# Theoretical foundations for calculating PDFs using quasi-parton operators


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Based on YQM, Qiu, 1404.6860, 1709.03018
Ishikawa, YQM, Qiu, Yoshida, 1609.02018, 1707.03107
Li, YQM, Qiu, 1809.01836
and works in preparation

**CFNS Workshop on Lattice Parton Distribution Functions**
**BNL, USA, Apr. 17-19, 2019**

# Outline

**I. Introduction to quasi-parton operators**

**II. Factorization at all orders**

**III. Multiplicatively renormalizability at all orders**

**IV. Nonperturbative renormalization and perturbative matching (new)**

# Difficulty of PDFs calculation

- LQCD: the main nonperturbative approach to solve QCD

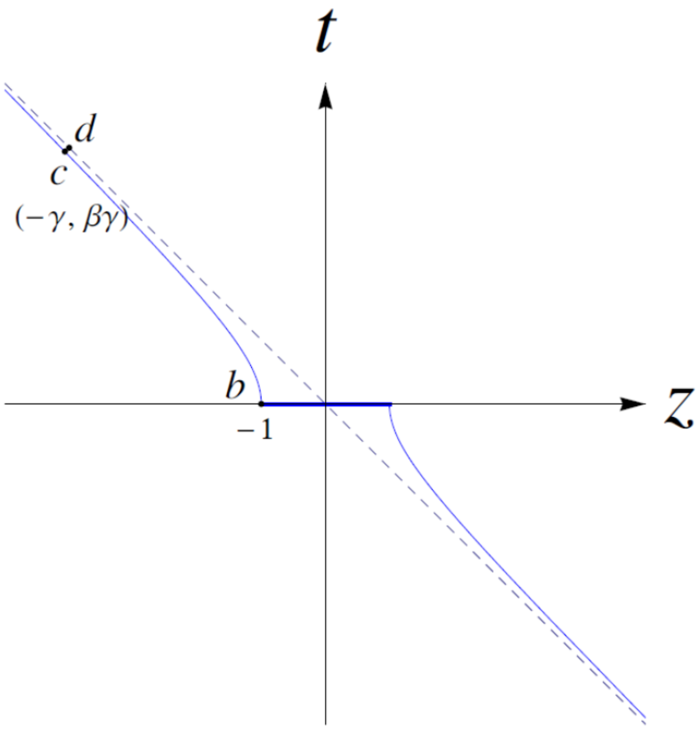
- Intrinsically Euclidean time: $\tau = i\, t$

- PDFs are on lightcone, have time dependence, lattice QCD cannot calculate directly

# Off light cone and equal time

- **If a quark bilinear is slightly off light cone**

Exist a frame where quark bilinear is equal time, but proton is moving fast




Chen, Cohen, Ji, Lin, Zhang, 1603.06664

# Quasi-PDFs: a classical picture

## ➢ Quasi-PDFs
Ji, 1305.1539, 1404.6680

$$\tilde{f}_{q/p}(x,\mu^2,P_z) = \int \frac{d\xi_z}{4\pi} e^{-ix\xi_z P_z} \langle P|\overline{\psi}(\xi_z)\,\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle$$

- **Classical picture:** When proton is moving fast enough, i.e. $P_z \to \infty$, there exists a frame so that quark bilinear in quasi-PDFs are only slightly off light cone, approach PDFs

## ➢ Pseudo-PDFs
Orginos, Radyushkin, Karpie, Zafeiropoulos, 1706.05373

- Similar to quasi-PDFs but integrated out $P_z$ with fixed $\xi_z$

# QPDFs in QFT

- **A good idea, clear picture**
  - Fields separated along the z-direction, no time dependence, calculable using standard lattice method
- **Things are much complicated in QFT, classical picture needs further justification**
  - Non-analyticity between on light cone and off light cone
  - Quantum fluctuation makes it impossible to take $P_z \to \infty$ first
  - Complicated UV divergences, mix with finite contributions
  - Is there a relation between qPDFs and PDFs (factorization)?

# Quasi-parton operators

## ➢ (Bare) quasi-parton operators

$$\mathcal{O}_{bq}^{\nu}(\xi) = \overline{\psi}_q(\xi)\,\gamma^{\nu}\,\Phi^{(f)}(\xi,0)\,\psi_q(0)$$

$$\mathcal{O}_{bg}^{\mu\nu\rho\sigma}(\xi) = F^{\mu\nu}(\xi)\,\Phi^{(a)}(\{\xi,0\})\,F^{\rho\sigma}(0)$$

- Path ordered gauge links

$$\Phi^{(a,f)}(\xi,0) = \mathcal{P}e^{-ig\int_0^1 \xi\cdot A^{(a,f)}(\lambda\xi)\,d\lambda}$$

## ➢ One type of "lattice cross sections"

See J. W. Qiu's talk

$$\sigma_n(\xi^2,\omega,P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle$$

- Quasi-PDFs can be obtained by F.T. $\xi_z$ with fixed $P_z$
- Pseudo-PDFs can be obtained by F.T. $P_z$ with fixed $\xi_z$

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# All-order proof for factorization

See also J. W. Qiu's talk

## ➢ Diagrammatic method
YQM, Qiu, 1404.6860

1. Assume mulitiplicative renormalizable
2. Decompose diagrams to 2PI diagrams
3. Extract collinear divergent contributions
4. Resum diagrams to factorize

## ➢ OPE method
YQM, Qiu, 1709.03018
Izubuchi, Ji, Jin, Stewart, Zhao, 1801.03917

1. Assume mulitiplicative renormalizable
2. Assume OPE is correct for quasi-parton operators
3. In the $\xi^2 \to 0$ limit, express nonlocal operators to local operators
4. Express local operators in terms of moments of PDFs
5. Analytical continuation to finite $\xi^2$

# Factorization structure

YQM, Qiu, 1709.03018

- **LCSs defined by quasi-parton operators can be factorized to PDFs (if renormalizable)**

$$\sigma_n(\xi^2,\omega,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2) K_n^a(\xi^2, x\omega, x^2P^2, \mu^2) + O(\xi^2\Lambda_{\rm QCD}^2)$$

where $f_{\bar{a}/h}(x,\mu^2) = -f_{a/h}(-x,\mu^2)$

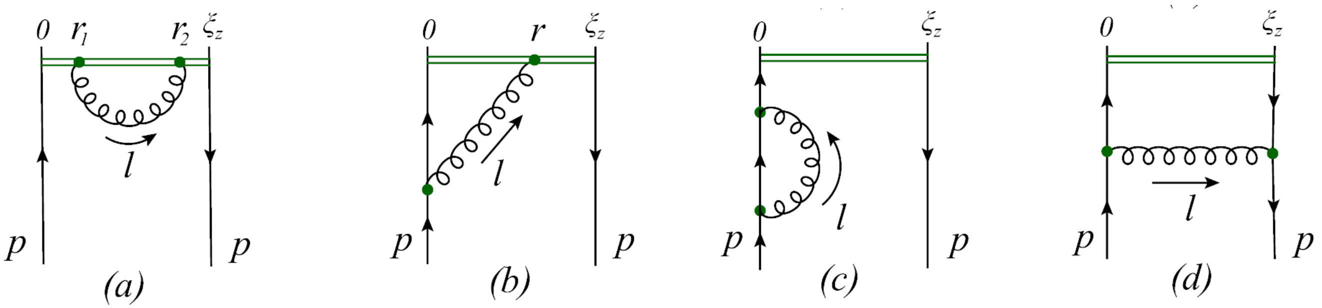
$$K_n^a = \sum_J 2W_n^{(J,a)}(\xi^2,\mu^2)\,\Sigma_J(x\omega, x^2P^2\xi^2)$$

- According to the proof: $K_n^a$ is analytical for all values of $\omega$ except the infinity.

# One-loop result: quark in quark

$$\sigma^{\nu}_{bq/h}(\omega,\xi^2) = \langle h(p)|\mathcal{O}^{\nu}_{bq}(\xi)|h(p)\rangle$$

- **One-loop calculation in a quark state**




$$\sigma^{\nu}_{bq/q}(\omega,\xi^2) = Z_{UV}\int_{-1}^{1}\frac{dx}{x}\,Z_{CO}(x,\mu_f^2)e^{ix\omega}\left[\frac{i\xi^{\nu}}{-\xi^2}H_1(x\omega,\mu_f^2)+xp^{\nu}H_2(x\omega,\mu_f^2)\right]$$

$$Z_{UV}(\xi^2) = 1+\frac{\alpha_s C_F}{\pi}\left[2+\frac{3}{4}\gamma_E+\frac{3}{4}\ln\pi+\frac{1}{3}\pi^2+\frac{3}{4\epsilon}(-\xi^2\mu^2)^{\epsilon}-\frac{|\xi|\mu}{1-2\epsilon}\right]$$

$$Z_{CO}(x,\mu_f^2) = \delta(1-x)-\frac{\alpha_s C_F}{\pi}\frac{(4\pi)^{\epsilon}\Gamma(1+\epsilon)}{2\epsilon}(\mu^2/\mu_f^2)^{\epsilon}\left(\frac{1+x^2}{1-x}\right)_+\theta(x)$$

$\overline{MS}$ **renormalized PDF**

# One-loop result: quark in quark

$$H_1(r,\mu_f^2) = \frac{\alpha_s C_F}{\pi} \frac{e^{ir}-1-ir}{ir} e^{-ir}$$

$$H_2(r,\mu_f^2) = 1 + \frac{H_1(r)}{2ir} + \frac{\alpha_s C_F}{\pi}\Bigg[-1-\frac{1}{3}\pi^2$$
$$+\int_0^1 \frac{dz}{z}(\ln(z^2)-1)\frac{e^{-izr}-1+izr}{izr}$$
$$+(\frac{1}{2}\ln(-4\xi^2\mu_f^2)+\gamma_E)\left(z^2-2z+2\right)\left(e^{-izr}-1\right)\Bigg]$$

- Confirm factorization structure
- As $\xi \to 0$, $\ln\xi^2$ is the UV regulator, it only appears in $Z_{UV}$
- Confirm analytical behavior regarding to $\omega$

$$r = x\,\omega$$

**Can do Taylor expansion of $\omega$ from the beginning, then calculation is much easier. Three-loop order calculation should have no difficulty.**

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# Renormalization: importance and difficulty

## ➢ Why proof is important?

- All-order proof of factorization needs multiplicative renormalization YQM, Qiu, 1404.6860, 1709.03018
- Need to take continuum limit for lattice calculation
- Find out all operators mixing under renormalization

## ➢ Difficulty

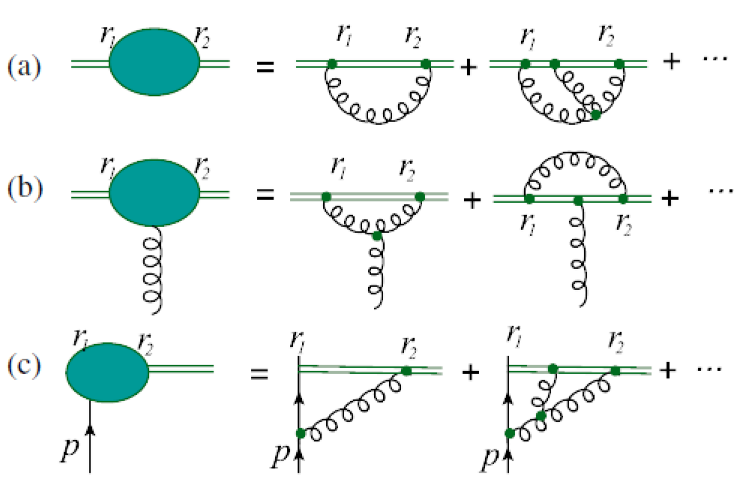
- Renormalization of nonlocal composite operator, not well studied
- Because of $z$-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences

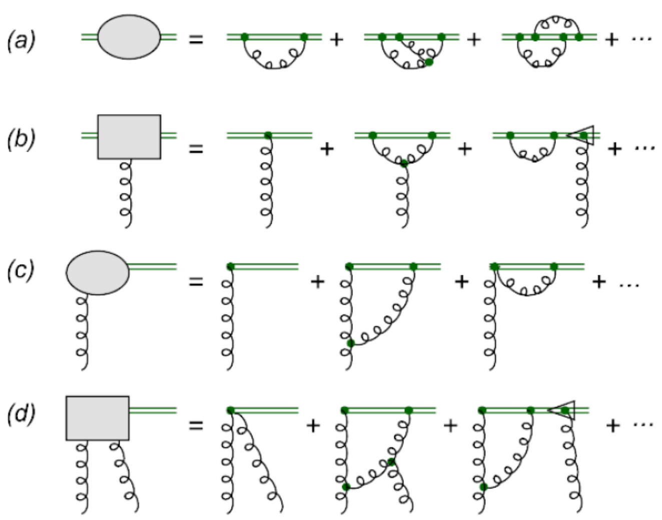
Normal PDFs, UV divergences from the region $(l_+, l_-, l_\perp) \sim (1, \lambda^2, \lambda)$ with $\lambda \to \infty$, nonlocal in '-' direction in coordinate space, leading to the mixing of operators along '-' direction.

# Locality of UV divergence

Ishikawa YQM, Qiu, Yoshida, 1707.03107
Li, YQM, Qiu, 1809.01836

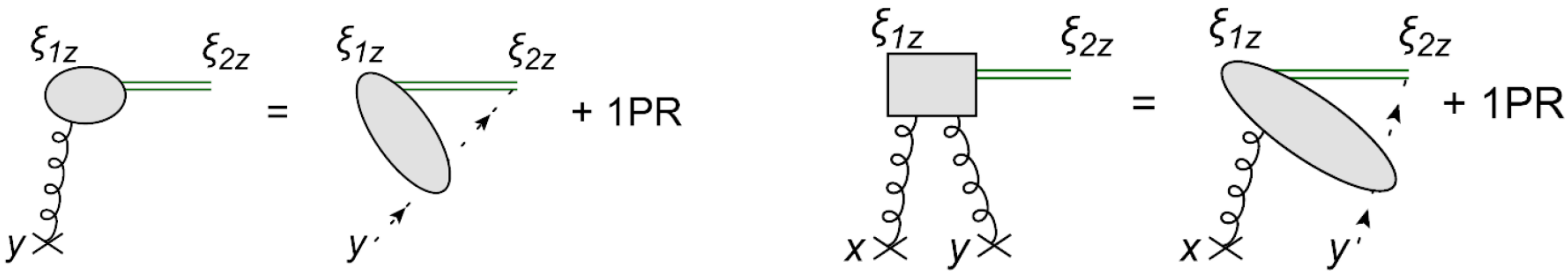
- Diagrammatic method: key is to show that UV divergences are local in space-time
  - Nontrivial conclusion! The most difficult part in the proof
- As UV is local, dimensional counting can give superficial divergence, find divergent topologies



# Gauge invariance

- Renormalization of gauge links is well-known
- Gauge invariance reduce UV divergence of vertex topologies by 1




$$\Gamma^{\lambda\mu\nu}(p,n) = c_1\Pi_1^{\lambda\mu\nu} + c_2(\Pi_2^{\lambda\mu\nu} - p\cdot n\Pi_4^{\lambda\mu\nu}) + c_3(\Pi_3^{\lambda\mu\nu} - p^2\Pi_4^{\lambda\mu\nu})$$

$$\Pi_1^{\lambda\mu\nu} = g^{\mu\lambda}p^\nu - g^{\nu\lambda}p^\mu,\ \Pi_2^{\lambda\mu\nu} = (p^\mu n^\nu - p^\nu n^\mu)n^\lambda$$

$$\Pi_3^{\lambda\mu\nu} = (p^\mu n^\nu - p^\nu n^\mu)p^\lambda,\ \Pi_4^{\lambda\mu\nu} = g^{\mu\lambda}n^\nu - g^{\nu\lambda}n^\mu$$

Remove this linear UV

# Renormalization structure

- **Logarithmic UV can be multiplicative ren.**

$$\mathcal{O}_q^{\nu}(\xi) = e^{-C_q|\xi_z|} Z_{wq}^{-1} Z_{vq}^{-1} \mathcal{O}_{bq}^{\nu}(\xi)$$

$$\mathcal{O}_g^{\mu\nu\rho\sigma}(\xi) = e^{-C_g|\xi_z|} Z_{wg}^{-1} Z_{vg1}^{-s/2} Z_{vg2}^{-(2-s)/2} \mathcal{O}_{bg}^{\mu\nu\rho\sigma}(\xi)$$

  - $s = 0,1,2$ is the number of $z$'s in $\{\mu,\nu,\rho,\sigma\}$, for each choice, coefficient of $c_2$ equals 0 or proportional to that of $c_1$
  - It is important for regular to guarantee gauge invariance
  - How about lattice regulator?

- **Auxiliary field method** See Jianhui Zhang's talk

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# Consequence of multiplicative

- **Quasi-quark operators can be renormalized as**

$$\mathcal{O}_q^\nu(\xi) = Z_{RS}^{-1}(\xi^2)\mathcal{O}_{bq}^\nu(\xi)$$

- **RI/MOM:** $Z_{RS} = \langle q(p)|O(\xi)|q(p)\rangle|_{p^2=-\mu_r^2,\ldots}$ See Yong Zhao's talk
  - Advantage: perturbative calculable
  - Disadvantage: involving too many scales, hard to keep the gauge invariance, high order perturbative calculation is hard

- **Pseudo-PDFs:** $Z_{RS} = \langle h(p)|O(\xi)|h(p)\rangle|_{p^\mu=0}$ See Anatoly Radyushkin's talk
  - Advantage: having less scales, gauge invariant
  - Disadvantage: nonperturbative

**Is there a way to combine the two advantages?**

# Our renormalization scheme

- **Choose vacuum expectation value** Li, YQM, Qiu, work in preparation

$$Z_{RS} = \frac{\langle \Omega | O(\xi) | \Omega \rangle}{\langle \Omega | O(\xi) | \Omega \rangle^{(0)}}$$

"(0)" means perturbative expansion to lowest order

  - Perturbatively calculable with the hard scale $1/\xi^2$
  - A single scale problem, gauge invariant
  - Can be easily calculated to high orders (like third order)

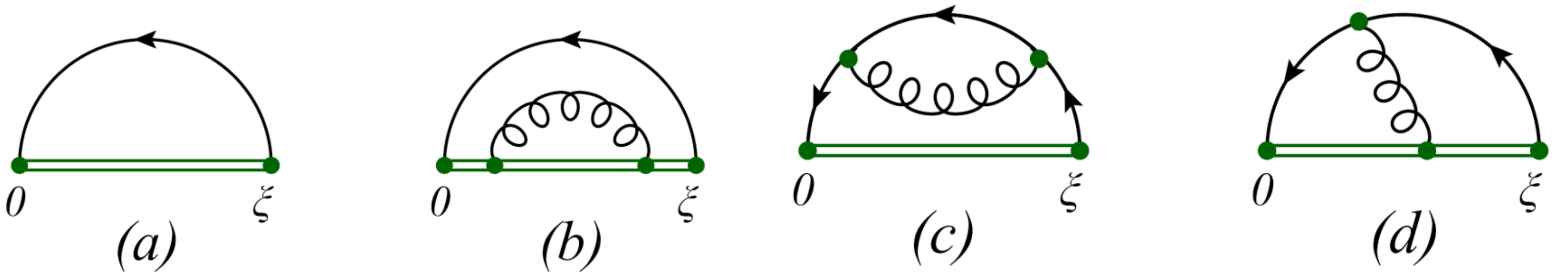
- **This general scheme can also be used to renormalize other non-local operators**

  - As far as $\xi^2$ is small to guarantee perturbation to work

# Example

## ➢ NLO calculation

Li, YQM, Qiu, work in preparation




$$Z_{UV} = \frac{\langle \Omega | O(\xi) | \Omega \rangle}{\langle \Omega | O(\xi) | \Omega \rangle^{(0)}}$$

$$\sigma^{\nu}_{q/q}(\omega, \xi^2) = Z^{-1}_{UV}(\xi^2)\sigma^{\nu}_{bq/q}(\omega, \xi^2)$$

$$Z_{UV}(\xi^2) = 1 + \frac{\alpha_s C_F}{\pi}\left[2 + \frac{3}{4}\gamma_E + \frac{3}{4}\ln\pi + \frac{1}{3}\pi^2 + \frac{3}{4\epsilon}(-\xi^2\mu^2)^{\epsilon} - \frac{|\xi|\mu}{1-2\epsilon}\right]$$

$$\sigma^{\nu}_{bq/q}(\omega, \xi^2) = Z_{UV}\int_{-1}^{1}\frac{dx}{x}\, Z_{CO}(x, \mu_f^2)e^{ix\omega}\left[\frac{i\xi^{\nu}}{-\xi^2}H_1(x\omega, \mu_f^2) + xp^{\nu}H_2(x\omega, \mu_f^2)\right]$$

- $H_1$ and $H_2$ are matching coefficients

# Summary

- Proof of factorization to all orders
- Proof of renormalization to all orders
- Propose a new renormalization scheme: is easy to be implemented nonperturbatively, simplifies perturbative calculation of the matching coefficients
    - Make the high order (beyond NLO) calculation of matching coefficients much easy to do and realistic

**Thank you!**