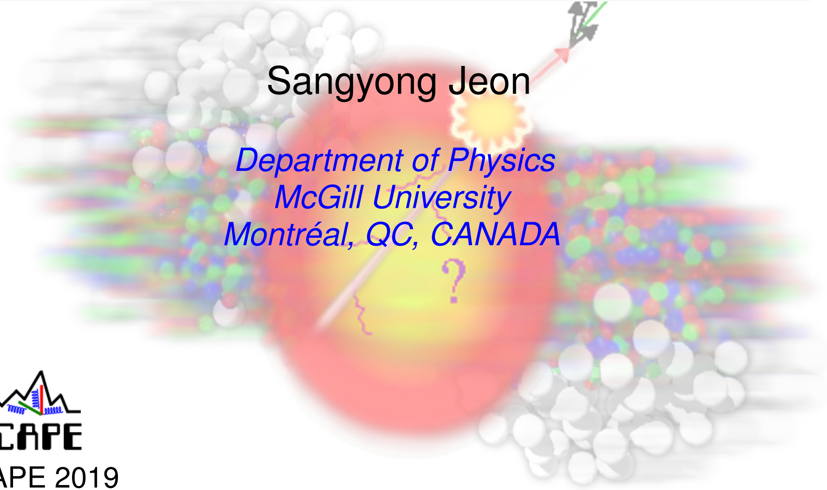
Theory Models in JETSCAPE



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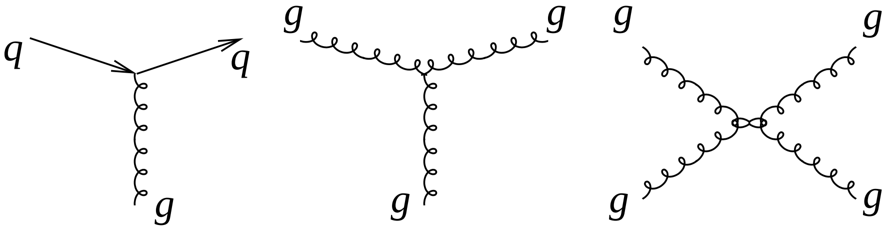
Topics

- Elastic energy loss
- MATTER
- LBT
- MARTINI
- Hybrid

Time's short. So I need to be brief.

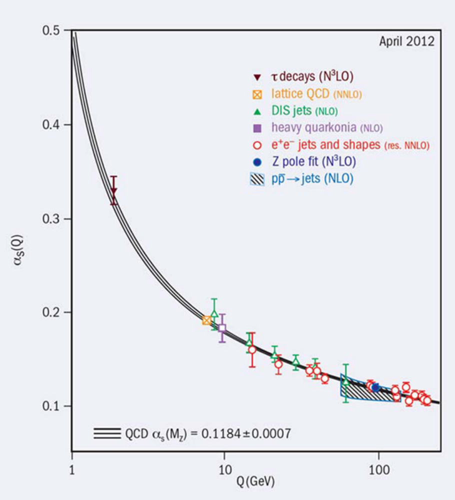
Perturbative QCD and Jets

QCD
– Interaction of quarks and gluons



- 3 colors for quarks and anti-quarks
- 8 gluons
- N_f flavors
- Perturbative when $g \ll 1$

Asymptotic Freedom



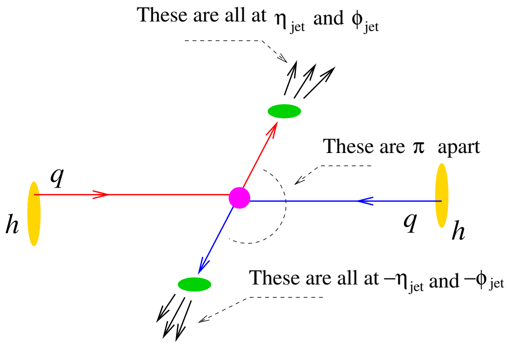
S. Bethke, arXiv:1210.0325.

- Perturbative expansion possible because of the asymptotic freedom
- $Q^2 \dfrac{\partial \alpha_S}{\partial Q^2} = -\beta_0 \alpha_S^2 - \beta_1 \alpha_S^3 + \cdots$
- $\alpha_S(Q^2) = \dfrac{g^2}{4\pi} \approx \dfrac{1}{((33 - 2n_f)/12\pi)\ln(Q^2/\Lambda_{\rm QCD}^2)}$
- pQCD reliable for $Q \gtrsim 1\,\text{GeV}$ $\Lambda_{\rm QCD} \approx 0.2\,\text{GeV} \approx 1/\text{fm}$
- Thermal pQCD may be reliable when $g/\pi \ll 1$ or $T \sim Q/(2\pi) \sim 100\,\text{GeV}$

Soft vs Hard collisions

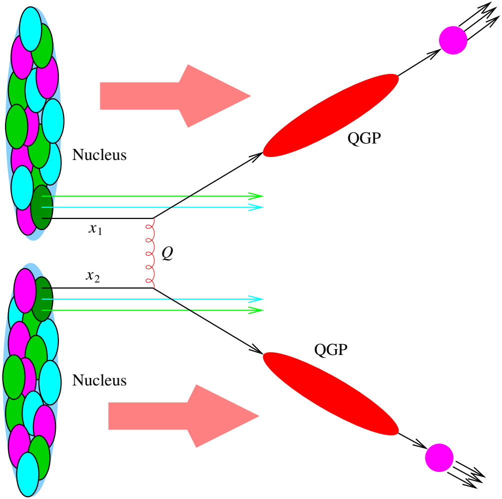
- Total cross-section: $\sigma_{NN} \sim a + b \ln \sqrt{s} + c(\ln \sqrt{s})^2$
- This is *nothing like* the pQCD cross-section: For instance,
  $$\frac{d\sigma_{ud\to ud}}{dt} = \frac{4\pi\alpha_S^2}{9}\left(\frac{s^2+u^2}{s^2 t^2}\right)$$
- *Soft gluons* with large coupling are responsible for the total cross-section
- Feynman's argument: Let the amplitude to emit small $x = 2p/\sqrt{s}$ gluon be $1/x^{1+\lambda}$. Then the soft-soft cross-section is
  $$\sigma \sim \left|\int \frac{dx_a}{x_a^{1+\lambda}}\right|^2 \left|\int \frac{dx_b}{x_b^{1+\lambda}}\right|^2 \sim (x_a x_b)^{-2\lambda} \sim s^{2\lambda}$$
  With $0 < \lambda \ll 1$ you get $\sigma \sim a + b \ln s$
- There must be a lot of soft gluons in a high energy hadron
- The pQCD cross-section applies only to very occasional *hard collisions* or *jets* $\Longrightarrow$ Scales are well separated

What is a jet?



- A jet is a phenomenon where a lot of final state energy is concentrated in a small angle around a common axis
- Origin: Hard collisions of partons $\Longrightarrow$ pQCD applies
- Usually dijet, sometimes triple-jet (Radiation of a hard gluon at a large angle)

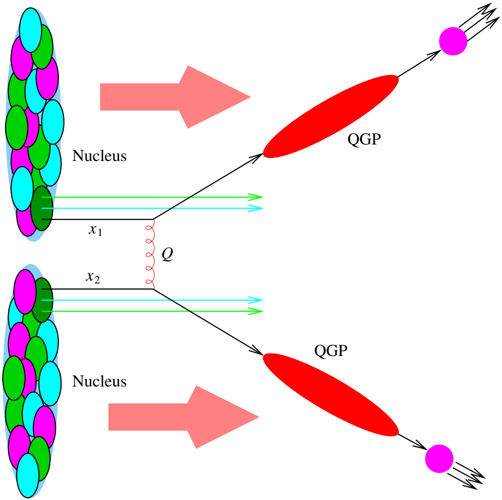
QCD in Heavy Ion Collisions



What we want to study:

- How does QGP modify jet property?

QCD in Heavy Ion Collisions



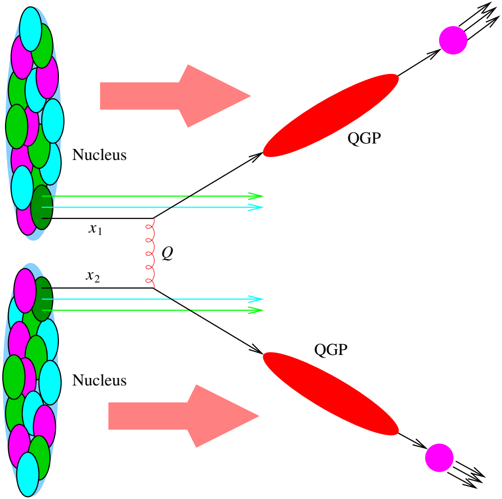
What we want to study:

- How does QGP modify jet property?

Complications:
How well do we know the *initial condition?*

- Nuclear initial condition?
- What happens to a jet between the production and the formation of (hydrodynamic) QGP?

QCD in Heavy Ion Collisions

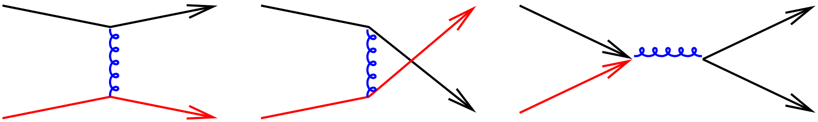


Schematically,

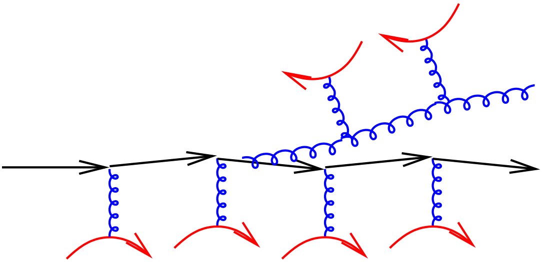
$$\frac{d\sigma_{AB}}{d\hat{t}} = \int_{\text{geometry}} \int_{abcd} \times f_{a/A}(x_a, Q_f) f_{b/B}(x_b, Q_f) \times \frac{d\sigma_{ab\to cd}(Q_R)}{d\hat{t}} \times \mathcal{P}(x_c \to x_c' | T, u^\mu) \times D(z_c', Q)$$

$\mathcal{P}(x_c \to x_c' | T, u^\mu)$: Medium modification of high energy parton property $\Longrightarrow$ Jet quenching via parton-QGP interactions

Relevant leading order processes for E-loss



Elastic scatterings with thermal particles



Collinear radiation

Why it is non-trivial

- Evolution of a many-body system
- A multi-scale problem $\Longrightarrow$ A big part of the system is *not* perturbative $\Longrightarrow$ Models based on LO-QCD may not be strictly valid (Recall: g has to be small)
- There are things we cannot calculate from first principles, such as hadronization, even in vacuum
- Fluctuations are as important as averages
- Physically motivated well calibrated Monte-Carlo models are essential

Why it is non-trivial

- A jet parton starts with a large virtuality $\Longrightarrow$ Different formalisms needed for different energy and virtuality regimes
  - MATTER: High E, high Q^2
  - MARTINI & LBT: High E, low Q^2
  - Hybrid: Low E, low Q^2
- Coherence matters – Requires resummation: HTL & LPM
- Finite size system
- Background is also evolving

What any MC evolution needs

1. Probabilities and rates
2. How to propagate in space and time
3. Prototype: Kinetic theory molecular dynamics

$$p^\mu \partial_\mu f(p_1) = \frac{1}{2}\int d\Gamma_{234} |\mathcal{M}_{12\leftrightarrow 34}|^2 (f_3 f_4 - f_1 f_2) + \frac{1}{2}\int d\Gamma_{23} |\mathcal{M}_{1\leftrightarrow 23}|^2 (f_2 f_3 - f_1) + \cdots$$

What we need

- Interaction rates and probabilities
- How to decide whether something should happen
- How to sample the differential rate
- How to propagate the partons

They are also what makes the models different.

Rates

Transition rate

- For $p_1 + p_2 \to k_1 + k_2$, the rate with which the incoming p_1 changes is

$$dR_{\rm el} = \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \frac{d^3 k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3 k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4 \delta^{(4)}(p_{\rm in} - k_{\rm out}) \left|\mathcal{M}_{\rm el}\right|^2}{2E_{p_1}}$$

- For the splitting process, $p_1 \to k_1 + k_2$,

$$dR_{\rm split} = \frac{d^3 k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3 k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4 \delta^{(4)}(p_{\rm in} - k_{\rm out}) \left|\mathcal{M}_{\rm split}\right|^2}{2E_{p_1}}$$

- In general

$$\text{Rate} = \text{Phase space volume}(s, t, E_{p_1}) \times |\mathcal{M}(s,t)|^2$$

Transition rate

- For $p_1 + p_2 \to k_1 + k_2$, the rate with which the incoming p_1 changes is

$$dR_{\rm el} = \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4 \delta^{(4)}(p_{\rm in} - k_{\rm out}) \left|\mathcal{M}_{\rm el}\right|^2}{2E_{p_1}}$$

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- Differential rate

$$\frac{dR}{dk} = \int dR_{\rm el}\, f_{\rm th}(p_2)\, \delta(k - k_1) + \int dR_{\rm split} \delta(k - k_1)$$

Transition rate

- For $p_1 + p_2 \to k_1 + k_2$, the rate with which the incoming p_1 changes is

$$dR_{\rm el} = \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4\delta^{(4)}(p_{\rm in} - k_{\rm out})\,|\mathcal{M}_{\rm el}|^2}{2E_{p_1}}$$

- For the splitting process, $p_1 \to k_1 + k_2$,

$$dR_{\rm split} = \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4\delta^{(4)}(p_{\rm in} - k_{\rm out})\,|\mathcal{M}_{\rm split}|^2}{2E_{p_1}}$$

- The total loss-rate (ignoring quantum statistics for simplicity)

$$\Gamma_L(p_1) = f_{\rm jet}(p_1)\left(\int dR_{\rm el}\, f_{\rm th}(p_2) + \int dR_{\rm split}\right)$$

Transition rate

- For $p_1 + p_2 \to k_1 + k_2$, the rate with which the incoming p_1 changes is

$$dR_{\rm el} = \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4\delta^{(4)}(p_{\rm in} - k_{\rm out})\,|\mathcal{M}_{\rm el}|^2}{2E_{p_1}}$$

- For the splitting process, $p_1 \to k_1 + k_2$,

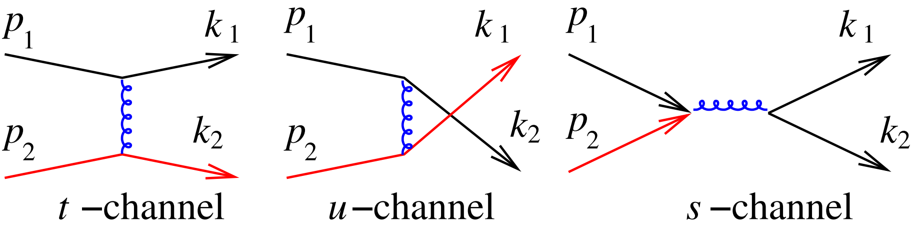
$$dR_{\rm split} = \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4\delta^{(4)}(p_{\rm in} - k_{\rm out})\,\left|\mathcal{M}_{\rm split}\right|^2}{2E_{p_1}}$$

- The total gain-rate

$$\Gamma_G(p_1) = \int dR_{\rm el}\, f_{\rm jet}(k_1) f_{\rm th}(k_2) + \int dR_{\rm split} f_{\rm jet}(k_1)$$

Elastic Energy Loss

Mandelstam variables



$\hat{t} = (p_1 - k_1)^2$ $\hat{u} = (p_1 - k_2)^2$ $\hat{s} = (p_1 + p_2)^2$

$$\hat{s} + \hat{t} + \hat{u} = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Elastic scattering rate

- For $p_1 + p_2 \to k_1 + k_2$, start with

$$dR_{\rm el} = \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{(2\pi)^4 \delta^{(4)}(p_{\rm in} - k_{\rm out}) \, |\mathcal{M}_{\rm el}|^2}{2E_{p_1}}$$

- The differential rate for an incoming particle p_1 to lose ω

$$\frac{dR_{\rm el}}{d\omega} = \int dR_{\rm el} f_{\rm th}(p_2) \delta(\omega - E_{p_1} + E_{k_1})$$

- The differential rate in $\hat{t}$

$$\frac{dR_{\rm el}}{d\hat{t}} = \int dR_{\rm el} f_{\rm th}(p_2) \delta\left(\hat{t} - (p_1 - k_1)^2\right)$$

Elastic scattering rate

- One can further show

$$\frac{dR_{\rm el}}{d\hat{t}} = \frac{\sqrt{\lambda(\sqrt{s}, m_1, m_2)}}{E_{p_1}} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} f_{\rm th}(p_2) \frac{d\sigma}{d\hat{t}}$$

- Leading order differential rate (in the fluid cell rest frame)

$$\frac{dR_{\rm el}}{d^2 q_\perp} = \frac{C_s g_s^2 T}{(2\pi)^2} \frac{m_D^2 F(q_\perp/T)}{q_\perp^2 (q_\perp^2 + m_D^2)}$$

  This is valid for any thermal $q_\perp^2 \sim -\hat{t}$ with Arnold and Xiao's correction factor $F(q_\perp/T)$ (varies from 1 to about 0.85)

- Simulation strategy
  1. Go to the fluid cell rest frame
  2. Sample $\mathbf{q}_\perp$ using the LO differential rate
  3. Sample $f_{\rm th}(p_2)$ so that $k_2 = k_1 - q$ is also on-shell
  4. Go back to the original frame

Radiational Energy Loss

Two Evolutions

- Evolution in the virtuality Q^2
  - This proceeds by successive shedding of excess Q^2 until low enough virtuality is achieved
- Evolution in time
  - This mainly concerns medium-induced radiation off of an *on-shell* particle
- Trouble:
  - They cannot be so cleanly separated.
  - Evolution in Q^2 is not exactly (although related) evolution in t.
- Need two radiation rates. One for radiation per unit time and another for radiation per unit virtuality.
  At the end, one needs to translate the latter to the former.

Evolution in virtuality $\hat{t}$

- Regularized DGLAP equation with the vacuum splitting function

$$\hat{t}\frac{\partial}{\partial \hat{t}} f(x, \hat{t}) = \int_x^1 dz \frac{\alpha_s(\hat{t})}{2\pi} P_v(z) \left(\frac{f(x/z, \hat{t})}{z} - f(x, \hat{t}) \right)$$

(i) (Gain − Loss) form. (ii) x independent splitting function.
$\Longrightarrow$ Exact solution possible.

- Let $\xi = \ln \hat{t}$. The Poisson solution is

$$\tilde{f}(x, \xi) = \Delta(\xi) \left[1 + \sum_{n=1} \frac{1}{n!} \prod_{j=1}^{n} \int_{\xi_0}^{\xi} d\xi_j \int_0^1 dz_j \frac{\alpha_s(\xi_j)}{2\pi} P_v(z_j) \delta(x - \prod_{k=1}^{n} z_k) \right]$$

with the Sudakov factor representing no-interaction prob.

$$\Delta(\xi) = \exp\left(-\int_{\xi_0}^{\xi} d\xi' \int dz' \frac{\alpha_s(\xi')}{2\pi} P_v(z') \right)$$

MATTER

- Basic idea: Medium-modified DGLAP

$$\tilde{t}\frac{\partial}{\partial \tilde{t}} f(x,\tilde{t}) = \int_x^1 dz \frac{\alpha_s(\tilde{t})}{2\pi} P_{v+m}(z,\tilde{t},\mathbf{r}) \left(\frac{f(x/z,\tilde{t})}{z} - f(x,\tilde{t}) \right)$$

with

$$P_{v+m}(z,\tilde{t},\mathbf{r}) = P_v(z) \left(1 + \frac{4}{z(1-z)\tilde{t}} \int_0^{\zeta^+_{\rm max}} d\zeta^+ \hat{q}(\mathbf{r}+\hat{\mathbf{n}}\zeta^+) \sin^2\left(\frac{\zeta^+}{2\tau_f^+}\right)\right)$$

and $\tau_f^+ = 2p^+/\tilde{t}$, $\hat{\mathbf{n}} = \mathbf{p}/|\mathbf{p}|$

- Medium-modified Sudakov factor

$$\Delta_m(\xi,\tilde{t},\mathbf{r}) = \exp\left(-\int_{\xi_0}^{\xi} d\xi' \int dz' \frac{\alpha_s(\xi')}{2\pi} P_{v+m}(z',\tilde{t},\mathbf{r}) \right)$$

Connecting $\hat{t}$ evolution with t evolution

- Uncertainty in time: $\tau_f^+ = 2p^+/\hat{t}$
- Given $\hat{t}$, the splitting time is sampled from a Gaussian

$$\rho(\zeta^+) = \frac{2}{\tau_f^+ \pi} \exp\left(- \left(\frac{\zeta^+}{\tau_f^+ \sqrt{\pi}} \right)^2 \right)$$

- That is, the parton propagates by ζ^+ and radiates.
- The parton is then ready to radiate again.

Evolution in time

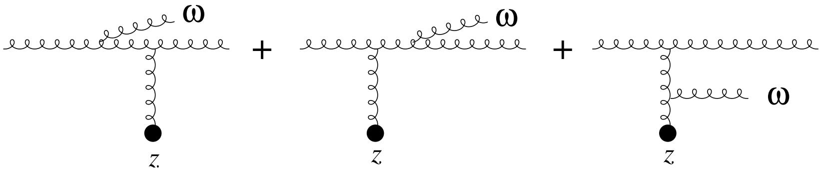
- The rate equation

$$\frac{dP_a(p)}{dt} = \int dk \frac{dR_{b,a}(p+k,k)}{dk} P_b(p+k) - \int dk \frac{dR_{b,a}(p,k)}{dk} P_a(p)$$

- In general, the rates depend on both the mother's momentum and the daughter's momentum $\Longrightarrow$ Poisson-like solution not available
- Use tabulated rates $dR_{b,a}(p,k)/dk$ in p/T and k/T
- This is for nearly on-shell partons
- Main difference:
  - MARTINI: AMY rates
  - LBT: Higher Twist rates

Radiation rate calculation

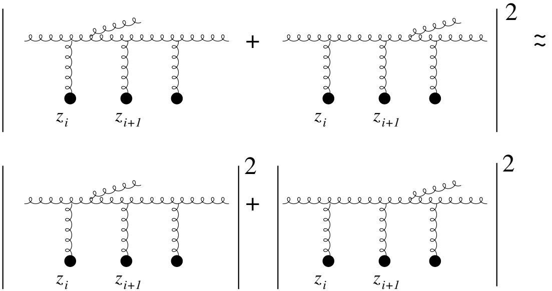
- A single scatterer radiation amplitude



- In the small ω limit, the radiation probability is (Bethe-Heitler)

$$\frac{dP_1}{d\omega} \sim \frac{\alpha_s N_c}{\pi \omega}$$

Multiple scatterers – Incoherent emission



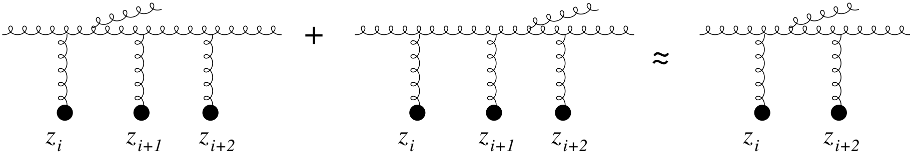
- $|\sum_n T_n|^2 \approx \sum |T_n|^2$
- Interference terms $T_n^* T_m$ with $n \neq m$ negligible. (Large phase change between scatterings)
- Average number of emissions scales like the number of scatterers:

$$\mathcal{P}_{\rm N_{sc}} \approx N_{\rm sc}\mathcal{P}_1$$

- In a unit length, there are $N_{\rm sc} = \dfrac{1}{\ell_{\rm mfp}}$ number of scatterers.

Coherent emission

- If there is a destructive interference,



- Average number of emissions scales like

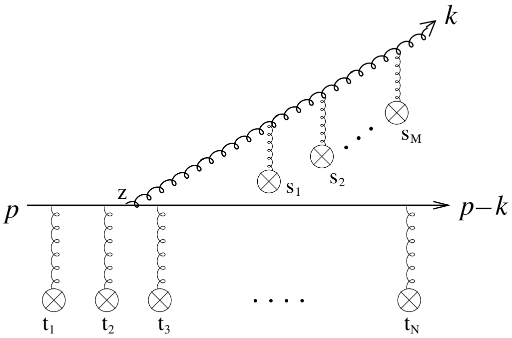
$$\mathcal{P}_{N_{\rm sc}} \approx \frac{N_{\rm sc}}{N_{\rm coh}} \mathcal{P}_1$$

where $N_{\rm coh}$ is the number of scattering centers that destructively interfere. (Small phase change between scatterings)

- The medium's power to induce radiation is *reduced*.
$\Longrightarrow$ Landau-Pomeranchuck-Migdal (LPM) effect
- Define the coherence length

$$\ell_{\rm coh} = \ell_{\rm mfp} N_{\rm coh}$$

All scatterings contribute



- If the radiation is strictly collinear, the parent parton and the offspring will never separate.
- In reality: p_T kicks from the medium separates them within $\ell_{\rm coh} \approx \omega_k / \langle k_\perp^2 \rangle$
- Main task: To sum over all such diagrams and then square. This gets you the radiation rate.

LBT Radiation rate

- Leading order without the infinite sum

$$\frac{dN}{dx dk_\perp^2 dt} = \frac{2\alpha_s C_A P(x)}{\pi \mathbf{k}_\perp^4} \hat{q} \sin^2\left(\frac{t - t_i}{2\tau_f}\right)$$

- The $\sin^2\left(\frac{t-t_i}{2\tau_f}\right)$ factor is the finite time correction.
- Formation time $\tau_f = \dfrac{2E_p x(1-x)}{k_\perp^2}$
- Medium information through $\hat{q} = \langle k_\perp^2 \rangle / \ell_{\rm mfp}$

McGill-AMY

- SD-Eq to resum all diagrams $\Longrightarrow$ Full leading order in g

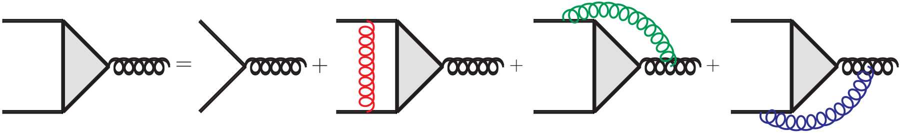


Figure from G. Qin

- Integral Eq:

$$2\mathbf{h} = i\delta E(\mathbf{h},p,k)\mathbf{F}(\mathbf{h}) + g^2 \int \frac{d^2 q_\perp}{(2\pi)^2} C(q_\perp) \Big\{ (C_s - C_a/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}-k\mathbf{q}_\perp)]$$

$$+ (C_a/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}+p\mathbf{q}_\perp)] + (C_a/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}-(p-k)\mathbf{q}_\perp)] \Big\}$$

$$\delta E(\mathbf{h},p,k) = \frac{\mathbf{h}^2}{2pk(p-k)} + \frac{m_k^2}{2k} + \frac{m_{p-k}^2}{2(p-k)} - \frac{m_p^2}{2p}, \quad C(q_\perp) = \frac{m_D^2}{q_\perp^2 (q_\perp^2 + m_D^2)}$$

For $g \to q\bar{q}$, $(C_s - C_a/2)$ term is the one with $\mathbf{F}(\mathbf{h} - p\mathbf{q}_\perp)$ rather tan $\mathbf{F}(\mathbf{h} - k\mathbf{q}_\perp)$

AMY Rates

Rate for $p > T, k > T$ (valid for $p \gg T$ and $k \gg T$ as well)

$$\frac{dN_g(p,k)}{dkdt} = \frac{g_s^2}{16\pi p^7} \frac{1}{1 \pm e^{-k/T}} \frac{1}{1 \pm e^{-(p-k)/T}} \times$$
$$\times \left\{ \begin{array}{cc} C_f \frac{1+(1-x)^2}{x^3(1-x)^2} & q \to qg \\ 2N_f T_f \frac{x^2+(1-x)^2}{x^2(1-x)^2} & g \to q\bar{q} \\ C_a \frac{1+x^4+(1-x)^4}{x^3(1-x)^3} & g \to gg \end{array} \right\} \times \int \frac{d^2\mathbf{h}}{(2\pi)^2} 2\mathbf{h} \cdot \mathrm{Re}\, \mathbf{F}(\mathbf{h}, p, k) \,,$$

These are tabulated in terms of $k/T, p/T$.

The Hybrid Model

The Hybrid Model

- Energy loss rate by AdS/CFT strong coupling calculations

$$\frac{1}{E_{in}}\frac{dE}{dx} = -\frac{4}{\pi}\frac{x^2}{x_{\rm stop}^2}\frac{1}{\sqrt{x_{\rm stop}^2 - x^2}}$$

with $x_{\rm stop} = \frac{1}{\kappa_{sc}}\frac{E_{\rm in}^{1/3}}{T^{4/3}}$

- Parameter κ_{sc} controls the "strength" of interaction
- Follow the branching history of a jet from perturbative (PYTHIA) calculations
- Similar issue and solution as in MATTER: Use the formation time as the time between branchings

Simulation Procedures

Time evolution procedures

At a given global time t,

1. Pick a parton from the population $f_{\rm jet}(p)$
2. Go to the fluid rest frame
3. Get the partial transition rates $R_\alpha(p)$ for the given particle with momentum p.
   Get total transition rate $R_{\rm tot} = \sum_\alpha R_\alpha$ and the total interaction probability $P_{\rm tot} = R_{\rm tot}\Delta t$
4. Decide whether to do an interaction
5. If yes, decide on a specific process according to the partial rates
6. Sample dR_α and carry out the scattering/branching including the recoil
7. Go back to the original frame and propagate all hard partons by Δt
8. Repeat for other partons in $f_{\rm jet}(p)$

MATTER – Procedure 1

1. Start with a hard parton at **r** and p^μ
2. Calculate the Sudakov factor for each channel i with the medium modified splitting function

$$\Delta_i(\hat{t}_{\max}, \hat{t}) = \exp\left(- \int_{\hat{t}}^{\hat{t}_{\max}} \frac{d\hat{t}'}{\hat{t}'} \frac{\alpha_s(\hat{t}')}{2\pi} \int_{z_c}^{1-z_c} dy\, P_i(y, \hat{t}') \right)$$

   with $z_c = \hat{t}_{\min}/\hat{t}'$

3. Calculate the no-splitting probability as

$$\Delta(\hat{t}_{\max}, \hat{t}) = \prod_i \Delta_i(\hat{t}_{\max}, \hat{t})$$

4. If splitting, then sample $\hat{t}$ from $P(t) = \Delta(\hat{t}_{\max}, \hat{t}_{\min})/\Delta(\hat{t}_{\max}, \hat{t})$
5. Determine which channel by the branching ratio

$$\mathrm{BR}_i(\hat{t}) = \int_{\hat{t}_{\min}/\hat{t}}^{1-\hat{t}_{\min}/\hat{t}} dy\, P_i(y, \hat{t})$$

MATTER – Procedure 2

1. Once the channel is determined, sample $P_i(y, \hat{t})$ to get y
2. Set the maximum virtuality to $\hat{t}_1^{\max} = y^2\hat{t}$ and $\hat{t}_2^{\max} = (1-y)^2\hat{t}$ for the daughters and sample $\hat{t}_1$ and $\hat{t}_2$.
3. Get

$$\mathbf{k}_\perp^2 = y(1-y)\hat{t} - (1-y)\hat{t}_1 - y\hat{t}_2$$

4. The splitting time is sampled from

$$\rho(\zeta^+) = \frac{2}{\tau_f^+ \pi} \exp\left(-\left(\frac{\zeta^+}{\tau_f^+\sqrt{\pi}}\right)^2\right)$$

with $\tau_f^+ = 2p^+/\hat{t}$

Hybrid Model procedures

- Start from a full PYTHIA jet shower structure
- Assign the time $\tau = 2\frac{E}{Q^2}$ between branchings
- Apply the AdS/CFT energy loss rate in between branchings

Summary

- Realistic jet simulations need to deal with different energy and virtuality regimes
- In most simulations, elastic collisions are treated more or less the same
- Treatment for radiations differ greatly in different simulations
  $\Longrightarrow$ *Unified treatment in JETSCAPE*
- To be added: JEWEL, ASW, ...

Backups

General MC procedure

1. Start with a parton with momentum $\mathbf{p}$ and position $\mathbf{x}$. Change to the local rest frame of the medium. You may also need to rotate to the frame where $\mathbf{p} = p\mathbf{e}_z$.
2. Calculate $\Delta P = R\Delta t$ where R is the total rate of something happening to $\mathbf{p}$ during next Δt.
   Alternatively, calculate P_0 which is the probability for nothing happening to $\mathbf{p}$ during Δt.
3. Decide whether anything should happen to $\mathbf{p}$.
4. If yes, then decide what should happen to $\mathbf{p}$ using the partial rates R_α for all possible processes.
5. Decide the outcome of that process using the differential rate dR_α/d^3k where $\mathbf{k}$ is the momentum of the daughter parton
6. Decide at what point in time during Δt the interaction happens.
7. Propagate all partons by Δt. You may need to change frame for this step.
8. Repeat.

The rate equation

- This is what the MC procedure tries to solve:

$$\frac{dP_a(p)}{dt} = \int dk\, P_b(p+k)\frac{dR_{b\to a}(p+k,k)}{dk} - P_a(p)\int dk\, \frac{dR_{a\to b}(p,k)}{dk}$$

- The rate $\frac{dR_{a\to b}(p,k)}{dk}$ is the rate for a particle of species a with momentum p to become a particle of species b with momentum k.
- For instance

$$\begin{aligned}\frac{dR_{q\to q}(p+k,k)}{dk} &= (q+g\to q+g)+(q+q\to q+q)\\ &\quad +(q+\bar{q}\to q+\bar{q})+(g+g\to q+\bar{q})\\ &\quad +(q\to q+g)+(q+g\to q)\end{aligned}$$

Simplest solution of the rate equation

Consider the case where the transition rate is independent of the incoming momentum and we have just a single process

$$\frac{dP(p,t)}{dt} = \int dk\, P(p+k,t)\frac{dR(k,t)}{dk} - P(p,t)\int dk\, \frac{dR(k,t)}{dk}$$

The solution with $P(p,0) = 1$ is

$$P(p,t) = e^{-\int_0^t dt' \int dk\, \frac{dR(k,t')}{dk}} \times \left[1 + \sum_{n=1} \frac{1}{n!}\left(\prod_{j=1}^{n}\int_0^t dt'_j \int dk_j\, \frac{dR(k_j,t'_j)}{dk_j}\right)\delta\left(k - \sum_{j=1}^{n} k_j\right)\right]$$

Note that the *no interaction* probability is

$$P_0(p,t) = \exp\left(-\int_0^t dt' \int dk\, \frac{dR(k,t')}{dk}\right)$$

Semi-classical argument for coherence

- Semi-classical photon radiation

$$E_k \frac{dN}{d^3k} = \frac{\alpha_{\rm EM}}{4\pi^2} \langle \tilde{\mathbf{J}}^\dagger(\omega_k, \mathbf{k}) \cdot \tilde{\mathbf{J}}(\omega_k, \mathbf{k}) \rangle$$

where the transverse part of the current

$$\tilde{\mathbf{J}}(\omega, \mathbf{k}) = \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{J}(\omega, \mathbf{k}))$$

- Current: A charged particle kicked by the medium at t_i

$$\mathbf{J}(t, \mathbf{x}) = \sum_{i=1}^{N} \mathbf{v}_{i-1} \delta^{(3)}(\mathbf{x} - \mathbf{x}_{i-1} - \mathbf{v}_{i-1}(t - t_i))\theta(t_{i-1} < t < t_i)$$

or

$$\mathbf{J}(\omega_k, \mathbf{k}) = \sum_{i=1}^{N} \mathbf{v}_{i-1} \left(\frac{e^{i\omega_k t_i - i\mathbf{k}\cdot\mathbf{x}_i} - e^{i\omega_k t_{i-1} - i\mathbf{k}\cdot\mathbf{x}_{i-1}}}{i(\omega_k - \mathbf{k} \cdot \mathbf{v}_{i-1})} \right)$$

Semi-classical argument for coherence

Suppose $N = 1$ with $t_0 = 0, \mathbf{x}_0 = 0, \mathbf{x}_1 = \mathbf{v}_0 t_1$

- Current

$$\tilde{\mathbf{J}}(\omega_k, \mathbf{k}) = (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}_0)) \left(\frac{e^{it_1(\omega_k - i\mathbf{k}\cdot\mathbf{v}_0)} - 1}{i(\omega_k - \mathbf{k}\cdot\mathbf{v}_0)} \right)$$

- Can show $\omega_k - \mathbf{k}\cdot\mathbf{v}_0 \approx \dfrac{\mathbf{k}_\perp^2}{2\omega_k}$
  and $(\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}_0)) \approx \left(\dfrac{\mathbf{k}_\perp}{\omega_k}\right)$

- Number of photons produced

$$\left|\tilde{\mathbf{J}}(\omega_k, \mathbf{k})\right|^2 \propto \frac{\sin^2(\Delta E t_1/2)}{\mathbf{k}_\perp^2}$$

  with $\Delta E = \dfrac{\mathbf{k}_\perp^2}{2\omega_k}$. Basically the same factor appears in the medium modified splitting function

Semi-classical argument for coherence

- If the kicks are soft, then this can be re-expressed as [BDMPS 9604327]

$$\omega_k \frac{dR}{d\omega_k} = \frac{\alpha_{\rm EM}}{\pi} \int \frac{d^2 k_T}{\omega_k^2} \left\langle 2 \sum_i \sum_{j>i} \mathbf{A}_i \cdot \mathbf{A}_j \left(e^{i(\Phi_j - \Phi_i)} - 1 \right) + \left(\sum_i \mathbf{A}_i \right)^2 \right\rangle$$

where

$$\mathbf{A}_i = \frac{\mathbf{u}_i}{\mathbf{u}_i^2} - \frac{\mathbf{u}_{i-1}}{\mathbf{u}_{i-1}^2}$$

with the relative transverse velocity

$$\mathbf{u}_i = \left(\frac{\mathbf{k}_T}{\omega_k} - \mathbf{v}_{i,T} \right)$$

The longitudinal direction is the direction of $\mathbf{v}_0$ and

$$\Phi_j - \Phi_i = \frac{\omega_k}{2} \sum_{l=i+1}^{j} \mathbf{u}_{l-1}^2 \Delta t_l$$

is the phase accumulated between t_i and t_j

Semi-classical argument for coherence

Photon spectrum:

$$\omega_k \frac{dR}{d\omega_k} = \frac{\alpha_{\rm EM}}{\pi} \int \frac{d^2k_T}{\omega_k^2} \left\langle 2\sum_i \sum_{j>i} \mathbf{A}_i \cdot \mathbf{A}_j \left(e^{i(\Phi_j - \Phi_i)} - 1 \right) + \left(\sum_i \mathbf{A}_i \right)^2 \right\rangle$$

- Incoherent limit: $|\Phi_j - \Phi_i| \gg 1 \implies$ The bracket becomes

$$\sum_j |\mathbf{A}_j|^2$$

- Coherent limit: $|\Phi_j - \Phi_i| \ll 1 \implies$ The bracket becomes

$$\left(\sum_i \mathbf{A}_i \right)^2 = \left(\frac{\mathbf{u}_N}{\mathbf{u}_N^2} - \frac{\mathbf{u}_0}{\mathbf{u}_0^2} \right)^2$$

Effective Emission rate

- Incoherent Emission rate:

$$\frac{d\mathcal{P}}{dt} \approx \frac{C}{\ell_{\rm mfp}} \mathcal{P}_1$$

- Coherent Emission rate:

$$\frac{d\mathcal{P}}{dt} \approx \frac{C}{\ell_{\rm coh}} \mathcal{P}_1$$

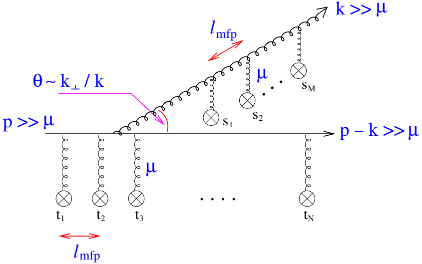
- $\mathcal{P}_1$: Bethe-Heitler (BH, Single emission off of one scatterer)

$$\mathcal{P}_1 \sim \left. \frac{dN_g}{d\omega} \right|_{BH} \approx \frac{\alpha_S N_c}{\pi\omega}$$

for small ω

Coherent scattering can be important

Following BDMPS



- What we need to calculate R_{AA}:

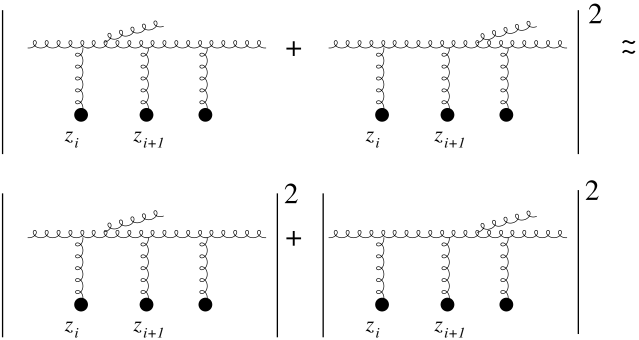
  Differential gluon radiation rate $\omega \dfrac{dN_g}{d\omega dt}$

  Medium dependence comes through the scattering time (length) scale

$$\omega \frac{dN_g}{d\omega dt} \approx \frac{\omega}{\ell_{sc}} \left. \frac{dN_g}{d\omega} \right|_{\mathrm{BH}}$$

Length Scales

Following BDMPS

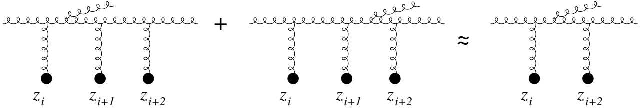


- If all scatterings are incoherent ($\ell_{\rm mfp} > \ell_{\rm coh}$),

$$\ell_{sc} = \ell_{\rm mfp} \approx \tau_{\rm mft}$$

Length Scales

Following BDMPS



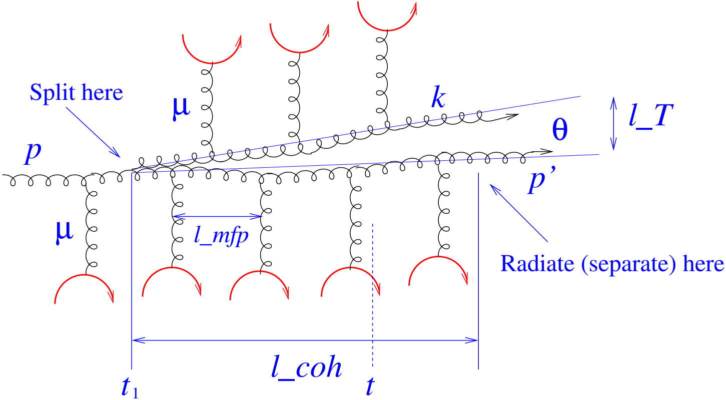
- If $\ell_{\rm coh} \geq \ell_{\rm mfp} \implies$ LPM effect:
  All scatterings within $\ell_{\rm coh}$ effectively count as a single scattering.
- $\ell_{sc} = \ell_{\rm coh}$

Estimation of ℓ_{mfp}

- Elastic cross-section (Coulombic) $\dfrac{d\sigma}{d\hat{t}} \approx C_R \dfrac{2\pi\alpha_s^2}{\hat{t}^2}$
- With thermal $f_{\mathrm{scatt}}(x,k)$, this yields

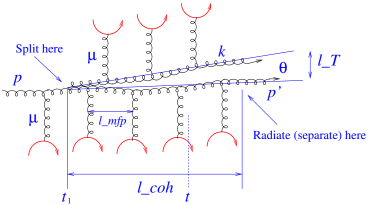
$$\frac{1}{\tau_{\mathrm{mft}}} \approx \int \frac{d^3k}{(2\pi)^3} f_{\mathrm{scatt}}(x,k)(1-\cos\theta_{pk}) \int d\hat{t}\, C_R \frac{2\pi\alpha_s^2}{\hat{t}^2} \sim \alpha_s T$$

Estimation of ℓ_{coh}



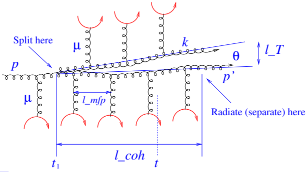
- *E*: Original parton energy
- ω: Energy of the radiated gluon
- μ: Typical transverse momentum transfer
- $E \gg \omega \gg \mu$

Estimation of $\ell_{\rm coh}$



- The radiated gluon random walks away from the original parton. Original parton's trajectory is less affected since $\omega \ll E$
- From the geometry $\theta \approx \frac{k_T^g}{\omega}$ and $\theta \approx \frac{\ell_T}{\ell_{\rm coh}}$
- Separation condition: ℓ_T is longer than the transverse size of the radiated gluon: $\ell_T \approx 1/k_T^g$
- Putting together,

$$\ell_{\rm coh} \approx \frac{\omega}{(k_T^g)^2}$$



- We have: $\ell_{\rm coh} \approx \dfrac{\omega}{(k_T^g)^2}$
- After suffering $N_{\rm coh}$ collisions (random walk),

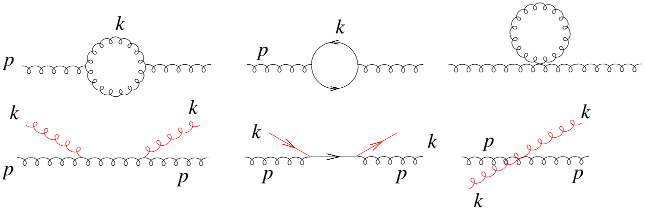
$$\langle (k_T^g)^2 \rangle = N_{\rm coh}\mu^2 = \frac{\ell_{\rm coh}}{\ell_{\rm mfp}}\mu^2 = \ell_{\rm coh}\left(\frac{\mu^2}{\ell_{\rm mfp}}\right) = \ell_{\rm coh}\,\hat{q}$$

- $\hat{q}$: Transport coefficient. Momentum transfer squared per elastic collision $\Longrightarrow$ QGP property
- $\ell_{\rm coh} \approx \dfrac{\omega}{(k_T^g)^2}$ becomes, with $\hat{q} = \mu^2/\ell_{\rm mfp}$ and $E_{\rm LPM} = \mu^2 \ell_{\rm mfp}$,

$$\ell_{\rm coh} \approx \ell_{\rm mfp}\sqrt{\frac{\omega}{E_{\rm LPM}}} = \sqrt{\frac{\omega}{\hat{q}}}$$

Estimation of μ^2

- Transverse scale set by the Debye mass in $G(q_\perp) = 1/(q_\perp^2 + m_D^2)$



- Second row: Physical forward scattering with particles in the medium
- The last term is easiest to calculate:

$$m_D^2 \propto g^2 \int \frac{d^3k}{E_k} f(k) \propto g^2 T^2$$

- Effectively, this adds $m_D^2 A_0^2$ to the Lagrangian $\Longrightarrow$ NOT gauge invariant $\Longrightarrow$ Gauge invariant formulation: Hard Thermal Loops

Length scales

Coherence length: $\dfrac{\ell_{\rm coh}}{\ell_{\rm mfp}} \approx \sqrt{\dfrac{\omega}{E_{\rm LPM}}}$

Key quantity: $E_{\rm LPM} = \mu^2 \ell_{\rm mfp} \sim T$ in pert. thermal QCD.

- L: The size of the medium
- $\mu^2 \sim m_D^2 \sim \alpha_s T^2$
- $\ell_{\rm mfp} \sim 1/(\alpha_s T)$: The mean free path for elastic collisions
- $\ell_{\rm coh} \sim \ell_{\rm mfp}\sqrt{\dfrac{\omega}{T}}$
- $\ell_{\rm coh} > \ell_{\rm mfp}$ when $\omega > T$
- $\ell_{\rm coh} > L$ when $\omega > E_L$ with $E_L = \alpha_s^2 T^3 L^2 = T(L/\ell_{\rm mfp})^2$

DGLAP and Sudakov

- This function satisfies the DGLAP equation

$$
\begin{aligned}
\frac{\partial}{\partial\xi}\tilde{f}(x,\xi) &= -\int dz \frac{\alpha_s(\xi)}{2\pi} P(z)\tilde{f}(x,\xi) \\
&\quad + \int dz \frac{\alpha_s(\xi)}{2\pi} P(z) \\
&\quad \sum_{n=1} \frac{\Delta(\xi)}{(n-1)!} \prod_{j=1}^{n-1} \int_{\xi_0}^{\xi} d\xi_j \int dz_j \frac{\alpha_s(\xi_j)}{2\pi} P(z_j)\delta(x - z\prod_{k=1}^{n-1} z_k \\
&= -\int dz \frac{\alpha_s(\xi)}{2\pi} P(z)\tilde{f}(x,\xi) + \int dz \frac{\alpha_s(\xi)}{2\pi} \frac{P(z)}{z} \tilde{f}(x/z,\xi)
\end{aligned}
$$

with the understanding that $\tilde{f}(x,\xi) = 0$ when $x > 1$.

- Interpretation: $P_0(x,\xi) = \Delta(\xi)$ is the probability for no branching to happen.

Sampling an arbitrary distribution

- We need to sample

$$\rho(x) = \frac{f(x)}{\int_a^b dy f(y)}$$

  which is normalized $\int_a^b dx \rho(x) = 1$

- Let $r = \int_a^x dy\, \rho(y)$

  Then $dr = \rho(x)dx$ and $\int_a^b dr = 1$

  That is, the variable r is uniformly distributed and $0 \le r \le 1$.

- Sample r and solve $r = \int_a^x dy \rho(y)$ for x. If the inverse function r^{-1} is known, use that. If not, solve it numerically.