Non-perturbative structure of hadrons: Wigner functions and GPDs

Initial Stages 2019

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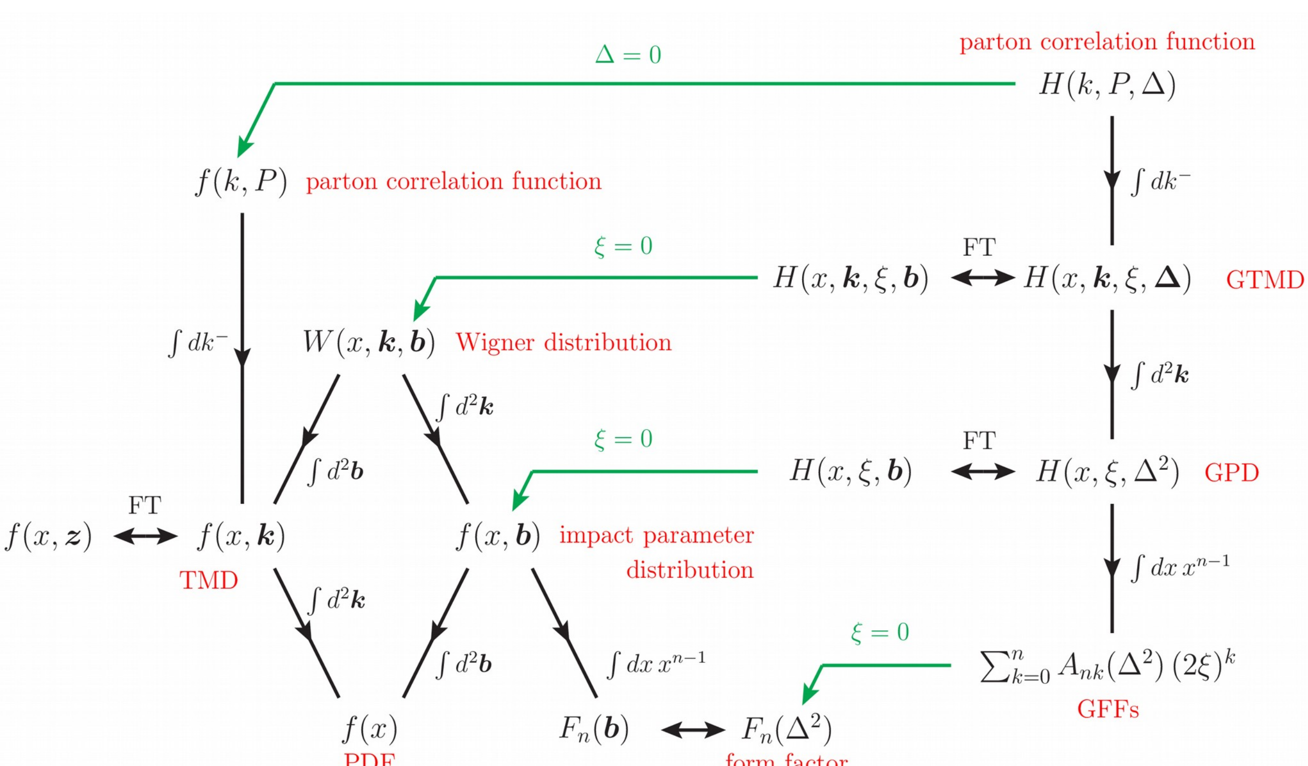
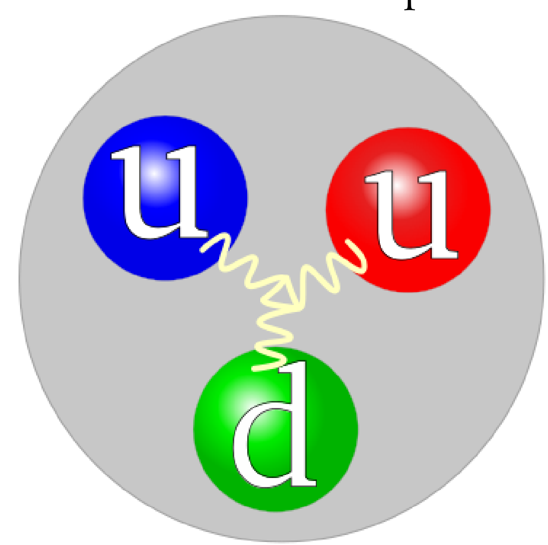


M. Diehl, 1512.01328

wikipedia



What's that ?

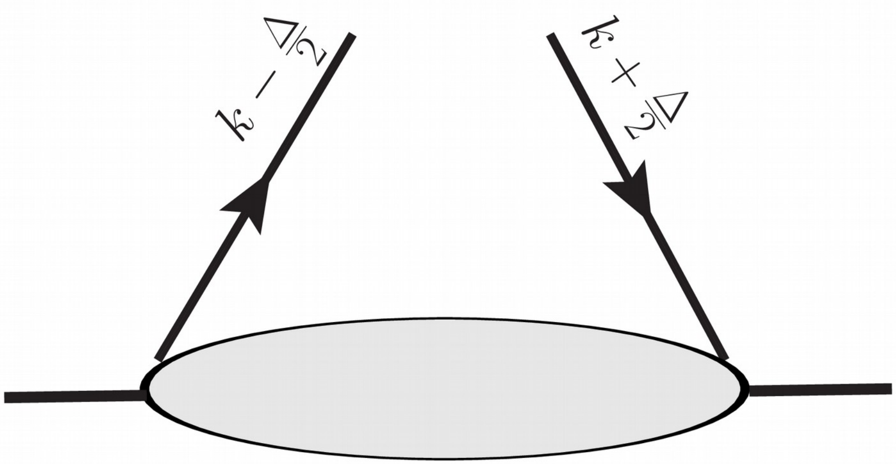
Quark field correlation function :

$$H(k,P,\Delta) = \int \frac{d^4z}{(2\pi)^4}\, e^{ikz} \left\langle P + \frac{\Delta}{2} \left| \overline{q}(-\frac{z}{2})\,\Gamma\, q(\frac{z}{2}) \right| P - \frac{\Delta}{2} \right\rangle$$

omitted proton spin, gauge links, renormalization, ...

$|\,P\rangle$: proton state of QCD Hamiltonian

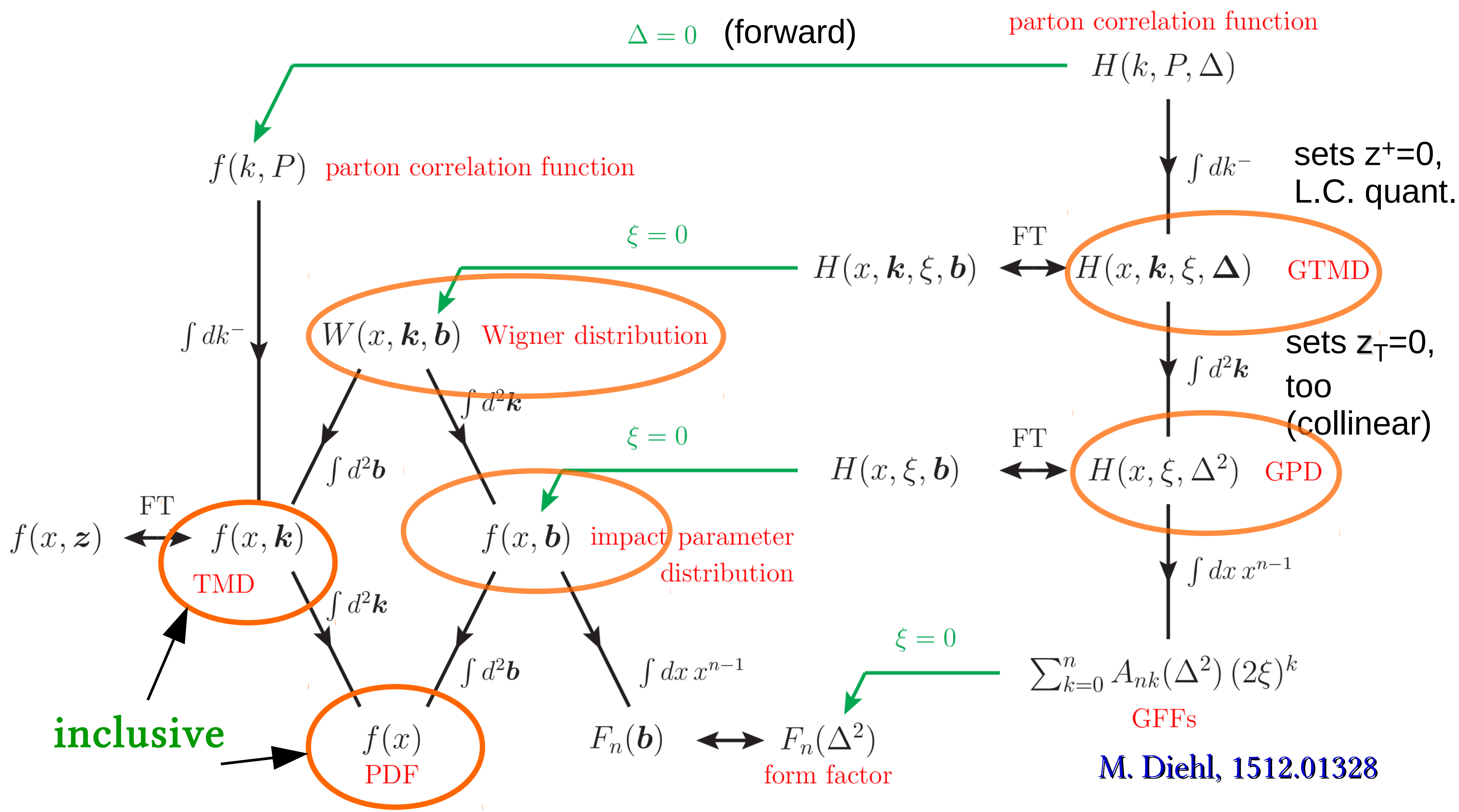
Γ : Dirac matrix selects quark spin configuration & twist; for example $\Gamma = \gamma^+,\ \gamma^+\gamma^5$ for twist 2 unpolarized quark distributions



Phase space distributions :

encode fundamental nonperturbative QCD dynamics of hadrons / nuclei

parton momentum $k^+ = xP^+, k^-, \vec{k}_T$ impact parameter $\vec{b}$



Wigner distribution over 5D phase space :

Belitsky, Ji, Yuan, PRD (2004)
Meissner, Metz, Schlegel, JHEP (2009)
Lorce, Pasquini, PRD (2011)
Y. Hatta, PLB (2012), ...

Quarks:

$$W(x,\vec{k}_T,\vec{b}_T) = \int \frac{d^2\Delta_T}{(2\pi)^2}\, e^{i\vec{b}\cdot\vec{\Delta}_T} \int \frac{dz^- \, d^2 z_T}{2(2\pi)^3}\, e^{ixP^+z^- - i\vec{k}_T\cdot\vec{z}_T} \left\langle P+\frac{\Delta}{2}\left|\overline{q}(-\frac{z}{2})\,\gamma^+\,q(\frac{z}{2})\right|P-\frac{\Delta}{2}\right\rangle$$

* Does not factorize into $f(x,\vec{k}_T)\,g(x,\vec{b})$: correlation of transv. momentum & coordinate $\longrightarrow$ angular momentum

Gluons:

replace quark bilinear $\overline{q}(-\frac{z}{2})\,\Gamma\,q(\frac{z}{2}) \to \Gamma^{ij}\,F^{+i}(-\frac{z}{2})\,F^{+j}(\frac{z}{2})$

(w/ gauge links which select dipole vs. WW distribution)

where $\Gamma^{ij} = \delta^{ij},\ -i\epsilon^{ij}$ etc.

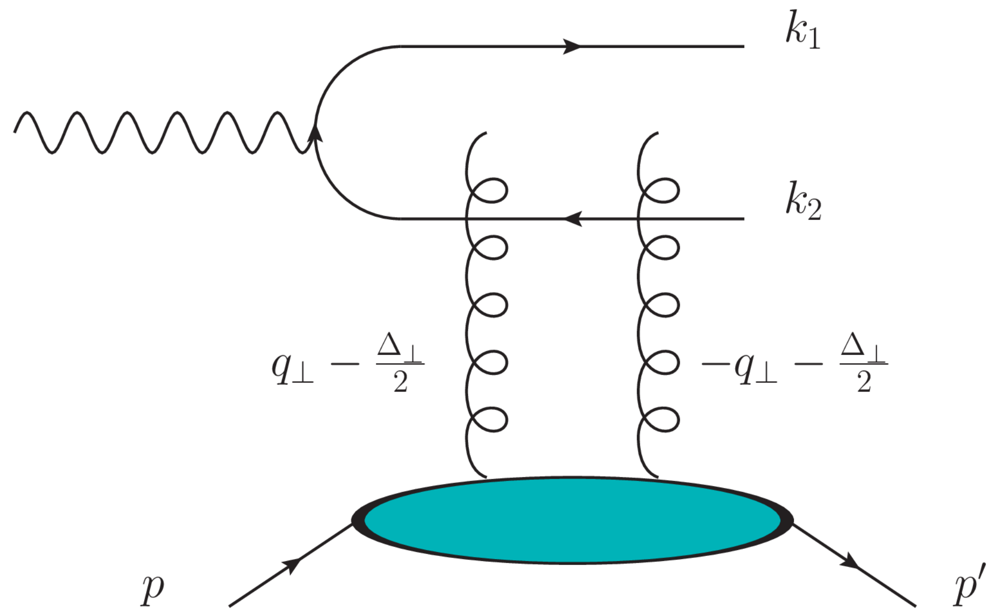
$$W(x,\vec{k}_T,\vec{b}_T) = \int \frac{d^2\Delta_T}{(2\pi)^2}\, e^{i\vec{b}\cdot\vec{\Delta}_T} \int \frac{dz^-\, d^2 z_T}{2(2\pi)^3}\, e^{ixP^+z^- - i\vec{k}_T\cdot\vec{z}_T}$$
$$\left\langle P+\frac{\Delta}{2}\left| F_a^{+i}(-\frac{z}{2})\, F_a^{+i}(\frac{z}{2})\right| P-\frac{\Delta}{2}\right\rangle$$

At small x :

Hatta, Xiao, Yuan, PRL (2016)

$$xG_{\mathrm{DP}}(x,\vec{k}_T,\vec{\Delta}_T) \sim \frac{1}{\alpha_s}\int d^2R_T\, d^2R'_T\, e^{i\vec{k}_T\cdot(\vec{R}_T-\vec{R}'_T)+\frac{i}{2}\vec{\Delta}_T\cdot(\vec{R}_T+\vec{R}'_T)}$$
$$\left(\vec{\nabla}_R\cdot\vec{\nabla}_{R'}\right)\left\langle \mathrm{tr}\ U(R_T)\, U^\dagger(R'_T)\right\rangle_x$$

Probe gluon Wigner distribution at small x: diffractive dijet in ep / eA



Cross section differential in

$$\vec{P} = \frac{1}{2}\left(\vec{k}_1 - \vec{k}_2\right)$$

$$-\vec{\Delta}_T = \vec{k}_1 + \vec{k}_2$$

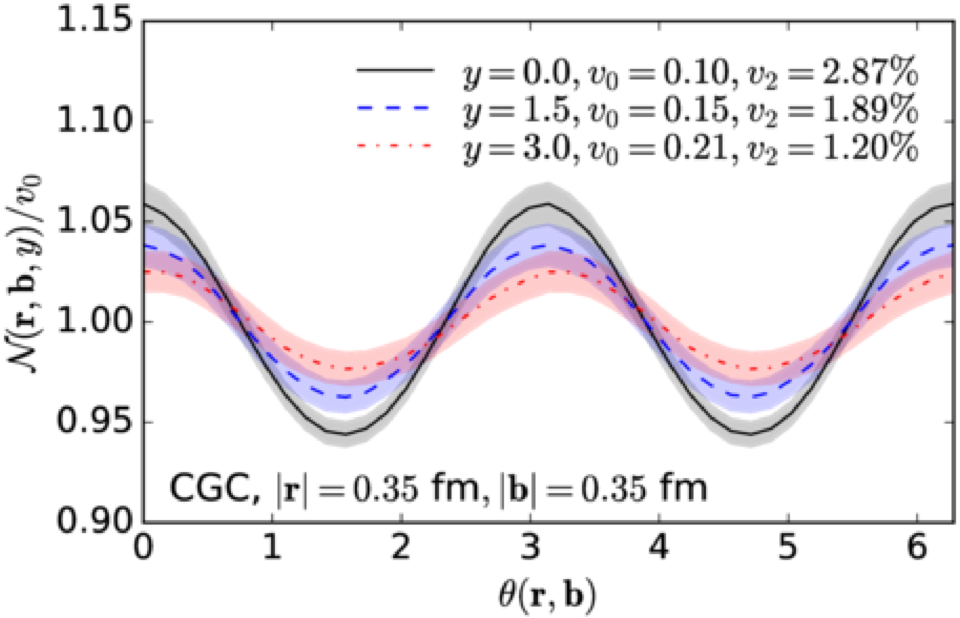
$$\boxed{xW_g^T(x, \vec{k}, \vec{b}) = xW_0(x, k, b) + 2\cos(2\phi)\, xW_2(x, k, b) + \cdots}$$

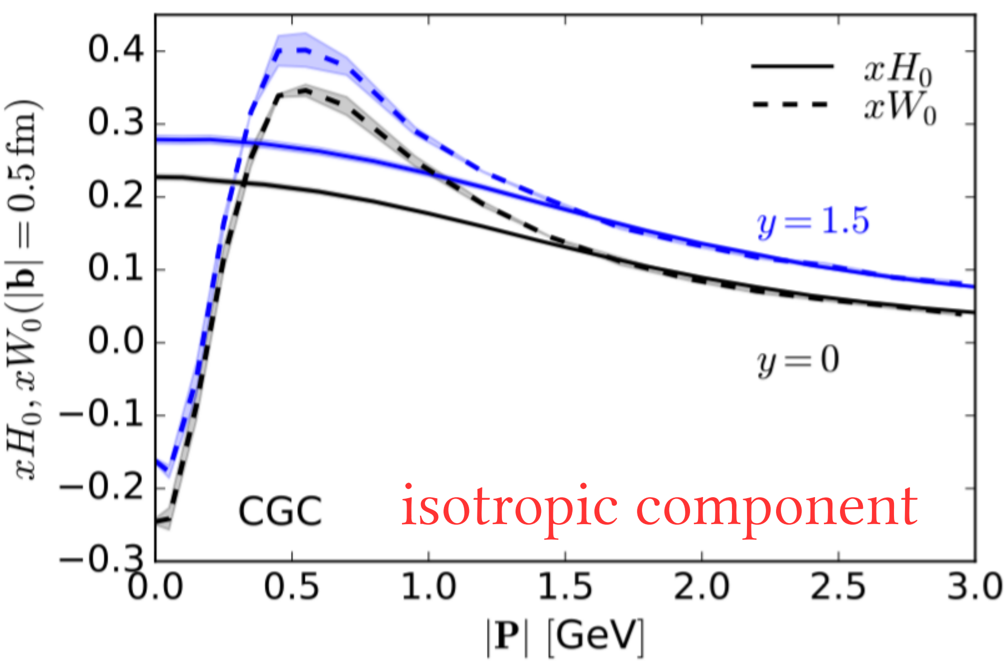
$\phi =$ angle between $\vec{k}, \vec{b}$

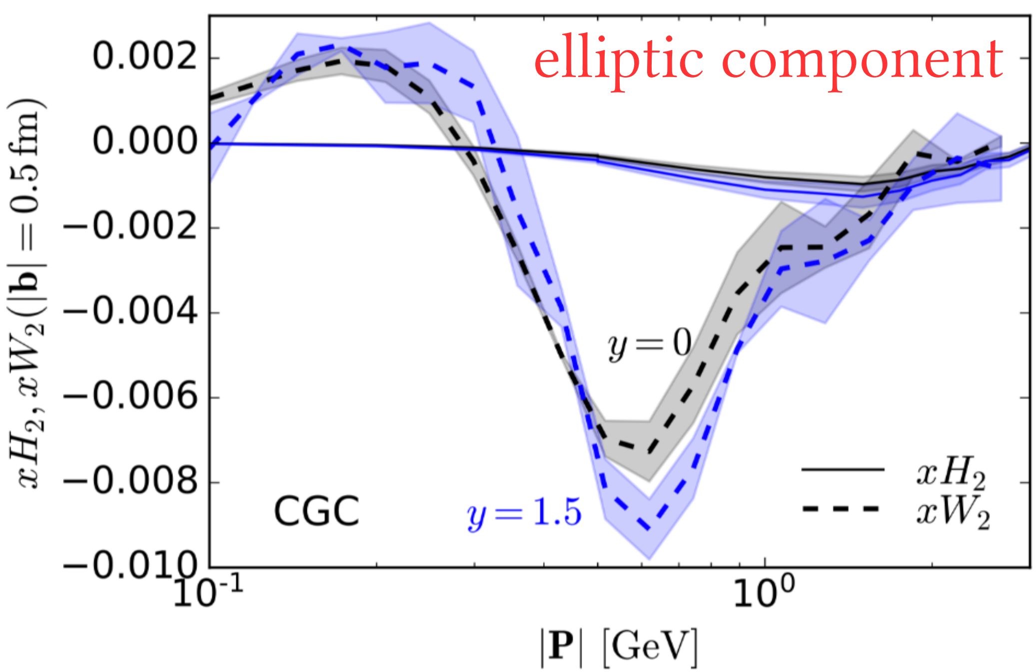
Hatta, Xiao, Yuan, PRL (2016)

Altinoluk et al, PLB (2016)

Mäntysaari, Mueller, Schenke, PRD (2019)
also see Salazar, Schenke, 1905.03763

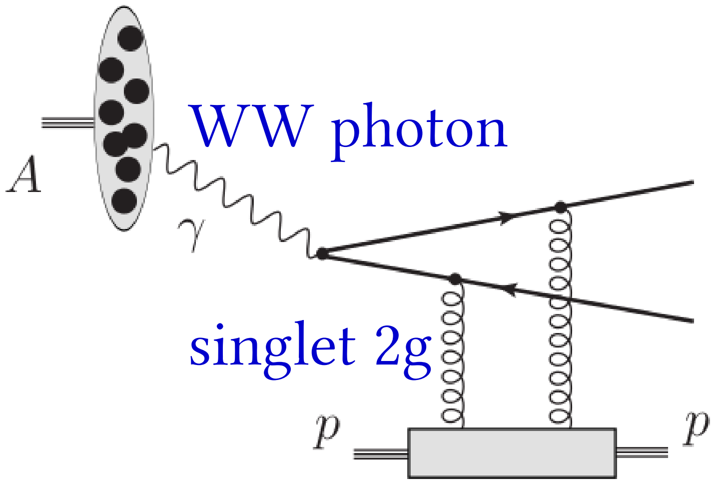






Gluon Wigner distribution at small x: Ultraperipheral pA, AA

Y. Hagiwara et al, PRD (2017)
P. Kotko et al, EPJ-C (2017)

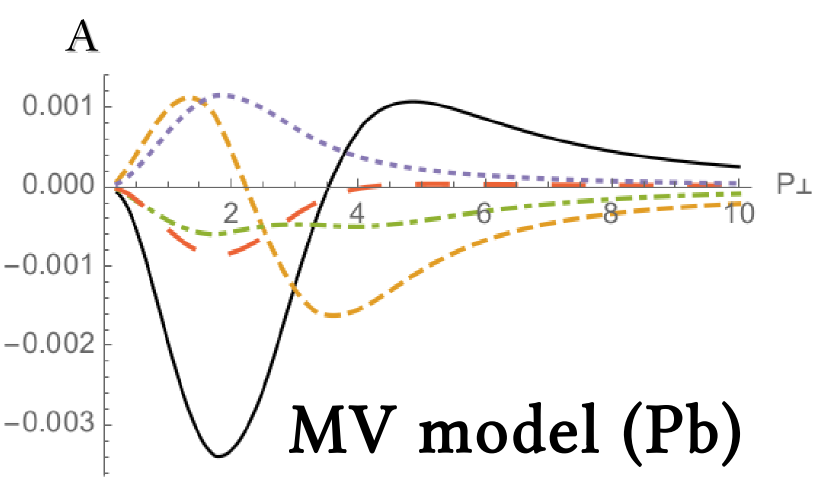


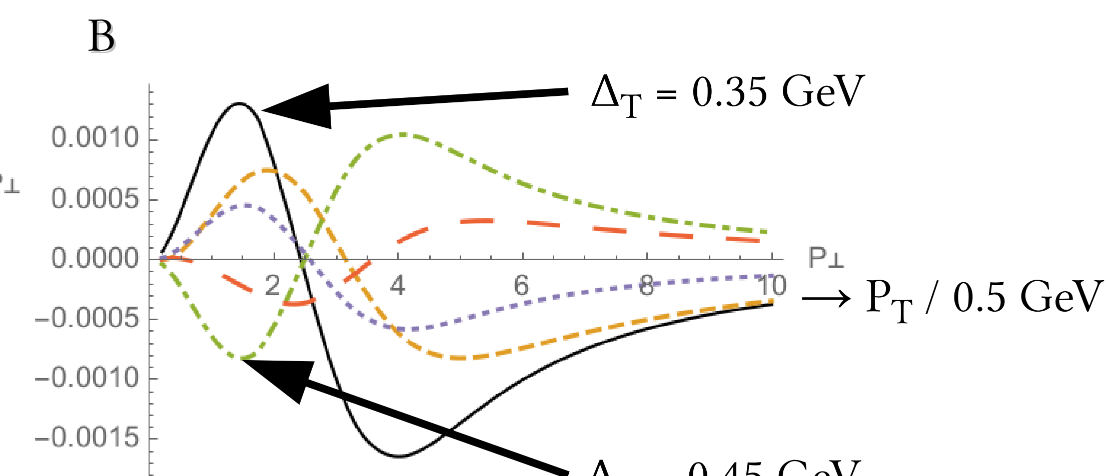
* requires intact proton, measurement of momentum transfer

* and reconstruction of dijet

$$\boxed{\frac{d\sigma^{pA}}{dy_1 dy_2 d^2\vec{k}_{1\perp} d^2\vec{k}_{2\perp}} \approx \omega\frac{dN}{d\omega}\frac{2(2\pi)^4 N_c\alpha_{em}}{P_\perp^2}\sum_f e_f^2 z(1-z)(z^2+(1-z)^2)\left(A^2+2\cos 2(\phi_P-\phi_\Delta)AB\right)}$$

(Wigner distribution can be reconstructed from A, B; see Hagiwara et al)



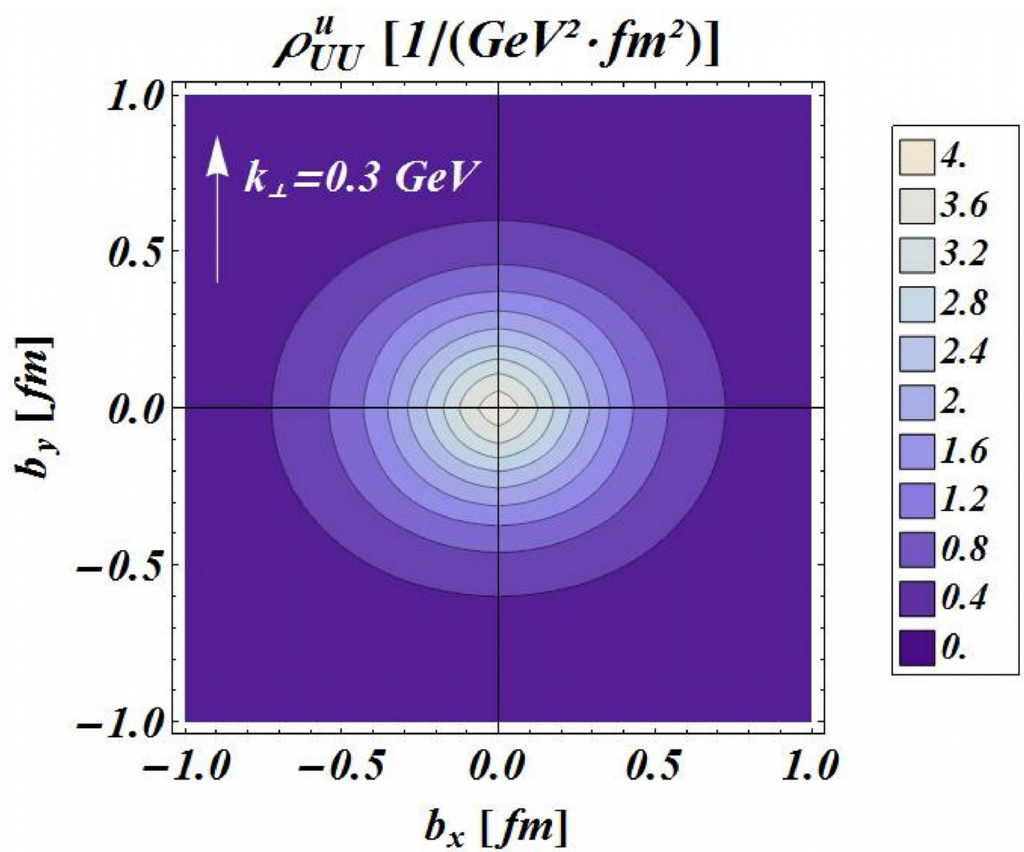


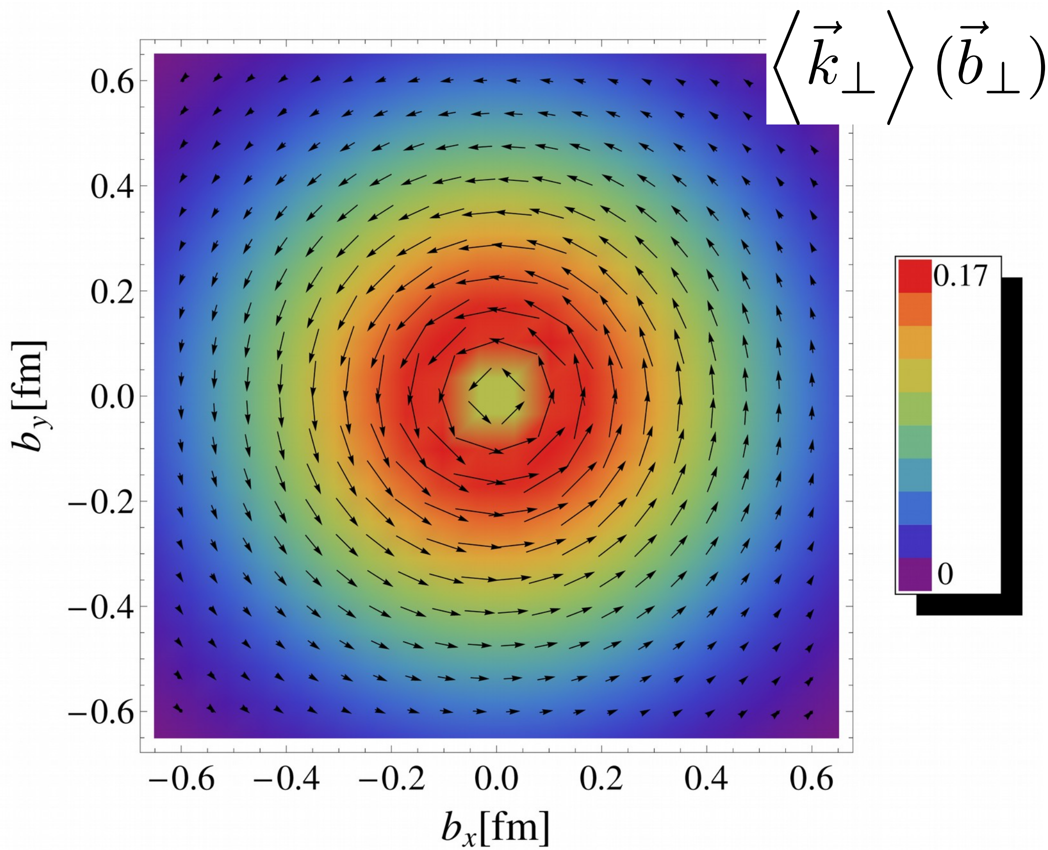
Wigner distribution from proton LF wave function :

Burkardt, Pasquini, EPJ-A (2016)

$$\rho^{[\Gamma]}_{\Lambda',\Lambda}(x,\vec{k}_\perp,\vec{b}_\perp,\mathcal{W}) \equiv \int \frac{d^2\Delta_\perp}{(2\pi)^2}\,\langle p^+,\frac{\vec{\Delta}_\perp}{2},\Lambda'|\widehat{W}^{[\Gamma]}(x,\vec{k}_\perp,\vec{b}_\perp,\mathcal{W})|p^+,-\frac{\vec{\Delta}_\perp}{2},\Lambda\rangle$$

$$\widehat{W}^{[\Gamma]}(x,\vec{k}_\perp,\vec{b}_\perp,\mathcal{W}) \equiv \frac{1}{2}\int \frac{dz^-\,d^2z_\perp}{(2\pi)^3}\,e^{i(xP^+z^- - \vec{k}_\perp\cdot\vec{z}_\perp)}\,\overline{\psi}(y-\frac{z}{2})\Gamma\mathcal{W}\,\psi(y+\frac{z}{2})\Big|_{z^+=0}$$





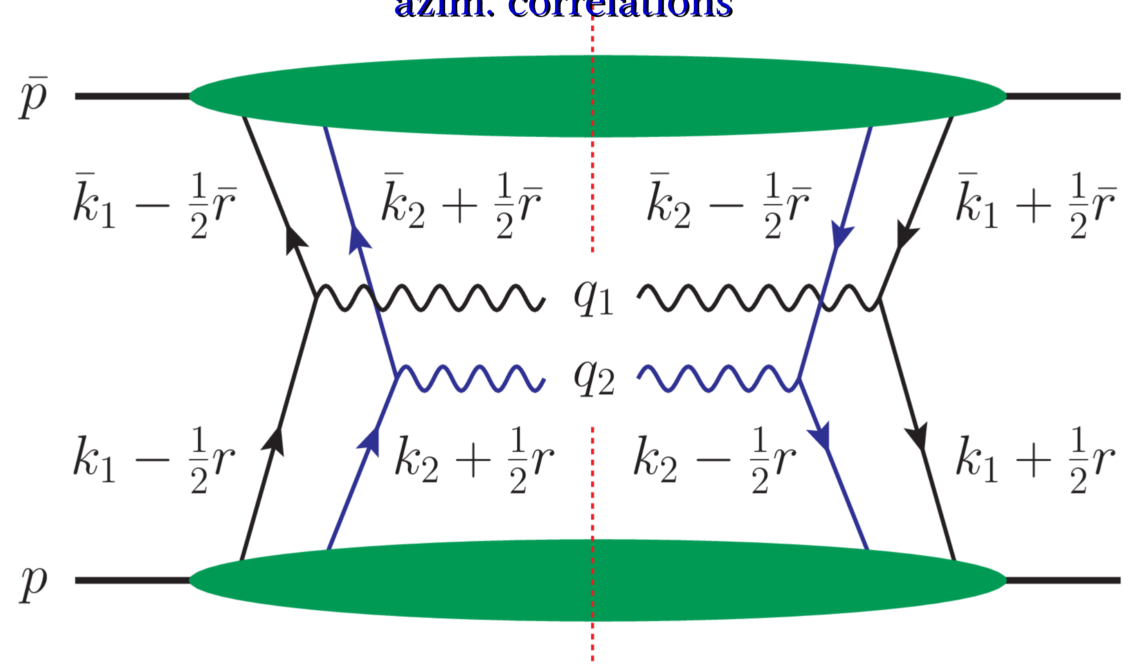
Two-particle Wigner distribution :

Diehl, Schäfer, PLB (2011)

also: Lappi, Schenke, Schlichting, Venugopalan, JHEP (2016), for 2-gluon azim. correlations

double parton scattering in p+p

X-section involves two-quark Wigner distribution



a

$$W(x_1, x_2, \vec{k}_1, \vec{k}_2, \vec{b}_1, \vec{b}_2) \sim \left[\prod_{i=1,2} \int dz_i^- \, d^2 z_i \, e^{i x_i P^+ z_i^- - i \vec{k}_i \cdot \vec{z}_i} \right]$$

$$\times 2P^+ \int db_1^- \, db_2^- \;\langle P \,| O(b_2, z_2)\, O(b_1; z_1) |\, P \rangle$$

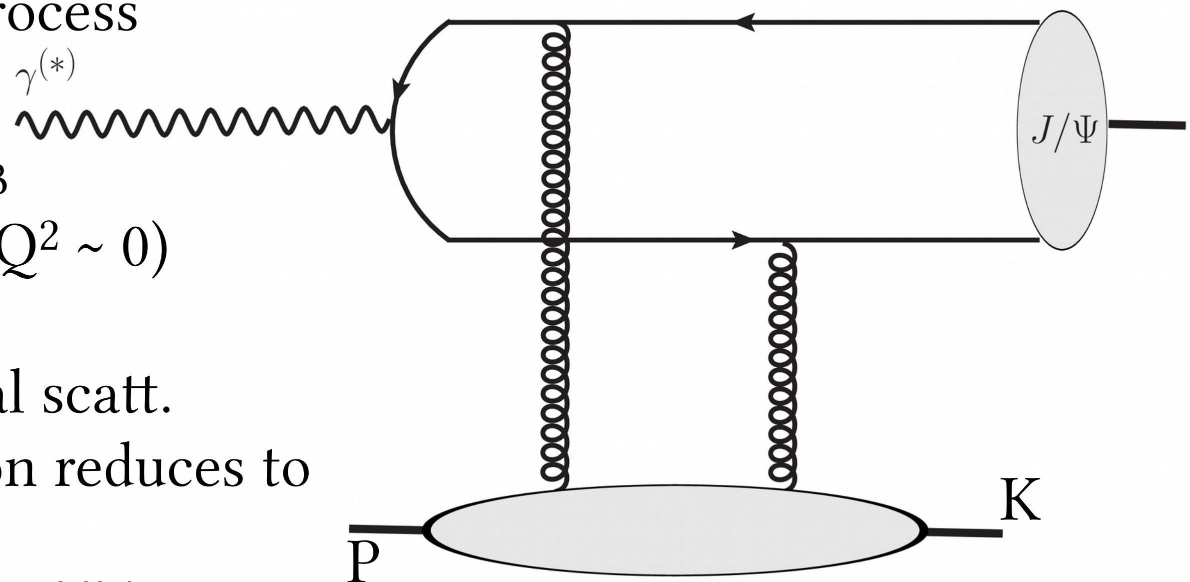
$$O(y, z) = \overline{q}(y - \frac{z}{2})\, \Gamma\, q(y + \frac{z}{2})$$

Generalized Parton Distributions (GPDs) :

reviews: Boffi, Pasquini, Riv. Nuovo Cim. (2007);
Favart, Guidal, Horn, Kroll, EPJ-A (2016)

* Exclusive vector meson
(photo- / electro-) production

* Factorization in hard partonic process
⊗ GPD has been shown for
- Bjorken limit, $Q^2 >> |t|$, fixed x_B
- Quarkonium production (even $Q^2 \sim 0$)



* High-energy limit: dipole eikonal scatt.
At leading twist, CGC expression reduces to

$$\left(\frac{d\sigma^{GPD}_{excl}}{dy_1 dy_2 d^2\boldsymbol{q} d^2\boldsymbol{\Delta}}\right)_{LT} = \frac{\alpha_s^2 \delta\left(p_1^+ + p_2^+ \ldots\right)}{8\left(2\pi\right) C_0 p_0^+}$$

$$\left[z\mathcal{J}^{ij}_{\mathcal{H}}\left(\boldsymbol{q}, -z\boldsymbol{\Delta}, \boldsymbol{0}\right) + \bar{z}\mathcal{J}^{ij}_{\mathcal{H}}\left(\boldsymbol{q}, \bar{z}\boldsymbol{\Delta}, \boldsymbol{0}\right)\right]\left[z\mathcal{J}^{kl*}_{\mathcal{H}}\left(\boldsymbol{q}, -z\boldsymbol{\Delta}, \boldsymbol{0}\right) + \bar{z}\mathcal{J}^{kl*}_{\mathcal{H}}\left(\boldsymbol{q}, \bar{z}\boldsymbol{\Delta}, \boldsymbol{0}\right)\right]$$

$$\int \frac{db^+}{2\pi P^-} \left\langle P - \boldsymbol{\Delta} \left| \mathrm{Tr}\, F^{i-}\left(b\right) F^{j-}\left(0\right) \right| P \right\rangle\Big|_{b^-=0, \boldsymbol{b}=\boldsymbol{0}}$$

$$\int \frac{db'^+}{2\pi P^-} \left\langle P \left| \mathrm{Tr}\, F^{l-}\left(b'\right) F^{k-}\left(0\right) \right| P - \boldsymbol{\Delta} \right\rangle\Big|_{b'^-=0, \boldsymbol{b}'=\boldsymbol{0}}$$

small-x LT GPDs

Altinoluk, Boussarie, 1902.07930

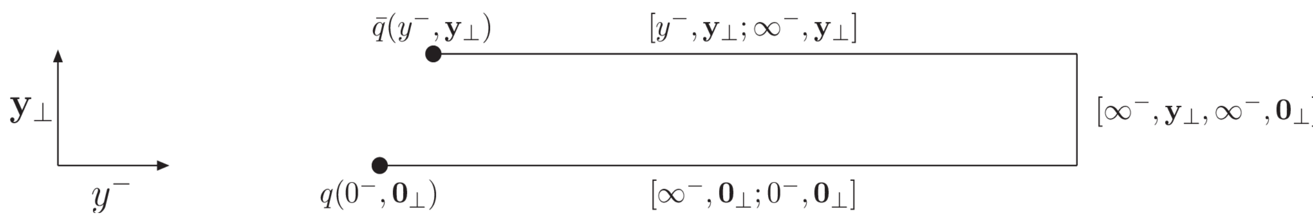
GTMDs / GPDs and color charge densities

M. Burkardt, PRD (2004)

Quark GTMD :

$(y^+ = 0)$

$$q(x, \vec{k}_T, \vec{b}_T) \;=\; \int \frac{dy^- d^2 y_T}{16\pi^3} e^{-ixp^+ y^- + i\vec{k}_T \cdot \vec{y}_T} \left\langle p \left| \bar{q}(y^-, \vec{y}_T) \right.\right. \gamma^+ \left[y^-, \vec{y}_T; \infty^-, \vec{y}_T\right] \left[\infty^-, \vec{y}_T, \infty^-, \vec{0}_T\right] \left[\infty^-, \vec{0}_T; 0^-, \vec{0}_T\right] q(0) \left| p \right\rangle$$



(for DIS)

In L.C. gauge :

$$q(x, \vec{k}_T, \vec{b}_T) \;=\; \int \frac{dy^- d^2 y_T}{16\pi^3} e^{-ixp^+ y^- + i\vec{k}_T \cdot \vec{y}_T} \left\langle p \left| \bar{q}(y^-, \vec{y}_T)\, \gamma^+ \left[\infty^-, \vec{y}_T, \infty^-, \vec{0}_T\right] q(0) \right| p \right\rangle$$

Expectation value of quark $\vec{k}_T$ (e.g. relevant for Sivers asymmetry):

$$\left\langle \vec{k}_T \right\rangle (x) = \int d^2k_T\, \vec{k}_T\, q(x, \vec{k}_T)$$
$$= -g \int \frac{dy^-}{4\pi} e^{-ixp^+y^-} \left\langle p \left| \bar{q}(y^-, \vec{0}_T)\gamma^+ \vec{A}_\perp(\infty^-, \vec{0}_T) q(0) \right| p \right\rangle$$

QCD L.C. Hamiltonian P⁻ acting on hadron state should be free of IR singularities at x⁻ = ±∞ :

$$-\partial^i \alpha^i(\vec{x}_T) = \rho(\vec{x}_T) \equiv \int dx^- J^+(x^-, \vec{x}_T)$$

$$\alpha^i(\vec{x}_T) \equiv A^i(\infty^-, \vec{x}_T) - A^i(-\infty^-, \vec{x}_T)$$

$$J^+ = ig\left[A^i, \partial_- A^i\right] + g\,\bar{q}\gamma^+ t^a q$$

$\longrightarrow$

$$\alpha^i(\vec{x}_T) = -\frac{i}{g} U^\dagger(\vec{x}_T)\, \partial^i\, U(\vec{x}_T)$$

$$\longrightarrow \quad \left\langle \vec{k}_T \right\rangle \equiv \int dx \left\langle \vec{k}_T \right\rangle (x)$$

$$\sim \quad g \int_{\vec{y}_T} \frac{\vec{y}_T}{y_T^2} \left\langle p \left| \bar{q}(0)\, \gamma^+\, t^a\, q(0) \rho_a(\vec{y}_T) \right| p \right\rangle + \mathcal{O}(\rho^2)$$

* no gauge link $\longrightarrow$ "parton model interpretation"
* can be calculated from L.C. wave function of the proton (and possibly on the lattice)
* my note: taking the "shockwave limit" one could write the above as a correlator of color charge densities :

$$\left\langle \vec{k}_T \right\rangle \sim g \int_{\vec{y}_T} \frac{\vec{y}_T}{y_T^2} \left\langle p \left| \rho_a(\vec{0}_T)\, \rho_a(\vec{y}_T) \right| p \right\rangle + \mathcal{O}(\rho^2)$$

Color charge correlators in the proton

Dumitru, Miller, Venugopalan, PRD (2018)

The proton on the light front (valence quark Fock state; L.C. time $x^+ = 0$)

$$
\begin{aligned}
|P\rangle \;=\; & \frac{1}{\sqrt{6}} \int \frac{dx_1 dx_2 dx_3}{\sqrt{x_1 x_2 x_3}} \delta(1 - x_1 - x_2 - x_3) \int \frac{d^2k_1 d^2k_2 d^2k_3}{(16\pi^3)^3} 16\pi^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \\
& \times \psi(k_1, k_2, k_3) \sum_{i_1, i_2, i_3} \epsilon_{i_1 i_2 i_3} |p_1, i_1; \, p_2, i_2; \, p_3, i_3\rangle
\end{aligned}
$$

+ higher Fock states

P. Lepage & Brodsky, 1979 - ….
Brodsky, Pauli, Pinsky, PR (1998)

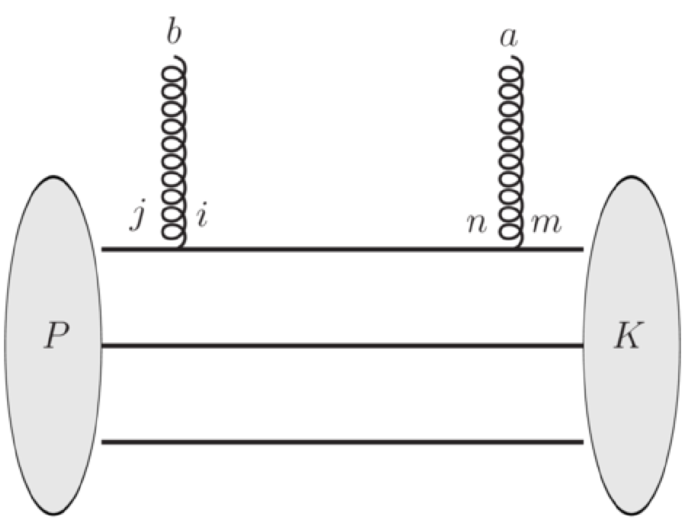
* Fock space amplitude ψ is gauge invariant, universal, and process independent

* encodes the non-perturbative structure of hadrons (QCD eigenstates)

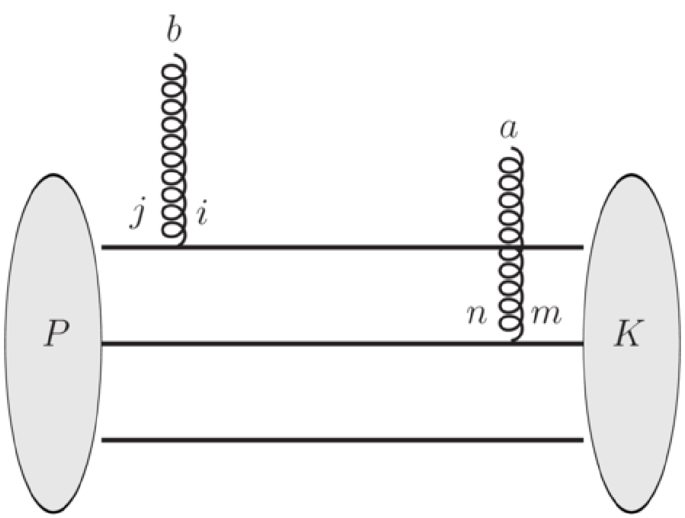
$\longrightarrow$ Evaluate color charge $\overline{q}\gamma^+ t^a q$ correlators explicitly !

note: $\langle \cdots \rangle_{K_T} \equiv \left\langle P^+, \vec{K}_T | \cdots | P^+, \vec{0}_T \right\rangle / \left\langle P^+, \vec{K}_T | P^+, \vec{0}_T \right\rangle$

$$\begin{aligned}
\langle\, \rho^a(\vec{q})\, \rho^b(\vec{k})\, \rangle_{K_\perp} &= (\mathrm{tr}\ t^a t^b) \int [dx_i][d^2 p_i] \\
&\left\{ \psi^* \left(\vec{p}_1 + (1-x_1)\vec{K}_T, \vec{p}_2 - x_2 \vec{K}_T, \vec{p}_3 - x_3 \vec{K}_T \right) \right. \\
&\left. -\psi^* \left(\vec{p}_1 - \vec{q} - x_1 \vec{K}_T, \vec{p}_2 - \vec{k} - x_2 \vec{K}_T, \vec{p}_3 - x_3 \vec{K}_T \right) \right\} \\
&\psi\left(\vec{p}_1, \vec{p}_2, \vec{p}_3\right) \\
&\equiv \frac{1}{2}\, \delta^{ab} \left[\mathcal{G}_1(\vec{K}_T) - \mathcal{G}_2(\vec{q}, \vec{k}) \right] \qquad (\vec{q} + \vec{k} + \vec{K}_T = 0)
\end{aligned}$$



1- and 2-particle GPDs,
sum vanishes in IR



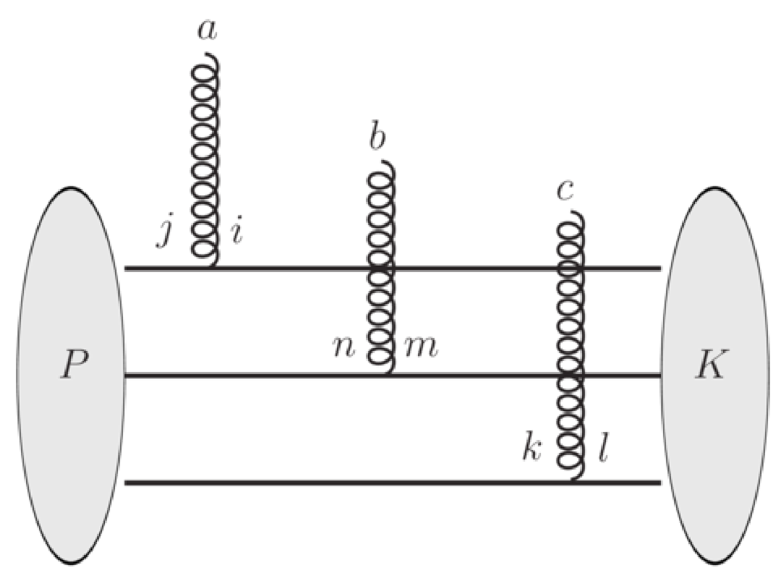
$\langle\rho^3\rangle$ does not vanish (color charge fluct. not Gaussian) :

$$\left\langle[\rho^a(\vec{q}_1)\rho^b(\vec{q}_2)\rho^c(\vec{q}_3)]_{\rm sym}\right\rangle_{K_\perp} \quad\equiv\quad \frac{1}{4}\,d^{abc}\,\mathcal{G}_O(\vec{q}_1,\vec{q}_2,\vec{q}_3;\vec{K}_\perp)$$

$$\begin{aligned}\mathcal{G}_O(\vec{q}_1,\vec{q}_2,\vec{q}_3;\vec{K}_T) \quad = \quad & \int [dx_i][dp_i] \\ & \Big[\psi_3^*(\vec{p}_1+(1-x_1)\vec{K}_\perp,\vec{p}_2-x_2\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi_3^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+(1-x_2)\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi_3^*(\vec{p}_1+\vec{q}_2+(1-x_1)\vec{K}_\perp,\vec{p}_2-\vec{q}_2-x_2\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi_3^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+\vec{q}_2+(1-x_2)\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & +2\,\psi_3^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+\vec{q}_2+(1-x_2)\vec{K}_\perp,\vec{p}_3-\vec{q}_2-x_3\vec{K}_\perp) \\ & \Big]\,\psi_3(\vec{p}_1,\vec{p}_2,\vec{p}_3)\end{aligned}$$

* 1-, 2- and 3-particle GPDs, sum vanishes in IR

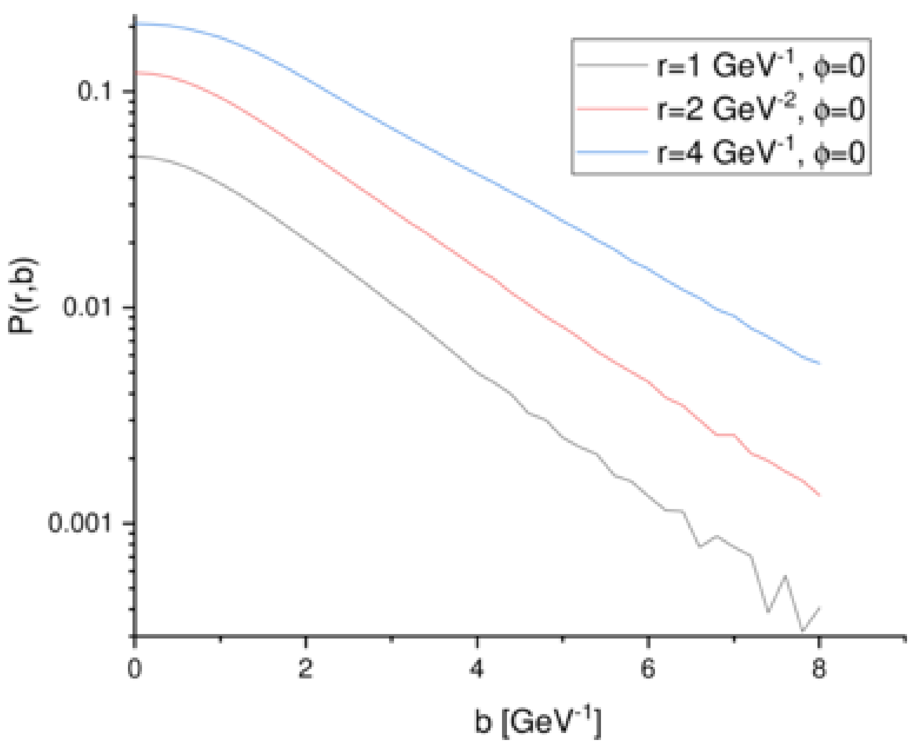
* "3-body" diagrams not (power-) suppressed when $\vec{q}_1\sim\vec{q}_2\sim\vec{q}_3\sim-\vec{K}_T/3\gg\Lambda_{\rm QCD}$ but actually dominant !

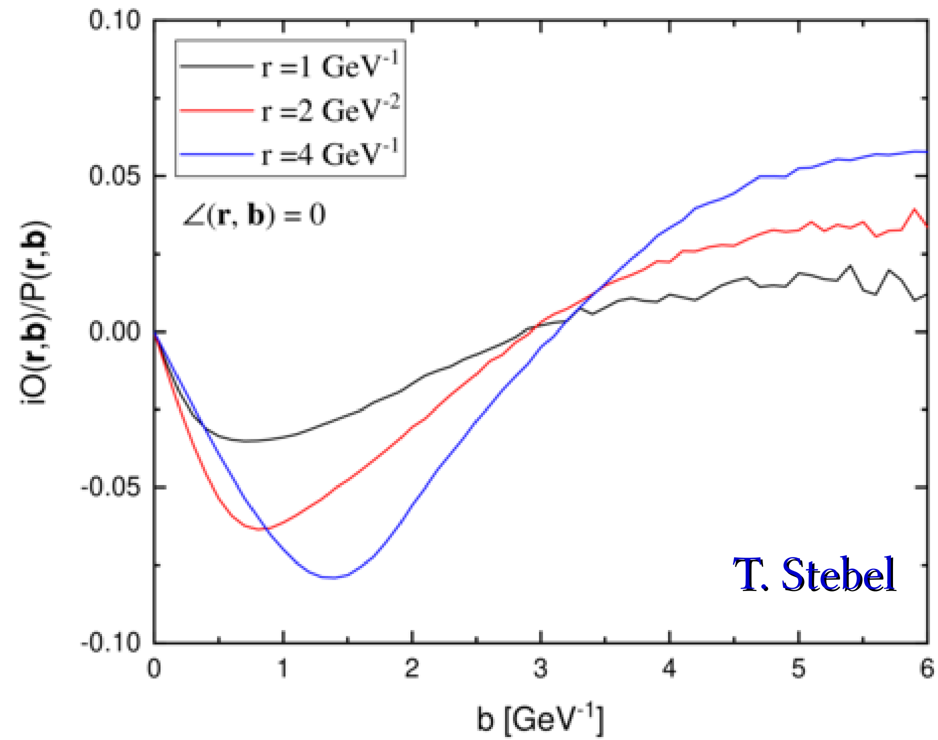


Dipole scattering amplitude (2g vs. 3g singlet exchanges)

$$\mathcal{T}(\vec{r},\vec{b}) = \left\langle 1 - \frac{1}{N_c} \mathrm{tr}\, U\left(\vec{b} + \frac{1}{2}\vec{r}\right) U^\dagger \left(\vec{b} - \frac{1}{2}\vec{r}\right) \right\rangle$$

$$\mathcal{P}(\vec{r},\vec{b}) = \mathrm{Re}\,\mathcal{T}(\vec{r},\vec{b}) \quad , \quad i\mathcal{O}(\vec{r},\vec{b}) = \mathrm{Im}\,\mathcal{T}(\vec{r},\vec{b})$$





* 2g exchange prefers b ~ 0 (intuition: highest density in target)
* 3g exchange prefers b ~ 0.2 – 0.3 fm (~ gradient of density)

J/Ψ: $J^{PC} = 1^{--}$
η_c: $J^{PC} = 0^{-+}$

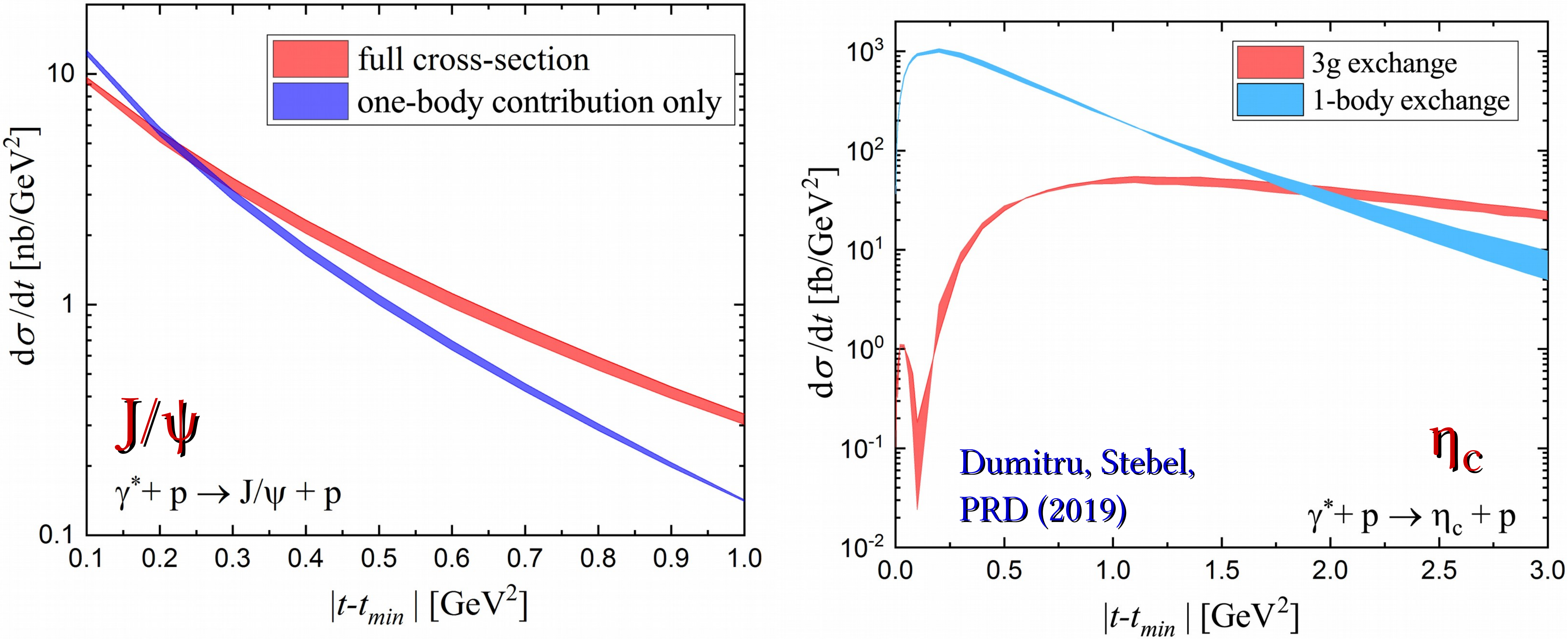
J/Ψ production :

$$\mathcal{A}^{\gamma^* p \to J/\psi\, p}(Q^2, \vec{K}_T) \sim i \int d^2r \int_0^1 \frac{dz}{4\pi} \left(\Psi_{\gamma^*} \Psi^*_{J/\psi}\right)(r, z, Q^2) \times e^{-i\frac{(1-2z)}{2}\vec{r}\cdot\vec{K}_T}\, \mathcal{P}(r, K_\perp)$$

η_c production :

$$\mathcal{A}^{\gamma^* p \to \eta_c\, p}(Q^2, \vec{K}_T) \sim i \int d^2r \int_0^1 \frac{dz}{4\pi} \left(\Psi_{\gamma^*} \Psi^*_{\eta_c}\right)(r, z, Q^2) \times e^{-i\frac{(1-2z)}{2}\vec{r}\cdot\vec{K}_T}\, iO(r, K_\perp)$$

- 3g / Odderon exchange predicted by QCD
- no clear evidence found at small-x (HERA)
- perhaps better to look at “moderately large” x ~ 0.1 and production of η_c using high luminosities of EIC



* For η_c production, "one-body" approximation (1-particle GPD, matrix element of single $\overline{q}\,\gamma^+ q$) gives much too steep slope

* Complete 3g exchange diagrams lead to striking, almost constant cross section for 0.5 GeV^2 < |t| < 3 GeV^2

* For |t| > 1 GeV^2 interpretation as 3g exchange appears reasonable

Summary

- Non-perturbative structure of hadrons is much richer than inclusive PDFs f(x) :
  - distribution in $x, \vec{b} \longrightarrow$ GPDs; in $x, \vec{k}_T \longrightarrow$ TMDs
  - *correlated* distribution in $x, \vec{k}_T, \vec{b} \longrightarrow$ GTMDs / Wigner

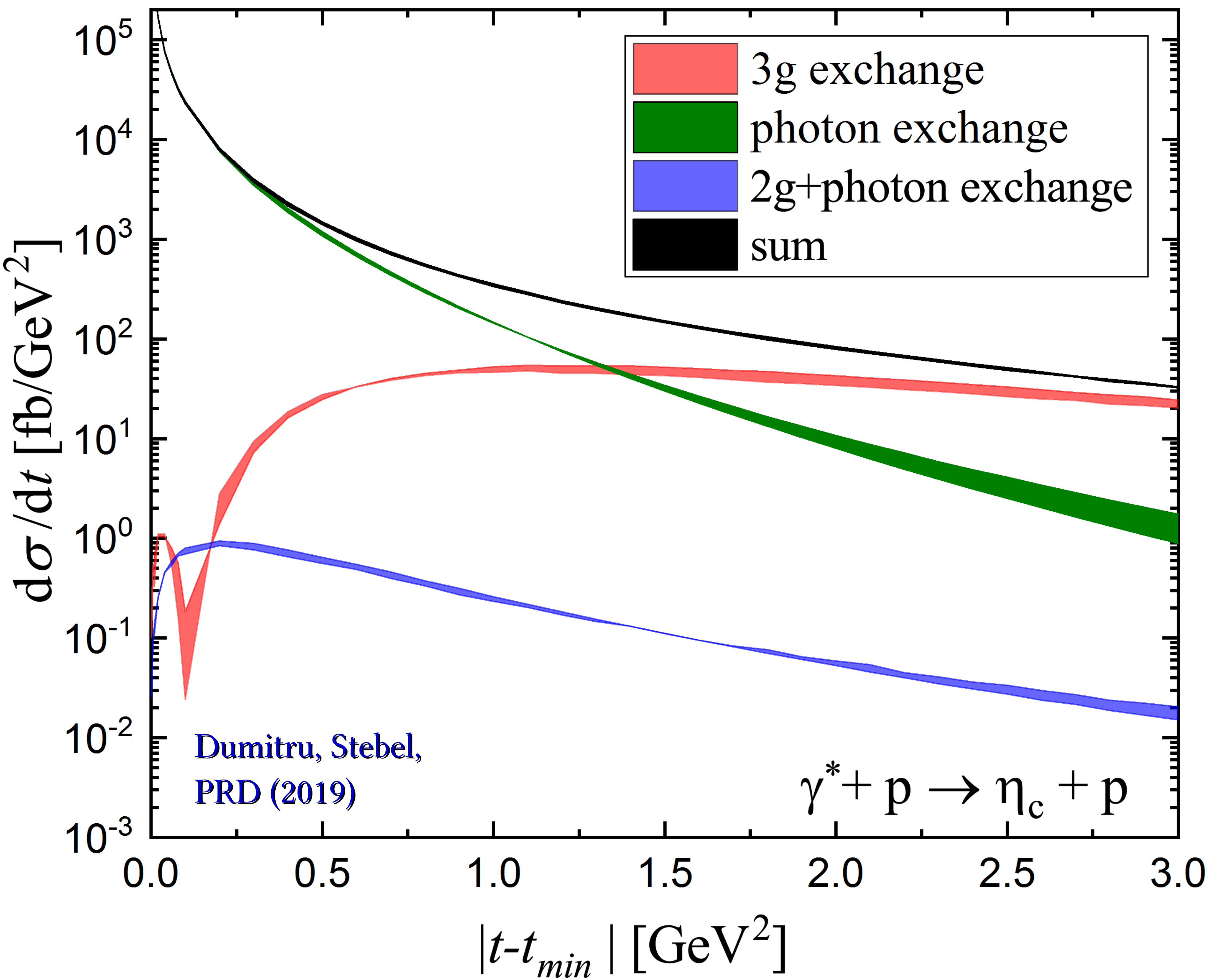
- Related to L.C. wave function of the proton
  [includes n-body GTMDs / GPDs !]

- Processes : exclusive dijets, J/ψ, η_c production in DIS/UPC
  [η_c at (moderately) large x:
  - potential discovery of 3g exchange at $|t| > 1$ GeV^2
  - corrections to Gaussian color charge correlations, ...]

- Color charge correlators in proton L.C. wave function provides a link to / initial conditions for small-x
  [non-vanishing $\langle\rho^3\rangle$, non-Gaussian $\langle\rho^4\rangle$, $\vec{k}_T \leftrightarrow \vec{b}$ correlations etc]

Backup Slides

η_c production : photon & photon + 2g exchanges



$$
\begin{aligned}
&\left\langle \rho^a(\vec{q}_1)\,\rho^b(\vec{q}_2)\,\rho^c(\vec{q}_3)\,\rho^d(\vec{q}_4)\,\right\rangle = \\
&\int dx_1\,dx_2\,dx_3\,\delta(1-x_1-x_2-x_3)\int \frac{d^2p_1\,d^2p_2\,d^2p_3}{(16\pi^3)^2}\,\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)\,\psi(\vec{p}_1,\vec{p}_2,\vec{p}_3) \\
&\Big\{ t^at^bt^ct^d\ \psi^*(\vec{p}_1+(1-x_1)\vec{K}_T,\vec{p}_2-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&+\left(t^at^b\,t^ct^d - t^at^bt^ct^d\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-x_1\vec{K}_T,\vec{p}_2-\vec{q}_3-\vec{q}_4-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&+\left(t^at^c\,t^bt^d - t^at^ct^bt^d\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_3-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-\vec{q}_4-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&+\left(t^at^d\,t^bt^c - t^at^dt^bt^c\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_4-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-\vec{q}_3-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&-t^at^bt^ct^d\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-\vec{q}_3-x_1\vec{K}_T,\vec{p}_2-\vec{q}_4-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&-t^at^bt^ct^d\ \psi^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-\vec{q}_3-\vec{q}_4-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&-t^at^bt^dt^c\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-\vec{q}_4-x_1\vec{K}_T,\vec{p}_2-\vec{q}_3-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&-t^at^ct^dt^b\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_3-\vec{q}_4-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-x_2\vec{K}_T,\vec{p}_3-x_3\vec{K}_T) \\
&+\left(t^at^bt^ct^d+t^at^bt^dt^c-t^at^b\,t^ct^d\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-x_1\vec{K}_T,\vec{p}_2-\vec{q}_3-x_2\vec{K}_T,\vec{p}_3-\vec{q}_4-x_3\vec{K}_T) \\
&+\left(t^at^ct^bt^d+t^at^ct^dt^b-t^at^c\,t^bt^d\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_3-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-x_2\vec{K}_T,\vec{p}_3-\vec{q}_4-x_3\vec{K}_T) \\
&+\left(t^at^dt^bt^c+t^at^dt^ct^b-t^at^d\,t^bt^c\right)\ \psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_4-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-x_2\vec{K}_T,\vec{p}_3-\vec{q}_3-x_3\vec{K}_T) \\
&+\left(t^at^bt^ct^d+t^at^dt^bt^c-t^at^d\,t^bt^c\right)\ \psi^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_T,\vec{p}_2-\vec{q}_2-\vec{q}_3-x_2\vec{K}_T,\vec{p}_3-\vec{q}_4-x_3\vec{K}_T)\Big\}\ ,
\end{aligned}
$$

where $\vec{K}_T = -(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)$

Note: $\neq \langle\rho^a(\vec{q}_1)\,\rho^b(\vec{q}_2)\rangle\,\langle\rho^c(\vec{q}_3)\,\rho^d(\vec{q}_4)\rangle + \text{perm.}$