Distribution amplitudes: moments and “direct” determination

Gunnar Bali

with

Vladimir M. Braun, Simon Bürger, Benjamin Gläßle, Meinulf Göckeler, Michael Gruber, Fabian Hutzler, Piotr Korcyl, Andreas Schäfer, André Sternbeck, Philipp Wein, Jian-Hui Zhang

Outline

- DAs: definition and moments of DAs
- Lattice calculation of the 2nd moment of the pion DA[1]
- Calculation of the pion DA in X-space[2]
- First X-space results on leading and higher twist DAs[2]
- Outlook

1) RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236; 1811.06050;
RQCD: GB, VM Braun, S Bürger, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, A Sternbeck, P Wein, 1903.08038.

2) RQCD: GB, VM Braun, B Gläßle, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, P Wein, J-H Zhang, 1709.04325;
RQCD: GB, VM Braun, B Gläßle, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, P Wein, J-H Zhang, 1807.06671.

Not covered:

ρ DAs [RQCD: VM Braun et al, 1612.02955]

octet baryon DAs [RQCD: GB et al, 1512.02050; 1903.12590]

What are distribution amplitudes?

Wavefunction of a hadron (here pion) near the infinite momentum frame, written as a superposition of different Fock states:

$$|\pi\rangle = c_1|\bar{q}q\rangle + c_2|\bar{q}gq\rangle + c_3|\bar{q}q\bar{q}q\rangle + \ldots$$

Light front wavefunction (Distribution amplitude, DA) describes the distribution of the longitudinal momentum among the partons.

Momentum fractions $0 \le u_i \le 1$, $\sum_{i\in\{q,\bar{q},g\}} u_i = 1$.

At leading twist (twist 2) only the valence quarks contribute:

$$u = u_q = 1 - u_{\bar{q}}, \quad \xi = u_q - u_{\bar{q}} = 2u - 1 \in [-1, 1].$$

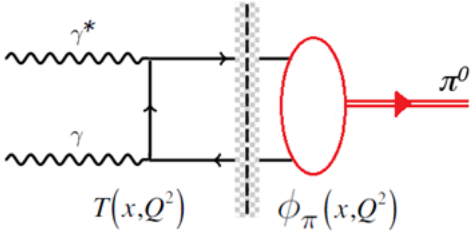
In hard processes higher Fock states are power suppressed.

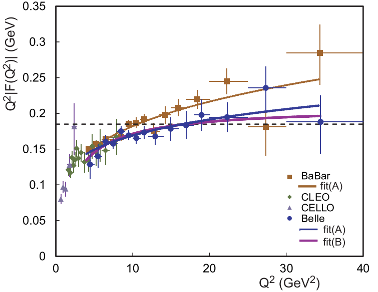
PDFs are (within the parton model) single particle probability densities and can **directly be extracted from fits to DIS and SIDIS data**.

DAs are wavefunctions and only appear within convolutions in hard exclusive processes. Due to other hadronic uncertainties (and experimental techniques), it is much **harder to extract these reliably from experimental data**.

DAs are needed for the theoretical description of hard exclusive processes.

Example: collinear factorization of the $\gamma\gamma^* \to \pi^0$ photoproduction formfactor ($Q \gtrsim \mu \gg \Lambda$) [Belle, 1205.3249]





$$F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3}\int_0^1 du\, \underbrace{H_{\bar{q}q\gamma}(u,\mu_F,Q^2)}_{\text{hard matching function}} \cdot \underbrace{\phi_\pi(u,\mu_F)}_{\text{soft factor (DA)}} + \underbrace{\text{higher twist}}_{F_\pi\cdot\mathcal{O}(1/Q^2)}.$$

μ_F is the factorization scale and we renormalize the hard coefficient function H at the scale $\mu_R^2 = Q^2$, $\mu_F^2 \sim Q^2/4$. $F_\pi \approx 92\,\text{MeV}$

Non-local light front matrix element at a separation n ($n^2 = 0$):

$$\left\langle 0 \left| \bar{d}\left(\frac{n}{2}\right) \not{n}\gamma_5 \left[\frac{n}{2}, -\frac{n}{2}\right] u\left(-\frac{n}{2}\right) \right| \pi^+(p) \right\rangle$$
$$= i\,F_\pi\, n\cdot p \int_0^1 du\, \exp\left\{ i\,\underbrace{[u-(1-u)]}_{=\xi} \frac{(n\cdot p)}{2} \right\} \phi_\pi(u,\mu)$$

$[n/2, -n/2]$ above denotes a gauge covariant connection.

The DA is not accessible in Euclidean spacetime but moments of DAs are:

$$\langle \xi^n \rangle = \int_0^1 du\,(2u-1)^n \phi_\pi(u,\mu)\,, \quad \langle \xi^0 \rangle = 1\,, \quad \langle \xi^1 \rangle = 0.$$

$\langle \xi^{0,2} \rangle$ can be extracted from local matrix elements $\langle 0 | O^{\pm}_{\mu\nu\rho} | \pi^+(p) \rangle$ with

$$O^{\pm}_{\mu\nu\rho} = \bar{d}\left\{ \left[\overleftarrow{D}_{(\mu} \overleftarrow{D}_{\nu} \pm 2 \overleftarrow{D}_{(\mu} \overrightarrow{D}_{\nu} + \overrightarrow{D}_{(\mu} \overrightarrow{D}_{\nu} \right] \gamma_{\rho)} \gamma_5 \right\} u,$$

where $(\cdots)$ gives a traceless, symmetrized expression.

Gegenbauer moments

Gegenbauer expansion:

$$\phi_\pi(u,\mu) = 6u(1-u)\left[1 + \sum_{n\in\mathbb{N}} a_{2n}^\pi(\mu)\, C_{2n}^{3/2}(2u-1)\right]$$

Collinear conformal symmetry: $C_n^{3/2}(\xi)$ in $SL(2,\mathbb{R})$ analogous to $Y_{\ell m}(\theta,\phi)$ in SO(3).
$\langle \xi^{2n}\rangle$ and a_{2n}^π are related by simple algebraic expressions ($n=1$ example):

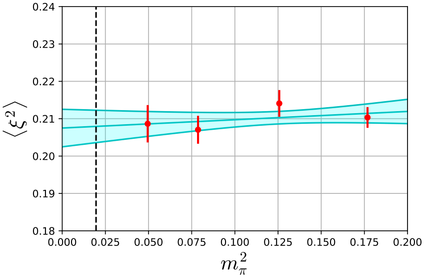
$$a_2^\pi(\mu) = \frac{7}{12}\left(5\langle\xi^2\rangle - 1\right) = \frac{7}{12}\left(5\langle\xi^2\rangle - \langle\xi^0\rangle\right)$$

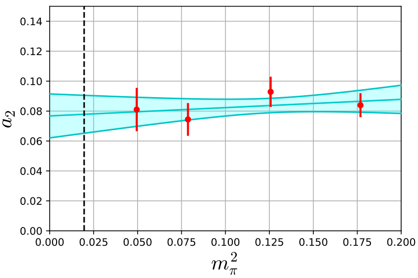
$a_{2n}^\pi(\mu) \to 0$ as $\mu \to \infty$: At large scales the lower moments will dominate.

Note the difference in the counting: 2nd DA-moment $\sim$ 3rd PDF-moment.

Using momentum smearing at one lattice spacing

Note that in one loop ChPT $\nexists$ chiral logs in this DA moment.





$N_f = 2 + 1$, where $m_s + 2m_{ud} = \mathrm{Tr}\, M = \mathrm{const}$

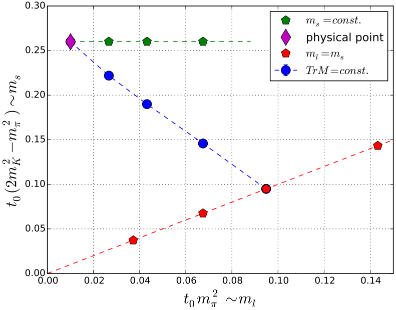
[RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236]

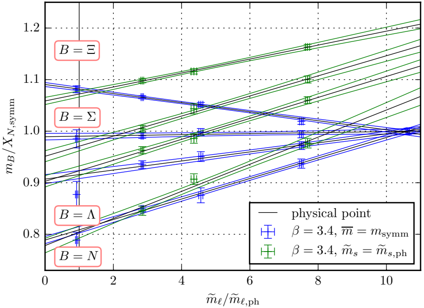
NNLO matching between RI'/SMOM and $\overline{\mathrm{MS}}$. NNNLO matching to $\overline{\mathrm{MS}}$?

Above was at $a \approx 0.086\,\mathrm{fm}$. **Next**: the continuum limit.

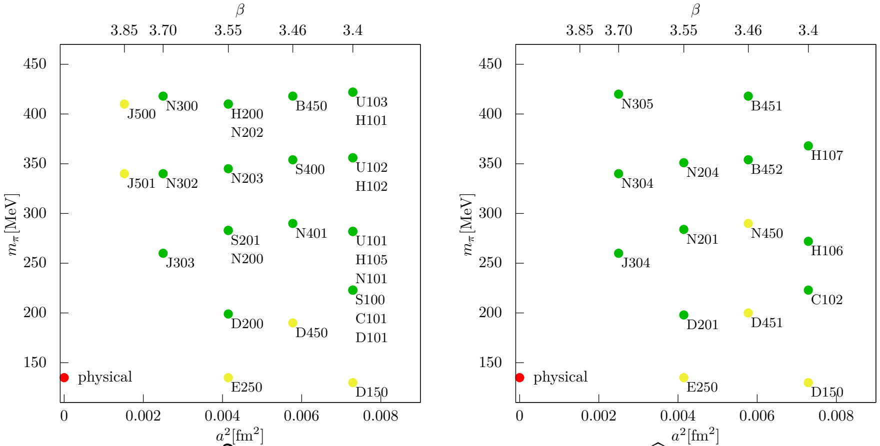
CLS simulation strategy

Simulate along $m_s + 2m_\ell = \text{const}$ [QCDSF+UKQCD: W Bietenholz et al, 1003.1114], **and** $\widehat{m}_s \approx \text{const}$ [G Bali et al, 1606.09039; 1702.01035], enabling Gell-Mann–Okubo/SU(3) and SU(2) χPT extrapolations.





(Only linear unconstrained baryon mass fits are shown.)



$m_s + 2m_\ell = \text{const}$ $\qquad$ $\widehat{m}_s \approx \text{const}$

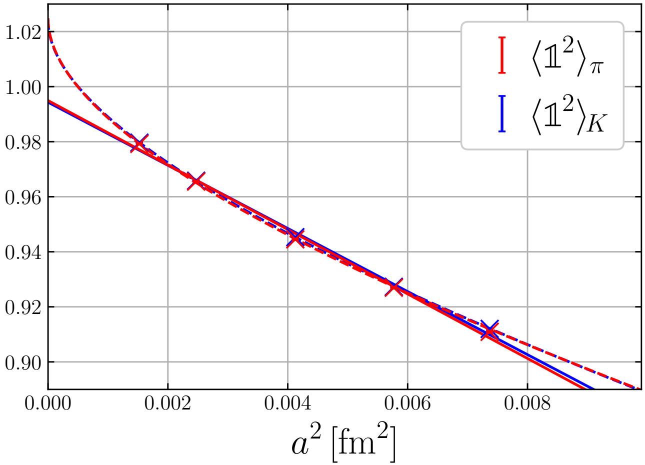
E: $192 \cdot 96^3$, J: $192 \cdot 64^3$, D: $128 \cdot 64^3$, N: $128 \cdot 48^3$, C: $96 \cdot 48^3$,
S: $128 \cdot 32^3$, H: $96 \cdot 32^3$, B: $64 \cdot 32^3$, U: $128 \cdot 24^3$.

$\exists$ additional ensembles with $m_s = m_\ell$.

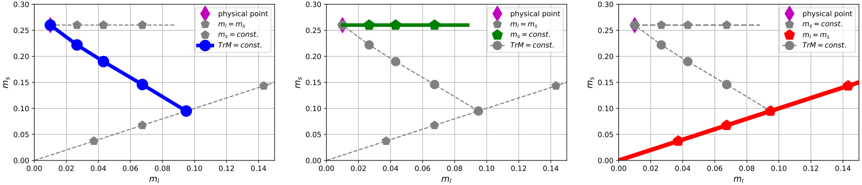
Results I: $\langle \xi^0 \rangle \to 1 + \mathcal{O}(\alpha_s^3) \quad (a \to 0)$?

Check of renormalization: $\langle \xi^0 \rangle = 1 + \mathcal{O}(a) \sim (\zeta_{22} - 5\zeta_{12}) O^+$

ζ_{ij}: renormalization from lattice to $\overline{\mathrm{MS}}$.



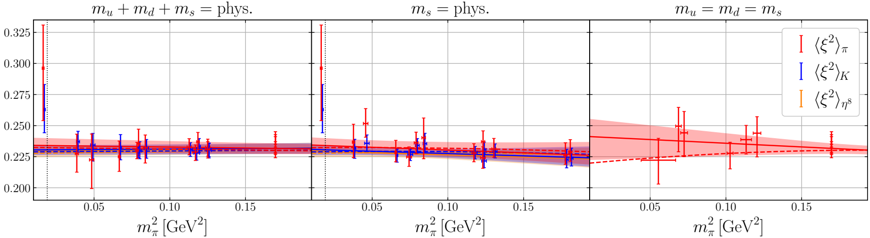
Results II: chiral extrapolation



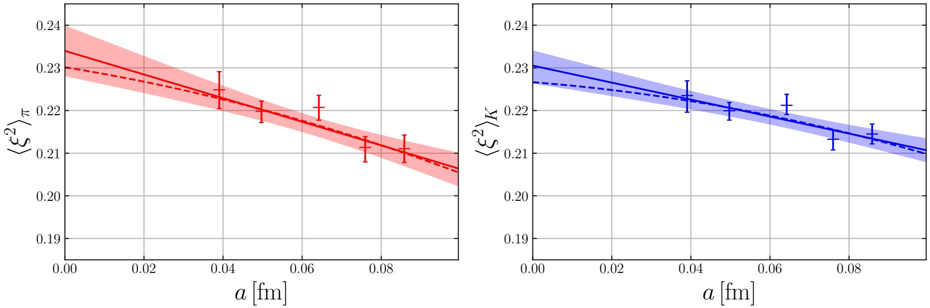
SU(3) NLO ChPT and linear lattice spacing effects (8 parameters):

$$\xi_X^2(M_\pi, M_K, a) = \left[\xi_0^2 + \bar{A}\overline{M}^2 + A_X \delta M^2\right]\left[1 + a\left(c_0 + \bar{c}\overline{M}^2 + c_X \delta M^2\right)\right],\ X \in \{\pi, K, \eta_8\},$$

$$A_\pi = -2A_K = -A_{\eta_8},\ \ \delta M^2 = M_K^2 - M_\pi^2,\ \ \overline{M}^2 = \frac{1}{3}\left(2M_K^2 + M_\pi^2\right).$$



Results III: continuum limit



Continuum limit, at the physical point: ($a_2 \propto 5\langle\xi^2\rangle - 1$)

$$a_2^{\pi,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.099^{+17}_{-17}(11)_r(11)_a(5)_m, \qquad \langle\xi^2\rangle_\pi = 0.234^{+6}_{-6}(4)_r(4)_a(2)_m,$$

$$a_2^{K,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.089^{+10}_{-12}(11)_r(11)_a(4)_m, \qquad \langle\xi^2\rangle_K = 0.231^{+4}_{-4}(4)_r(4)_a(2)_m.$$

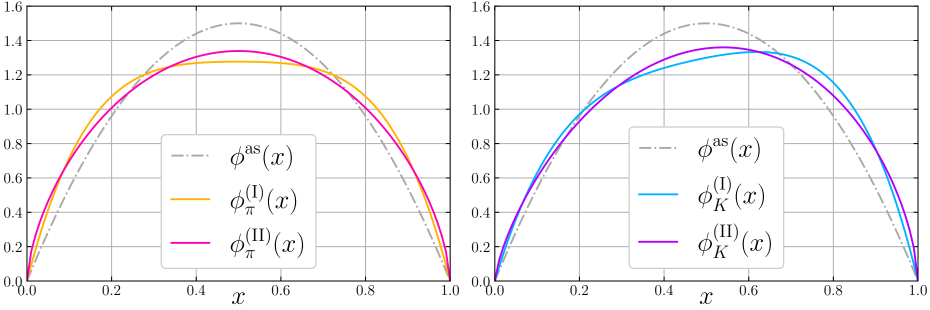
$$a_1^{K,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.0533^{+18}_{-19}(9)_r(23)_a(17)_m = \frac{5}{3}\langle\xi^1\rangle_K$$

Higher moments

Model I: $\phi^{(\mathrm{I})}(u) = 6u(1-u)\left[1 + a_1 C_1^{3/2}(\xi) + a_2 C_2^{3/2}(\xi)\right]$

Model II: $\phi^{(\mathrm{II})}(u) = \frac{\Gamma(2+\alpha^+ + \alpha^-)}{\Gamma(1+\alpha^+)\Gamma(1+\alpha^-)}\, u^{\alpha^+}(1-u)^{\alpha^-}$

with a_1, a_2 fixed from our analysis



$x \mapsto u$

Note that in model I $\langle \xi^4 \rangle_\pi = 0.108^{+6}_{-5}$ and in model II $\langle \xi^4 \rangle_\pi = 0.112^{+7}_{-7}$.
Discrimination requires very high precision of $\langle \xi^4 \rangle_\pi$!

Is a 1% error on $\langle \xi^4 \rangle_\pi$ achievable? The present error on $\langle \xi^2 \rangle_\pi$ is 3.6%.

Euclidean spacetime determination of the DA

Large momentum effective theory (LaMET) [X Ji,1305.1539]: compute "quasi-distribution", in analogy to "quasi-PDF".

$$\tilde{\phi}_\pi(u, a^{-1}, p_z) = \frac{i}{F_\pi}\int_{-\infty}^{\infty}\frac{dZ}{2\pi}e^{-i(u-1)p_z Z}\langle\pi(p)|\bar{\psi}(0)\gamma_z\gamma_5[0,z]\psi(z)|0\rangle,$$

with quark fields separated along the spatial z direction $((z^\mu) = (0,0,0,Z))$. Then match to pion DA (like [X Ji, 1506.00248] for PDFs):

$$\tilde{\phi}_\pi(u, a^{-1}, p_z) = \int_u^1 \frac{dv}{v} Z_\phi\left(\frac{u}{v}, a^{-1}, \mu, p_z\right)\phi_\pi(v,\mu) + \overbrace{\mathcal{O}\left(\Lambda^2/p_z^2, M_\pi^2/p_z^2\right)}^{\text{no } u^{-1},\,(1-u)^{-1}\text{ ???}}.$$

Practical problems: Z_ϕ has many arguments and is power divergent.

However: [K Orginos, A Radyushkin, J Karpie, S Safeiropoulos, 1706.05373]

Perturbative matching to $\overline{\mathrm{MS}}$ at large Z? Contribution suppressed for large p_z!

Discard large Z data and integrate over $d(p_z Z)/p_z$ to minimize corrections?

Does $\exists$ enough data in relevant $p_z Z$ region to carry out Fourier transform?

Window and statistics: $\pi/a \gg p_z \ggg m_N$ for nucleon PDF.

The pion DA is an interesting test case.

What do we do differently?

We will compute the DA in X-**space**. The method is related to "quasi-PDFs", however, there are **essential** differences:

- We **do not** employ a Wilson line $[z/2, -z/2]$ and we **throw away** $2/|z| < 1\,\mathrm{GeV}$ data. This simplifies renormalization and matching.
- Most importantly: we **do not** transform the DA to the longitudinal momentum fraction space.

We compute an object $(z = (0, \vec{z}) \Rightarrow z^2 = -\vec{z}^2 < 0\,,\ |p \cdot z| = |-\vec{p} \cdot \vec{z}|)$

$$\sqrt{2}\,T(p \cdot z, z^2) = \langle 0|\bar{d}(z/2)\Gamma_A\, \overbrace{q(z/2)\ \bar{q}(-z/2)}\,\Gamma_B u(-z/2)|\pi^+(p)\rangle$$

q is an auxiliary field. We use $J_{\Gamma_A} J_{\Gamma_B} = SP + PS,\ VA + AV,\ VV + AA$.
$T(z^2)$ can then be factorized just like

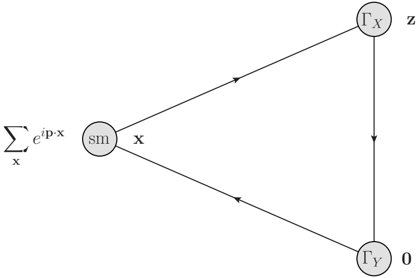
$$F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3}\int_0^1 du\, H_{\bar{q}q\gamma}(u, \mu, Q^2)\,\phi_\pi(u, \mu) \quad + \quad \text{higher twist}$$

into a hard, perturbative function and the DA, as long as $2/|z| \gg \Lambda$.
We set $\mu_R = 2/|z|$, which "plays" the role of $|Q|$. $(|z| = \sqrt{-z^2})$

$T(p \cdot z, z^2)$

Following [VM Braun, D Müller, 0709.1348], we use a light quark propagator for $q(z/2)\bar{q}(-z/2)$. Advantage: Renormalization factor of $\bar{q}\Gamma u$ known. Other suggestions:

- "Static" propagator: Large Energy Effective Theory (LEET) [MJ Dugan, B Grinstein, PLB255(91)583], Large Momentum Effective Theory (LaMET) [X Ji, 1305.1539],
- Scalar propagator: [U Aglietti et al, hep-ph/9806277]; [A Abada et al, hep-ph/0105221],
- Heavy quark propagator: [W Detmold, CJD Lin, hep-lat/0507007].



Tree level result ($\Gamma_A, \Gamma_B = \gamma_5, \mathbb{1}$):

$$T(p \cdot z, z^2) = F_\pi \frac{p \cdot z}{2\pi^2 z^4} \Phi_\pi(p \cdot z),$$

with the X-space (Ioffe time) DA

$$\Phi_\pi(p \cdot z) = \int_0^1 du\, e^{i(u-1/2)p\cdot z} \phi_\pi(u).$$

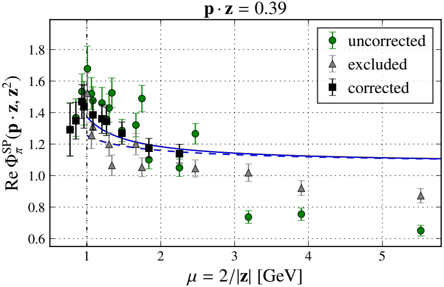
The DA in momentum and in X-space

In practice we compute for large t (example: $\Gamma_A\Gamma_B = \frac{1}{2}\{\gamma_5, \mathbb{1}\}$)

$$\frac{T(p\cdot z, z^2)}{F_\pi} = \frac{Z_S Z_P}{Z_A} \frac{\langle 0|[\bar{d}\mathbb{1}q](t,\vec{z}/2)\ [\bar{q}\gamma_5 u](t,-\vec{z}/2)\ O_\pi^\dagger(0,\vec{p})|0\rangle}{\langle 0|[\bar{d}\gamma_0\gamma_5 u](t,\vec{0})\ O_\pi^\dagger(0,\vec{p})|0\rangle} E(\vec{p}),$$

where the lattice currents are renormalized to the $\overline{\mathrm{MS}}$ scheme via $Z_S(\mu_R a, g^2)$, $Z_P(\mu_R a, g^2)$ and $Z_A(g^2)$ and $\mu_R = \mu_F = 2/|z|$.
We tree-level correct for $\vec{z}$-dependent lattice artefacts:

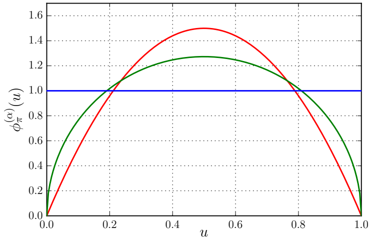
$$T(p\cdot z, z^2) \mapsto T(p\cdot z, z^2)\frac{\mathrm{tr}\,[\not{z}\, G^{\mathrm{tree}}_{\mathrm{cont}}(z)]}{\mathrm{tr}\,[\not{z}\, G^{\mathrm{tree}}_{\mathrm{latt}}(z,a)]} \quad \text{(for the chiral even part)}$$

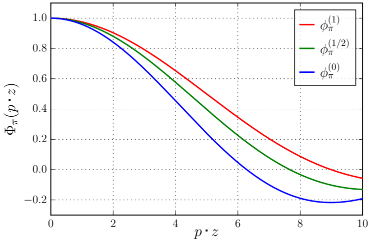


Three illustrative models of the DA (taken at a scale $\mu_0 = 1\,\text{GeV}$):

$$\phi_\pi^{(1)}(u) = 6u(1-u), \quad \phi_\pi^{(1/2)}(u) = \frac{8}{\pi}\sqrt{u(1-u)}, \quad \phi_\pi^{(0)}(u) = 1$$

$$\phi_\pi^{(\alpha)} = \frac{\Gamma[2(\alpha+1)]}{[\Gamma(\alpha+1)]^2}\,[u(1-u)]^\alpha$$





$|z| < 2/\text{GeV} \approx 0.4\,\text{fm}$ is **necessary** for factorization, $\overline{\text{MS}}$ scheme matching!

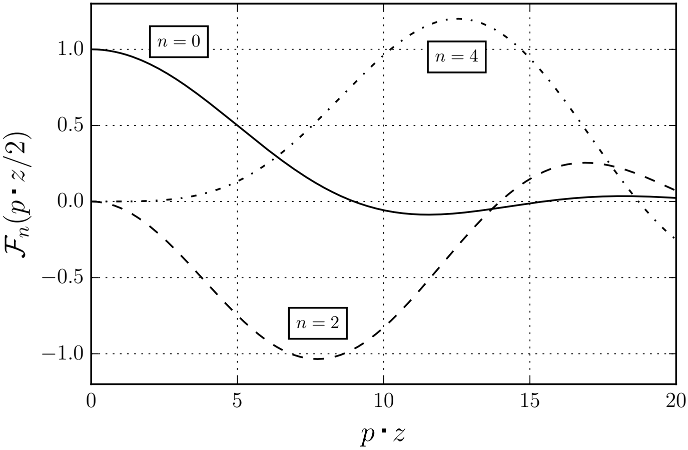
Large $|\vec{p}\cdot\vec{z}|/2$ values are **desirable** (conjugate to $\xi = 2u - 1$).

$|z|$ small $\Rightarrow$ large $|\vec{p}|$. No other reason why $|\vec{p}|$ has to be large!

Conformal partial waves: how large should $|p \cdot z|$ be?

Ioffe time DA: $$\Phi_\pi(p \cdot z, \mu) = \sum_{n \geq 0} a_{2n}^\pi \mathcal{F}_{2n}\left(\frac{1}{2}|p \cdot z|\right)$$

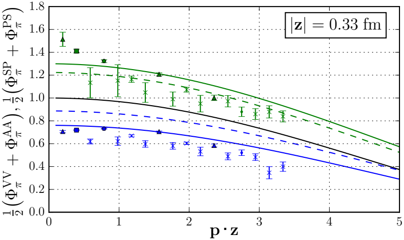
No sensitivity to a_4 for $|p \cdot z| < 4$!!!

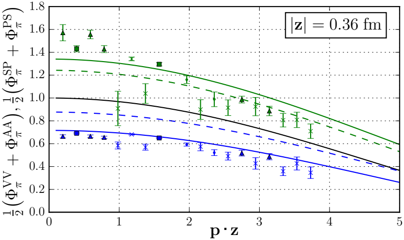


XDA Results I: comparison to expectation: VV and SP

$N_f = 2$ NP improved Wilson-clover quarks (QCDSF ensemble)

$a^{-1} \approx 2.76\,\text{GeV}$, $M_\pi \approx 290\,\text{MeV}$, $L = 32a \approx 3.4/M_\pi$, $a_2^\pi \approx 0.136$





Large $|\vec{p}|a$, $a/|z| \to$ bigger lattice corrections. QCD factorization:

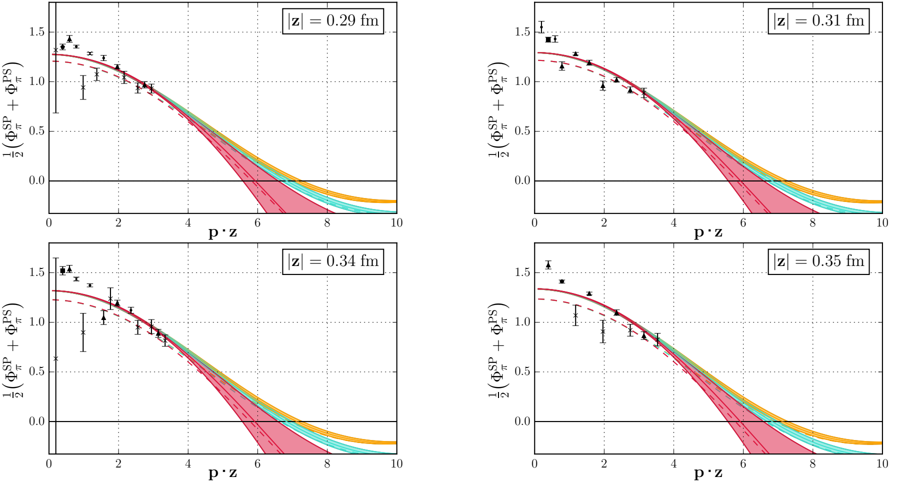
$$\frac{T(p\cdot z, z^2)}{F_\pi} = \frac{p\cdot z}{2\pi^2 z^4} \int_0^1 du\, e^{i(u-1/2)p\cdot z}\, H(u,\mu,z^2)\phi_\pi(u,\mu) + T^{\mathrm{HT}}\,,$$

We have computed $H(u,\mu,z^2)$ to order α_s. $T^{\mathrm{HT}} = \mathcal{O}(M_\pi^2 z^2, \Lambda^2 z^2)/z^4$ can be estimated from LCSR: $\delta_2^\pi \approx 0.17\,\text{GeV}^2$.

[VA Novikov et al, NPB237(84)525; VM Braun,A Khodjamirian,M Maul,hep-ph/9907495]

XDA Results II: SP+PS ($\mu = \mu_R = 2/|z|$) with fits

Dashed lines: higher twist effects subtracted from the fits.

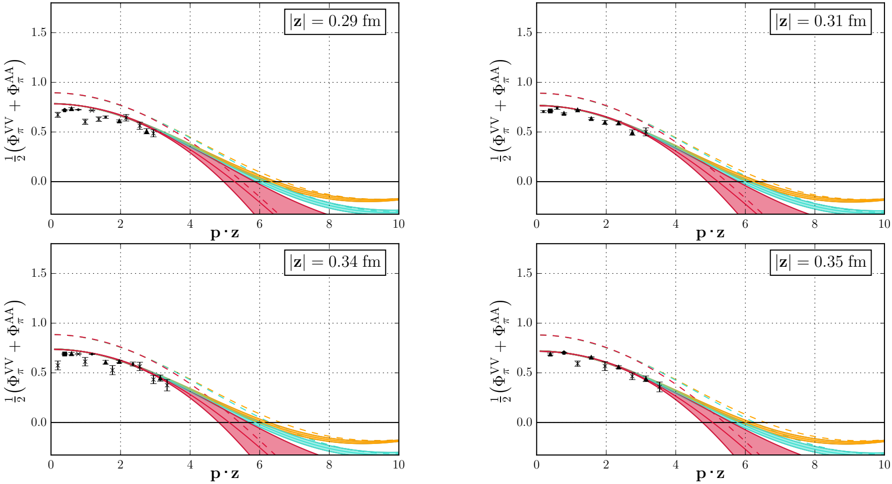


The future: smaller a, larger $|\vec{p}|$.

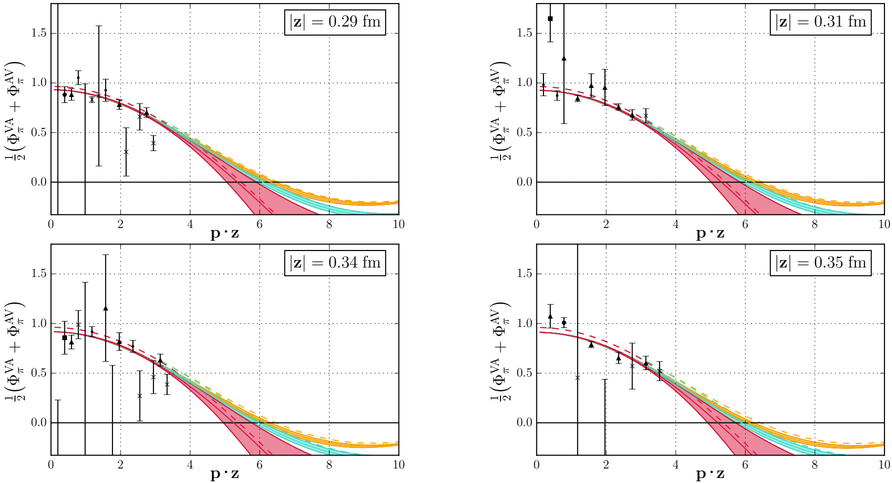
XDA Results III: Ioffe time DA from VV+AA

VV+AA current:

Higher twist effects (and data!) go in the different direction.



XDA Results IV: Ioffe time DA from VA+AV

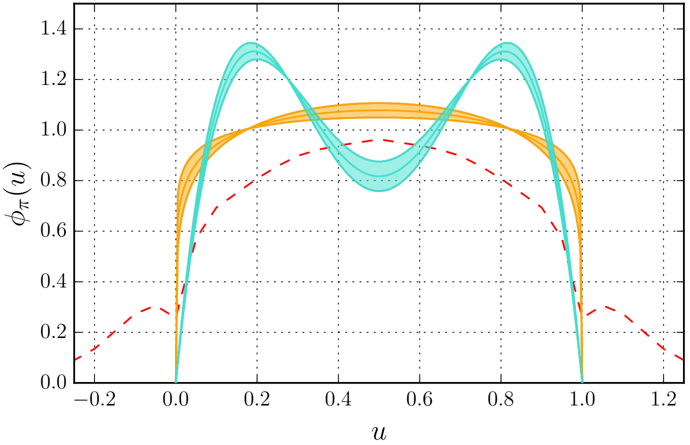


All fits give $a_2^\pi \sim 0.2 - 0.3$ (instead of ~ 0.14 from the local method).

But higher twist $0.2\,\mathrm{GeV}^2 \lesssim \delta_2^\pi \lesssim 0.25\,\mathrm{GeV}^2$ is well constrained.

XDA Results V: longitudinal momentum fraction DA

Two fitted DAs at $\mu = 2\,\text{GeV}$. Different shapes but similar a_2^π!
Also discrimination from experimental data is difficult!



Dashed: $N_f \approx 2+1+1$, $M_\pi = 310\,\text{MeV}$, $a = 0.12\,\text{fm}$, $LM_\pi = 4.5$
From quasi-DA: [LP^3: J-W Chen et al, 1712.10025]

Outlook

- Gegenbauer/Mellin moments: Problem solved. What next?
  - So far NP matching to RI'/SMOM (RI'/MOM) scheme and then to $\overline{\text{MS}}$ at NNLO (NNNLO). Exploring NNNLO X-space matching.
  - Further reduce statistical errors and increase number of data points?
  - How useful can novel methods for higher moments be, given the required statistics?
- Euclidean X-space method:
  - For $2/|z| \gtrsim 1\,\text{GeV}$ we need $|\vec{p}| \gtrsim 3\,\text{GeV}$ to reach "Ioffe times" $|p \cdot z|$ large enough to discriminate between DA parametrizations
    $\Rightarrow$ small lattice spacings are necessary.
  - Different current combinations depend differently on higher twist contributions: First lattice determination of higher twist DA!
  - Ideal: smaller $|z|$ to suppress higher twist effects $\Rightarrow$ even higher $|\vec{p}|$.
  - Already at the limit of the state-of-the art for the pion DA: what about nucleon PDFs?
  - Good news: matching to the $\overline{\text{MS}}$ scheme is easier for currents without derivatives and the matching function requires "only" a continuum calculation $\Rightarrow$ NNLO calculation ongoing.
- The future: combine different approaches within global fits.