Hydrodynamics with spin / Connection to vorticity

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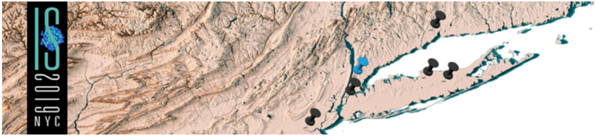
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

5th International Conference on the Initial Stages
in High-Energy Nuclear Collisions
8-12 April 2019
Columbia University, NYC, US



PHYSICS MOTIVATION

Angular momentum and spin in non-central heavy-ion collisions

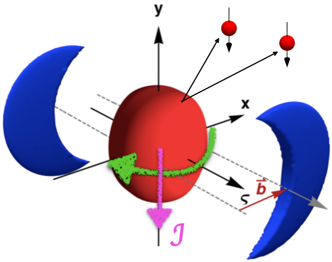
Non-central heavy-ion collisions create fireballs with large global angular momenta $L_{\rm init} \sim 10^5 \hbar$
F. Becattini, F. Piccinini, J. Rizzo, PRC 77 (2008) 024906

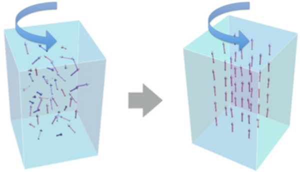
Some part of the angular momentum can be transferred from the orbital to the spin part
Barnett, Rev. Mod. Phys. 7, 129 (1935)

$$J_{\rm init} = L_{\rm init} \qquad = \qquad L_{\rm final} + S_{\rm final}$$

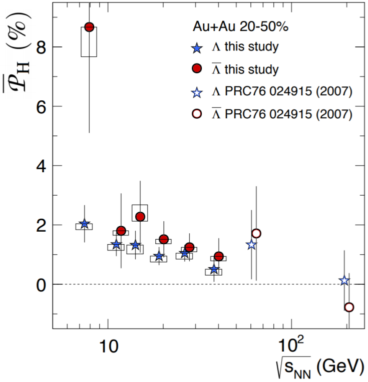
Should be reflected in the polarization of observed hadrons
(e.g. parton scattering polarizes quarks due to spin-orbit coupling)
Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301; PLB 629 (2005) 20–26
J.-H. Gao, et al., PRC 77 (2008) 044902
S.-W. Chen, J. Deng, J.-H. Gao, Q. Wang, Front. Phys. China 4 (2009) 509–516
B. Betz, M. Gyulassy, G. Torrieri, PRC 76 (2007) 044901

First HI experiments that measured spin polarization in Dubna, CERN and BNL reported negative results
M. K. Anikina, et al., Z. Phys. C25 (1984) 1–11
J. Bartke, et al., Z. Phys. C48 (1990) 191–200
B. I. Abelev, et al., PRC 76 (2007) 024915





First positive measurements of global spin polarization of Λ hyperons by STAR



thermal approach $\longrightarrow$ $P_\Lambda \approx \frac{1}{2}\frac{\omega}{T} + \frac{\mu_\Lambda B}{T}$ $\quad P_{\overline{\Lambda}} \approx \frac{1}{2}\frac{\omega}{T} - \frac{\mu_\Lambda B}{T}$

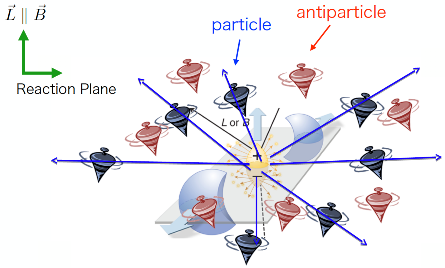
Becattini, F., Karpenko, I., Lisa, M., Upsal, I. , Voloshin, S. , PRC 95, 054902 (2017)

...the hottest, least viscous – and now, most vortical – fluid produced in the laboratory ...

$$\omega = \left(P_\Lambda + P_{\overline{\Lambda}}\right) k_B T/\hbar \sim 0.6 - 2.7 \times 10^{22}\mathrm{s}^{-1}$$

L. Adamczyk et al. (STAR) (2017), Nature 548 (2017) 62-65

Measurement of global spin polarization of Λ hyperons

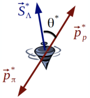


Parity-violating decay of hyperons

Daughter baryon is preferentially emitted in the direction of hyperon's spin (opposite for anti-particle)

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_{\rm H} \mathbf{P}_{\rm H} \cdot \mathbf{p}_{\mathbf{p}}^*)$$

P_H: Λ polarization
p_p^*: proton momentum in the Λ rest frame
α_H: Λ decay parameter
($\alpha_\Lambda = -\alpha_{\bar{\Lambda}} = 0.642 \pm 0.013$)



$$\Lambda \rightarrow p + \pi^-$$

(BR: 63.9%, cτ ~7.9 cm)

C. Patrignani et al. (PDG), Chin. Phys. C 40, 100001 (2016)

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

Theory succees - global polarization

Spin DoF are locally equilibrated
F.Becattini, F.Piccinini,Ann.Phys.323,2452(2008)
F.Becattini, F.Piccinini, J. Rizzo, PRC 77, 024906 (2008)

Polarization given by **thermal vorticity** (GEQ)
F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013)

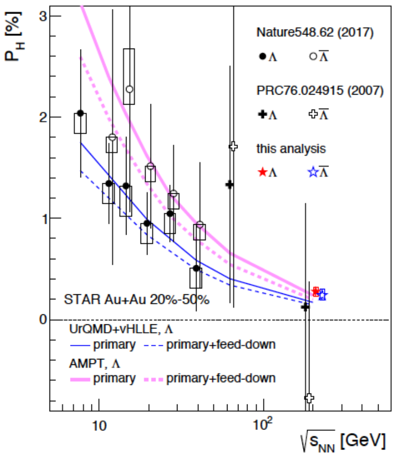
$$\omega^{\mu\nu} \quad \Leftrightarrow \quad \varpi^{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

$$\beta_\nu = \frac{u_\nu}{T}$$

Frozen at the hadronization stage using hydrodynamics/transport without spin

Possible 15%-20% dilution of primary Λ polarization due to **feed-down effect**
F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, PRC 95 (2017) no.5, 054902
X-L Xia, H. Li, X-G Huang, H. Z. Huang [1905.03120]
F. Becattini, G. Cao, E. Speranza [1905.03123]

J. Adam, et al., Phys. Rev. C 98, 014910 (2018)



UrQMD+vHLLE: I. Karpenko, F. Becattini, EPJC 77, 213 (2017)
AMPT: H. Li, L. Pang, Q. Wang, and X. Xia, PRC 96, 054908 (2017)

also agree with:
Y. Xie, D. Wang, and L. P. Csernai, PRC 95, 031901 (2017)
Y. Sun and C. M.Ko, PRC 96, 024906 (2017)

also more recently
D-X Wei, W-T. Deng., X-G.Huang, PRC 99 (2019) no.1, 014905

Phenomenological prescription used to describe the data

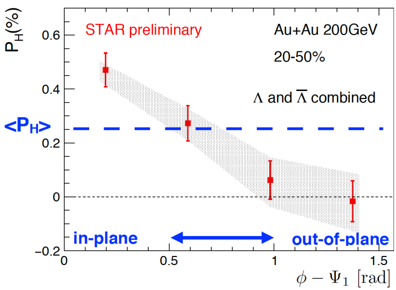
Statistical approach quite appealing!

Algorithm is:

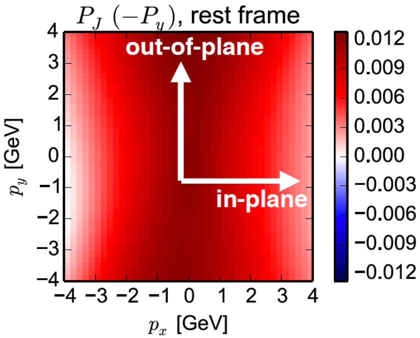
1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin
2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface (defined often by the condition T=const)
3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const
4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$
5) Make predictions about spin polarization

As we have seen such a method describes very well the global polarization of Λ but let us look differential ...

Theory failures - azimuthal angle dependence of P_y



Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

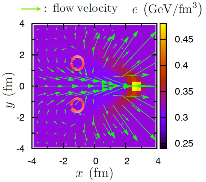


I. Karpenko, F. Becattini, EPJC 77, 213 (2017)

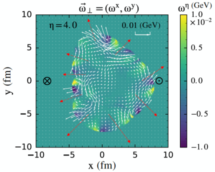
"thermal vorticity–based" approach fails to describe the azimuthal dependence of P_y component

Theory failures - azimuthal angle dependence of P_z

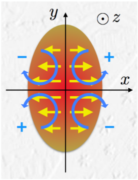
Non-trivial flow structure in the transverse plane (jet, ebe fluctuations etc)
generates longitudinal polarization



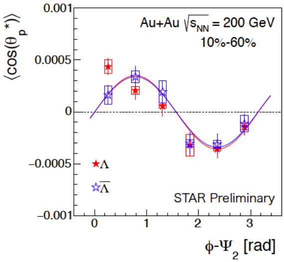
Y. Tachibana and T. Hirano, NPA904-905 (2013) 1023

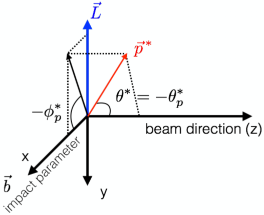


L.-G. Pang, H. Peterson, Q. Wang, and X.-N. Wang PRL117, 192301 (2016)



F. Becattini and I. Karpenko, PRL120.012302 (2018)
S. Voloshin, EPJ Web Conf.171, 07002 (2018)





$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_H \mathbf{P_H} \cdot \mathbf{p}_p^*)$$

$$\langle \cos\theta_p^* \rangle = \int \frac{dN}{d\Omega^*} \cos\theta_p^* d\Omega^*$$

$$= \alpha_H P_z \langle (\cos\theta_p^*)^2 \rangle$$

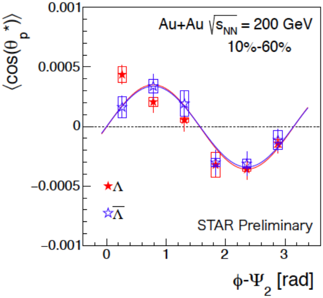
$$\therefore \; P_z = \frac{\langle \cos\theta_p^* \rangle}{\alpha_H \langle (\cos\theta_p^*)^2 \rangle}$$

$$= \frac{3\langle \cos\theta_p^* \rangle}{\alpha_H}$$ (if perfect detector)

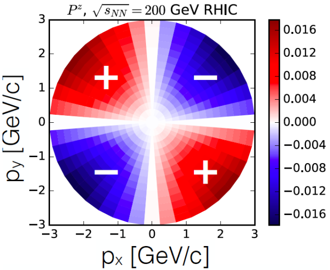
α_H: hyperon decay parameter
θ_p^*: θ of daughter proton in Λ rest frame

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

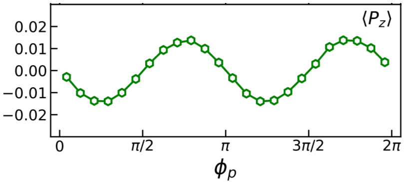
Theory failures - azimuthal angle dependence of P_z



T. Niida, NPA 982 (2019) 511514



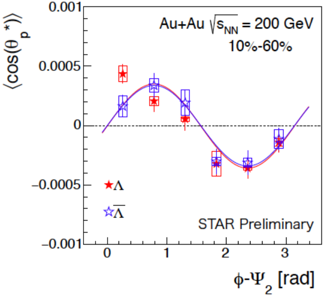
UrQMD+vHLLE: F. Becattini, I. Karpenko, PRL 120 (2018) no.1, 012302,



AMPT: X. Xia, H. Li, Z. Tang, Q. Wang, PRC98.024905 (2018)

"thermal vorticity–based" approach fails to describe the quadrupole structure of P_z component

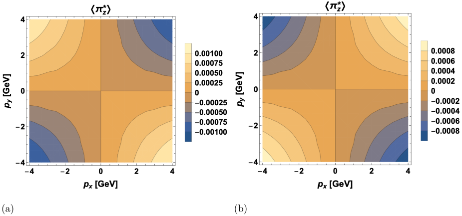
Longitudinal polarization - some explanations



T. Niida, NPA 982 (2019) 511514

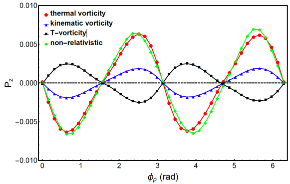
thermal model with projected vorticity $\omega_{\mu\nu} = \varpi_{\alpha\beta}\overline{\Delta}^{\alpha}_{\mu}\overline{\Delta}^{\beta}_{\nu}$

W.Florkowski, A. Kumar, A. Mazeliauskas, R.R., [1904.00002]



3D VH + AMPT IC with T-vorticity $\omega^{(T)}_{\mu\nu} = -\frac{1}{2}\left[\partial_\mu (Tu_\nu) - \partial_\nu \left(Tu_\mu\right)\right]$

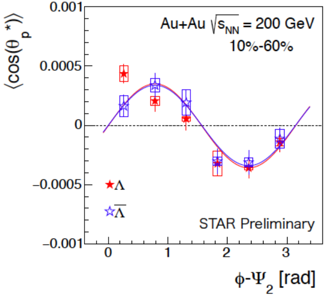
H-Z Wu, L-G Pang, X-G Huang, Q. Wang [1906.09385]



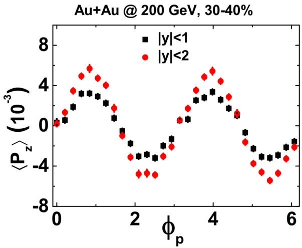
Longitudinal polarization - some explanations

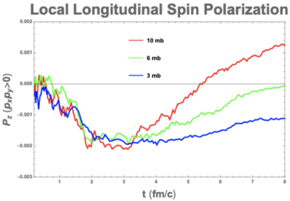
chiral kinetic approach with non-equilibrium spin dynamics

Y. Sun, C-M. Ko, Phys.Rev. C99 (2019) no.1, 011903



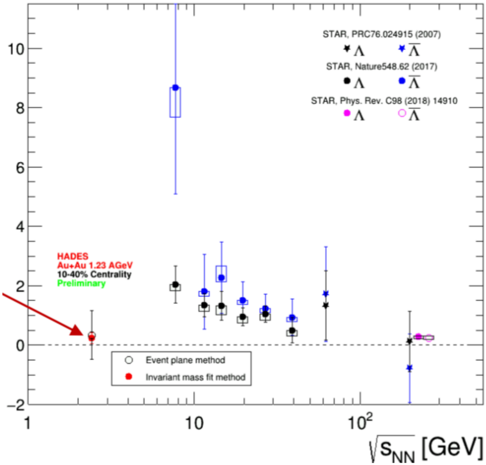
T. Niida, NPA 982 (2019) 511514





the dynamics of polarization may be decoupled from the space-time behavior of the thermal vorticity

Surprise from HADES - vanishing global polarization in Au+Au collisions at $\sqrt{s_{NN}}$= 2.4 GeV



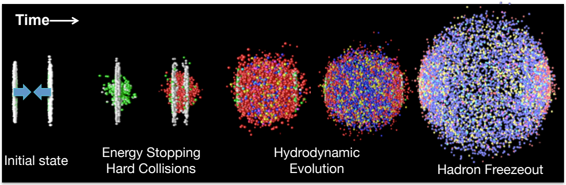
Credit: F. Kornas, International Workshop XLVII on Gross Properties of Nuclei and Nuclear Excitations, 2019

is there a threshold effect at very low energies ?

Fluid dynamics with spin?

space-time dynamics of spin polarization?

relativistic hydrodynamics forms the basis of HIC models



T. K. Nayak, Lepton-Photon 2011 Conference

perfect fluid dynamics = local equilibrium + conservation laws

for particles with spin the conservation of angular momentum is not trivial
new hydrodynamic variables should be introduced

STATISTICAL APPROACH

Conservation laws of canonical currents

Constructions of the hydrodynamic frameworks rely on the local conservation laws

Noether's theorem:

continuous symmetry of the action $\longleftrightarrow$ conserved current

Conservation of charge (baryon number, electric charge, ...)

$$\partial_\mu \widehat{N}^\mu(x) = 0 \qquad \text{(1 equation/charge)}$$

Conservation of energy and momentum

$$\partial_\mu \widehat{T}^{\mu\alpha}_C(x) = 0 \qquad \text{(4 equations)}$$

$\widehat{T}^{\mu\alpha}_C(x) \neq \widehat{T}^{\alpha\mu}_C(x)$ **– canonical energy-momentum tensor is not symmetric**

Conservation of total angular momentum

$$\partial_\mu \widehat{J}^{\mu,\alpha\beta}_C(x) = 0 \qquad \widehat{J}^{\mu,\alpha\beta}_C(x) = -\widehat{J}^{\mu,\beta\alpha}_C(x) \qquad \text{(6 equations)}$$

Spin and orbital angular momentum

Total angular momentum $\widehat{J}_C^{\mu,\alpha\beta}(x)$ may be decomposed into **orbital** $\widehat{L}_C^{\mu,\alpha\beta}(x)$ and **spin** $\widehat{S}_C^{\mu,\alpha\beta}(x)$ parts

$$\widehat{J}_C^{\mu,\alpha\beta}(x) = x^\alpha \widehat{T}_C^{\mu\beta}(x) - x^\beta \widehat{T}_C^{\mu\alpha}(x) + \widehat{S}_C^{\mu,\alpha\beta}(x)$$

Conservation of total angular momentum gives

$$\boxed{\partial_\mu \widehat{J}_C^{\mu,\alpha\beta}(x) = 0 \qquad \Rightarrow \qquad \partial_\mu \widehat{S}_C^{\mu,\alpha\beta}(x) = \widehat{T}_C^{\beta\alpha}(x) - \widehat{T}_C^{\alpha\beta}(x)}$$

$\widehat{T}_C^{\beta\alpha}(x) \neq \widehat{T}_C^{\alpha\beta}(x) \qquad \Rightarrow \qquad$ **canonical spin tensor is not conserved**

Pseudo-gauge transformations

Inclusion of the conservation of angular momentum in field-theoretical frameworks is connected with the problem of the localization of energy and spin densities

Pseudo-gauge transformation: stress-energy and spin tensors are not uniquely defined

W. Hehl, Rept. Math. Phys. 9 (1976) 55–82

F. Becattini, L. Tinti, Phys. Rev. D 84 (2011) 025013; Phys. Rev. D 87(2) (2013) 025029

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}\right)$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

$\rightsquigarrow$ preserve $\widehat{P}^\mu = \int d^3\Sigma_\lambda \widehat{T}^{\lambda\mu}(x)$ $\quad \widehat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \widehat{J}^{\lambda,\mu\nu}(x)$

$\rightsquigarrow$ conservation laws not changed

Belinfante-Rosenfeld construction (choosing superpotential $\widehat{\Phi} = \widehat{S}_C^{\lambda,\mu\nu}$)

Belinfante, F. J. (1939): Physica 6. 887-898, (1940); Rosenfeld, L. (1940): Mem. Acad. Roy. Belgique, cl. SC., tome 18, fasc. 6

$$\widehat{T}_B^{\mu\nu} = \widehat{T}_C^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\mu,\nu\lambda} - \widehat{S}_C^{\nu,\lambda\mu}\right) \qquad \widehat{S}_B^{\lambda,\mu\nu} = 0$$

$\rightsquigarrow$ $\widehat{T}_B^{\beta\alpha}(x) = \widehat{T}_B^{\alpha\beta}(x)$ gives exactly Hilbert $T^{\mu\nu}$ acting as the source of gravity in GR

$\rightsquigarrow$ **long-standing problem of physical significance of the spin tensor**
(however: thermodynamic formulas for transport coefficients do depend on the presence of spin tensor)
F. Becattini, L. Tinti, Phys. Rev. D 84 (2011) 025013; Phys. Rev. D 87(2) (2013) 025029.

$\rightsquigarrow$ **spin tensor is used by the community that studies the spin of proton**
X.S. Chen, X.F. Lu, W.M. Sun, F. Wang, T. Goldman, Phys. Rev. Lett. 100 (2008) 232002;
E. Leader, C. Lorce, Phys. Rep. 541 (2014) 163.

Local equilibrium density operator - Belinfante

Hydrodynamics is, in essence, defined by the continuity equations of the mean value of the stress-energy tensor and charge current operators, which does not require knowledge of distribution function

Densities must significantly vary over distances much larger than the typical microscopic scale

In quantum statistical mechanics LTE can be defined as a maximum of the von Neumann entropy

D.N. Zubarev, Sov. Phys. Dokl. 10 (1966) 850; D.N. Zubarev, A.V. Prozorkevich, S.A. Smolyanskii, Theor. Math. Phys. 40 (1979) 821.
Ch.G. Van Weert, Ann. Phys. 140 (1982) 133;
F. Becattini, Phys. Rev. Lett. 108 (2012) 244502.;F. Becattini, L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75(5) (2015) 191;
M. Hongo, Ann. Phys. 383 (2017) 1.

$$S = -\mathrm{tr}(\widehat{\rho}_B \log \widehat{\rho}_B)$$

Constraints of given mean densities of conserved currents over some spacelike hypersurface Σ

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{T}^{\mu\nu}_B(x)\right] = n_\mu T^{\mu\nu}_B(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{J}^{\mu,\lambda\nu}_B(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_B \left(x^\lambda \widehat{T}^{\mu\nu}_B(x) - x^\nu \widehat{T}^{\mu\lambda}_B(x)\right)\right] = n_\mu J^{\mu,\lambda\nu}_B(x)$$

n^μ - vector orthogonal to Σ

$\widehat{S}^{\lambda,\mu\nu}_B = 0 \quad \Rightarrow$ **condition on total angular momentum is redundant - follows from the constraint on energy and momentum**

Resulting local equilibrium density operator is

$$\widehat{\rho}_B = \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x)\right)\right] \qquad \beta_\nu = \frac{u_\nu}{T} \qquad \xi = \frac{\mu}{T}$$

Local equilibrium - canonical

Again, use maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_C \log \widehat{\rho}_C)$$

Constraints on charge, energy, momentum and total angular momentum

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{T}^{\mu\nu}_C(x)\right] = n_\mu T^{\mu\nu}_C(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{J}^{\mu,\lambda\nu}_C(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_C \left(x^\lambda \widehat{T}^{\mu\nu}_C(x) - x^\nu \widehat{T}^{\mu\lambda}_C(x) + \widehat{S}^{\mu,\lambda\nu}_C\right)\right] = n_\mu J^{\mu,\lambda\nu}_C(x)$$

Spin tensor gives non-trivial constraint on total angular momentum
One has to introduce another Lagrange multiplier functions (spin polarization tensor) $\omega_{\lambda\nu}$

Resulting local equilibrium density operator is

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_C(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C(x)\omega_{\lambda\nu}(x)\right)\right]$$

$$\boxed{\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C(\omega_{\lambda\nu} - \varpi_{\lambda\nu})\right)\right]}$$

$$\omega_{\lambda\nu} = \varpi_{\lambda\nu} = -\tfrac{1}{2}\left(\partial_\lambda\beta_\nu - \partial_\nu\beta_\lambda\right) \quad \Rightarrow \quad \widehat{\rho}_C = \widehat{\rho}_B$$

description based on the Belinfante tensors is reduced compared to the description employing the canonical tensors.

Global equilibrium conditions - canonical

In global equilibrium the density operator becomes stationary
- divergence of the integrand must vanish

$$\widehat{\rho}_{\mathrm{LEQ}} = \frac{1}{Z}\exp\left[-\int_{\Sigma} \mathrm{d}\Sigma_{\mu}\left(\widehat{T}^{\mu\nu}_{C}\left(\beta_{\nu}-\omega_{\nu\lambda}x^{\lambda}\right)-\frac{1}{2}\omega_{\lambda\nu}\widehat{J}^{\mu,\lambda\nu}_{C}-\xi\widehat{j}^{\mu}\right)\right]$$

Using conservation laws it leads to the conditions

$$\partial_{\mu}\omega_{\nu\lambda}=0, \qquad \partial_{\mu}\beta_{\nu}=\omega_{\nu\mu}, \qquad \partial_{\mu}\xi=0$$

and

$$\boxed{\partial_{\mu}\beta_{\nu}+\partial_{\nu}\beta_{\mu}=0 \quad \text{(Killing equation)}}$$

which means that

$$\xi = \text{const.}$$

$$\omega_{\lambda\nu} = \varpi_{\lambda\nu} = -\frac{1}{2}(\partial_{\lambda}\beta_{\nu}-\partial_{\nu}\beta_{\lambda}) = \text{const}$$

The spin polarization tensor is equal to thermal vorticity

Global equilibria include:

- rigid rotation (special boundary conditions)
- constant acceleration along the stream lines

F. Becattini, Phys. Rev. D 97, 085013 (2018)
W. Florkowski, E. Speranza, F. Becattini, Acta Phys.Polon. B49 (2018) 1409
G. Prokhorov, O. Teryaev, V. Zakharov, PRD98 (2018) no.7, 071901; JHEP 1902 (2019) 146; PRD99 (2019) no.7, 071901

General concept of perfect-fluid hydrodynamics with spin

If GE conditions are not satisfied, the integral over Σ depends on its choice. The integrals over hypersurfaces Σ_1 and Σ_2 differ by the volume integral which describes dissipative phenomena

If we neglect dissipation we may treat the local-equilibrium operator $\widehat{\rho}_{\rm LEQ}$ as constant.
In this case we define the expectation values of the conserved currents through the expressions

$$T^{\mu\nu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{T}^{\mu\nu}\right), \quad S^{\mu,\lambda\nu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{S}^{\mu,\lambda\nu}\right), \quad j^{\mu} = \mathrm{tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{j}^{\mu}\right)$$

these tensors are all functions of the hydrodynamic variables β_μ, $\omega_{\mu\nu}$, and ξ which enter constitutive equations

$$T^{\mu\nu} = T^{\mu\nu}[\beta,\omega,\xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta,\omega,\xi], \quad j^{\mu} = j^{\mu}[\beta,\omega,\xi]$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^{\mu} = 0$$

These are 11 equations for 11 unknown functions, which represent a generalization of the standard perfect-fluid hydrodynamics to the case including spin.

first steps in this direction have been done in:
W. Florkowski, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C 97(4) (2018) 041901
W. Florkowski, B. Friman, A. Jaiswal, R. R., E. Speranza, Phys. Rev. D 97 (2018) 116017

The dissipative processes must drive the system to GE, hence ω should converge to ϖ eventually

as argued in:
D. Montenegro, L. Tinti, G. Torrieri, Phys. Rev. D 96 (2017) 076016

Canonical vs Belinfante – is it measurable?

F. Becattini, W. Florkowski, E. Speranza, PLB 789, (2019) 419-425 [1807.10994],

Is there any consequence of the choice between Canonical and Belinfante tensors on some measurable quantities?

$$T^{\mu\nu} = \left\langle \widehat{T}^{\mu\nu}_{\rm C} \right\rangle \qquad \longleftrightarrow \qquad T^{\mu\nu} = \left\langle \widehat{T}^{\mu\nu}_{\rm B} \right\rangle$$

Depends on what we measure!
Actual measurements involve momentum and polarization of asymptotic particle states, not densities.

Measurable quantities can be generally expressed as expectation values of some number of creation and destruction operators of asymptotic states.

Example: single-particle polarization matrix

$$\Theta(p)_{\sigma,\sigma'} = \mathrm{tr}(\widehat{\rho}\, a^\dagger(p)_\sigma a(p)_{\sigma'})$$

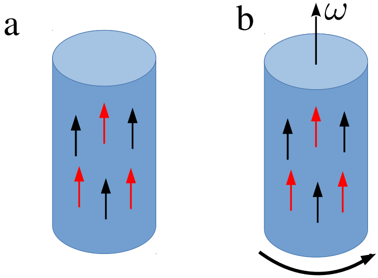
$$\Theta(p)_C = \Theta(p)_B?$$

Only $\widehat{\rho}$ can depend on pseudo-gauge!

$$\widehat{\rho}_B = \widehat{\rho}_C?$$

Answer: Equivalence only in global equilibrium

Polarized neutral system - physical meaning of spin chemical potential



a) $\beta^\mu = (1/T)(1,0,0,0)$ $\Rightarrow$ $\varpi^{\mu\nu} = 0$ $\Rightarrow$ can not be described within Belinfante gauge

$\Rightarrow$ can be described within Canonical gauge

b) if the spin tensor vanishes the polarization requires the fluid to rotate $\varpi^{\mu\nu} \neq 0$

Belinfante's "gauge" does not imply that polarization vanishes, but rather it is locked to thermal vorticity

In the Canonical "gauge" one needs the spin potential to describe hydro evolution, but only if spin density relaxes "slowly" to equilibrium

WIGNER FUNCTION APPROACH

Equilibrium distribution functions for particles with spin-1/2

Alternatively, hydrodynamics may be derived from the underlying microscopic theory (talks by G.S. Denicol, J. Noronha)

Distribution functions for spin-1/2 particles (+) and antiparticles (-) (2×2 spin density matrices)

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi , Annals Phys. 338 (2013) 32

$$f^+_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+u_s(p), \quad f^-_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^-v_r(p)$$

where

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

$\beta^\mu = u^\mu/T$

$\xi = \mu/T$

m - mass of particles

T - temperature

μ - chemical potential

u^μ - four velocity ($u^2 = 1$)

$\omega_{\mu\nu}$ - spin-polarization tensor

$\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ - spin operator

however, keep in mind that it has not been derived from any underlying microscopic model or theory

Equilibrium Wigner functions

One can study consequences of using spin-density matrices á la Becattini et al. as an input for construction of the equilibrium Wigner function.

W. Florkowski, A. Kumar, R. R., Phys. Rev. C98 (2018) 044906

Equilibrium Wigner functions - GLW framework

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*, 1980.

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k-p)u^r(p)\bar{u}^s(p) f^{+}_{rs}(x,p)$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k+p)v^s(p)\bar{v}^r(p) f^{-}_{rs}(x,p)$$

$$\mathcal{W}_{\rm eq}(x,k) = \mathcal{W}^{+}_{\rm eq}(x,k) + \mathcal{W}^{-}_{\rm eq}(x,k).$$

$$\begin{aligned}\mathcal{W}^{\pm}_{\rm eq}(x,k) \quad &= \quad \frac{e^{\pm\xi}}{4m}\int dP\, e^{-\beta\cdot p}\;\delta^{(4)}(k\mp p) \\ &\times\left[2m(m\pm \not{p})\cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\,(\not{p}\pm m)\Sigma^{\mu\nu}(\not{p}\pm m)\right].\end{aligned}$$

The presence of the Dirac delta functions indicates that, to large extent, $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ describe classical motion - cannot be regarded as complete, quantum equilibrium distributions.

Nevertheless, the functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ incorporate spin degrees of freedom and may serve to construct the formalism of hydrodynamics with spin (in the LO in $\hbar$).

Clifford-algebra expansion

Clifford-algebra expansion
(expansion in terms of 16 independent generators of the Clifford algebra with real coefficients)
C. Itzykson, J. B. Zuber, Quantum Field Theory, McGraw-Hill, New York, 1980.

$$\mathcal{W}^{\pm}(x,k) \quad = \quad \frac{1}{4}\left[\mathcal{F}^{\pm}(x,k) + i\gamma_5\mathcal{P}^{\pm}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{\mu}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{\mu\nu}(x,k)\right]$$

The coefficient functions $\mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_{\mu}, \mathcal{A}_{\mu}, \mathcal{S}_{\nu\mu}\}$ in the expansion can be obtained by calculating the trace of $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ multiplied first by the matrices: $\mathbf{1}$, $-i\gamma_5, \gamma_{\mu}, \gamma_{\mu}\gamma_5$ and $2\,\Sigma_{\mu\nu}$.

Formulation of the transport equations for:

abelian plasmas
D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173 (1987) 462–492
P. Zhuang, U. W. Heinz, Annals Phys. 245 (1996) 311–338.

the quark-gluon plasma
H. T. Elze, M. Gyulassy, D. Vasak, Phys. Lett. B177 (1986) 402–408; Nucl. Phys. B276 (1986) 706–728.

chiral models
W. Florkowski, J. Hufner, S. P. Klevansky, L. Neise, Annals Phys. 245 (1996) 445–463..

Studies on polarization, chirality, and vorticity:

J.-H. Gao, Z.-T. Liang, S. Pu, Q.Wang, X.-N.Wang, Phys. Rev. Lett. 109 (2012) 232301
R.-H. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, Phys. Rev. C94 (2) (2016) 024904
R.-H. Fang, J.-Y. Pang, Q.Wang, X.-N.Wang, Phys. Rev. D95 (1) (2017) 014032

Global-equilibrium Wigner function

The Wigner function satisfies the kinetic equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = \mathrm{C}[\mathcal{W}(x,k)] \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu$$

In **global equilibrium** the Wigner function $\mathcal{W}(x,k)$ exactly satisfies the equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0$$

Semi-classical expansion
(expansion in powers of $\hbar$ of the coefficient functions of the Clifford algebra expansion)

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots . \qquad \mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\nu\mu}\}$$

LO in $\hbar$:

$\mathcal{F}_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ may be treated as the only independent coefficients provided $k_\nu\, \mathcal{A}^\nu_{(0)}(x,k) = 0$

Algebraic relations imposed by the kinetic equation are satisfied by $\mathcal{W}^\pm_{\rm eq}(x,k)$

$\mathcal{W}^\pm_{\rm eq}(x,k)$ can be identified with the LO terms in $\hbar$ of the "true" equilibrium Wigner functions $\mathcal{W}^\pm(x,k)$ which satisfy the quantum kinetic equation.

NLO in $\hbar$:

Kinetic equations to be satisfied by the scalar and axial-vector coefficient are

$$k^\mu\partial_\mu\mathcal{F}_{(0)}(x,k) = 0 \qquad k^\mu\partial_\mu\, \mathcal{A}^\nu_{(0)}(x,k) = 0$$

From global to local equilibrium

Global equilibrium — the following equations are exactly fulfilled

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0, \quad k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{\rm eq}(x,k) = 0.$$

Constraints imposed on the equilibrium functions mean that the β_μ **field has to be the Killing vector while the parameters ξ and $\omega_{\mu\nu}$ are constant**

Local equilibrium — **only moments of the kinetic equations are satisfied allowing for space-time dependence of the hydrodynamic variables β_μ, ξ, and $\omega_{\mu\nu}$ (11 unknown functions), standard method for going from the kinetic-theory description to hydrodynamics**

G. S. Denicol, E. Molnr, H. Niemi and D. H. Rischke, Eur. Phys. J. A 48, 170 (2012)
G. S. Denicol, J. Phys. G 41, 124004 (2014)

$$\boxed{\partial_\alpha N^\alpha_{\rm eq}(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\rm GLW}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0}$$

$\widehat{T}^{\beta\alpha}_{GLW}(x) = \widehat{T}^{\alpha\beta}_{GLW}(x) \quad \Rightarrow$ **GLW spin tensor is conserved**

GLW and canonical tensors are connected by the pseudo-gauge transformation.

These are equations defining perfect-fluid hydrodynamics with spin (11 equations) which may be solved numericaly and confronted with the data

W. Florkowski, A. Kumar, R. R., Phys. Rev. C 98 (2018) 044906
W. Florkowski, A. Kumar, R. R., R.Singh, Phys.Rev. C99 (2019) no.4, 044910
W. Florkowski, A. Kumar, R. R., [1811.04409]

CLASSICAL TREATMENT OF SPIN

Internal angular momentum tensor

Lets introduce **internal angular momentum tensor**
M. Mathisson, APPB 6 (1937) 163–2900

$$\boxed{s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta}p_{\gamma}s_{\delta}}$$

Since $s \cdot p = 0$ the spin four-vector is

$$s^{\alpha} = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta}p_{\beta}s_{\gamma\delta} \qquad \overset{PRF\ \left[p^{\mu}=(m,0,0,0)\right]}{\Longrightarrow} \qquad s^{\alpha} = (0, \boldsymbol{s}_{*})$$

For spin-1/2 particles the length of $\boldsymbol{s}_{*}$ is given by the value of the Casimir operator,

$$\boxed{|\boldsymbol{s}_{*}|^2 = \not{\mathfrak{s}}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}}$$

Angular momentum conservation

identify collisional invariants of the Boltzmann eq. (BE) $\Rightarrow$ construct the equilibrium distribution

Apart from four-momentum and charge(s) one can include the **total angular momentum** conservation

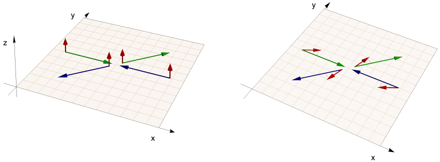
$$j_{\alpha\beta} = l_{\alpha\beta} + s_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha + s_{\alpha\beta}$$

The **locality of the standard BE** suggests that the **orbital part can be eliminated**, and **the spin part can be considered separately**.

For elastic binary collisions of particles 1 and 2 going to $1'$ and $2'$, this suggests that

C. G. van Weert, Henkes- Holland N.V. – Haarlem, 1970.

$$s_1^{\alpha\beta} + s_2^{\alpha\beta} = s_{1'}^{\alpha\beta} + s_{2'}^{\alpha\beta}$$



These equations admit two types of simple solutions: either the **sum of two spin three-vectors or their difference (before and after the collision) vanishes** (interpreted as collisions in the spin **singlet** and **triplet** states)

Departure from the locality assumption is most likely a necessary condition to include dissipative interactions which eventually could make the spin polarization tensor equal to the thermal vorticity.

S. Hess, L. Waldmann, Zeitschrift fuer Naturforschung 21a (1966) 1529

A. Jaiswal, R. S. Bhalerao, S. Pal, PLB 720 (2013) 347–351

Spin-dependent distribution function and phase-space measure

One can introduce a **spin-dependent equilibrium distribution functions for particles (+) and antiparticles (-)**
W. Florkowski, R. R. [1811.04409] to appear in *Progress in Particle and Nuclear Physics*

$$f^{\pm}_{\rm eq}(x,p) \quad \rightarrow \quad f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p^{\alpha}\beta_{\alpha}(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right)$$

A classical spin dependent distribution function was considered also in the context of deriving the kinetic equation for massive Dirac fermions in electromagnetic fields
J.-W. Chen, J.-y. Pang, S. Pu, Q. Wang, Phys. Rev. D89 (9) (2014) 094003.

Different orientations of spin can be integrated out with the help of a **covariant measure**

$$\int dS \ldots = \frac{m}{\pi \mathfrak{s}} \int d^4s \, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) \ldots$$

The prefactor $m/(\pi\mathfrak{s})$ may be chosen to obtain the **normalization**

$$\int dS = \frac{m}{\pi \mathfrak{s}} \int d^4s \, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) = 2$$

Conserved currents

Conserved currents are obtained from the generalizations of the standard kinetic-theory definitions

$$N^{\mu}_{\rm eq} = \int dP \int dS \; p^{\mu} \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s) \right]$$

$$T^{\mu\nu}_{\rm eq} = \int dP \int dS \; p^{\mu} p^{\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s) \right]$$

$$S^{\lambda\mu\nu}_{\rm eq} = \int dP \int dS \; p^{\lambda} \, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s) \right]$$

For $|\omega_{\mu\nu}| < 1$ one obtains the formalism that agrees with that based on the quantum description of spin (in the GLW version).

Arbitrarily large values of ω induce momentum anisotropy
(for instance, $T^{\mu\nu}$ has no longer perfect fluid form)
$\Downarrow$
anisotropic hydrodynamics with spin

W. Florkowski, R. R., PRC 83 (2011) 034907. [1007.0130]
M. Martinez, M. Strickland, NPA 848 (2010) 183–197. [1007.0889]

Pauli-Lubański vector

For **arbitrary values of the polarization** one gets

$$\pi_\mu = -\mathfrak{s}^2 \frac{\tilde{\omega}_{\mu\beta} p^\beta}{m} \frac{L(P\mathfrak{s})}{P\mathfrak{s}}$$

In PRF

$$\pi_*^0 = 0, \quad \boldsymbol{\pi}_* = -\mathfrak{s}^2 \boldsymbol{P} \frac{L(P\mathfrak{s})}{P\mathfrak{s}}$$

For small and large P one obtains two important results:

$$\boldsymbol{\pi}_* = -\mathfrak{s} \frac{\boldsymbol{P}}{P}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s} = \sqrt{\frac{3}{4}}, \quad \text{if} \quad P \gg 1$$

The normalization of the PL vector cannot exceed the value of $\mathfrak{s}$.

$$\boldsymbol{\pi}_* = -\mathfrak{s}^2 \frac{\boldsymbol{P}}{3}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s}^2 \frac{P}{3} = \frac{P}{4}, \quad \text{if} \quad P \ll 1$$

For small values of P the classical treatment of spin reproduces the quantum mechanical result

PHENOMENOLOGICAL DERIVATION

Hydrodynamics with spin based on entropy-current analysis

Phenomenological derivation of hydrodynamics based on the conservation laws

K. Hattori, M. Hongo, X-G Huang, M. Matsuo, H. Taya, PLB 795 (2019) 100-106

$$\partial_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} \equiv T^{\mu\nu}_{(s)} + T^{\mu\nu}_{(a)}$$
$$\partial_\mu J^{\mu\alpha\beta} = 0 \qquad J^{\mu\alpha\beta} = \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\right) + \Sigma^{\mu\alpha\beta}$$

Dynamical variables near local thermal equilibrium are assumed to satisfy the first law of thermodynamics generalized with finite spin density S. (spin potential ω is conjugate to the spin density S)

$$Ts = e + p - \omega_{\mu\nu} S^{\mu\nu}, \qquad Tds = de - \omega_{\mu\nu} dS^{\mu\nu}$$

One may organize the constitutive relations on the basis of a derivative expansion $T^{\mu\nu}_{(1)} \sim \mathcal{O}(\partial^1)$

$$T^{\mu\nu} = e u^\mu u^\nu + p \Delta^{\mu\nu} + T^{\mu\nu}_{(1)}$$
$$\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + \Sigma^{\mu\alpha\beta}_{(1)}$$

Lowest-order hydrodynamic equations of motion do not conserve the lowest-order entropy current (in contrast to the case of a fluid without spin)

$$\partial_\mu s^\mu_{(0)} = 2\beta \omega_{\alpha\beta} T^{\alpha\beta}_{(1a)}$$

The entropy production implies that spin density is inherently a dissipative quantity

$\Downarrow$

no counterpart of ideal spin-less hydrodynamics ?

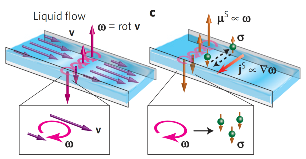
new transport coefficients appear that control the relaxation time of spin density (rotational viscosity, boost heat conductivity)

$\Downarrow$

relativistic generalization of a non-relativistic micropolar hydrodynamics

R. Takahashi, et al. Nat. Phys. 12(1) (2016) 52

M. Matsuo, Y. Ohnuma, S. Maekawa, PRB 96 (2017) 020401



1st order dissipative corrections

$\Downarrow$

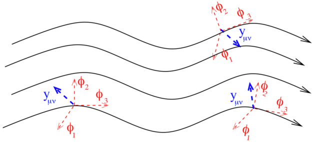
causality, stability?

EFFECTIVE FIELD THEORY APPROACH

Lagrangian formulation of relativistic fluid mechanics for spin-polarized systems

D. Montenegro, G. Torrieri, Phys.Rev. D94 (2016) no.6, 065042
D. Montenegro, L. Tinti, G. Torrieri, Phys. Rev. D 96(5) (2017) 056012; Phys. Rev. D 96(7) (2017) 076016
D. Montenegro, G. Torrieri, [1807.02796]

One constructs lagrangian which contains the information of the EoS including an entropy term derived from the fluid coordinate d.o.f. $b = \left(\det_{IJ}\left[\partial_\mu \phi_I \partial^\mu \phi_J\right]\right)^{1/2}$ **as well as polarization tensor** $y^{\mu\nu}$



D. Montenegro, R. Ryblewski, G. Torrieri, [1903.08729]

For small polarizations, the EoS reduces to

$$\mathcal{L} = F(b, y) = F\left(b\left(1 - c y_{\mu\nu} y^{\mu\nu}\right)\right)$$

$c > 0$ - ferromagnet, $c < 0$ - antiferromagnet

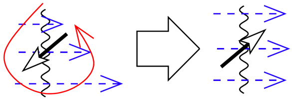
The ideal limit of hydrodynamics with polarization is generally non-causal (due to Ostrogradski theorem)

$$y_{\mu\nu} = \chi\left(b, \Omega_{\mu\nu}\Omega^{\mu\nu}\right)\Omega_{\mu\nu}, \qquad \Omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]}$$

Causality can be cured by using Israel-Stewart type lagrangian, written in doubled coordinates and employing non-equilibrium polarization d.o.f. Y giving Maxwell-Cataneo type relation

$$\tau_Y \partial_\tau \delta Y_{\mu\nu} + \delta Y_{\mu\nu} = y_{\mu\nu}$$

causality can be fixed by a relaxation type term
$\Downarrow$
any material with a non-zero spin must have a minimum amount of dissipation



Other developments

- world-line approach to constructing phase space distributions in systems with internal symmetries (color and spin d.o.f.) using Grassmann algebra
  N. Mueller, R. Venugopalan, Phys.Rev. D99 (2019) no.5, 056003
- spin current hydrodynamics in holographic first order gravity
  Banados, Miskovic and Theisen (2006), Cvetkovic, Miskovic and Simic (2017), D. Gallegos, U. Gursoy (2019)
- quantum kinetic equation for spin polarization of massive quarks from perturbative QCD
  Shiyong Li, Ho-Ung Yee , [1905.10463]
- kinetic models of spin dynamics
  J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 109 (2012) 232301
  J.-W. Chen, S. Pu, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 110 (26) (2013) 262301
  R.-H. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, Phys. Rev.C94 (2) (2016) 024904
  R.-H. Fang, J.-Y. Pang, Q. Wang, X.-N. Wang, Phys. Rev. D95 (1) (2017) 014032
- anomalous hydrodynamics
  D. T. Son, P. Surowka, Phys. Rev. Lett. 103 (2009) 191601
  D. E. Kharzeev, D. T. Son, Phys.Rev. Lett. 106 (2011) 062301
- study of Wigner equation for massive fermions including the EM field
  J.H. Gao, and Z.T. Liang, [1902.06510]
  N. Weickgenannt, X.l. Sheng, E. Speranza, Q. Wang, D. Rischke, [1902.06513]
  Z. Wang, X. Guo, S. Shi and P. Zhuang, [1903.03461]
  K. Hattori, Y. Hidaka, DiLun Yang, [1903.01653]
- and others ...

Conclusions and Summary

- At the moment the differential polarization data is a chalange for theory
- It is established that global equilibrium polarization is given by thermal vorticity
- Fails and successes of the current theories suggest that polarization evolves independently of the thermal vorticity and maybe finally converges to it
- Descriptions with different pseudo-gauge tensor pairs are equivalent only in global equilibrium – they should give different values of physical observables
- GLW forms of the energy-momentum and spin tensors are natural candidates for formulating hydrodynamic equations (symmetric energy-momentum tensor; the spin tensor strictly conserved; follow from kinetic theory)
- Separate conservation of the spin tensor may be traced back to the locality of the collision term. Departure from the locality is necessary to include dissipative effects.
- The inclusion of dissipation is an obvious demand for future studies. Some work in this direction has been already done.
- Future looks exciting!

Thank you for your attention!