Matching for gluon quasi distribution in LaMET

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Based on arXiv:1904.00978 by Wei Wang, Jian-Hui Zhang, SZ and Ruilin Zhu

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Large Momentum Effective Theory

- Parton distribution functions:

$$f_{q/H}(x,\mu^2) = \int \frac{d\xi^-}{4\pi} e^{-i\xi^- xP^+} \langle P|\overline{\psi}(\xi^-)\gamma^+ \mathcal{P}\exp\left(-ig\int_0^{\xi^-} d\eta A^+(\eta)\right)\psi(0)|P\rangle,$$

- Large momentum effective theory (LaMET): A novel approach of accessing parton physics on the lattice (Ji,2013, 2014)

- The quasi-PDFs: light-cone correlation matrix elements->equal-time correlation matrix element

$$\tilde{q}(x,\Lambda,P_z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{-ixzP_z} \left\langle P\middle|\bar{\psi}(z)\,\gamma_z \exp\left(-ig\int_0^z dz'\,A^z(z')\right)\psi(0)\middle|P\right\rangle$$

- The light-cone PDF is related to quasi-PDF by a matching equation:

$$\tilde{f}_{i/H}(x,P^z) = \int_0^1 \frac{dy}{y} Z_{ij}\left(\frac{x}{y},\frac{\mu}{P^z}\right) f_{j/H}(y,\mu) \qquad Z_{ij}\left(\xi,\frac{\mu}{P^z}\right) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{2\pi}\right)^n Z_{ij}^{(n)}\left(\xi,\frac{\mu}{P^z}\right)$$

- Lattice cross-sections see Jianwei's talk

- Pseudo-PDFs see Anatoly's talk

Gluon quasi-PDF

- Gluon PDFs are key input parameters for physics at hadron colliders
- Important for various physical processes, e.g., higgs and quarkonium production at hadron colliders.
- At ultrahigh energy, gluon PDF dominates.
- Crucial in the study of hadron structure and spin physics.
- Gluon quasi PDF is also important to extract quark PDFs.

Gluon quasi-PDF

- Gluon PDF

$$f_{g/H}(x,\mu) = \int \frac{d\xi^-}{2\pi x P^+} e^{-i\xi^- x P^+} \langle P | G^+_{\ i}(\xi^-) W(\xi^-, 0; L_{n^+}) G^{i+}(0) | P \rangle$$

- Gluon quasi PDF

$$\tilde{f}_{g/H}(x,P^z) = \int \frac{dz}{2\pi x P^z} e^{izxP^z} \langle P | G^z_{\ i}(z) W(z, 0; L_{n^z}) G^{iz}(0) | P \rangle$$

G: gluon field strength tensor

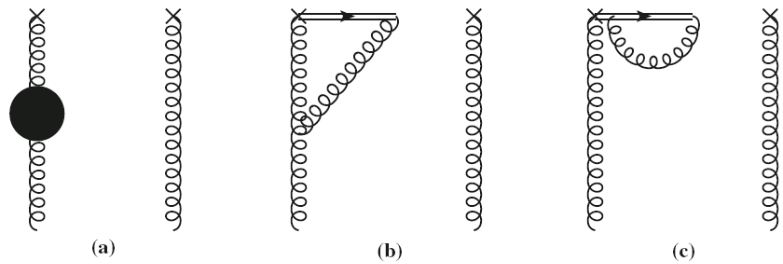
W: Wilson line, links 0 and ξ^- (or z), adjoint representation

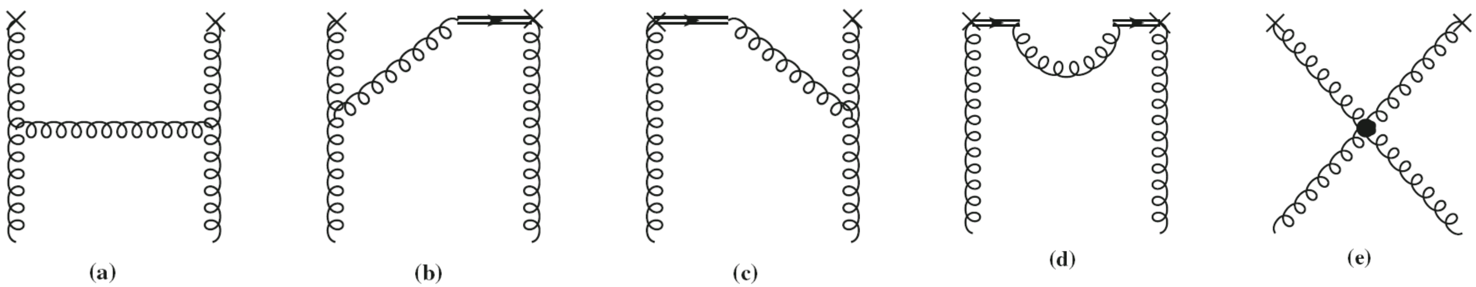
- Lattice simulation:

Z.Y. Fan, Y. B. Yang, A. Anthony, H.W. Lin, K.F. Liu, Phys.Rev.Lett. 121 (2018) no.24, 242001

See Yi-Bo's talk

Gluon quasi-PDF: one-loop matching





- One loop matching with dimensional regularization (DR) and UV cutoff

  W.Wang, SZ, R.L.Zhu, Eur. Phys. J. C (2018)78: 147
  W.Wang, SZ, JHEP 1805 (2018) 142

$$\tilde{f}_{i/H}(x,P^z) = \int_0^1 \frac{dy}{y} Z_{ij}\left(\frac{x}{y},\frac{\mu}{P^z}\right) f_{j/H}(y,\mu) \equiv Z_{ij}\left(\xi,\frac{\mu}{P^z}\right)\otimes f_{j/H}(y),$$

One-loop matching coefficients

❖ Matching coefficients

$$Z^{(1)}_{qq}\left(\xi,\frac{\mu}{P^z}\right)=C_F\begin{cases}\left[-\frac{1+\xi^2}{1-\xi}\ln\frac{\xi-1}{\xi}+1\right]_+, & \xi>1\\ \left[-\frac{1+\xi^2}{1-\xi}\ln\frac{\mu^2}{4(P^z)^2\xi(1-\xi)}+\frac{2-5\xi+\xi^2}{1-\xi}\right]_+, & 0<\xi<1\\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi-1}{\xi}-1\right]_+, & \xi<0\end{cases}$$

$$Z^{(1)}_{qg}\left(\xi,\frac{\mu}{P^z}\right)=T_F\begin{cases}(\xi^2+(1-\xi)^2)\ln\frac{\xi}{\xi-1}-2\xi+1, & \xi>1\\ -(\xi^2+(1-\xi)^2)\ln\frac{\mu^2}{4\xi(1-\xi)(P^z)^2} & \\ +1+2\xi-4\xi^2, & 0<\xi<1\\ -(\xi^2+(1-\xi)^2)\ln\frac{\xi}{\xi-1}+2\xi-1, & \xi<0\end{cases}$$

$$Z^{(1)}_{gq}\left(\xi,\frac{\mu}{P^z}\right)=C_F\begin{cases}\frac{1+(1-\xi)^2}{\xi}\ln\frac{\xi}{\xi-1}-1+\frac{5}{2\xi}, & \xi>1\\ -\frac{1+(1-\xi)^2}{\xi}\ln\frac{\mu^2}{4\xi(1-\xi)(P^z)^2} & \\ +3-\frac{1}{2\xi}, & 0<\xi<1\\ -\frac{1+(1-\xi)^2}{\xi}\ln\frac{\xi}{\xi-1}+1-\frac{5}{2\xi}, & \xi<0\end{cases}$$

$$Z^{(1)}_{gg}\left(\xi,\frac{\mu}{P^z}\right)=C_A\begin{cases}\frac{2\xi^3-3\xi^2+2\xi-2}{\xi}\ln\frac{\xi-1}{\xi} & \\ +\xi\left[\frac{1+\xi}{\xi-1}\ln\frac{\xi-1}{\xi}+1\right]_++\xi-1+\frac{8}{3\xi}, & \xi>1\\ \frac{2\xi^3-3\xi^2+2\xi-2}{\xi}\ln\frac{\mu^2}{4\xi(1-\xi)(P^z)^2} & \\ +\xi\left[\frac{1+\xi}{\xi-1}\ln\frac{\mu^2}{4\xi(1-\xi)(P^z)^2}\right]_+ & \\ +\delta(1-\xi)\left(\frac{5}{3}C_A-\frac{4}{3}T_Fn_f\right)\ln\frac{(P^z)^2}{\mu^2} & \\ -\left[\frac{2\xi^2-\xi+1}{1-\xi}\right]_++\frac{10\xi^2}{3}-4\xi+4-\frac{2}{3\xi}, & 0<\xi<1\\ -\frac{2\xi^3-3\xi^2+2\xi-2}{\xi}\ln\frac{\xi-1}{\xi} & \\ -\xi\left[\frac{1+\xi}{\xi-1}\ln\frac{\xi-1}{\xi}+1\right]_+-\xi+1-\frac{8}{3\xi}, & \xi<0\end{cases}$$

$$\xi\equiv x/y,$$

x is the momentum fraction in quasi PDF, y is the momentum fraction in PDF.

W.Wang,SZ, R.L.Zhu, Eur. Phys. J. C (2018)78: 147

One-loop matching: DR vs cutoff

DR

- Preserve gauge invariance
- Lattice evaluation needs cutoff. Unpractical
- Power divergences only exist in Wilson line's self energy

Naïve cutoff

- Lattice provides cutoff. More "practical" than DR
- Power divergence. Mixes with other operators
- Gauge symmetry broken

Renormalization of quasi-PDFs

❖Well defined continuum limit call for renormalization

❖Renormalization of Quasi PDF on the Lattice

- Lattice perturbation:

  Complicated Feynman rules, hard to evaluate

  See Xiaonu's talk

- Nonperturbative methods

  gradient flow (K. Orginos and C. Monohan, 2016)

  Regularization independent momentum subtraction (RI/MOM)

  I.Stewart and Y.Zhao, Phys.Rev. D97 (2018) no.5, 054512

  LP3, Phys.Rev. D97 (2018) no.1, 014

  M. Constantinou and H. Panagopoulos, Phys.Rev. D96 (2017) no.5, 054506

  ETMC, Nucl.Phys. B923 (2017) 394-415

RI/MOM

- Renormalization condition

$$\tilde{Z}(z, p_z^R, a^{-1}, \mu_R) = \left. \frac{\mathrm{Tr}[\not{p} \sum_s \langle ps|O_{\gamma_t}(z)|ps\rangle]}{\mathrm{Tr}[\not{p} \sum_s \langle ps|O_{\gamma_t}(z)|ps\rangle_{tree}]} \right|_{\substack{p^2 = -\mu_R^2 \\ p_z = p_z^R}}$$

- Renormalized quark quasi-PDF

$$\tilde{h}^R(z, P_z, p_z^R, \mu_R) = \tilde{Z}^{-1}(z, p_z^R, a^{-1}, \mu_R)\tilde{h}(z, P_z, a^{-1})\Big|_{a\to 0},$$

$$\tilde{h}(z, P_z, a^{-1}) = \frac{1}{2P^0}\langle P|O_{\gamma_t}(z)|P\rangle$$

$$\tilde{q}(x, P_z, p_z^R, \mu_R) = P_z \int \frac{dz}{2\pi}\, e^{ixP_z z}\tilde{h}^R(z, P_z, p_z^R, \mu_R)\,.$$

- The renormalized matrix element does not depend on regularization scheme
- Matching in RI/MOM: convert lattice regularized quasi-distribution to MSbar light-cone distribution
- Has been used in: quark PDFs, meson DAs, GPDs, …

See Yong and Yu-Sheng's talk

Renormalization of quasi-PDFs

- Quark quasi-PDF: multiplicatively renormalizable

  X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. Lett. 120, 112001 (2018)

  Ishikawa, Ma, Qiu, Phys.Rev. D96 (2017) no.9, 094019

  J. Green, K. Jansen and F. Steffens, Phys. Rev. Lett. 121, 022004 (2018)

$$\overline{\psi}_B(z)\gamma^z W^B(z,0)\psi_B(0) = Z_{\psi,z} e^{\delta m|z|}\overline{\psi}_R(z)\gamma^z W^R(z,0)\psi_R(0)$$

- Gluon quasi-PDF:

  Li, Ma, Qiu, Phys.Rev.Lett. 122 (2019) no.6, 062002

  Zhang, Ji, Schäfer, Wang, SZ, Phys.Rev.Lett. 122 (2019) no.14, 142001

Four multiplicatively renormalizable operators

$$O_R^1(z_2,z_1) = Z_{11}^2 e^{\overline{\delta m}\Delta z} F^{ti}(z_2)L(z_2,z_1)F^{ti}(z_1), \quad ✓$$

$$O_R^2(z_2,z_1) = Z_{22}^2 e^{\overline{\delta m}\Delta z} F^{zi}(z_2)L(z_2,z_1)F^{zi}(z_1), \quad ✓$$

$$O_R^3(z_2,z_1) = Z_{11}Z_{22} e^{\overline{\delta m}\Delta z} F^{ti}(z_2)L(z_2,z_1)F^{zi}(z_1) \quad ✓$$

$$O_R^4(z_2,z_1) = Z_{22}^2 e^{\overline{\delta m}\Delta z} F^{z\mu}(z_2)L(z_2,z_1)F^{z}{}_{\mu}(z_1), \quad ✓$$

$$O_{g,R}^{(5)}(z_2,z_1) \equiv (F^{t\mu}(z_2)\mathcal{W}(z_2,z_1)F^{t}{}_{\mu}(z_1))_R = -O_{g,R}^{(1)}(z_2,z_1) - O_{g,R}^{(2)}(z_2,z_1) - O_{g,R}^{(4)}(z_2,z_1) \quad ✗$$

See Jian-Hui's talk

Renormalization in RI/MOM

- Renormalization equation

$$\begin{pmatrix} O_g^{(n)}(z,0) \\ O_q^s(z,0) \end{pmatrix} = \begin{pmatrix} Z_{11}(z) & Z_{12}(z)/z \\ zZ_{21}(z) & Z_{22}(z) \end{pmatrix} \begin{pmatrix} O_{g,R}^{(n)}(z,0) \\ O_{q,R}^s(z,0) \end{pmatrix} \qquad \bar{\mathcal{Z}} = \begin{pmatrix} \bar{Z}_{11}(z) & \bar{Z}_{12}(z)/z \\ z\bar{Z}_{21}(z) & \bar{Z}_{22}(z) \end{pmatrix} = \begin{pmatrix} Z_{11}(z) & Z_{12}(z)/z \\ zZ_{21}(z) & Z_{22}(z) \end{pmatrix}^{-1}$$

- Renormalization condition:

$$\frac{\mathrm{Tr}[\Lambda_{22}(p,z)\mathcal{P}]_R}{\mathrm{Tr}[\Lambda_{22}(p,z)\mathcal{P}]_{\mathrm{tree}}}\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 1, \qquad \frac{[P_{ij}^{ab}\Lambda_{11}^{ab,ij}(p,z)]_R}{[P_{ij}^{ab}\Lambda_{11}^{ab,ij}(p,z)]_{\mathrm{tree}}}\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 1,$$

$$\mathrm{Tr}[\Lambda_{12}(p,z)\mathcal{P}]_R\Big|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 0, \qquad [P_{ij}^{ab}\Lambda_{21}^{ab,ij}(p,z)]_R\Big|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 0,$$

P: projectors
Λ: offshell amputated Green's functions

- Renormalized matrix element

$$h_{g,R}^{(n)}(z,P^z,\mu_R,p_z^R) = \bar{Z}_{11}(z,\mu_R,p_z^R,1/a)h_g^{(n)}(z,P^z,1/a) + \bar{Z}_{12}(z,\mu_R,p_z^R,1/a)/z\; h_q^s(z,P^z,1/a),$$
$$h_{q,R}^s(z,P^z,\mu_R,p_z^R) = \bar{Z}_{22}(z,\mu_R,p_z^R,1/a)h_q^s(z,P^z,1/a) + z\bar{Z}_{21}(z,\mu_R,p_z^R,1/a)\; h_g^{(n)}(z,P^z,1/a).$$

Matching equation

❖ Matching equation:

$$\tilde{f}^{(n)}_{g/H}(x, P^z, p^R_z, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} \Big[C_{gg}\Big(\frac{x}{y}, \frac{\mu_R}{p^R_z}, \frac{yP^z}{\mu}, \frac{yP^z}{p^R_z}\Big) f_{g/H}(y,\mu) + C_{gq}\Big(\frac{x}{y}, \frac{\mu_R}{p^R_z}, \frac{yP^z}{\mu}, \frac{yP^z}{p^R_z}\Big) f_{q_j/H}(y,\mu) \Big]$$
$$+ \mathcal{O}\Big(\frac{M^2}{(P^z)^2}, \frac{\Lambda^2_{\rm QCD}}{(P^z)^2}\Big),$$
$$\tilde{f}_{q_i/H}(x, P^z, p^R_z, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} \Big[C_{q_i q_j}\Big(\frac{x}{y}, \frac{\mu_R}{p^R_z}, \frac{yP^z}{\mu}, \frac{yP^z}{p^R_z}\Big) f_{q_j/H}(y,\mu) + C_{qg}\Big(\frac{x}{y}, \frac{\mu_R}{p^R_z}, \frac{yP^z}{\mu}, \frac{yP^z}{p^R_z}\Big) f_{g/H}(y,\mu) \Big]$$
$$+ \mathcal{O}\Big(\frac{M^2}{(P^z)^2}, \frac{\Lambda^2_{\rm QCD}}{(P^z)^2}\Big),$$

❖ Polarized PDFs also have same matching equation but with different matching coefficients

❖ Can be derived by OPE

Unpolarized PDF: quark in quark

- The amputated Green's function has structure

$$\Lambda_{\gamma^t}(p,z) = \widetilde{f}_t(p,z)\gamma^t + \widetilde{f}_z(p,z)\frac{p^t\gamma^z}{p^z} + \widetilde{f}_p(p,z)\frac{p^t \not{p}}{p^2}$$

- Minimal Projection:

Project out $\widetilde{f}_t(p,z)$ LP3, 1807.06566

Project out $\widetilde{f}_t(p,z)$ and $\widetilde{f}_z(p,z)$ This work

- Bare quark quasi PDF

$$\left[\tilde{f}^{(1)}_{q/q}(x,\rho\to 0)\right]_+ = \left[\tilde{f}^{(1)}_{q/q,t}(x,\rho\to 0)\right]_+ + \left[\tilde{f}^{(1)}_{q/q,z}(x,\rho\to 0)\right]_+ \qquad \rho = -p^2/p_z^2$$

- Matching coefficient

$$C^{(1)}_{qq}\left(x, r, \frac{p_z}{\mu}, \frac{p_z}{p_z^R}\right) = \left[\tilde{f}^{(1)}_{q/q}(x,\rho\to 0) - f^{(1)}_{q/q}\left(x, \frac{\mu^2}{-p^2}\right) - (\tilde{f}^{(1)}_{q/q})_{C.T.}\right]_+$$

$$(\tilde{f}^{(1)}_{q/q})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| \tilde{f}^{(1)}_{q/q,t}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right) + \left|\frac{p_z}{p_z^R}\right| \tilde{f}^{(1)}_{q/q,z}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right)$$

$$r = \mu_R^2/(p_z^R)^2.$$

Unpolarized PDF : gluon in gluon

- Projector: $P_{ij}^{ab} = \delta^{ab} g_{\perp,ij}/(2-D)$
- At one-loop level, the partonic quasi-PDF can be written as

$$x\tilde{f}_{g/g}^{(n)}(x,\rho) = \left[x\tilde{f}_{g/g}^{(n)}(x,\rho)\right]_+ + \tilde{c}^{(n)}\delta(x-1), \qquad \tilde{c}^{(n)} = \frac{1}{p_z^2} N^{(n)} \langle g(p)|O_g^{(n)}(0,0)|g(p)\rangle$$

- Offshell gluon matrix element can mix with gauge variant operators

$$\begin{aligned}
\tilde{c}^{(1,g)} &= \frac{\alpha_s C_A}{12\pi\epsilon} \frac{p^2}{p^2+p_z^2} + \mathcal{O}(\epsilon^0), \\
\tilde{c}^{(2,g)} &= -\frac{\alpha_s C_A}{12\pi\epsilon} \frac{p^2}{p_z^2} + \mathcal{O}(\epsilon^0), \\
\tilde{c}^{(3,g)} &= \mathcal{O}(\epsilon^0), \\
\tilde{c}^{(4,g)} &= \frac{\alpha_s C_A}{3\pi\epsilon} \frac{p^2}{p_z^2} + \mathcal{O}(\epsilon^0),
\end{aligned}$$

- O_3 is a “good operator” because it is the “t z” component of the gluon energy-momentum tensor
- For O_1, O_2, O_4, the large x behavior depends on the offshellness of external gluon

Unpolarized PDF : gluon in gluon

- Asymptotic region of quark quasi-PDF

$$\lim_{\xi\to\infty} C_B\left(\xi, \frac{p^z}{\mu}\right) = -\frac{3}{2|\xi|} \qquad \lim_{\xi\to\infty} \left|\frac{p^z}{p_z^R}\right| h\left(1+\frac{p^z}{p_z^R}(\xi-1), r\right) = -\frac{3}{2|\xi|}$$

From Yong's talk

- For gluon quasi-PDF defined by O_1

$$\lim_{\xi\to\infty} C_B\left(\xi, \frac{p^z}{\mu}\right) \propto -\frac{1}{|\xi|} \qquad \lim_{\xi\to\infty} \left|\frac{p^z}{p_z^R}\right| h\left(1+\frac{p^z}{p_z^R}(\xi-1), r\right) \propto -\frac{1}{(1-r)|\xi|}$$

The UV divergence is subtracted but the asymptotic behavior is not

- Mixing with gauge variant operators, even for local operator

J. C. Collins and R. J. Scaliset, PRD50,1994

Unpolarized PDF : gluon in gluon

- Bare gluon quasi-PDF

$$\left[x\tilde{f}^{(3,1)}_{g/g}(x,\rho\to 0)\right]_+ = \frac{\alpha_s C_A}{2\pi}\begin{cases}\left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{x-1}{x}+\frac{4x^3-6x^2+8x-5}{2(x-1)}\right]_+, & x>1\\ \left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{\rho}{4}+\frac{12x^4-24x^3+30x^2-17x+5}{2(x-1)}\right]_+, & 0<x<1\\ \left[-\frac{2(1-x+x^2)^2}{x-1}\ln\frac{x-1}{x}-\frac{4x^3-6x^2+8x-5}{2(x-1)}\right]_+, & x<0.\end{cases}$$

- Light-cone PDF

$$\left[xf^{(1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right)\right]_+ = \theta(x)\theta(1-x)\left\{\frac{\alpha_s C_A}{2\pi}\left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{-p^2x(1-x)}{\mu^2}+2x^3-2x^2+3x-2\right]_+ - \frac{\alpha_s C_A}{4\pi}\left[\frac{x}{1-x}\right]_+\right\},$$

- Matching coefficient

$$xC^{(3,1)}_{gg}\left(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}\right) = \left[x\tilde{f}^{(3,1)}_{g/g}(x,\rho\to 0) - xf^{(1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right) - (x\tilde{f}^{(3,1)}_{g/g})_{C.T.}\right]_+ + \left(\tilde{c}^{(3,g)}_{\mathrm{RI/MOM}} - c^{3,g}_{\overline{\mathrm{MS}}}\right)\delta(x-1),$$

$$(x\tilde{f}^{(3,1)}_{g/g})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| x\tilde{f}^{(3,1)}_{g/g}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right)$$

Unpolarized PDF : gluon in gluon

- The counter term

$$
\begin{aligned}
&\left[x\tilde{f}^{(3,1)}_{g/g}(x,\rho)\right]_+ \\
&=\frac{\alpha_s C_A}{2\pi}\begin{cases}
\Bigg[\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\
+\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{8x^4-16x^3-22x^2+34x-9}{4(\rho-1)(x-1)(2x-1)}-\frac{8x^3(x-1)}{(\rho+4x^2-4x)^2}+\frac{3(2x-1)x}{2(\rho-1)^2}-\frac{4x+1}{4(x-1)}\Bigg]_+, & x>1 \\
\Bigg[\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}} & \\
+\frac{-30x^2+34x-9}{4(\rho-1)(x-1)}+\frac{3\left(4x^3-4x^2+x\right)}{2(\rho-1)^2}+\frac{6x+1}{4(x-1)}\Bigg]_+, & 0<x<1 \\
\Bigg[-\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\
-\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{-8x^4+16x^3+22x^2-34x+9}{4(\rho-1)(x-1)(2x-1)}+\frac{8x^3(x-1)}{(\rho+4x^2-4x)^2}-\frac{3(2x-1)x}{2(\rho-1)^2}+\frac{4x+1}{4(x-1)}\Bigg]_+, & x<0.
\end{cases}
\end{aligned}
$$

Unpolarized PDF :Gluon in quark

- Light-cone splitting function

$$xf^{(1)}_{g/q}\left(x,\frac{\mu^2}{-p^2}\right)=\frac{\alpha_s C_F}{2\pi}\left[(1+(1-x)^2)\ln\frac{\mu^2}{-p^2x(1-x)}+x(1-x)-2\right].$$

- Bare splitting function

$$\mathrm{Tr}\left[\left(xf^{(3,1)}_{g/q,t}(x,\rho)\gamma^t+xf^{(3,1)}_{g/q,z}(x,\rho)\frac{p^t}{p^z}\gamma^z+xf^{(3,1)}_{g/q,p}(x,\rho)\frac{p^t\not{p}}{p^2}\right)\mathcal{P}\right],$$

$$x\tilde{f}^{(3,1)}_{g/q}(x,\mu,P^z)=\frac{\alpha_s C_F}{2\pi}\begin{cases}-\left(1+(1-x)^2\right)\ln\frac{x-1}{x}-x+2, & x>1\\ -\left(1+(1-x)^2\right)\ln\frac{\rho}{4}-4x^2+6x-2, & 0<x<1\\ \left(1+(1-x)^2\right)\ln\frac{x-1}{x}+x-2, & x<0.\end{cases},$$

- Matching coefficient

$$xC^{(3,1)}_{g/q}\left(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}\right)=\left[x\tilde{f}^{(3,1)}_{g/q}(x,\rho\to 0)-xf^{(1)}_{g/q}\left(x,\frac{\mu^2}{-p^2}\right)-(x\tilde{f}^{(3,1)}_{g/q})_{C.T.}\right],$$

$$(x\tilde{f}^{(3,1)}_{g/q})_{C.T.}=\left|\frac{p_z}{p_z^R}\right|x\tilde{f}^{(3,1)}_{g/q}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right).$$

Unpolarized PDF :Quark in gluon

- Light-cone splitting function

$$f^{(1)}_{q/g}\left(x,\frac{\mu^2}{-p^2}\right)=\frac{\alpha_s T_f}{2\pi}\left[(x^2+(1-x)^2)\ln\frac{\mu^2}{-p^2x(1-x)}-1\right],$$

- Bare splitting function

$$\tilde{f}^{(1)}_{q/g}(x,\rho\to 0)=\frac{\alpha_s T_f}{2\pi}\begin{cases}-(x^2+(1-x)^2)\ln\frac{x-1}{x}-2x+1, & x>1\\ -(x^2+(1-x)^2)\ln\frac{\rho}{4}-6x^2+6x-2, & 0<x<1\\ (x^2+(1-x)^2)\ln\frac{x-1}{x}+2x-1, & x<0.\end{cases}$$

- Matching coefficient

$$C^{(1)}_{qg}\left(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}\right)=\left[\tilde{f}^{(1)}_{q/g}(x,\rho\to 0)-f^{(1)}_{q/g}\left(x,\frac{\mu^2}{-p^2}\right)-(\tilde{f}^{(1)}_{q/g})_{C.T.}\right],$$

$$(\tilde{f}^{(1)}_{q/g})_{C.T.}=\left|\frac{p_z}{p_z^R}\right|\tilde{f}^{(1)}_{q/g}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right).$$

Polarized quasi-PDFs

- Polarized gluon PDF

$$\Delta f_{g/H}(x,\mu) = i\epsilon_{\perp ij}\int \frac{d\xi^-}{2\pi x P^+} e^{-i\xi^- x P^+} \langle P|F^{+i}(\xi^- n_+)\mathcal{W}(\xi^- n_+, 0; L_{n_+})F^{j+}(0)|P\rangle,$$

- Multiplicatively renormalizable polarized gluon quasi-PDFs:

$$\Delta O_g^1(z,0) = i\epsilon_{\perp,ij}F^{ti}(z_2)\mathcal{W}(z_2,z_1)F^{tj}(z_1),$$
$$\Delta O_g^2(z,0) = i\epsilon_{\perp,ij}F^{zi}(z_2)\mathcal{W}(z_2,z_1)F^{zj}(z_1),$$
$$\Delta O_g^3(z,0) = i\epsilon_{\perp,ij}F^{ti}(z_2)\mathcal{W}(z_2,z_1)F^{zj}(z_1),$$

$$\mathcal{P}_{\perp,ij} = \frac{i}{D-2}\epsilon_{\mu\nu ij}n_t^\mu n^\nu$$

Zhang, Ji, Schäfer, Wang, SZ, Phys.Rev.Lett. 122 (2019) no.14, 142001

Polarized PDF: gluon in gluon

- Projector: $\mathcal{P}_{\perp,ij} = \frac{i}{D-2}\epsilon_{\mu\nu ij} n_t^\mu n^\nu.$

- At one-loop level, the partonic quasi-PDF can be written as

$$x\Delta\tilde{f}^{(n)}_{g/g}(x) = [x\Delta\tilde{f}]_+ + \Delta\tilde{c}^{(n)}\delta(x-1)$$

$$\Delta\tilde{c}^{(n)} = \frac{1}{(p^z)^2}\Delta N^{(n)}\langle g(p)|\Delta O^{(n)}_{g,R}(0,0)|g(p)\rangle,$$

$$\Delta N^{(1)} = \frac{(p^z)^2}{(p^t)^2},\ \ \Delta N^{(2)} = 1,\ \ \Delta N^{(3)} = \frac{p^z}{p^t}.$$

with

$$\Delta\tilde{c}^{(1)} = -\frac{\alpha_s C_A(p^2+6(p^z)^2)}{24\pi\epsilon(p^2+(p^z)^2)},$$

$$\Delta\tilde{c}^{(2)} = -\frac{\alpha_s C_A(5p^2+6(p^z)^2)}{24\pi\epsilon(p^z)^2},$$

$$\Delta\tilde{c}^{(3)} = -\frac{\alpha_s C_A}{4\pi\epsilon},$$

Offshell gluon matrix element may mix with gauge variant operators

- We use ΔO_g^3 to define quasi-PDF

Polarized PDF: gluon in gluon

- Light-cone PDF

$$x\Delta f_{g/g}^{(1)}(x,\mu) = \frac{\alpha_s C_A}{2\pi}\left\{\frac{x}{x-1}\left[\left(4x^2-6x+4\right)\ln\frac{-p^2(1-x)x}{\mu^2}+8x^2-11x+7+\frac{(1-\xi)}{2}\right]\right\}_+$$

- Bare quasi-PDF

$$x\Delta\tilde{f}_{g/g}^{(3,1)}(x,\rho) = \frac{\alpha_s C_A}{2\pi}\begin{cases}\frac{8x^2+4\left(2x^2-3x+2\right)x\ln\frac{x-1}{x}-8x+1}{2(x-1)}, & x>1\\ \frac{4\left(2x^2-3x+2\right)x\ln\frac{\rho}{4}+20x^3-28x^2+15x-1}{2(x-1)}, & 0<x<1\\ -\frac{8x^2+4\left(2x^2-3x+2\right)x\ln\frac{x-1}{x}-8x+1}{2(x-1)}, & x<0.\end{cases}$$

The virtual contribution is the same as the unpolarized case

- Matching coefficient

$$x\Delta C_{gg}^{(3,1)}\left(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}\right) = xC_{gg}^{(3,1)}\left(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}\right)+\Bigg[\left(x\Delta\tilde{f}_{g/g}^{(3,1)}(x,\rho\to 0)-x\tilde{f}_{g/g}^{(3,1)}(x,\rho\to 0)\right)$$
$$-\left(x\Delta f_{g/g}^{(3,1)}\left(x,\frac{\mu^2}{-p^2}\right)-xf_{g/g}^{(3,1)}\left(x,\frac{\mu^2}{-p^2}\right)\right)-(x\Delta\tilde{f}_{g/g}^{(3,1)})_{C.T.}\Bigg]$$

$$(x\Delta\tilde{f}_{g/g}^{(3,1)})_{C.T.} = \left|\frac{p_z}{p_z^R}\right|\left[x\Delta\tilde{f}_{g/g}^{(3,1)}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right)-x\tilde{f}_{g/g}^{(3,1)}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right)\right]$$

polarized PDF: Quark in quark

- Light-cone PDF

$$\Delta f^{(1)}_{q/q}\left(x, \frac{\mu^2}{-p^2}\right) = f^{(1)}_{q/q}\left(x, \frac{\mu^2}{-p^2}\right)$$

- Quasi PDF

$$\Delta \tilde{f}^{(1)}_{q/q,z}(x,\rho)|_{\rho\to 0} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\left(x^2+1\right)\ln\frac{x-1}{x}+x-1}{x-1}, & x>1 \\ \frac{2\left(x^2+1\right)\ln\frac{\rho}{4}+4x^2+1}{2(x-1)}, & 0<x<1 \\ -\frac{\left(x^2+1\right)\ln\frac{x-1}{x}+x-1}{x-1}, & x<0. \end{cases}$$

- Matching coefficient

$$\Delta C^{(1)}_{qq}(x, r, \frac{p_z}{\mu}, \frac{p_z}{p_z^R}) = \left[\Delta \tilde{f}^{(1)}_{q/q,z}(x,\rho\to 0) - \Delta f^{(1)}_{q/q}\left(x, \frac{\mu^2}{-p^2}\right) - (\Delta \tilde{f}^{(1)}_{q/q})_{C.T.}\right]_+$$

$$(\Delta \tilde{f}^{(1)}_{q/q})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| \Delta \tilde{f}^{(1)}_{q/q,z}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right)$$

LP3, 1807.06566, 1807.07431

polarized PDF :Gluon in quark

- Light-cone PDF

$$x\Delta f^{(1)}_{g/q}(x,\mu) = \frac{\alpha_s C_F}{2\pi}\left(x(x-2)\ln\frac{-p^2(1-x)x}{\mu^2} + x^2 - 5x\right)$$

- Quasi PDF

$$x\Delta \tilde{f}^{(3,1)}_{g/q}(x,\rho\to 0) \;=\; \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{1}{2}\left(2x + 2(x-2)x\ln\frac{x-1}{x} - 1\right), & x>1 \\ \frac{1}{2}\left(2(x-2)x\ln\frac{\rho}{4} + 6x^2 - 8x + 1\right), & 0<x<1 \\ \frac{1}{2}\left(-2x - 2(x-2)x\ln\frac{x-1}{x} + 1\right), & x<0. \end{cases}$$

- Matching coefficient

$$x\Delta C^{(3,1)}_{g/q}(x,r,\frac{p_z}{\mu},\frac{p_z}{p_z^R}) \;=\; \left[x\Delta\tilde{f}^{(3,1)}_{g/q}(x,\rho\to 0) - x\Delta f^{(1)}_{g/q}\left(x,\frac{\mu^2}{-p^2}\right) - (x\Delta\tilde{f}^{(3,1)}_{g/q})_{C.T.}\right]$$

$$(x\Delta\tilde{f}^{(3,1)}_{g/q})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| x\Delta\tilde{f}^{(3,1)}_{g/q}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right)$$

polarized PDF : Quark in gluon

- Light-cone PDF

$$\Delta f^{(1)}_{q/g}(x,\mu) = \frac{\alpha_s T_f}{2\pi}\left((1-2x)\ln\frac{-p^2(1-x)x}{\mu^2} - 4x + 1\right)$$

- Quasi PDF

$$\Delta \tilde{f}^{(1)}_{q/g}(x,\rho\to 0) = \frac{\alpha_s T_f}{2\pi}\begin{cases}(1-2x)\ln\frac{x-1}{x} - 2, & x>1\\ (1-2x)\ln\frac{\rho}{4} - 4x + 1, & 0<x<1\\ (2x-1)\ln\frac{x-1}{x} + 1, & x<0.\end{cases}$$

- Matching coefficient

$$\Delta C^{(1)}_{qg}(x, r, \frac{p_z}{\mu}, \frac{p_z}{p_z^R}) = \left[\Delta \tilde{f}^{(1)}_{q/g}(x,\rho\to 0) - \Delta f^{(1)}_{q/g}\left(x, \frac{\mu^2}{-p^2}\right) - (\Delta \tilde{f}^{(1)}_{q/g})_{C.T.}\right]$$

$$(\Delta \tilde{f}^{(1)}_{q/g})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| \Delta \tilde{f}^{(1)}_{q/g}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right)$$

Summary

- The RI/MOM matching for gluon and singlet quark quasi PDF is studied

- Operators that can avoid mixing with gauge variant operators are identified

- The matching coefficient in RI/MOM scheme is determined at one-loop accuracy

- Similar calculation can be performed to the GPDs

- This work provides the possibility of extracting quark and gluon PDFs from lattice simulation

Thank you!

Backup: counter terms

$$
\tilde{f}^{(1)}_{q/q,t}(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases}
\frac{2x^2}{(2x-1)(\rho+4x^2-4x)} + \frac{4x-3}{2(\rho-1)(2x-1)} - \frac{3}{2(x-1)} - \frac{\left(3\rho+4x^2+(\rho-8)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x>1\\
\frac{4x-3}{2(\rho-1)} + \frac{3}{2(x-1)} - \frac{\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}\left(3\rho+4x^2+(\rho-8)x\right)}{4(1-\rho)^{3/2}(x-1)}, & 0<x<1\\
-\frac{2x^2}{(2x-1)(\rho+4x^2-4x)} + \frac{3-4x}{2(\rho-1)(2x-1)} + \frac{3}{2(x-1)} + \frac{\left(3\rho+4x^2+(\rho-8)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x<0,
\end{cases}
$$

$$
\tilde{f}^{(1)}_{q/q,z}(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases}
\frac{\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)} + \frac{2\left(3x^2-2x\right)}{(2x-1)^3(\rho+4x^2-4x)} & \\
-\frac{8\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)^2} + \frac{8x^4-34x^3+40x^2-17x+2}{(\rho-1)(x-1)(2x-1)^3} + \frac{3(4x-3)}{2(\rho-1)^2(2x-1)}, & x>1\\
\frac{\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)}{4(1-\rho)^{5/2}(x-1)} + \frac{2-3x}{(\rho-1)(x-1)} + \frac{3(4x-3)}{2(\rho-1)^2}, & 0<x<1\\
-\frac{\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)} - \frac{2\left(3x^2-2x\right)}{(2x-1)^3(\rho+4x^2-4x)} & \\
+\frac{8\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)^2} + \frac{-8x^4+34x^3-40x^2+17x-2}{(\rho-1)(x-1)(2x-1)^3} - \frac{3(4x-3)}{2(\rho-1)^2(2x-1)}, & x<0.
\end{cases}
$$

$$
x\tilde{f}^{(3,1)}_{g/q}(x,\mu,P^z) = \frac{\alpha_s C_F}{2\pi}\begin{cases}
-\frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\
-\frac{(\rho-4)\rho+8(2\rho+1)x^3-4\left(\rho^2+2\rho+6\right)x^2+2\left(3\rho^2-2\rho+8\right)x}{2(1-\rho)^2(\rho+4x^2-4x)}, & x>1\\
-\frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}} & \\
-\frac{(2x-1)(\rho+2(\rho+2)x-4)}{2(1-\rho)^2}, & 0<x<1\\
\frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\
+\frac{(\rho-4)\rho+8(2\rho+1)x^3-4\left(\rho^2+2\rho+6\right)x^2+2\left(3\rho^2-2\rho+8\right)x}{2(1-\rho)^2(\rho+4x^2-4x)}, & x<0.
\end{cases}
$$

Backup: counter terms

$$\tilde{f}_{q/g}^{(1)}(x,\rho)$$

$$=\frac{\alpha_s T_f}{2\pi}\begin{cases} -\frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}-\frac{(2x-1)\left(-(\rho-4)\rho+4(\rho+2)x^2-4(\rho+2)x\right)}{2(1-\rho)^{3/2}(\rho+4x^2-4x)}, & x>1 \\ -\frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}}-\frac{-\rho+12x^2-12x+4}{2(1-\rho)^{3/2}}, & 0<x<1 \\ \frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}-\frac{(2x-1)\left((\rho-4)\rho-4(\rho+2)x^2+4(\rho+2)x\right)}{2(1-\rho)^{3/2}(\rho+4x^2-4x)}, & x<0. \end{cases}$$

$$x\Delta\tilde{f}_{g/g}^{(3,1)}(x,\rho)=\frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ +\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{-8x^3-8x^2+14x-3}{4(\rho-1)(x-1)(2x-1)} \\ -\frac{8\left(x^4-x^3\right)}{(\rho+4x^2-4x)^2}+\frac{3(2x-1)}{2(\rho-1)^2}-\frac{4x+1}{4(x-1)}, & x>1 \\ -\frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}} \\ +\frac{3\left(4x^2-4x+1\right)}{2(\rho-1)^2}+\frac{-16x^3+8x^2+6x-3}{4(\rho-1)(x-1)}+\frac{6x+1}{4(x-1)}, & 0<x<1 \\ \frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ -\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{8x^3+8x^2-14x+3}{4(\rho-1)(x-1)(2x-1)} \\ +\frac{8\left(x^4-x^3\right)}{(\rho+4x^2-4x)^2}-\frac{3(2x-1)}{2(\rho-1)^2}+\frac{4x+1}{4(x-1)}, & x<0. \end{cases}$$

Backup: counter terms

$$
\Delta\tilde{f}^{(1)}_{q/q,z}(x,\rho)=\frac{\alpha_s C_F}{2\pi}\begin{cases}
-\frac{3\rho-2x^2-2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\
+\frac{4x^2}{(2x-1)(\rho+4x^2-4x)}+\frac{1-2x^2}{(\rho-1)(x-1)(2x-1)}-\frac{8\left(x^3-x^2\right)}{(\rho+4x^2-4x)^2}-\frac{3}{2(x-1)}, & x>1 \\
-\frac{3\rho-2x^2-2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}+\frac{1-2x^2}{(\rho-1)(x-1)}+\frac{3}{2(x-1)}, & 0<x<1 \\
-\frac{-3\rho+2x^2+2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\
-\frac{4x^2}{(2x-1)(\rho+4x^2-4x)}+\frac{2x^2-1}{(\rho-1)(x-1)(2x-1)}+\frac{8\left(x^3-x^2\right)}{(\rho+4x^2-4x)^2}+\frac{3}{2(x-1)}, & x<0.
\end{cases}
$$

$$
x\Delta\tilde{f}^{(3,1)}_{g/q}(x,\rho)=\frac{\alpha_s C_F}{2\pi}\begin{cases}
\frac{(2x-1)\left(\rho(\rho+2)+4(\rho+2)x^2+4\left(\rho^2-2\rho-2\right)x\right)}{4(\rho-1)^2(\rho+4x^2-4x)} \\
+\frac{\left(-(\rho-4)\rho+4(\rho+2)x^2-4\left(\rho^2-2\rho+4\right)x\right)\ln\left(\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}\right)}{8(1-\rho)^{5/2}}, & x>1 \\
\frac{\rho+12x^2+4(\rho-4)x+2}{4(\rho-1)^2}+\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(-(\rho-4)\rho+4(\rho+2)x^2-4\left(\rho^2-2\rho+4\right)x\right)}{8(1-\rho)^{5/2}}, & 0<x<1 \\
-\frac{(2x-1)\left(\rho(\rho+2)+4(\rho+2)x^2+4\left(\rho^2-2\rho-2\right)x\right)}{4(\rho-1)^2(\rho+4x^2-4x)} \\
+\frac{\left((\rho-4)\rho-4(\rho+2)x^2+4\left(\rho^2-2\rho+4\right)x\right)\ln\left(\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}\right)}{8(1-\rho)^{5/2}}, & x<0.
\end{cases}
$$

$$
\Delta\tilde{f}^{(1)}_{q/g}=\frac{\alpha_s T_f}{2\pi}\begin{cases}
-\frac{\rho+8x^2+2(\rho-4)x}{1-\rho(\rho+4x^2-4x)}-\frac{\rho+4x-2}{2(1-\rho)^{3/2}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}, & x>1 \\
\frac{1-4x}{1-\rho}-\frac{\rho+4x-2}{2(1-\rho)^{3/2}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}, & 0<x<1 \\
\frac{\rho+8x^2+2(\rho-4)x}{1-\rho(\rho+4x^2-4x)}+\frac{\rho+4x-2}{2(1-\rho)^{3/2}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}, & x<0.
\end{cases}
$$