Quasi-PDF in Lattice Perturbation Theory

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BNL lattice PDF workshop
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Lattice Action

- **Wilson fermion action (naïve fermion $r=0$)**

$$S_q = a^4 \sum_x \left\{ -\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x)\,(r-\gamma_\mu)\,U_\mu(x)\psi(x+a\hat{\mu}) \right.\right.$$
$$\left. +\overline{\psi}(x+\mu)\,(r+\gamma_\mu)\,U_\mu^\dagger(x-a\hat{\mu})\psi(x) \right]$$
$$\left. +\frac{1}{a}(4r+am)\overline{\psi}(x)\psi(x) \right\}$$

- **Plaquette gauge field action**

$$S_g = -\frac{\beta}{3} \sum_x \sum_{\mu>\nu} \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)\right]$$

- **Clover action**

$$S_{\mathrm{C}} = -\sum_{\mu<\nu} c_{\mathrm{SW}} g_s \frac{a}{4} \bar{\psi}\,(x)\,\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi(x)$$

EOM and dispersion relation for free quark

$$\left[i\sum_{\mu}\gamma_{\mu}\boxed{\widehat{2P_{\mu}}}+r\sum_{\mu}\left(\frac{2}{a}-\boxed{\widetilde{2P_{\mu}}}\right)+2m\right]U(P)=0$$

$$\widehat{k}_{\mu}=\frac{2}{a}\sin\frac{ak_{\mu}}{2}\qquad\widetilde{k}_{\mu}=\frac{2}{a}\cos\frac{ak_{\mu}}{2}$$

$$\begin{aligned}P_4 =&\frac{1}{a}\sinh^{-1}\left(\frac{1}{\sqrt{2}}\left\{\frac{1}{\left(1-r^2\right)^2}\left[2\left(r^2+1\right)\left(am+2r\right)\left(a^2r\widehat{P}_3^2+am\right)\right.\right.\right.\\&-\frac{1}{2}a^2\left(r^4+2r^2-1\right)\widehat{2P}_3^2+4r^2-\left(a^2r^2\widehat{P}_3^2+2ram+2r^2\right)\\&\times\sqrt{a^4\left(2r^2-1\right)\widehat{P}_3^4+4a^2\widehat{P}_3^2(amr+1)+4am(am+2r)+4}\}\}^{\frac{1}{2}})\end{aligned}$$

$$P_4\big|_{r=0}=\frac{1}{a}\sinh^{-1}\frac{\sqrt{4a^2m^2-a\widetilde{4P}_3+2}}{2}$$

$$\lim_{a\to 0}P_4=\sqrt{P_3^2+m^2}$$

Feynman Rules

- **Propagators**

quark:

$$S_F(k) = 2\left[\frac{-i\sum_\mu \gamma_\mu \widehat{2k}_\mu + r\sum_\mu\left(\frac{2}{a} - \widetilde{2k}_\mu\right) + 2m}{\widehat{2k}^2 + \left(r\sum_\mu\left(\frac{2}{a} - \widetilde{2k}_\mu\right) + 2m\right)^2}\right]$$

gluon:

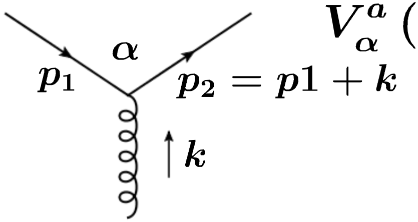
$$D_{g,\mu\nu}(k) = \frac{1}{\hat{k}^2}\left[\delta_{\mu\nu} - (1 - \boxed{\xi})\frac{a^2}{4}\hat{k}_\mu\hat{k}_\nu\right]$$

Feynman gauge: $\xi = 1$

R. Horsley, H. Perlt, P. E. L. Rakow, G. Schierholz, A. Schiller, PRD 78, 054504 (2008)

- $\mathcal{O}(\alpha_s^1)$ **Vertices**

$q-g-q$ vertex (naïve fermion $r=0$)

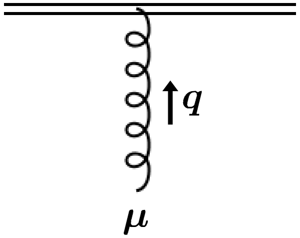


$$V^a_\alpha(p_2, p_1, k) = -ig_s T^a \frac{a}{2}\left(\widetilde{p_2 + p_1}\right)_\alpha \gamma_\alpha - g_s T^a r \frac{a}{2}\left(\widehat{p_2+p_1}\right)_\alpha$$

$$-ig_s T^a r c_{\mathrm{SW}} \frac{a^2}{8} \tilde{k}_\alpha \sum_\mu \sigma_{\alpha\mu} \widehat{2k}_\mu$$

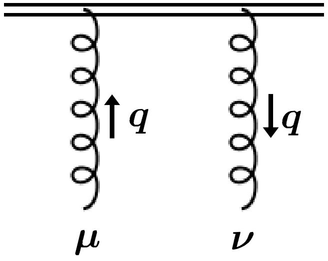
R. Horsley, H. Perlt, P. E. L. Rakow, G. Schierholz, A. Schiller, PRD 78, 054504 (2008)

$g-$gauge link (in 3-direction) vertices



$$O^A_{1,\mu}(q) = \frac{g_s a T^A \gamma_3 \delta_{\mu 3}}{\hat{q}_3}$$

$$O^{AB}_{2,\mu\nu}(q) = -g_s^2 a^2 \left\{T^A, T^B\right\} \gamma_3 \frac{\delta_{\mu 3}\delta_{\nu 3}}{\hat{q}_3^2}$$



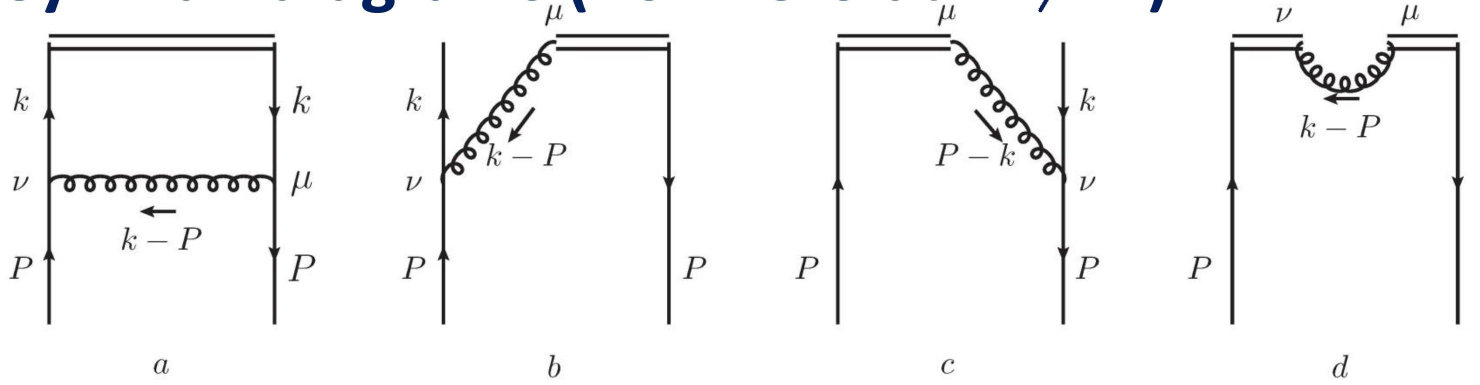
$$O^{AB}_{3,\mu\nu}(q) = -g_s^2 a^2 \left\{T^A, T^B\right\} \gamma_3 \delta_{\mu 3}\delta_{\nu 3} \mathcal{F}\left[e^{-ip_3 z}\left(\frac{z}{i\hat{q}_3} e^{\frac{z}{|z|} i \frac{a q_3}{2}} - \frac{a|z|}{2}\right)\right]$$

$\propto \delta'(x-1)$ only contribute at $x=1$, omitted!

T. Ishikawa,Y.-Q. Ma,b,c,d J.-W. Qiu,S. Yoshida, arXiv:1609.02018v1

One-Loop Diagrams

- **Feynman diagrams (non-zero at $x \neq 1$)**



$$\tilde{q}_a(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} \frac{\bar{U}(P)V_\mu(P,k,P-k)S_F(k)\gamma_3 S_F(k)V_\nu(k,P,k-P)U(P)}{\overline{U}(P)\gamma_3 U(P)} \times D_{g,\mu\nu}(P-k)\delta\left(x - \frac{k^3}{P^3}\right)$$

$$\tilde{q}_b(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} \frac{\overline{U}(P)O_{1,\mu}(P,k,P-k)S_F(k)V_\nu(k,P,k-P)U(P)}{\overline{U}(P)\gamma_3 U(P)} \times D_{g,\mu\nu}(P-k)\delta\left(x - \frac{k^3}{P^3}\right)$$

$$\tilde{q}_d(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} \frac{\overline{U}(P)O_{2,\mu\nu}(P,P,k-P)U(P)}{\overline{U}(P)\gamma_3 U(P)} D_{g,\mu\nu}(P-k)\delta\left(x - \frac{k^3}{P^3}\right)$$

Loop Integration

- $$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \to \overset{\text{Numerical}}{\boxed{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k_\perp}{(2\pi)^2}}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_4}{2\pi} \overset{\delta(k_3 - xP_3)}{\boxed{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_3}{2\pi}}}$$

- $$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 f\left(k_4\right) = \frac{-i}{a} \oint_{|z|=a^{-2}} \frac{dz}{z} f\left(\frac{-i}{a} \ln\left(a^2 z\right)\right)$$

Poles of propagators

$$S_F(k) = \frac{\cdots}{-a^{-2} z^{-2}\left(a^2 z^2 - \Gamma_+\right)\left(a^2 z^2 - \Gamma_-\right)}$$

$$\Gamma_\pm = \frac{\kappa \pm \sqrt{\kappa^2 - \frac{4}{a^4}}}{2},$$

$$D_{g,\mu\nu}(k) = \frac{\cdots}{-e^{-iaP_4} z^{-1}\left(z - \Pi_+\right)\left(z - \Pi_-\right)}$$

$$\Pi_\pm = e^{iaP_4} \frac{\eta \pm \sqrt{\eta^2 - \frac{4}{a^4}}}{2}$$

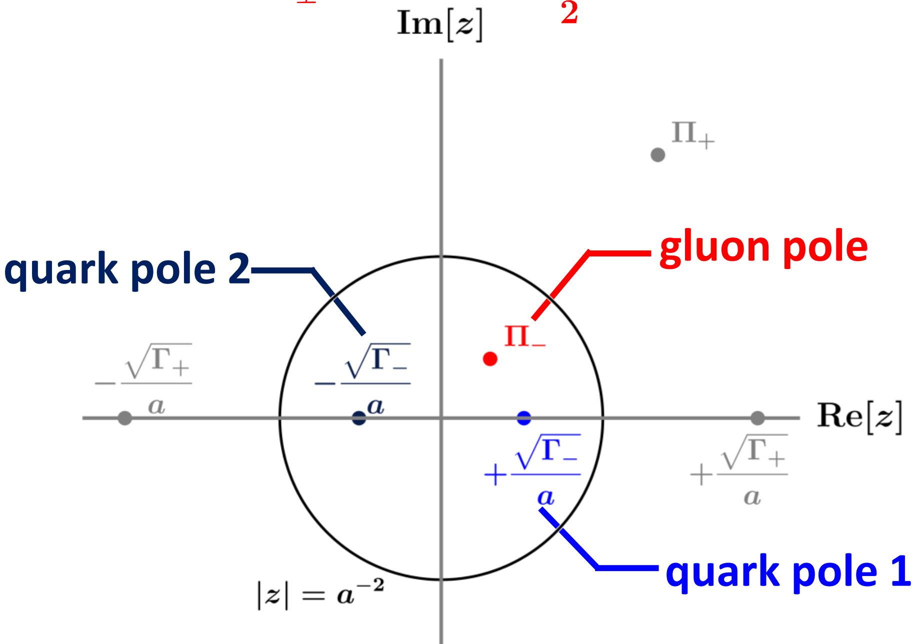
κ, η are positive definite functions of $a, P_3, m, k_\perp, x$

- **Position of *z*-poles**

$$P_4 \to -iP^0$$

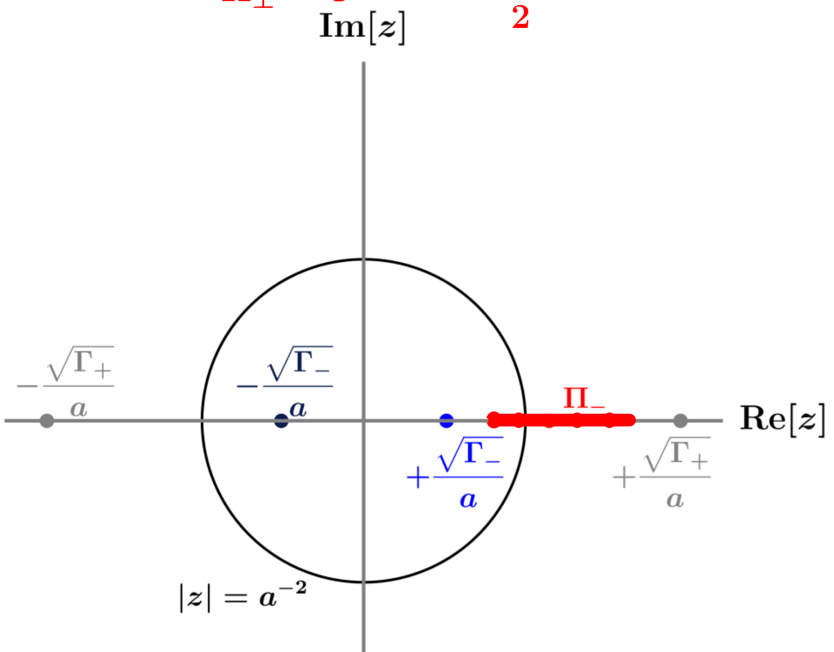
Before Wick rotation

$$\Pi_\pm = e^{iaP_4}\frac{\eta \pm \sqrt{\eta^2 - \frac{4}{a^4}}}{2}$$



After Wick rotation

$$\Pi_\pm = e^{aP^0}\frac{\eta \pm \sqrt{\eta^2 - \frac{4}{a^4}}}{2}$$



$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 \tilde{q}\left(k_4\right) = \frac{-i}{a}\oint_{|z|=a^{-2}} \frac{dz}{z} f(z)$$

$$=2\pi i\left\{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), \frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), \Pi_-\right]\right\}$$

● inside/outside depends on $k_\perp, x$
numerical integration on $k_\perp$
needs to be divided into regions
according to the position

- Poles (*k*-space) in continuum limit

$$k_4^g = -\frac{i}{a}\log\left(a^2\Pi_-\right) \quad \rightarrow k_g^0 = P^0 - \sqrt{k_\perp^2 + (k_3 - P_3)^2 - i\epsilon}$$

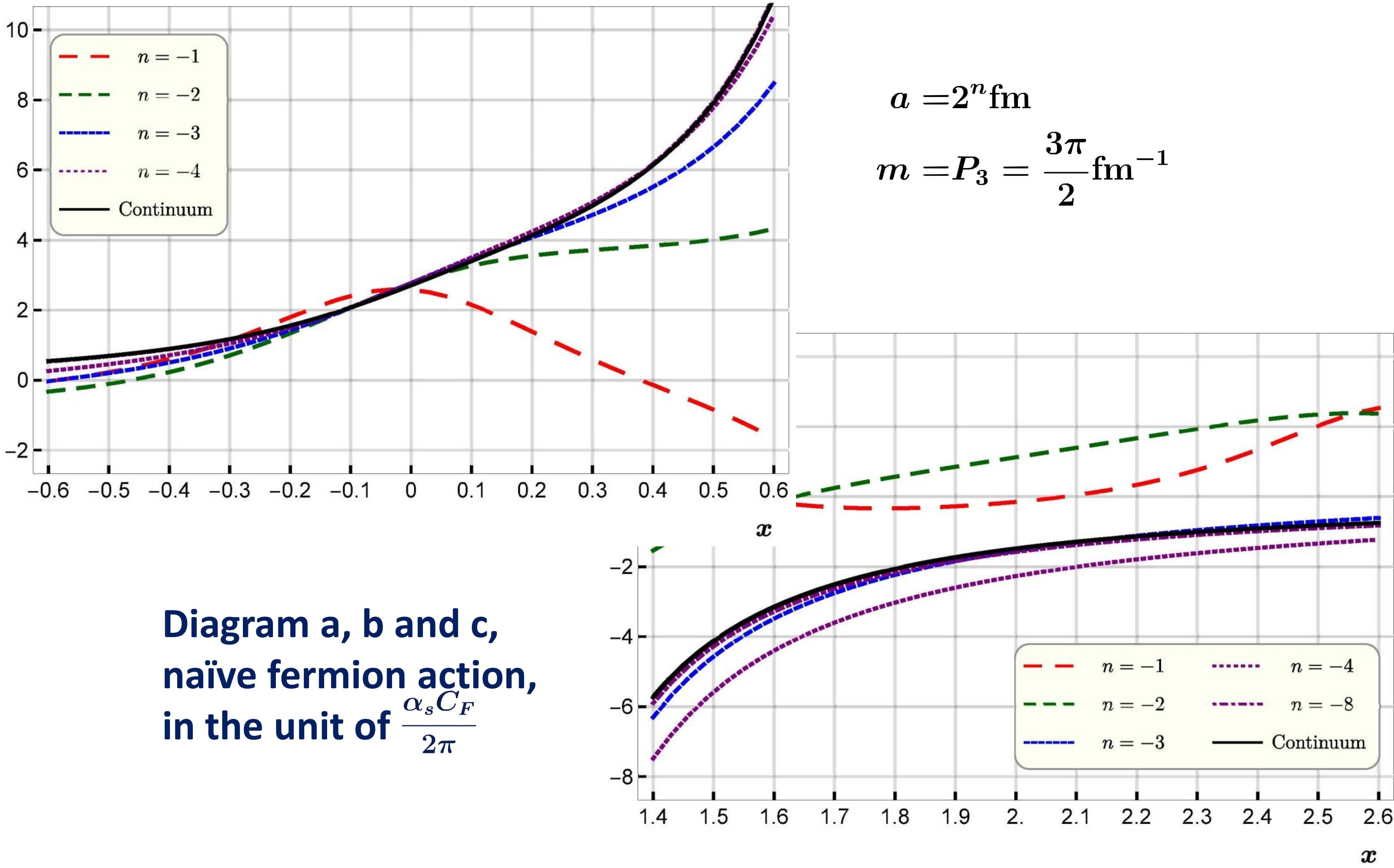
$$k_4^{q,+} = -\frac{i}{a}\log\left(a\sqrt{\Gamma_-}\right) \quad \rightarrow k_{q,+}^0 = -\sqrt{k_\perp^2 + k_3^2 + m^2 - i\epsilon}$$

$$k_4^{q,-} = -\frac{i}{a}\log\left(-a\sqrt{\Gamma_-}\right) \quad \rightarrow k_{q,-}^0 = \frac{i\pi}{a} - \sqrt{k_\perp^2 + k_3^2 + m^2 - i\epsilon}$$

unphysical pole, decouples
in continuum limit

- $k_\perp$ integrand in $a \rightarrow 0$ reproduces continuum result
- The same residue integration technic could be applied to Wilson-Clover fermion case, but much much more complicated…

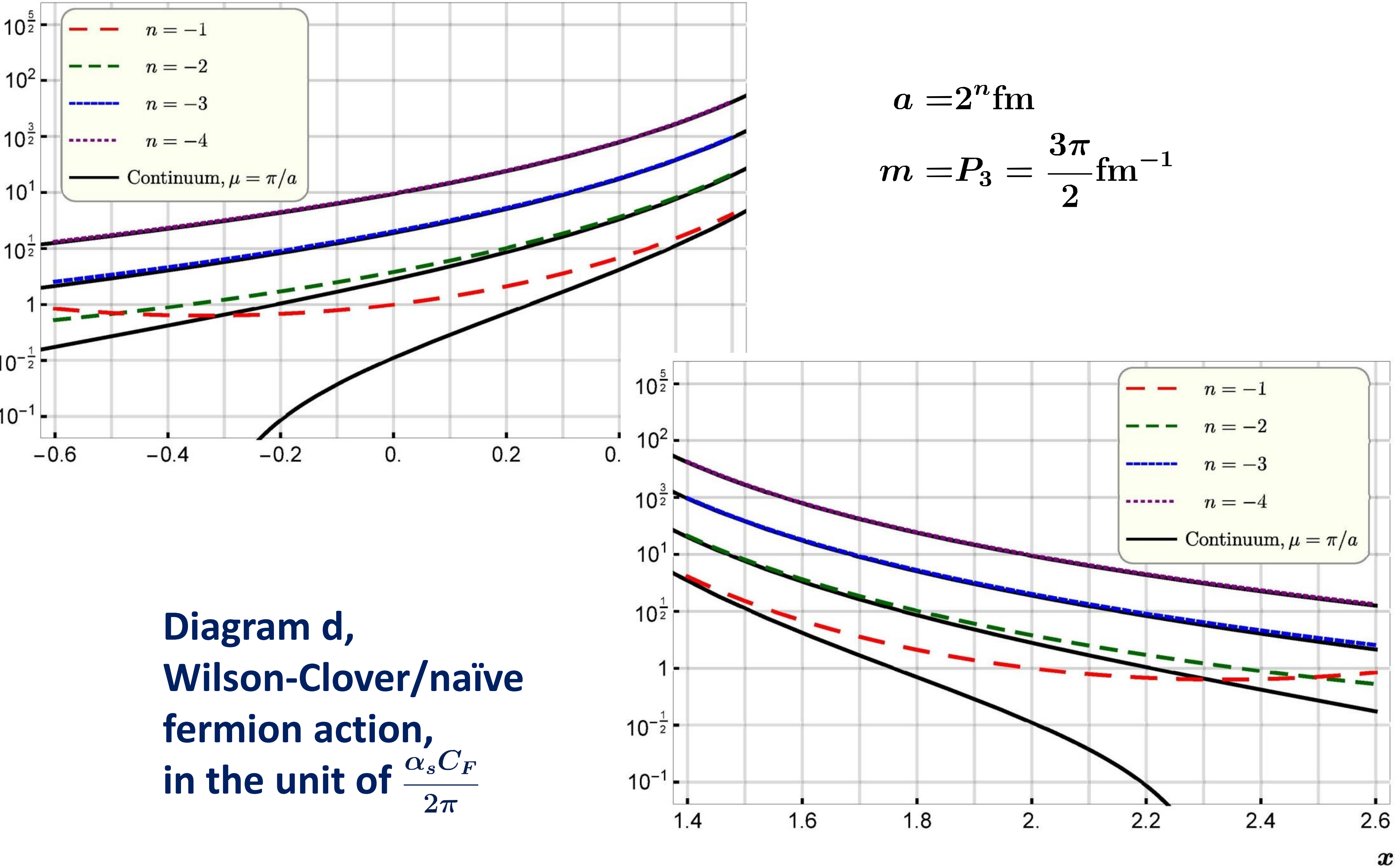
- **Numerical comparison with continuum QCD results**



$$a = 2^n \mathrm{fm}$$

$$m = P_3 = \frac{3\pi}{2}\mathrm{fm}^{-1}$$

Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$



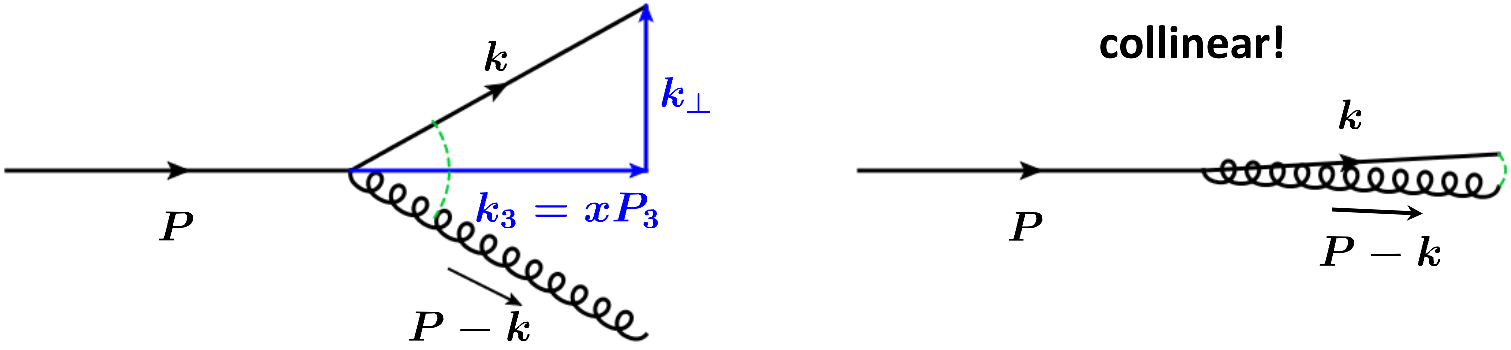
$$a = 2^n \text{fm}$$

$$m = P_3 = \frac{3\pi}{2}\text{fm}^{-1}$$

Diagram d, Wilson-Clover/naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

- **Extract collinear behavior of quasi-PDF**

expand $\begin{matrix}\text{numerator}\\ \text{denominator}\end{matrix}$ to $\begin{matrix}\mathcal{O}\left(k_\perp^0\right)\\ \mathcal{O}\left(k_\perp^2\right)\end{matrix}$ around $k_\perp \approx 0_\perp$



1. Continuum QCD case

$$\lim_{k_\perp \to 0} \tilde{q}_b\left(x, k_\perp\right) = \begin{cases} \frac{x}{1-x}\frac{1}{k_\perp^2+(1-x)^2 m^2} & 0 < x < 1 \\ \cdots & \text{otherwise} \end{cases}$$

collinear regulator

finite scale $\mu \gg m$

$$\int_0^\mu d^2k_\perp \lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}\left(x, k_\perp\right) = \begin{cases} -\frac{x}{1-x}\ln m^2 & 0 < x < 1 \\ \cdots & \text{otherwise} \end{cases}$$

2. Lattice perturbation

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 \tilde{q}_b^{\mathrm{nv}}(x,k_4,k_\perp) = \mathrm{Res}\left[f(z), \Pi_-\right] + \mathrm{Res}\left[f(z), +\frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right]$$

⬇

$$\lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}(x, k_\perp) = \left(\frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(0)} + \mathcal{D}_{b,1}^{(1)} k_\perp^2}\right) + \left(\frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(0)} + \mathcal{D}_{b,2}^{(1)} k_\perp^2}\right) + \left(\frac{\mathcal{N}_{b,3}^{(0)}}{\mathcal{D}_{b,3}^{(0)} + \mathcal{D}_{b,3}^{(1)} k_\perp^2}\right)$$

2. Lattice perturbation

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 \tilde{q}_b^{\mathrm{nv}}(x,k_4,k_\perp) = \mathrm{Res}\left[f(z), \Pi_-\right] + \mathrm{Res}\left[f(z), +\frac{1}{a}\sqrt{\Gamma_-}\right] + \cancel{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right]}$$

$$\Downarrow$$

$$\lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(0)} + \mathcal{D}_{b,1}^{(1)} k_\perp^2}\right) + \left(\frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(0)} + \mathcal{D}_{b,2}^{(1)} k_\perp^2}\right) + \cancel{\left(\frac{\mathcal{N}_{b,3}^{(0)}}{\mathcal{D}_{b,3}^{(0)} + \mathcal{D}_{b,3}^{(1)} k_\perp^2}\right)}$$

Unphysical, $\mathcal{O}(a^2)$,
no collinear divergence

$$\Downarrow$$

$$\int_0^\mu d^2k_\perp \lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\pi \frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(1)}} \ln \frac{\mu^2 \mathcal{D}_{b,1}^{(1)}}{\mathcal{D}_{b,1}^{(0)}}\right) + \left(\pi \frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(1)}} \ln \frac{\mu^2 \mathcal{D}_{b,2}^{(1)}}{\mathcal{D}_{b,2}^{(0)}}\right) + \cdots$$

2. Lattice perturbation

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 \tilde{q}_b^{\mathrm{nv}}(x,k_4,k_\perp) = \mathrm{Res}\left[f(z), \Pi_-\right] + \mathrm{Res}\left[f(z), +\frac{1}{a}\sqrt{\Gamma_-}\right] + \cancel{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right]}$$

$$\Big\downarrow$$

$$\lim_{k_\perp\to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(0)} + \mathcal{D}_{b,1}^{(1)} k_\perp^2}\right) + \left(\frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(0)} + \mathcal{D}_{b,2}^{(1)} k_\perp^2}\right) + \cancel{\left(\frac{\mathcal{N}_{b,3}^{(0)}}{\mathcal{D}_{b,3}^{(0)} + \mathcal{D}_{b,3}^{(1)} k_\perp^2}\right)}$$

Unphysical, $\mathcal{O}(a^2)$,
no collinear divergence

$$\Big\downarrow$$

$$\int_0^{\mu} d^2k_\perp \lim_{k_\perp\to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\pi\frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(1)}} \ln\frac{\mu^2\mathcal{D}_{b,1}^{(1)}}{\mathcal{D}_{b,1}^{(0)}}\right) + \left(\pi\frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(1)}} \ln\frac{\mu^2\mathcal{D}_{b,2}^{(1)}}{\mathcal{D}_{b,2}^{(0)}}\right) + \cdots$$

$$\Big\downarrow\; a \to 0 \text{ then } m \to 0$$

$$\left[\theta(-x)\frac{2x}{1-x}\ln m^2 + \cdots\right]$$
$$+\left[\theta(1-x)\frac{-2x}{1-x}\ln m^2 + \cdots\right]$$

2. Lattice perturbation

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_4 \tilde{q}_b^{\mathrm{nv}}(x,k_4,k_\perp) = \mathrm{Res}\left[f(z), \Pi_-\right] + \mathrm{Res}\left[f(z), +\frac{1}{a}\sqrt{\Gamma_-}\right] + \cancel{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right]}$$

$$\lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(0)} + \mathcal{D}_{b,1}^{(1)} k_\perp^2}\right) + \left(\frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(0)} + \mathcal{D}_{b,2}^{(1)} k_\perp^2}\right) + \cancel{\left(\frac{\mathcal{N}_{b,3}^{(0)}}{\mathcal{D}_{b,3}^{(0)} + \mathcal{D}_{b,3}^{(1)} k_\perp^2}\right)}$$

Unphysical, $\mathcal{O}(a^2)$, no collinear divergence

$$\int_0^{\mu} d^2k_\perp \lim_{k_\perp \to 0} \tilde{q}_b^{\mathrm{nv}}(x,k_\perp) = \left(\pi \frac{\mathcal{N}_{b,1}^{(0)}}{\mathcal{D}_{b,1}^{(1)}} \ln \frac{\mu^2 \mathcal{D}_{b,1}^{(1)}}{\mathcal{D}_{b,1}^{(0)}}\right) + \left(\pi \frac{\mathcal{N}_{b,2}^{(0)}}{\mathcal{D}_{b,2}^{(1)}} \ln \frac{\mu^2 \mathcal{D}_{b,2}^{(1)}}{\mathcal{D}_{b,2}^{(0)}}\right) + \cdots$$

$a \to 0$ then $m \to 0$

$m \to 0$ then $a \to 0$

$$\left[\theta(-x)\frac{x}{1-x}\ln m^2 + \cdots\right] + \left[\theta(1-x)\frac{-x}{1-x}\ln m^2 + \cdots\right] = \left[\theta(x)\theta(1-x)\frac{-x}{1-x}\ln m^2 + \cdots\right]$$

$$\left[\theta(x)\frac{x}{1-x}\ln\frac{\mu^2}{a^2P_3^4p_1(x)+m^2(1-x)^2}\right] + \left[\theta(1-x)\frac{x}{1-x}\ln\frac{\mu^2}{a^2P_3^4p_2(x)+m^2(1-x)^2}\right]$$

$p1(x)$, $p_2(x)$ are polynomial function of x

- **Influence of lattice artifact**

$m \rightarrow 0$ then $a \rightarrow 0$

$$\left[\frac{x}{1-x}\ln\frac{\mu^2}{\boxed{a^2P_3^4p_2(x)}+m^2(1-x)^2}\right]$$

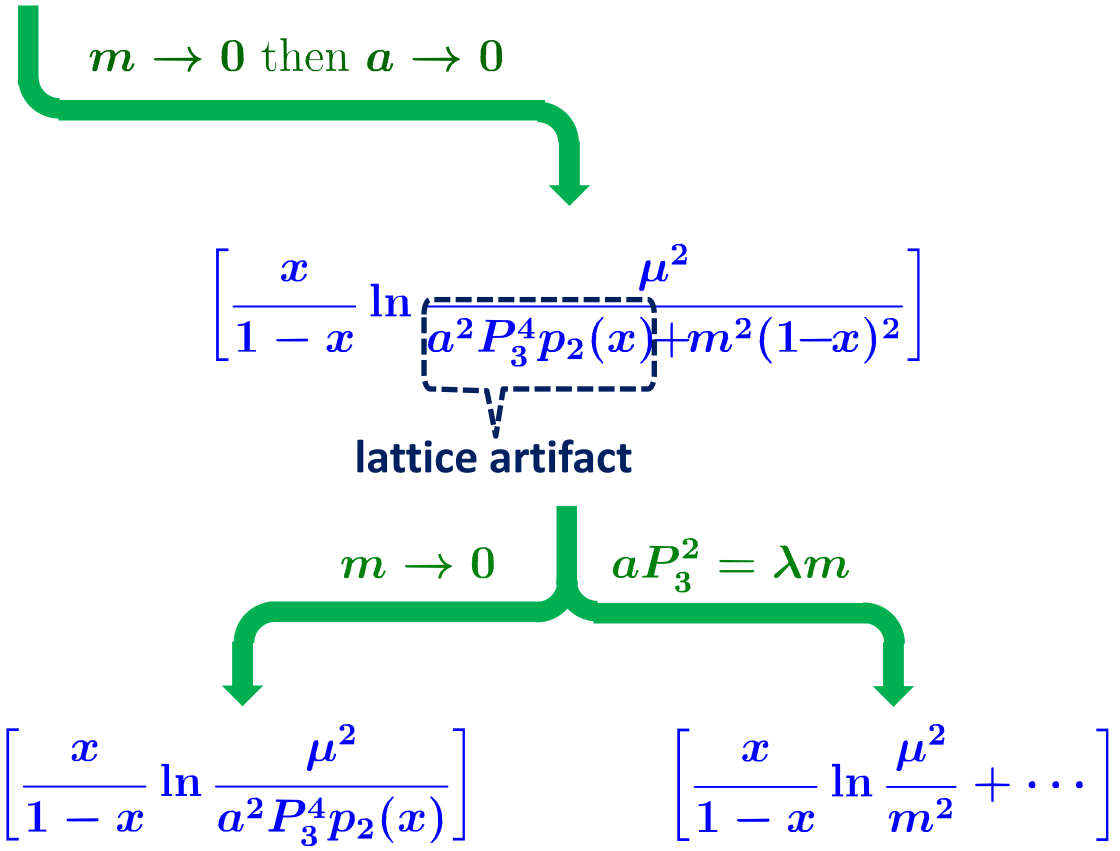
lattice artifact

$m \rightarrow 0$

$$\left[\frac{x}{1-x}\ln\frac{\mu^2}{a^2P_3^4p_2(x)}\right]$$

Collinear divergence has been regulated by lattice artifact

- **Influence of lattice artifact**



$m \to 0$ then $a \to 0$

$$\left[\frac{x}{1-x}\ln\frac{\mu^2}{\boxed{a^2P_3^4p_2(x)}+m^2(1-x)^2}\right]$$

lattice artifact

$m \to 0$

$aP_3^2 = \lambda m$

$$\left[\frac{x}{1-x}\ln\frac{\mu^2}{a^2P_3^4p_2(x)}\right]$$

$$\left[\frac{x}{1-x}\ln\frac{\mu^2}{m^2}+\cdots\right]$$

Collinear divergence has been regulated by lattice artifact

Collinear divergence has been reproduced

- **Numerical Results**

$$r = 0,\ P_3 = \frac{3\pi}{2}\mathrm{fm}^{-1}$$

$$a = 2^n \mathrm{fm},\ m = 5 \times 10^{-3}\pi \mathrm{fm}^{-1}$$

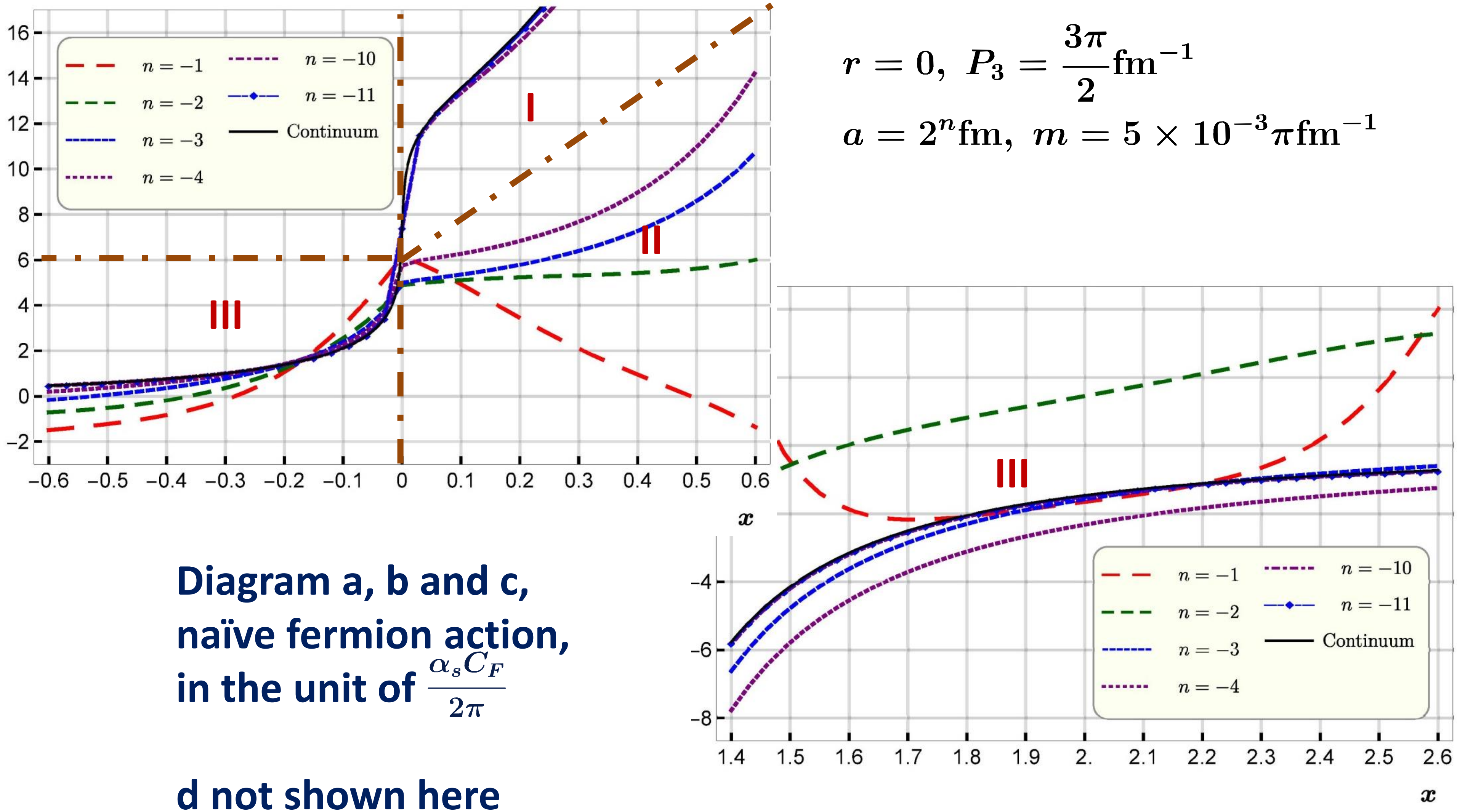
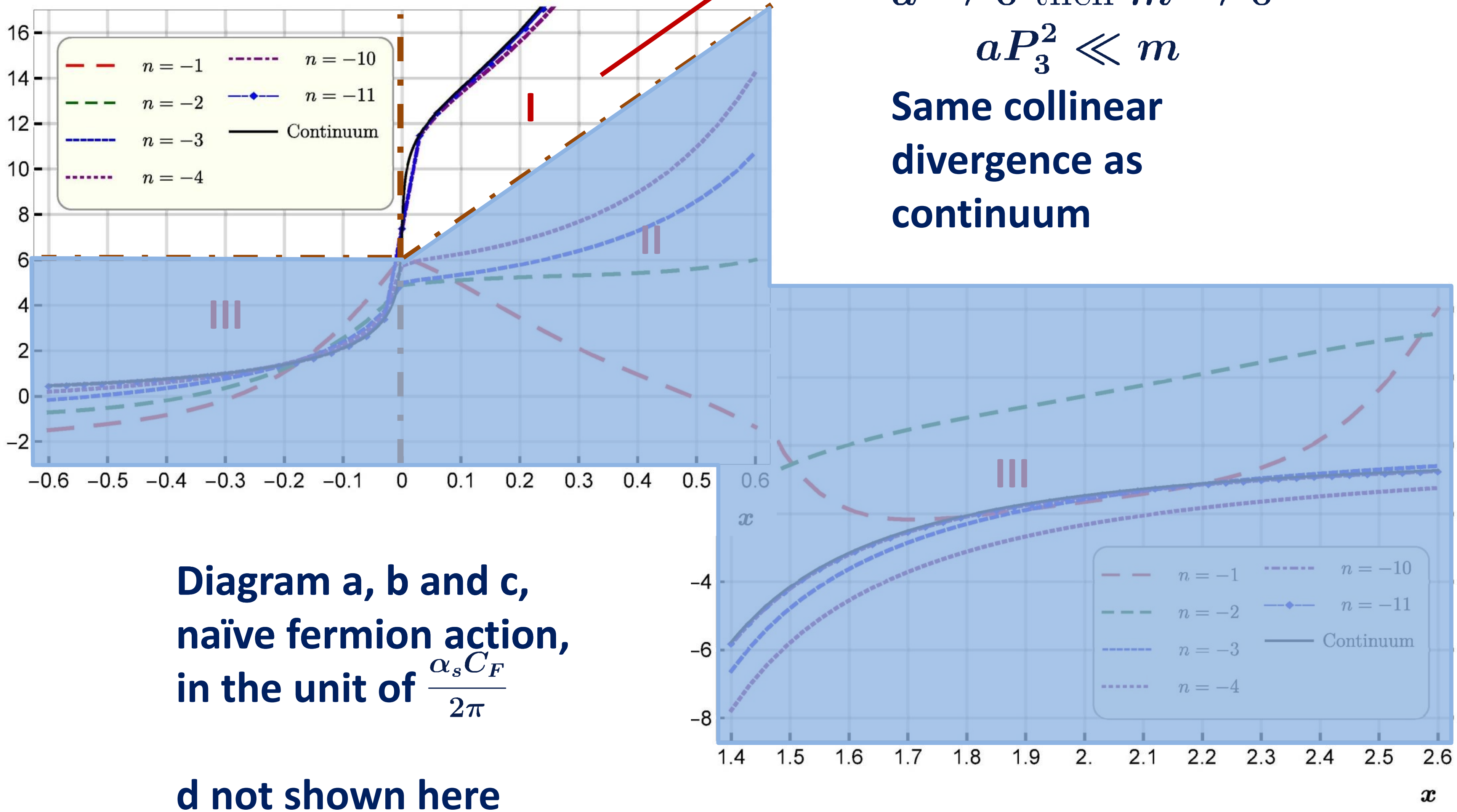


Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

d not shown here (does not contain collinear divergence)

- **Numerical Results**



$a \to 0$ then $m \to 0$

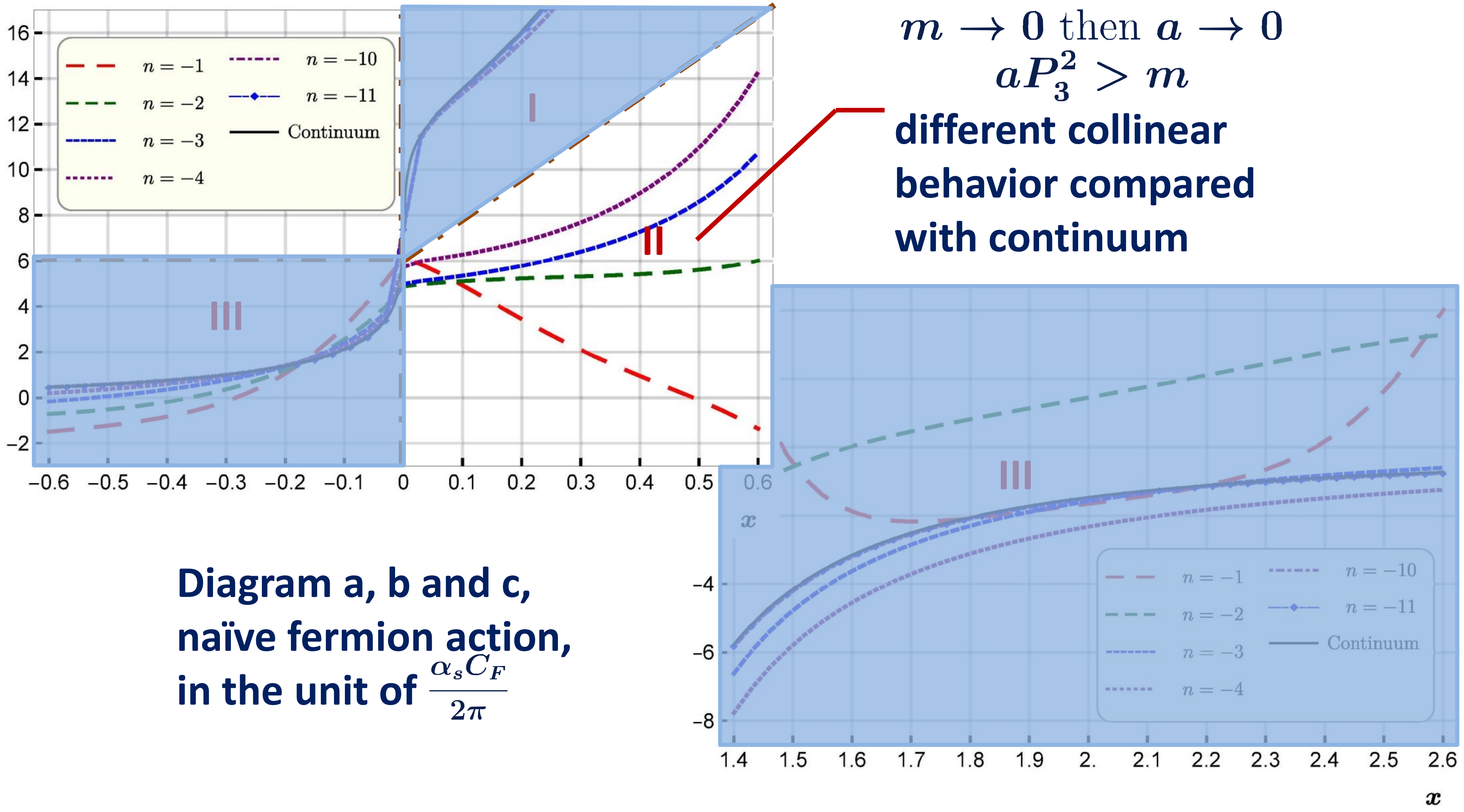
$aP_3^2 \ll m$

Same collinear divergence as continuum

Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

d not shown here (does not contain collinear divergence)

- **Numerical Results**



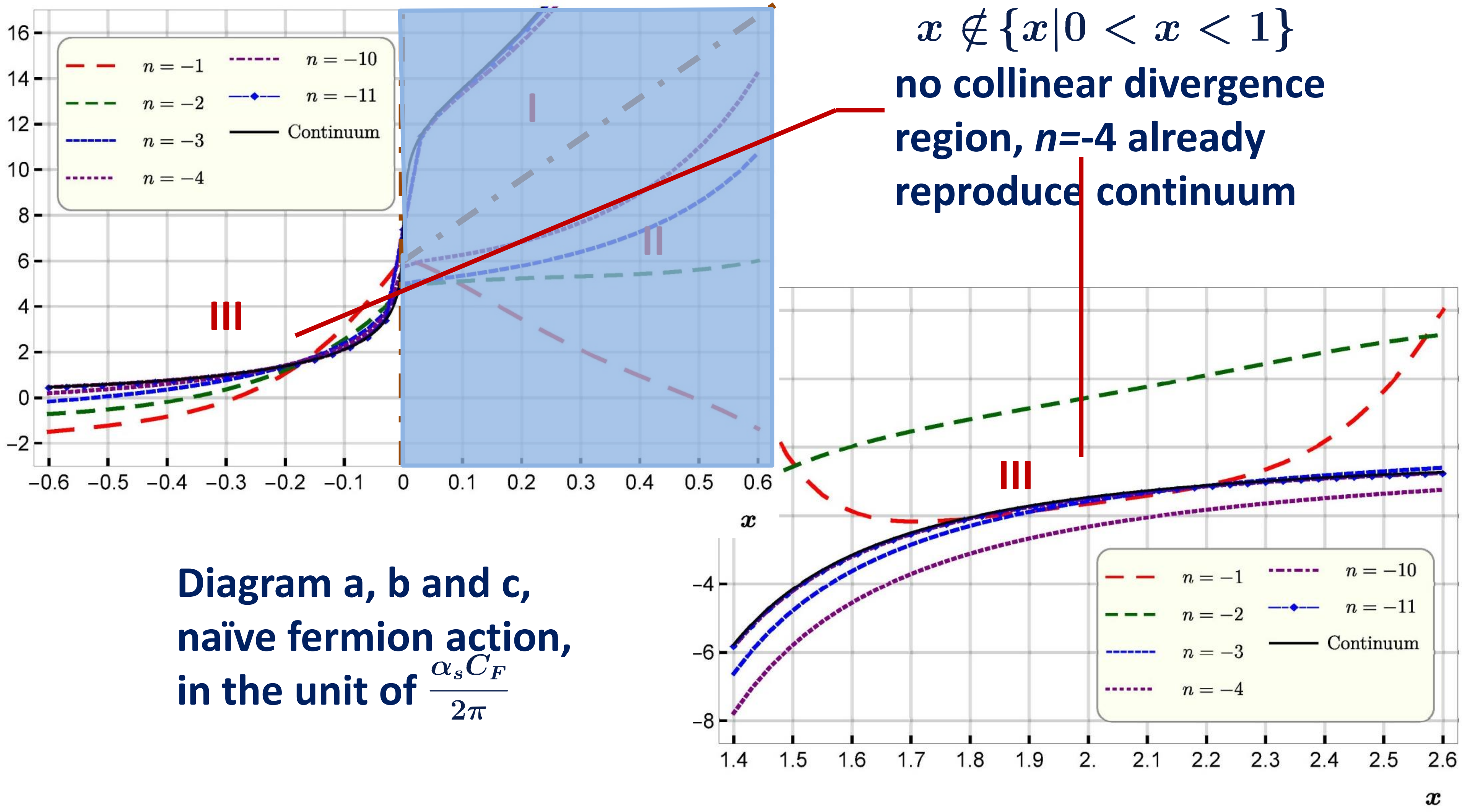
$m \to 0$ then $a \to 0$

$aP_3^2 > m$

different collinear behavior compared with continuum

Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

- **Numerical Results**



$x \notin \{x|0 < x < 1\}$

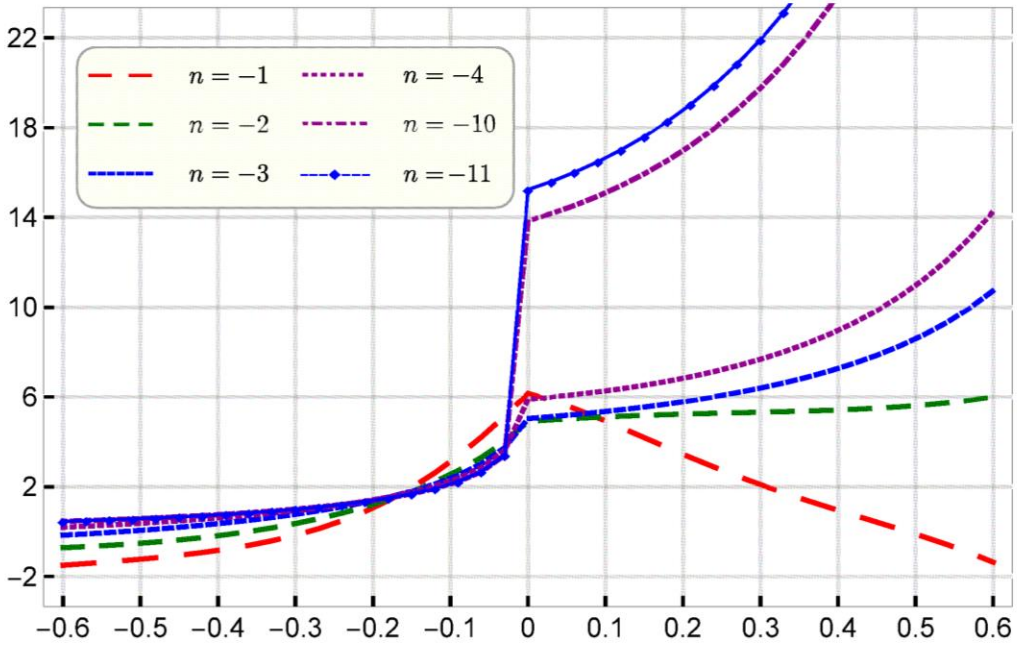
no collinear divergence region, *n*=-4 already reproduce continuum

Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

- **Numerical Results (massless quark)**

$$r = 0,\ P_3 = \frac{3\pi}{2}\mathrm{fm}^{-1}$$

$$a = 2^n\mathrm{fm},\ m = 0$$



No collinear divergence in all regions, lattice artifacts have regulated it

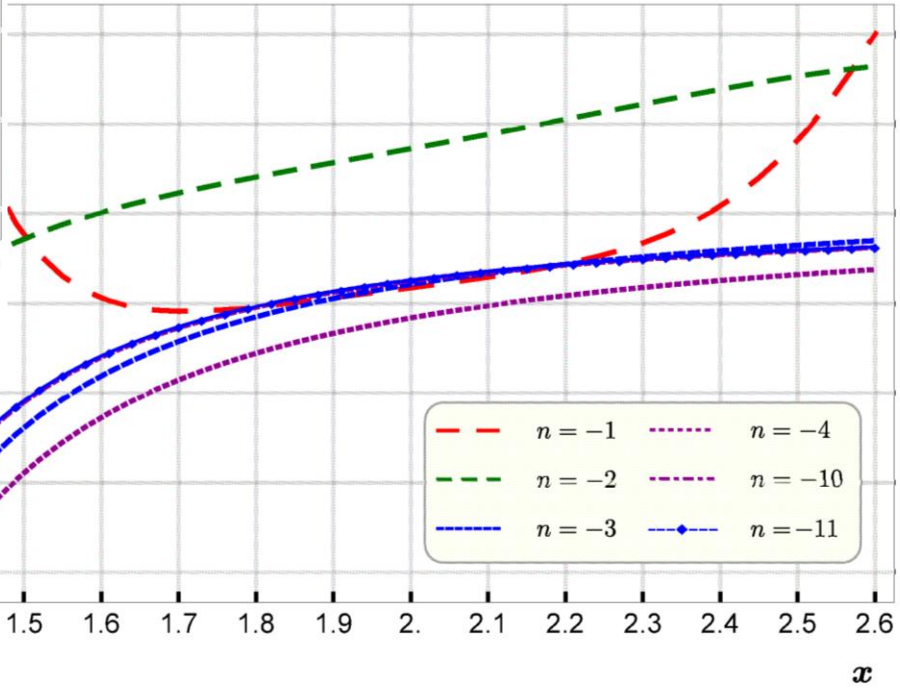
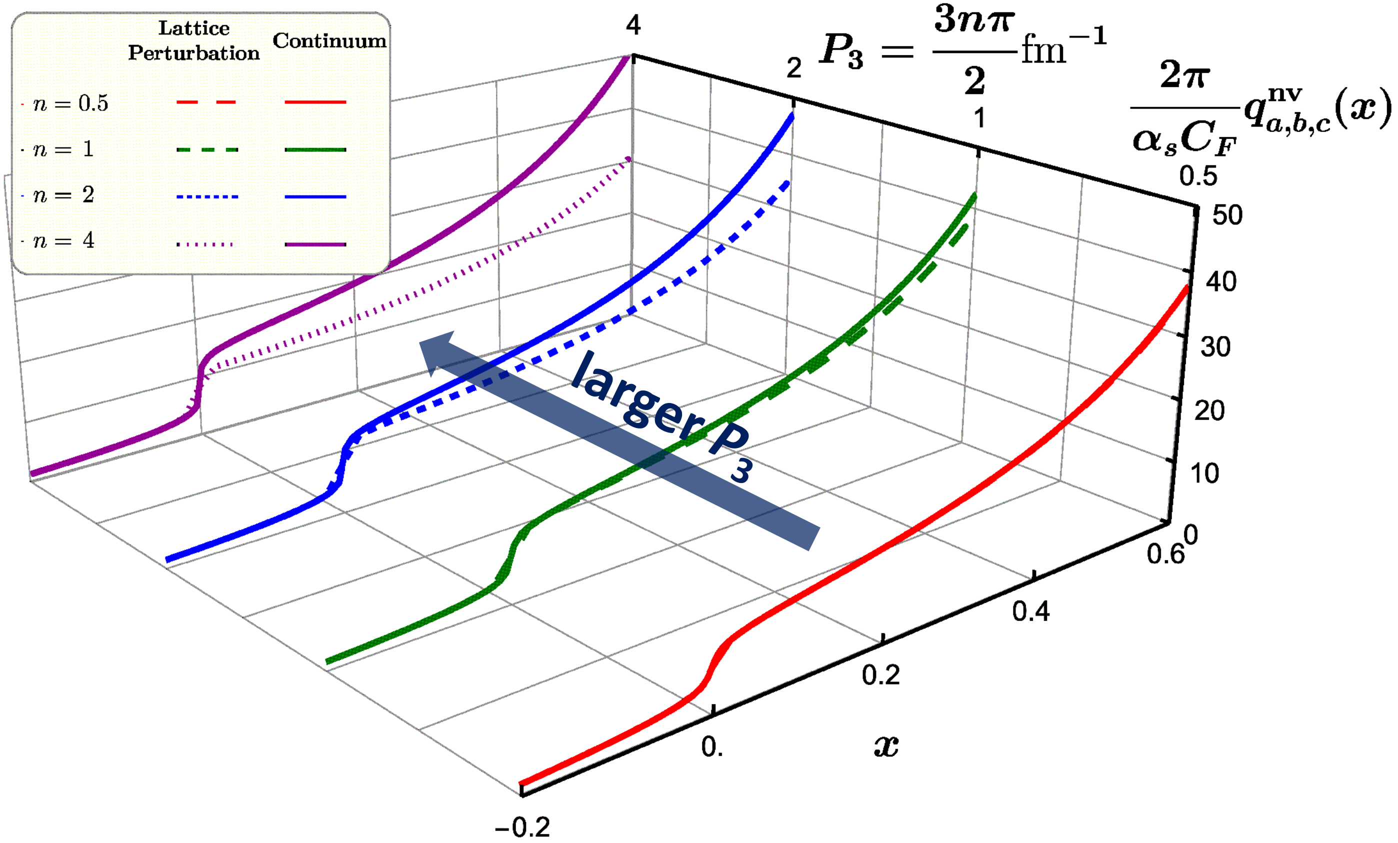


Diagram a, b and c, naïve fermion action, in the unit of $\frac{\alpha_s C_F}{2\pi}$

- **Numerical Results (increasing P_3)**

Lattice artifacts $\sim (aP_3^2)^n$
boost will increase the influence of lattice artifacts, resulting larger discrepancy to continuum quasi-PDF

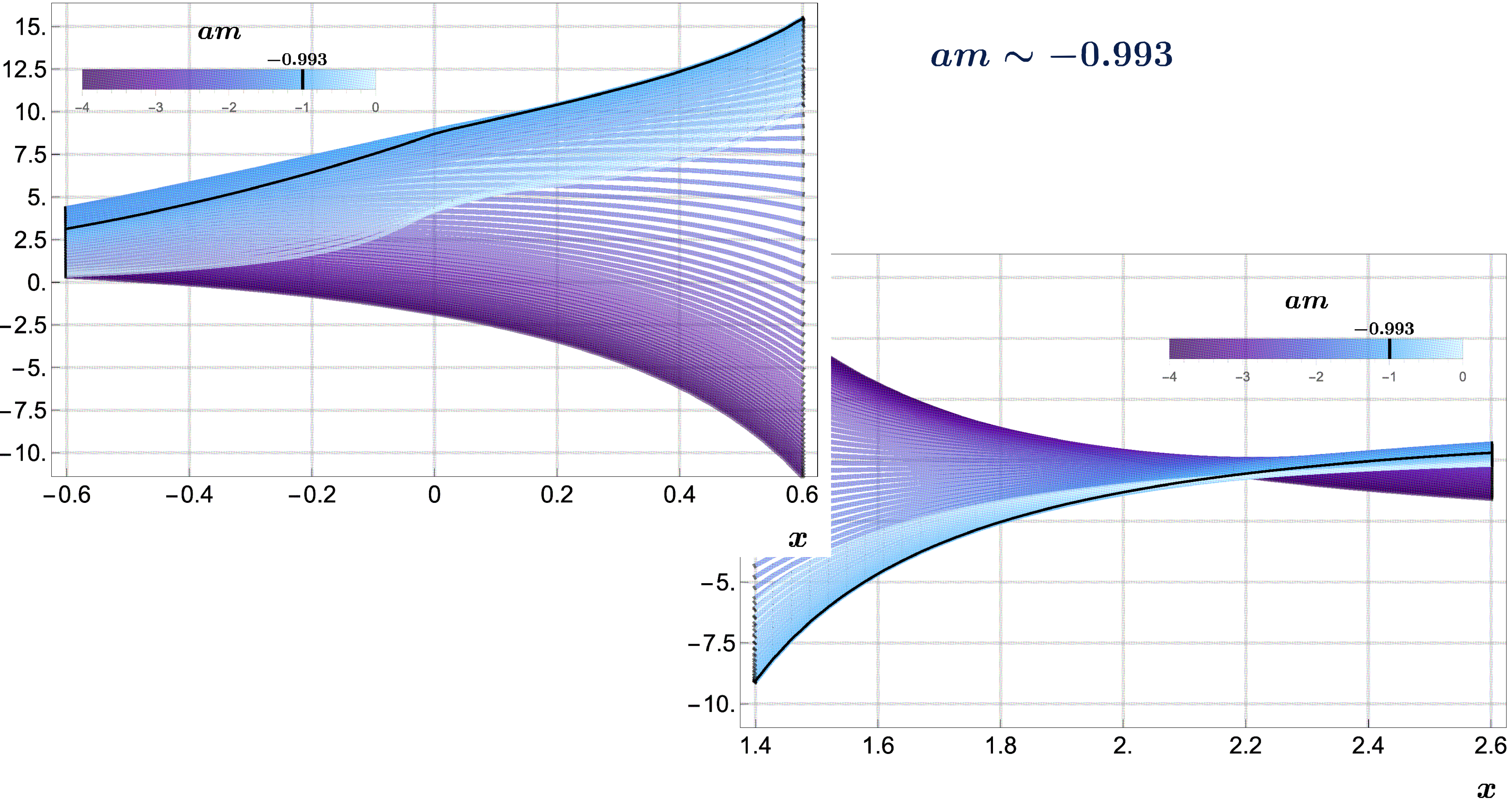


- **Numerical Results (Wilson-Clover fermion action)**

  Wilson-Clover fermion: additive mass renormalization,

  Negative bare quark mass to get vanishing renormalized quark mass

$$am \sim -0.993$$



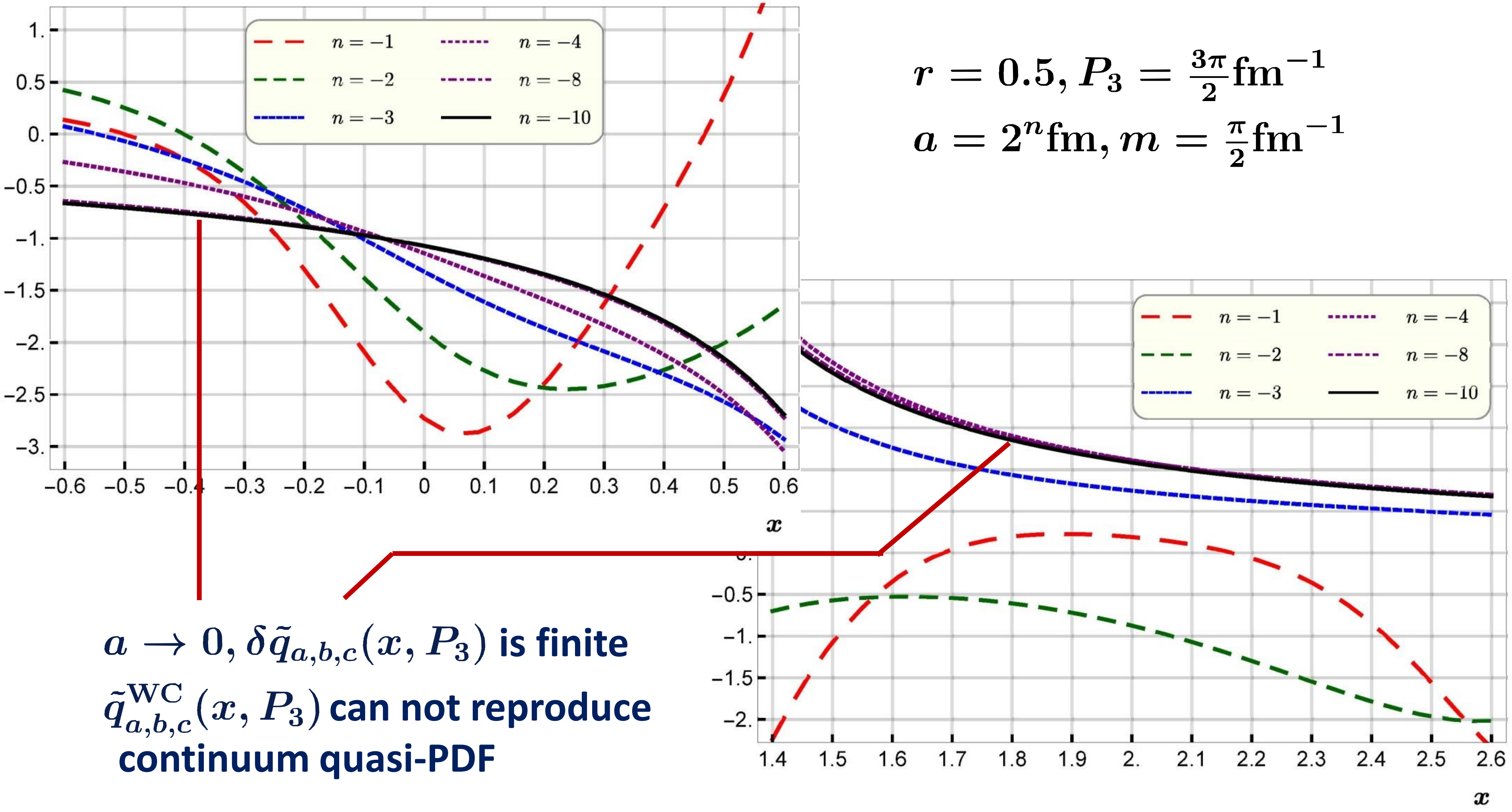
- **Numerical Results (difference between Wilson-Clover fermion and naïve fermion)**

$$\delta\tilde{q}_{abc}\left(x,P_3\right)=\tilde{q}^{\mathrm{WC}}_{abc}\left(x,P_3\right)-\tilde{q}^{\mathrm{nv}}_{abc}\left(x,P_3\right)$$

$$\delta\tilde{q}_{d}\left(x,P_3\right)=\tilde{q}^{\mathrm{WC}}_{d}\left(x,P_3\right)-\tilde{q}^{\mathrm{nv}}_{d}\left(x,P_3\right)=0$$

$$r=0.5,P_3=\frac{3\pi}{2}\mathrm{fm}^{-1}$$

$$a=2^n\mathrm{fm},m=\frac{\pi}{2}\mathrm{fm}^{-1}$$



$a\rightarrow 0, \delta\tilde{q}_{a,b,c}(x,P_3)$ is finite

$\tilde{q}^{\mathrm{WC}}_{a,b,c}(x,P_3)$ can not reproduce continuum quasi-PDF

Conclusion

- $a \to 0$ does not commute with $m \to 0$
- Condition to reproduce collinear behavior in continuum:

  $a \to 0$ and $aP_3^2 \approx m$
- Lattice artifact can regulate collinear divergence
- Boost increases the influence of lattice artifact $\sim aP_3^2$
- Quasi-PDF (naive fermion) $\tilde{q}^{\mathrm{nv}}(x, P_3)$ reproduces continuum quasi-PDF in $a \to 0$

  Quasi-PDF (Wilson-Clover) $\tilde{q}^{\mathrm{WC}}(x, P_3)$ can not reproduce continuum quasi-PDF $a \to 0$, due to the power UV divergent terms after k_4 integration, e.g.

$$\lim_{k_\perp \to \infty} \tilde{q}^{\mathrm{WC}}(x, k_\perp, P_3) = \int^{\frac{\pi}{a}} \frac{d^2 k_\perp}{16\pi^3} \frac{g_s^2 C_F m r P_3}{P_0^2} \frac{a}{|k_\perp|} + \cdots \sim \mathcal{O}\left(a^0\right)$$

Thanks !

Backup Slides

- **Definition of κ, η**

$$\kappa = \sum_{j=1}^{3} \widehat{2k}_j^{\,2} + 4m^2 + \frac{2}{a^2}, \eta = \sum_{j=1}^{3} \widehat{k-P}_j^{\,2} + \frac{2}{a^2}$$

- **$k_\perp$ integration after Wick rotation**

$$\Pi_-|_{P_4 \to -iP^0} = \frac{e^{aP^0}}{2}\left(\eta - \sqrt{\eta^2 - \frac{4}{a^4}}\right) < \frac{1}{a^2} \implies k_\perp \in \mathcal{R}_1\left(a, m, P_3, x\right)$$

$$\Pi_-|_{P_4 \to -iP^0} = \frac{e^{aP^0}}{2}\left(\eta - \sqrt{\eta^2 - \frac{4}{a^4}}\right) > \frac{1}{a^2} \implies k_\perp \in \mathcal{R}_2\left(a, m, P_3, x\right)$$

$$\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} d^2k_\perp \oint_{|z|=a^{-2}} dz f(z)$$

$$=2\pi i \int_{k_\perp \in \mathcal{R}_1} dk_\perp \left\{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), \frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), \Pi_-\right]\right\}$$

$$=2\pi i \int_{k_\perp \in \mathcal{R}_2} dk_\perp \left\{\mathrm{Res}\left[f(z), -\frac{1}{a}\sqrt{\Gamma_-}\right] + \mathrm{Res}\left[f(z), \frac{1}{a}\sqrt{\Gamma_-}\right]\right\}$$

- **In continuum limit**

$$\lim_{a\to 0}\mathcal{R}_1 = 1 + a\left(\sqrt{m^2+P_3^2} - \sqrt{k_\perp^2 + P_3^2(1-x)^2}\right) < 1$$

$$\Longrightarrow \; k_\perp^2 > m^2 + x(2-x)P_3^2$$

$$\lim_{a\to 0}\mathcal{R}_2 = 1 + a\left(\sqrt{m^2+P_3^2} - \sqrt{k_\perp^2 + P_3^2(1-x)^2}\right) > 1$$

$$\Longrightarrow \; k_\perp^2 < m^2 + x(2-x)P_3^2$$