Non-linear evolution in QCD at high-energy beyond leading order

Bertrand Ducloué
(IPhT Saclay)

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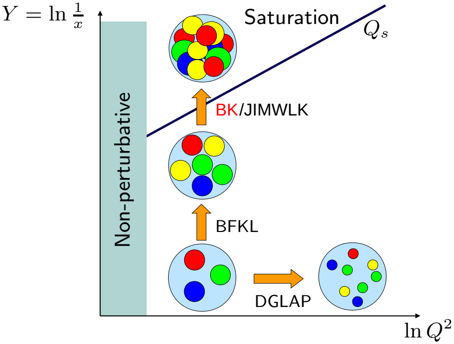
B. D., E. Iancu, A.H. Mueller, G. Soyez, D.N. Triantafyllopoulos, JHEP **1904** (2019) 081 [arXiv:1902.06637]

At high energy, the evolution of gluon densities is governed by:

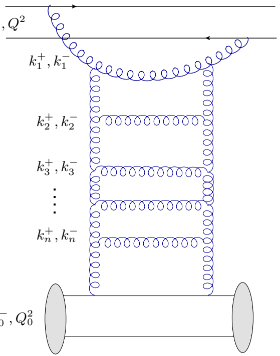
- the BFKL equation in the linear regime
- the BK / JIMWLK equations in the saturation regime



Our goal here is to go beyond the leading order (+ running coupling corrections) approximation used until now in the saturation regime (BK)

The LO BK equation

At high energy, DIS can be viewed as a virtual photon (virtuality Q^2, flying almost along P^+) splitting into a $q\bar{q}$ pair which then interacts eikonally with the target (transverse scale Q_0^2, flying almost along P^-)



Kinematics of interest: $Q^2 \gg Q_0^2$

Leading logarithmic approximation: resum any number of gluons strongly ordered in longitudinal momentum (rapidity)

Can look at the evolution

- in p^-: $q_0^- \gg k_n^- \gg \cdots \gg k_1^- \gg q^-$
  "η evolution": resum $(\alpha_s \eta)^n$
- in p^+: $q^+ \gg k_1^+ \gg \cdots \gg k_n^+ \gg q_0^+$
  "Y evolution": resum $(\alpha_s Y)^n$

The corresponding rapidity intervals are:

$$\eta = \ln \frac{q_0^-}{q^-} = \ln \frac{s}{Q^2} = \ln \frac{1}{x_{\rm Bj}}$$

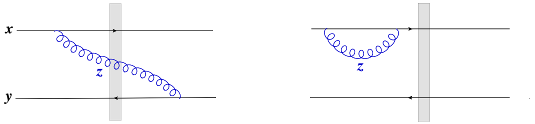
$$Y = \ln \frac{q^+}{q_0^+} = \ln \frac{s}{Q_0^2} = \eta + \ln \frac{Q^2}{Q_0^2} \equiv \eta + \rho > \rho$$

Note that the difference between Y and η is relevant only at NLO and beyond

Resummation of all soft emissions: Balitsky-Kovchegov (BK) equation:

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})\,, \qquad S_{\boldsymbol{xy}} \equiv \frac{1}{N_c}\left\langle \mathrm{Tr}\, U_{\boldsymbol{x}} U_{\boldsymbol{y}}^\dagger \right\rangle$$

Possibility for a parent dipole with size $r = |\boldsymbol{x} - \boldsymbol{y}|$ to emit two daughter dipoles with sizes $|\boldsymbol{x} - \boldsymbol{z}|, |\boldsymbol{z} - \boldsymbol{y}|$ or to remain intact



Starting with a given initial condition at $Y = 0$ (e.g. the simple GBW model $S^{(0)}_{\boldsymbol{xy}} = e^{-(\boldsymbol{x}-\boldsymbol{y})^2 Q_0^2}$), solve the BK equation numerically to larger rapidities

Can then compute various observables, e.g. $F_L(x_{\mathrm{Bj}}, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}\sigma_L(x_{\mathrm{Bj}}, Q^2)$

with $\sigma_L(x_{\mathrm{Bj}}, Q^2) \propto \sum_f e_f^2 \int \mathrm{d}z_1 \mathrm{d}^2\mathbf{r}\, Q^2 z_1^2 (1-z_1)^2 K_0^2\left(Q\sqrt{z_1(1-z_1)\mathbf{r}^2}\right)\left(1 - S_{\mathbf{r}}\left(Y = \ln\frac{1}{x_{\mathrm{Bj}}}\right)\right)$

The NLO BK equation in Y

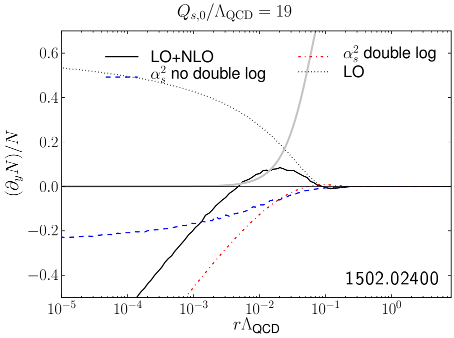
NLO BK for Y evolution has been derived by Balitsky, Chirilli:

$$
\begin{aligned}
\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right) \\
\times \Bigg\{ 1 + \bar{\alpha}_s \Bigg[\bar{b} \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2 - (\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2} \\
+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \Bigg] \Bigg\} \\
+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{u}\, \mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4} \left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right) \\
\times \Bigg\{ -2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \\
+ \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \right] \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \Bigg\}
\end{aligned}
$$

- green: leading order
- violet: running coupling corrections
- red: double collinear logarithm
- blue: single collinear logarithm (DGLAP)

The NLO BK equation in Y

First numerical solution of the NLO BK equation (Lappi, Mäntysaari):



Very large and negative NLO corrections which make the evolution unstable

The main source of the instability is the collinear double log. A similar issue arises with NLO BFKL. Solved long time ago by a resummation to all orders (Salam et al.)

Physical origin of the instability: time ordering problem

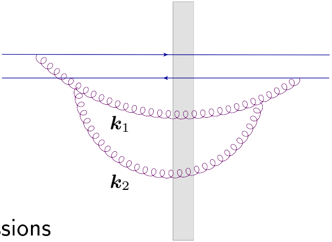
Two successive emissions $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ must have ordered lifetimes $\tau \sim \frac{1}{p^-} \sim \frac{p^+}{p_\perp^2}$

i.e. one should have ordering in both p^+ and p^-

Typical Y evolution: $k_1^+ > k_2^+$, $k_{1\perp} \gtrsim k_{2\perp}$

$\rightarrow$ the ordering $\frac{k_1^+}{k_{1\perp}^2} > \frac{k_2^+}{k_{2\perp}^2}$ is not guaranteed

The ordering in lifetimes is not automatic and must be imposed by hand ("kinematical constraint")



Problem with large daughter dipoles, i.e. small $k_\perp$ emissions

The double logs can be resummed in several ways:

- A non-local equation, similar to "kinematically-improved BK" proposed by Beuf

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[S_{\boldsymbol{xz}}(Y-\Delta_{\boldsymbol{xz};r})S_{\boldsymbol{zy}}(Y-\Delta_{\boldsymbol{zy};r})-S_{\boldsymbol{xy}}(Y)\right]$$

with $\Delta_{\boldsymbol{xz};r} \sim \ln\dfrac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}$ when $(\boldsymbol{x}-\boldsymbol{z})^2 \gg r^2$ and $\Delta_{\boldsymbol{xz};r} \to 0$ when $(\boldsymbol{x}-\boldsymbol{z})^2 \ll r^2$

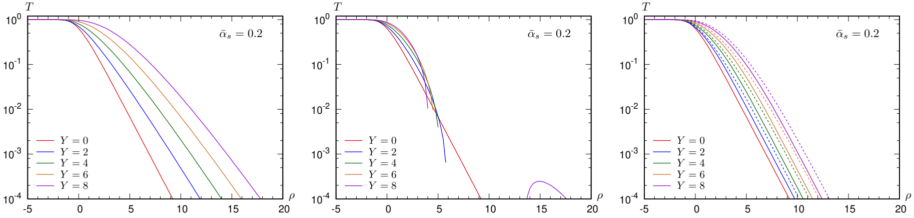
- A local equation with a modified kernel (Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos)

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\, \mathcal{K}_{\mathrm{DLA}}\left(\sqrt{\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}}\right) (S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})$$

with $\mathcal{K}_{\mathrm{DLA}}(\rho) = \mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)/\sqrt{\bar{\alpha}_s\rho^2}$. Expansion in powers of $\bar{\alpha}_s$:

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left(1 - \frac{\bar{\alpha}_s}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} + \cdots\right) (S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})$$

The resummation of the double logs indeed makes the evolution stable:



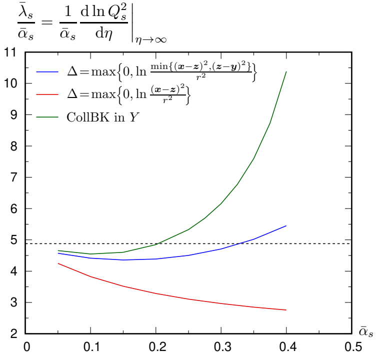
LO | LO+double logs | LO+resummation
solid: non-local
dashed: modified kernel

Good fits to HERA data obtained with these resummations
(Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos; Albacete)

But not consistent:

- Evolved over the rapidity interval η instead of $Y = \eta + \rho$
- Did not treat the initial condition properly: only values $Y > \rho$ are physical ($\Leftrightarrow x_{\text{Bj}} < 1$). These equations should not be solved with a standard GBW or MV-like initial condition at $Y = 0$
- Very large scheme dependence

The results for quantities such as the saturation exponent show a very large resummation scheme dependence when expressed as a function of the target rapidity $\eta = \ln 1/x_{\rm Bj}$:

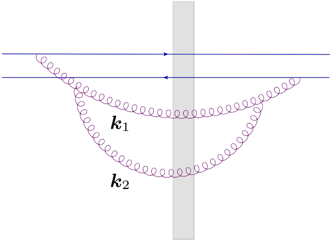


The resummed evolution is stable, but it lacks predictive power

Because of these issues, it appears that working in Y is not the best choice

This is confirmed by looking at the typical evolution in η : $k_1^- < k_2^-$, $k_{1\perp} \gtrsim k_{2\perp}$

$\Rightarrow \frac{k_{1\perp}^2}{k_1^-} > \frac{k_{2\perp}^2}{k_2^-} \Leftrightarrow k_1^+ > k_2^+$: both p^+ and p^- are correctly ordered for the typical η evolution



This is in contrast to what happens for Y evolution and motivates the use of η as the "right" variable

In addition, for η evolution the initial condition at $\eta = 0$ is just the physical IC (GBW, MV, ...)

Why did we start with Y evolution?

Collinear-improved BK in η

The NLO BK equation was derived for Y evolution (Balitsky, Chirilli):

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$- \frac{\bar{\alpha}_s^2}{4\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$+ \bar{\alpha}_s^2 \times \text{“regular”}.$$

But we can obtain NLO BK in η with the change $Y \to \eta + \rho$. At NLO:

- Such a change only affects the LO piece. In the $\mathcal{O}(\bar{\alpha}_s^2)$ terms we can just replace $S(Y) \to \bar{S}(\eta)$
- We can use LO BK to evaluate $\partial \bar{S}_{\boldsymbol{xz}}(\eta)/\partial\eta$ in

$$S_{\boldsymbol{xz}}(Y) = S_{\boldsymbol{xz}}(\eta+\rho) \equiv \bar{S}_{\boldsymbol{xz}}\left(\eta + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right) \simeq \bar{S}_{\boldsymbol{xz}}(\eta) + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\frac{\partial \bar{S}_{\boldsymbol{xz}}(\eta)}{\partial\eta}$$

This leads to NLO BK for η evolution:

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$- \frac{\bar{\alpha}_s^2}{4\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$+ \frac{\bar{\alpha}_s^2}{2\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}^2\boldsymbol{u}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\bar{S}_{\boldsymbol{xu}}(\eta)\left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{uy}}(\eta)\right]$$

$$+ \bar{\alpha}_s^2 \times \text{“regular”}$$

The extra term (3rd line) coming from the change $Y \to \eta$ cancels the double logs for large daughter dipoles

But this term creates new large double logs for small daughter dipoles!

Such atypical configurations are allowed by BFKL diffusion

Similarly to Y evolution, the large double logs can be resummed by a rapidity shift in the LO piece:

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{xz};r})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zy};r}) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

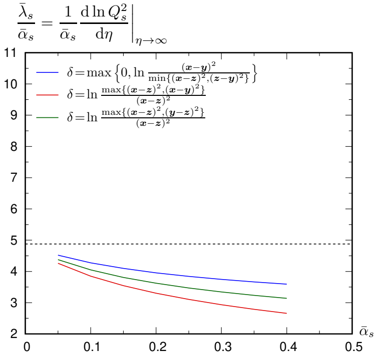
But this time the resummation only affects small daughter dipoles:

- $\delta_{\boldsymbol{xz};r} \sim \ln \frac{r^2}{(\boldsymbol{x}-\boldsymbol{z})^2}$ when $(\boldsymbol{x}-\boldsymbol{z})^2 \ll r^2$
- $\delta_{\boldsymbol{xz};r} \to 0$ when $(\boldsymbol{x}-\boldsymbol{z})^2 \gg r^2$

This is the equation we will study numerically in the following

The missing $\mathcal{O}(\bar{\alpha}_s^2)$ terms can be added (being careful to avoid double counting) to get full NLO accuracy + resummation of double logs

Saturation exponent as a function of the coupling for different resummations:

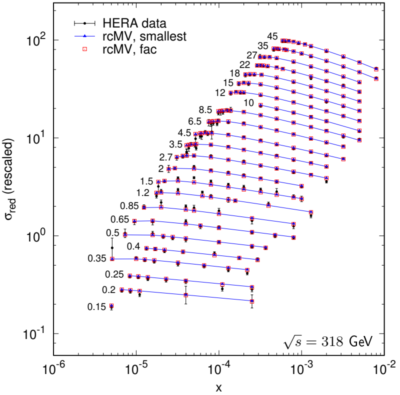


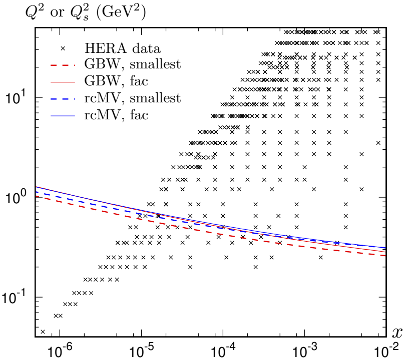
The results depend much less on the scheme than when considering Y evolution

This is likely due to the fact that the double logs do not become large for the typical evolution

Expect an even smaller dependence after adding the missing $\mathcal{O}(\bar\alpha_s^2)$ terms

Comparison with inclusive HERA DIS data (LO γ^* impact factor + evolution in η with resummation of double and single logs + running coupling):





Good description of the data ($\chi^2/\mathrm{ndf} < 1.2$) with two types of initial conditions (GBW and rcMV) and two different prescriptions for the running coupling (smallest dipole prescription and fastest apparent convergence)

Conclusions

NLO equation in Y: unstable because of double collinear logarithms which become large for the typical evolution. These logs can be resummed but:

- Not a simple initial condition problem
- Large scheme dependence

On the contrary, η is the "right" variable to consider the evolution: also double logarithms, but they become large only for the atypical evolution:

- Milder instability
- Small resummation scheme dependence
- Initial condition problem formulated at $x_{\mathrm{Bj}} = 1$

We propose a non-local equation for η evolution which resums the double logarithms to all orders and can be promoted to full NLO accuracy

LO BK supplemented by the resummation of double and single logs: good description of inclusive HERA data