Toward a full mapping of the hydro response to initial conditions

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Initial → Final Mapping

In usual hydro simulations, the distribution of energy/momentum when hydro becomes valid ("initial condition") completely determines the final result, which is a distribution function for independent hadrons.

$$T^{\mu\nu}(\tau=\tau_0,\vec{x}) \to E\frac{dN}{d^3p}\Big|_{\text{final}} = \frac{N(p_T,Y)}{2\pi}\sum_{n=-\infty}^{\infty} V_n(p_T,Y)e^{-in\phi}$$

This complicated process can be represented by relatively simple relations, such as

$$\boxed{V_n = \kappa\varepsilon_n} \qquad (1)$$

that identify the relevant properties of the initial state (ε_n) and separate them from the effects of subsequent evolution (κ). This allows for a deeper insight into the system dynamics and is a powerful way to constrain physical properties from measured observables.

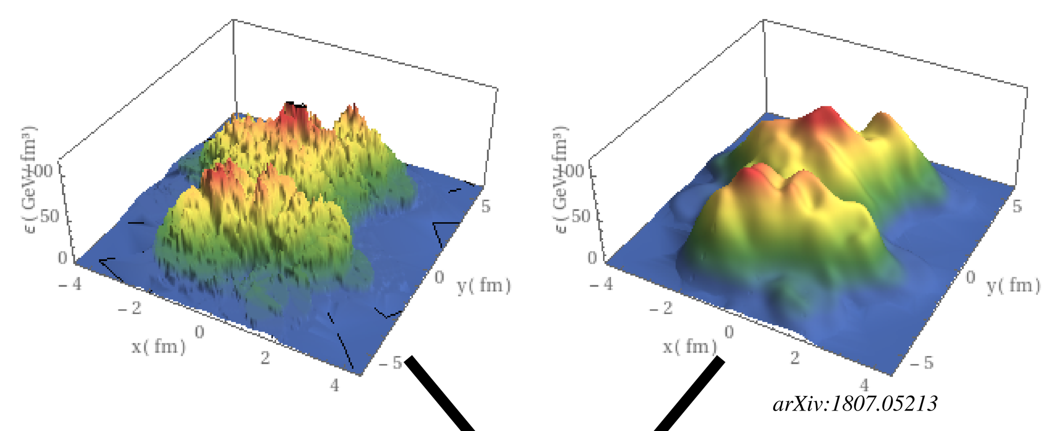
Normally, only the geometric distribution of energy (or entropy) is considered. We propose a way to incorporate the other components of $T^{\mu\nu}$ that comprise initial conditions.

These may be particularly important in small systems.

First we revisit the known case, so we can use the same considerations to construct estimators that include new contributions.A similar approach should allow for inclusion of, e.g., conserved currents in the future.

Eccentricity Scaling as a Controlled Expansion in Length Scales

The most important guideline is the ansatz that large-scale structures in the initial condition contribute more to the final particle distribution than structures at small scale



similar global structure ⟹ similar final state

The natural way to separate scales is with a Fourier transform.

$$\rho(\vec{x}) \equiv T^{\tau\tau}(\tau=\tau_0,\vec{x}) \qquad (2)$$
$$e^{W(\vec{k})} = \frac{1}{2\pi}\int d^2x e^{i\vec{x}\cdot\vec{k}}\rho(\vec{x})$$
$$W(\vec{k}) = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} W_{n,m}k^m e^{-in\phi_k}$$

Cumulants $W_{n,m}$ with smaller m represent larger length scales and have larger contribution to the final particle distribution. Cumulants above some maximum m can be disregarded.

To predict dimensionless flow coefficients, it is useful to compare to a common scale to make dimensionless ratios. A natural choice for this scale is the system size $W_{0,2} = \langle r^2\rangle_\epsilon - |\langle r^{i\phi}\rangle_\epsilon|^2$:

$$\varepsilon_{n,m} \equiv -\frac{W_{n,m}}{(W_{0,2})^{\frac{m}{2}}}$$

We can construct an estimator for each harmonic V_n as a power series in these cumulants:

$$V_n \simeq \sum_{m=n}^{m_{max}}\kappa_{n,m}\varepsilon_{n,m} + \sum_{l=1}^{m_{max}}\sum_{m=l}^{m_{max}}\sum_{m'=|n-l|}^{m_{max}}\kappa_{l,m,m'}\varepsilon_{l,m}\varepsilon_{n-l,m'} + O(W^3).$$

The lowest cumulant corresponding to rotational mode n is $W_{n,n}$, and so the largest contribution is typically $\varepsilon_n \equiv \varepsilon_{n,n}$.

Eccentricity scaling (1) means that only global, large-scale structure is important, and small-scale granularity is negligible.

Note that these arguments are quite general, and this behavior may apply even if hydrodynamics is not applicable. The essential assumptions are

1. The energy momentum tensor at some early time $T^{\mu\nu}(\tau=\tau_0)$ contains sufficient information to predict future evolution.
2. The final particle properties can be captured by a one body distribution for each event.
3. The structure of $T^{\mu\nu}(\tau=\tau_0)$ at large length scales generally has a larger affect on the final results than small-scale structure.

Including other $T^{\mu\nu}$ components

Using the same considerations as for the case of energy density only, we can include other components of the initial $T^{\mu\nu}$.

Here we focus on the contribution from the transverse momentum density $T^{\tau i}$ and stress tensor T^{ij}, with $i,j\in(1,2)$.

The proposal is to add terms to Eq. (2), and generate the cumulant expansion exactly as before.

$$\boxed{\rho(\vec{x}) = T^{\tau\tau}(\vec{x}) + \alpha\partial_i T^{\tau i}(\vec{x}) - \beta\partial_i\partial_j T^{ij}(\vec{x})} \qquad (3)$$

This ansatz introduces only two unknown response coefficients α, β, which are constants that do not depend on harmonic n or order m, for example, and should also depend only weakly on centrality. Thus it is a highly restrictive and nontrivial ansatz.

This gives estimators with the correct expected properties:

- Ordered in length scale
- Translation invariant
- Decomposed into definite rotational modes n
- In the limit when the extra components of $T^{\mu\nu}$ are uniformly scaled to 0, $\alpha,\beta\to 0$, the contribution of these terms vanish.*

**(Note that one can have energy density without momentum density or stress, but one cannot have the latter without energy density. In contrast to this criterion, anisotropic flow does not necessarily go smoothly to zero as the initial energy is scaled to zero.)*

An example of the result for the leading order term ϵ_n, in a particular centered coordinate system $W_{1,1}(\alpha,\beta)=0$, is:

$$\varepsilon_2(\alpha,\beta) = -\frac{\left\langle r^2e^{i2\phi}\right\rangle_\epsilon - 2\alpha\left\langle re^{i\phi}\right\rangle_U - 4\beta\langle 1\rangle_C}{\left\langle r^2\right\rangle_\epsilon - \left|\left\langle re^{i\phi}\right\rangle_\epsilon\right|^2}$$
$$\varepsilon_3(\alpha,\beta) = -\frac{\left\langle r^3e^{i3\phi}\right\rangle_e - 3\alpha\left\langle r^2e^{i2\phi}\right\rangle_U - 12\beta\left\langle r^{i\phi}\right\rangle_C}{\left(\left\langle r^2\right\rangle_\epsilon - \left|\left\langle re^{i\phi}\right\rangle_\epsilon\right|^2\right)^{\frac{3}{2}}}$$

using the following notation

$$U(\vec{x}) \equiv T^{\tau x} + iT^{\tau y}$$
$$C(\vec{x}) \equiv \frac{1}{2}(T^{xx} - T^{yy}) + iT^{xy}$$
$$\langle\ldots\rangle_\epsilon = \frac{\int d^2x\ldots T^{\tau\tau}(\vec{x})}{\int d^2x\ldots T^{\tau\tau}(\vec{x})}$$
$$\langle\ldots\rangle_U = \frac{\int d^2x\ldots U(\vec{x})}{\int d^2x\ldots T^{\tau\tau}(\vec{x})}$$
$$\langle\ldots\rangle_C = \frac{\int d^2x\ldots C(\vec{x})}{\int d^2x\ldots T^{\tau\tau}(\vec{x})}$$

The usual quantities are obtained for $\alpha,\beta\to 0$.

Physical Motivation

Let's return for the moment to the case where only initial energy density contributes, such that the evolution beginning at some time τ_0 depends on the distribution of energy density only.

We know that estimators derived from Eq. (2) give an excellent description of the final results (even momentum-dependent flow fluctuations — **see poster by Mauricio Hippert**).

Now imagine that we don't know the value of τ_0, and we instead generate estimators from the state of the system a short time before or after, $\tau = \tau_0 + \delta\tau$:

$$T^{\tau\tau}(\tau) = T^{\tau\tau}(\tau_0) + \delta\tau\partial_\tau T^{\tau\tau}|_{\tau_0} + \frac{\delta\tau^2}{2}\partial_\tau^2 T^{\tau\tau}|_{\tau_0} + O(\delta\tau^3)$$

The final state of the system is the same, and so our estimator should also be approximately unchanged.

Conservation of energy/momentum tells us that, to first order, any change in energy density is compensated by momentum density. We can continue to second order, which adds the stress tensor.

$$\partial_\tau T^{\tau\tau} = -\partial_i T^{\tau i}$$
$$\partial_\tau^2 T^{\tau\tau} = -\partial_i\partial_\tau T^{\tau i} = \partial_i\partial_j T^{ij}$$

Thus, our original scalar ρ can be approximated by these terms

$$\rho(x) = T^{\tau\tau}(\tau_0) \simeq T^{\tau\tau}(\tau) + \delta\tau\partial_i T^{\tau i}(\tau_0) - \frac{\delta\tau^2}{2}\partial_i\partial_j T^{ij}(\tau_0)$$

So, while we can't make a rigorous derivation of, e.g., the values of α and β, which clearly depend on the system, it is natural to construct a field with a some of the three quantities in the form of Eq. (3) ($T^{\tau\tau}, \partial_i T^{\tau i}, \partial_i\partial_j T^{ij}$) to make an estimator for the final flow.

While not all transverse degrees of freedom of $T^{\mu\nu}$ appear, these three quantities can be expected to be the most important.

Numerical Validation

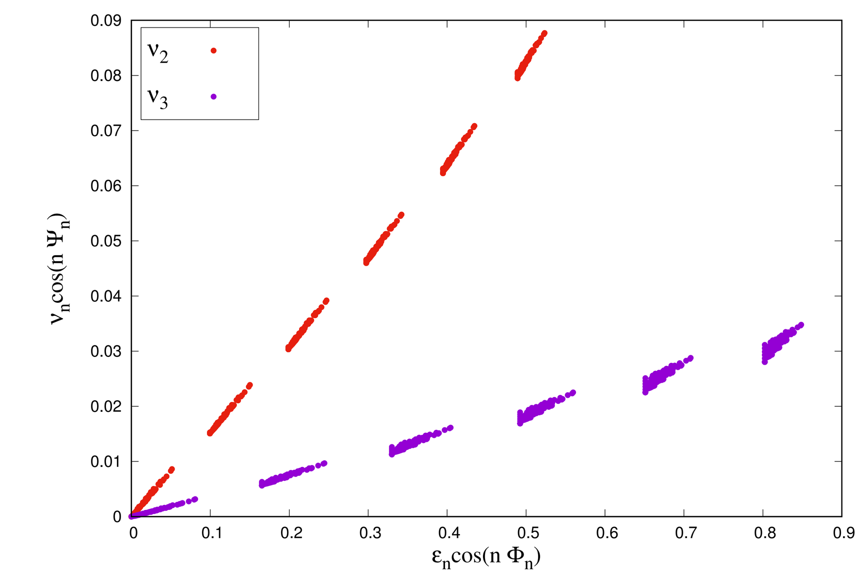
Toy Initial Conditions

We start with a simple Gaussian distribution, which can be distorted in various ways to obtain a rotational asymmetry in mode n in the distribution of energy density, magnitude and direction of momentum density, and magnitude and orientation of stress.

$$T^{\tau\tau} = Ae^{-\frac{r^2}{2\sigma_\epsilon^2}[1+a_n\cos(n\phi)]}$$
$$|U| = rBe^{-\frac{r^2}{2\sigma_U^2}[1-b_n\cos(n\phi)]}$$
$$\arg U = \phi_U = \phi - c_n\sin(n\phi)$$
$$|C| = r^2Pe^{-\frac{r^2}{2\sigma_C^2}(1-p_n\cos(n\phi))}$$
$$\arg C = \theta = \phi - q_n\sin(n\phi)$$

We follow the initial conditions with viscous hydrodynamics and calculate flow of direct pions. We find an approximate linear relation between V_n and the new $\epsilon_n(\alpha,\beta)$.

As an example,we show a scatter plot testing the contribution from the stress tensor. I.e., for parameters in ranges $a_n\in[0,0.5]$, $P\in[0,10]fm^{-6}$, $p_n\in[0,0.4]$, $q_n\in[0,0.8]$, and with$(\alpha,\beta)=(0,5fm^2)$.



We find that the lowest order estimator (1) gives in general a very good description of the results (until non-linearities appear, when the size and/or asymmetry of the extra components of $T^{\mu\nu}$ become large).

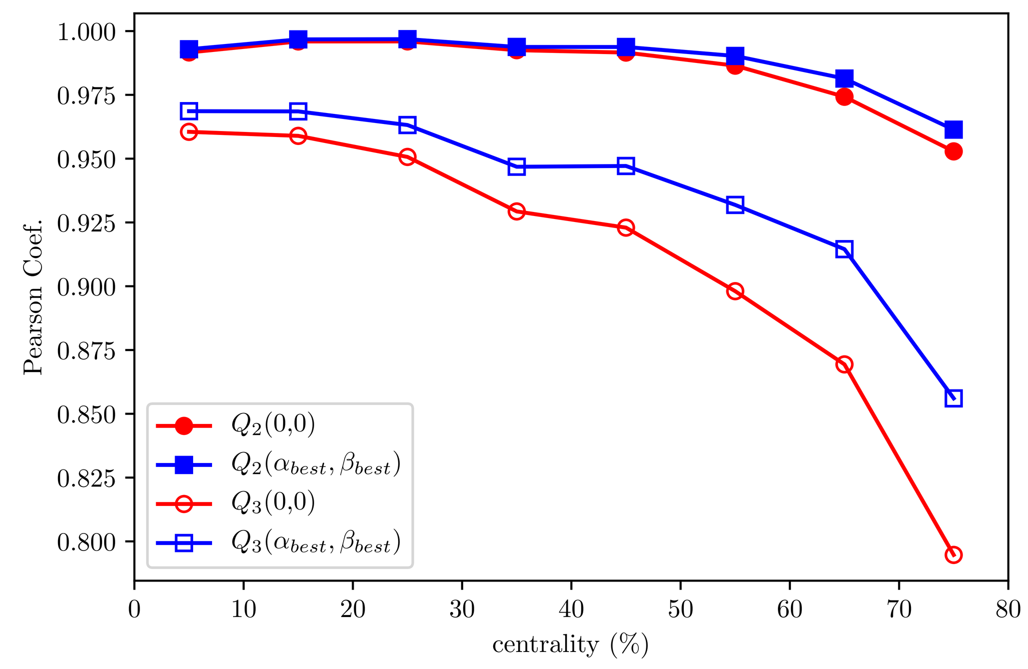
IP-Glasma Initial Conditions

To test whether the size and asymmetries of the momentum density and stress that we tested are realistic, we perform state-of-the-art simulations using IP-Glasma initial conditions of Pb-Pb collisions, viscous hydrodynamics, and UrQMD afterburner. We take the fluid properties from a previous Bayesian analysis (see thesis of Jonah Bernhard, arXiv:1804.06469).

The success of an estimator (ε_n) can be measured by the linear (Pearson) correlation coefficient between the estimator and the final flow vector V_n over the ensemble of events

$$Q_n = \frac{\mathrm{Re}\,\langle V_n\varepsilon_n^*\rangle}{\sqrt{\left\langle|V_n|^2\right\rangle\left\langle|\varepsilon_n|^2\right\rangle}}$$

Here we have the result with $(\alpha,\beta)=(1fm,15fm^2)$ compared to the original estimator $(\alpha,\beta)=(0,0)$



We can see that, while the majority of flow in this large system is generated as a response to the spatial anisotropy of the energy density, **including the effects of other components of the energy-momentum tensor improves the estimator**.

Additionally, approximately the **same value of** α, β (1 fm, 15 fm^2) maximize the Pearson coefficient for v_2 and v_3 and for all centralities, giving validation to our nontrivial ansatz (3).

With this information, it should be possible to estimate the importance of initial flow/correlations in a given system or model and to better understand the interplay between this initial flow/correlation and hydrodynamic evolution. This in turn can allow to constrain properties of the initial state and the QGP.