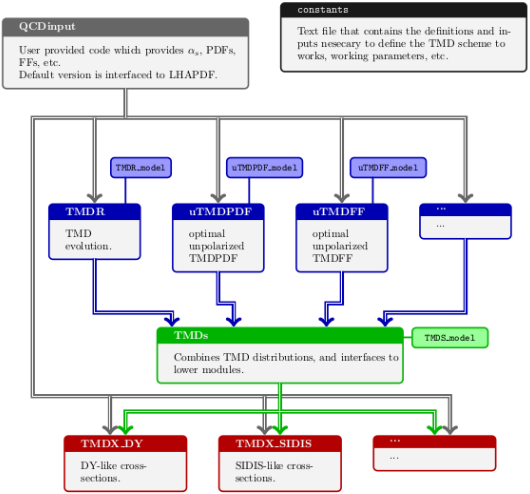


Uncertainties of TMD cross-section (2)

$$\text{E288 (200) } Q = 6 - 7 \text{ GeV}$$



arTeMiDe v1.3



- Variety of evolutions
- LO, NLO, NNLO
- No restriction for NP models
- Fast code
- DY cross-sections
- SIDIS cross-sections (not tuned yet)
- Theory uncertainty bands

https://teorica.fis.ucm.es/artemide/

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Conclusion

Main message:
TMD evolution is a double scale evolution.
Therefore, it should be considered with care, and then it grants many simplifications.

Message 1:
In truncated PT there is the solution-dependence of evolution

- It could be strong.
- There is no unique way to fix it.

Message 2:
TMD distributions on a same equipotential line are equivalent. Enumerate them with lines!

- Guarantied absence of (large) logarithms in coefficient function
- Universal for all quantum numbers
- Very simple practical formula (no integrations!)

Double-scale evolution is not unique for TMD case. It also appears in jet functions, k_T-resummation, joint resummation, DPDs, etc.

Backup

Collinear overlap

There are collinearly divergent subgraphs (then gluon is parallel to Wilson line), which result to overlap of UV and rapidity divergent sectors. It gives interdependance of anomalous dimension on "opposite" scale

$$\zeta\frac{d}{d\zeta}\gamma_F(\mu,\zeta) = -\Gamma(\mu),$$
$$\mu\frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu),$$

where Γ is the (light-like) cusp anomalous dimension.

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$$\mu \frac{d}{d\mu}\mathcal{D}(\mu,b) = \Gamma(\mu),$$

where Γ is the (light-like) cusp anomalous dimension.

Thus the logarithmic part of AD's could be fixed

(exact) $$\gamma_F(\mu,\zeta) = \Gamma(\mu) \ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_V(\mu)$$

(order-by-order) $$\mathcal{D}(\mu,b) = a_s(\mu)\frac{\Gamma_0}{2}\mathbf{L}_\mu + a_s^2\left(\frac{\Gamma_0\beta_0}{4}\mathbf{L}_\mu^2 + \frac{\Gamma_1}{2}\mathbf{L}_\mu + d^{(2,0)}\right) + ...$$

standard notation: $\mathbf{L}_X = \ln(C_0^{-2}b^2X^2)$, $C_0 = 2e^{-\gamma_E}$

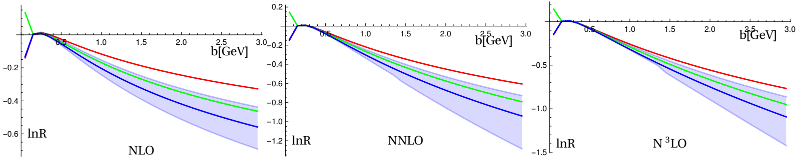
Test of solution independence

$$(Q, Q^2) \to (\mu_b, \mu_b^2) \qquad \mu_b = \frac{C_0}{b} + 2\text{GeV}$$

Test of solution independence

$$(Q, Q^2) \to (\mu_b, \mu_b^2) \qquad \mu_b = \frac{C_0}{b} + 2\text{GeV}$$

$Q = 10\text{GeV}$ (perturbation theory could work not very well)

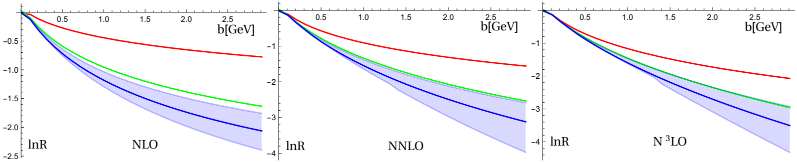


- Typical range of Fourier integration $b \in (0, 3)\text{GeV}^{-1}$
- The difference between $\ln R$ at $b = 1\text{GeV}^{-1}$ (1.74,1.39,1.23)
- The difference between R at $b = 1\text{GeV}^{-1}$ (1.09,1.08,1.06)
- Effect is almost negligible but non-zero(!)
- Improvement NLO→NNLO (~ 1.11) is (a bit) bigger then solution dependence
- Improvement NNLO→NNNLO (~ 1.04) is of the same order as solution dependence
- NP model for $\mathcal{D}$ could compensate the effect

Test of solution independence

$$(Q, Q^2) \to (\mu_b, \mu_b^2) \qquad \mu_b = \frac{C_0}{b} + 2\text{GeV}$$

$Q = M_Z$ (perturbation theory should work well)



- Typical range of Fourier integration $b \in (0, 1)\text{GeV}^{-1}$
- The difference between $\ln R$ at $b = 0.5\text{GeV}^{-1}$ (2.6,1.5,1.23)
- The difference between R at $b = 0.5\text{GeV}^{-1}$ (1.6,1.35,1.18)
- Effect is very sizable, $a_s \simeq 0.009$, b in perturbative region.
- Improvement NLO→NNLO (~ 1.22) is of the same order as solution dependence
- Improvement NNLO→NNNLO (~ 1.10) is smaller then solution dependence
- NP model for $\mathcal{D}$ could not compensate the effect, it is too large in PT region.

Effects of truncation of PT

Synopsis of the problem

- There is a solution dependence of TMD evolution
- It is almost negligible at smaller Q, but large at larger Q.
- It is not disappear (or disappear very slowly) with the increase of PT order.
- At 3-loop order it is the largest uncertainty that comes from perturbation theory

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The source of solution dependence is the violation of integrability condition.

In (truncated) perturbation theory

$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} \neq -\mu^2 \frac{d}{d\mu^2} \mathcal{D}(\mu,b) \qquad \Leftrightarrow \qquad \boldsymbol{\nabla} \times \mathbf{E} \neq 0 \qquad (3)$$

The evolution flow is *non-conservative*, the scalar potential is undetermined

The TMD evolution equation has not a unique solution.

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To measure perturbative uncertainties, we typically vary scales μ.

- In exact PT, μ-dependence is absent, but at finite PT there is the perturbative mismatch between the evolution exponent and the fixed order coefficient function.
- In TMD case there is an additional source of scale-dependence, solution dependence

A TMD cross-section

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2 b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f) \times \{R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i),\mu_0]\}^2 F_{f\leftarrow h}(x_1,b;\mu_i,\zeta_i) F_{f'\leftarrow h}(x_2,b;\mu_i,\zeta_i),$$

$$\mu_0 \to c_1\mu_0, \qquad \mu_f \to c_2\mu_f, \qquad \mu_i \to c_3\mu_i, \qquad \mu_{\text{OPE}} \to c_4\mu_{\text{OPE}}.$$

$$c_i \in (0.5, 2)$$

Some of these scales measure the solution dependence, some perturbative mismatch, some both.

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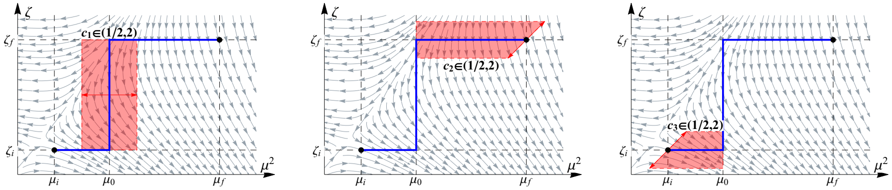

- c_1 measure only solution dependence
- c_2 measure mismatch between H and R + solution dependence
- c_3 measure mismatch between F and R + solution dependence
- c_4 measure mismatch between C and f

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Cross-section in the improved γ

In the improved γ there is no solution dependence

$$\frac{d\sigma}{dX} = \sigma_0 \sum_f \int \frac{d^2b}{4\pi} e^{i(b\cdot q_T)} H_{ff'}(Q,\mu_f)$$
$$\{R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)]\}^2 F_{f\leftarrow h}(x_1,b;\mu_i,\zeta_i) F_{f'\leftarrow h}(x_2,b;\mu_i,\zeta_i),$$

where

$$R^f[b;(\mu_f,\zeta_f)\to(\mu_i,\zeta_i)] = \exp\Big\{-\int_{\mu_i}^{\mu_f}\frac{d\mu}{\mu}\Big(2\mathcal{D}^f_{\text{NP}}(\mu,b)+\gamma^f_V(\mu)\Big)$$
$$+\mathcal{D}^f_{\text{NP}}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right) - \mathcal{D}^f_{\text{NP}}(\mu_i,b)\ln\left(\frac{\mu_i^2}{\zeta_i}\right)\Big\}.$$

There are 3 scales and no solution dependence

Cross-section in the ζ-prescription

$$\frac{d\sigma}{dX}=\sigma_0\sum_f\int\frac{d^2b}{4\pi}e^{i(b\cdot q_T)}H_{ff'}(Q,\mu_f)\{R^f[b;(\mu_f,\zeta_f)]\}^2F_{f\leftarrow h}(x_1,b)F_{f'\leftarrow h}(x_2,b),$$

where

$$R^f[b;(\mu_f,\zeta_f)] = \exp\left\{-\int_{\mu_{\text{saddle}}}^{\mu_f}\frac{d\mu}{\mu}\Big(2\mathcal{D}^f_{\text{NP}}(\mu,b)+\gamma_V^f(\mu)\Big)+\mathcal{D}^f_{\text{NP}}(\mu_f,b)\ln\left(\frac{\mu_f^2}{\zeta_f}\right)\right\}$$

WARNING: Special line boundary condition should be taken into account in the coefficient function (details in private)

However, we can exponentiate boundary conditions and get a simple **practical formula**

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ζ-prescription in PT

$$\text{TMD}(x, b; \mu_i, \zeta_i) = C(x, \mathbf{L}_\mu, \mathbf{L}_{\sqrt{\zeta}}, \mu_{\text{OPE}}) \otimes \text{PDF}(x, \mu_{\text{OPE}})$$

Practically, μ_i and μ_{OPE} are both set to single μ.
1-loop example

$$C(\mu, \zeta) = \delta(\bar{x}) + a_s C_F \Bigg[-2 \underbrace{\mathbf{L}_\mu p(x)}_{\substack{\text{never large} \\ \text{thanks to} \\ \text{charge} \\ \text{conservation}}} + 2\bar{x} + \delta(\bar{x}) \Big(\overbrace{-\mathbf{L}_\mu \mathbf{L}_{\sqrt{\zeta}} + 3\mathbf{L}_\mu}^{\text{usually large}} - \zeta_2 \Big) \Bigg]$$

ζ-prescription in PT

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We set $\zeta \to \zeta_\mu$:

$$\zeta_\mu = \frac{2\mu}{b} e^{-\gamma_E} \overbrace{e^{3/2 + a_s \ldots}}^{\text{PT-calculable}}$$

It has been used in [Scimemi,AV,1706.01473]

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